Equations of Motion of Charged Particles in Insertion Devices

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In this report the equations of motion of charged particles subject to magnetostatic fields of Insertion Devices (IDs) are derived and solved for specific cases within the linear approximation.

INTRODUCTION

Third generation synchrotron rings are populated with various Insertion Devices (IDs) that produce radiation with tailored characteristics. These devices generate complex field profiles and may have a profound impact on beam dynamics. Their effects maybe be linear, (e.g. tune shift, closed-orbit distortions, transverse coupling, etc) as well as non-linear, such as reduction of dynamical aperture or degradation of injection efficiency.

LORENTZ FORCE

The time evolution of a particle with charge e in the presence of a magnetostatic field $\vec{B} = (b_x, b_y, b_z)$ is governed by the Lorentz Force,

$$\frac{d\vec{p}}{dt} = e\vec{v} \times \vec{B} \tag{1}$$

expressed in the MKS system. Neglecting reactive forces, as was done in the equation above, the particle's kinematic mass is invariant and the equation of motion can be written as

$$\frac{d\vec{v}}{dt} = \frac{ec^2}{E}\vec{v} \times \vec{B} \tag{2}$$

Instead of using the time t as the independent parameter, a useful choice for circular accelerators is to take the longitudinal coordinate y as independent. This way both horizontal x and vertical z coordinates can be expressed in terms of it. Using the compact notation $\vec{r} = (x(y), y, z(y))$ and the definition $\vec{r}\,' \equiv d\vec{r}/dy$, it is simple to express the acceleration as

$$\frac{d\vec{v}}{dt} = v_y^2 \, \frac{d\vec{r}\,'}{dy} + \frac{dv_y}{dt} \, \vec{r}\,' \tag{3}$$

or, isolating $d\vec{r}\,'/dy$, as

$$\frac{d\vec{r}\,'}{dy} = \frac{ec^2}{v_y E} \left\{ \vec{r}\,' \times \vec{B} - \vec{r}\,' \, (\vec{r}\,' \times \vec{B})_y \right\} \tag{4}$$

The fact that the scalar velocity is constant can be used to substitute the longitudinal velocity v_y by the transverse angles $x' \equiv dx/dy$ and $z' \equiv dz/dy$ appearing in $\vec{r}\,'$: $v_y^{-1} = |\vec{r}\,'|/\beta c$. This substitution leads to

$$\frac{d\vec{r}\,'}{dy} = \alpha|\vec{r}\,'| \left\{ \vec{r}\,' \times \vec{B} - \vec{r}\,' \, (\vec{r}\,' \times \vec{B})_y \right\} \tag{5}$$

with the rigidity parameter $\alpha \equiv ec/\beta E$. This expression is exact, no approximation whatsoever has been made so far. Moreover, the above equation of motion (Eq.M.) works for an arbitrary fields $\vec{B} = (b_x, b_y, b_z)$, not necessarily ones from IDs. The y-component of the vector equation is just an identity. The other two components are

$$\begin{aligned} \frac{dx'}{dy} &= -\alpha \sqrt{1 + x'^2 + z'^2} \, \left\{ (1 + x'^2) b_z - z' b_y - x' z' b_x \right\} \\ \frac{dz'}{dy} &= +\alpha \sqrt{1 + x'^2 + z'^2} \, \left\{ (1 + z'^2) b_x - x' b_y - x' z' b_z \right\} \end{aligned} \tag{6}$$

The parameter $\alpha = 3\times 10^{-5}/E\ [(G\cdot m)^{-1}]$, with E in GeV, can be thought as a small parameter and the equation of motion above can be solved perturbatively in its power expansion. This is reasoning behind P.Ellaume's approach[1] in which he derives an expression for the solution of the equations of motion that is correct up to second order in $1/E$. His approximate solutions form the basis upon which a few software codes implement the effect of IDs on particle tracking calculations. Despite that, we will take a different approach in this report. The term *linear* in our case means expansion in terms of small angle deviation x' and z'.

IDEAL, ERROR-FREE AND REAL INSERTION DEVICES

To go further in solving the general Eq.(7) for the field of a planar ID it is important in our approach to define properly the approximations being used. In order to do that we can distinguish three kinds of IDs to be considered in the formalism: an *ideal ID* is one without any contruction error and no roll-off, in order words, it has an infinite width. This ID has no horizontal field component, $b_x = 0$. In addition, its field obeys a midplane symmetry condition: $b'_z(y, z = 0) = 0$. The 3D field reduces to a 2-component field that can be combined and described in the complex plane by an analytic function

$$b_z + ib_y = \sum_{n=0}^{\infty} b_n(y + iz)^n \tag{7}$$

This complex-plane formalism goes in the lines of the very insightful work of K.Halbach for PPM IDs[2, 3]. In the midplane, the exact solution of the Eqs.M. can be expressed in terms of first and second integrals of $b_z(y)$. Off the midplane a general analytic solution is not known. It is possible to shown that even for an ideal ID a vertical focusing effect does take place simply due to a combination of finite gap and Maxwell's equations in free space.

The second group of IDs is composed of *error-free IDs*. These are IDs with no construction errors but with finite width. This is the one most relevant for us because it is manageable and yet relatively simple to model and solve. It will be the subject of the next sections.

Finally, the third kind: *real IDs*. Fields of these IDs differ from modeling expectations due to construction errors like block's magnetization, positioning, dimension error, and so on. Deformations of the supporting mechanical structure may also have a considerable impact on field quality. The only way to take into account all these effects on beam dynamics calculations is to construct a model ID based on measurement data. It is my personal opinion that robust techniques for this level of modeling refinement are yet to be developed. There are a few tentative trials, though.

LINEAR MAPS OF INSERTION DEVICES

At this point we take it from Eq.(7). As mentioned previously we will proceed by expanding the r.h.s of the equations in powers of small angles and displacements. It will suffice for us to stop in first order:

$$\begin{aligned} \frac{dx'}{dy} &= -\alpha\left\{b_z - z'b_y\right\} \\ \frac{dz'}{dy} &= +\alpha\left\{b_x - x'b_y\right\} \end{aligned} \tag{8}$$

The region of interest for beam dynamics is near the device axis, that is, $x \approx 0$ and $z \approx 0$. In this region the field component b_z is dominant over the other two for planar devices, as both $b_x \propto \sin(k_x x)$ and $b_y \propto \sinh(k_z z)$. This implies that the second term of the r.h.s of equation for dx'/dy is very small compared to the first and it may be dropped. As for the equation of dz'/dy it is harder to compare both r.h.s terms and both should be kept. As a matter of fact the second term proportional to the horizontal angle x' is the one that produces vertical focusing:

$$\begin{aligned} x'(y; \{x_0, z_0, x'_0, z'_0\}) &= -\alpha \int_{-\infty}^{y} ds\ b_z[x(s), s, z(s)] \\ z'(y; \{x_0, z_0, x'_0, z'_0\}) &= +\alpha \int_{-\infty}^{y} ds\ b_x[x(s), s, z(s)] - x'(s)b_y[x(s), s, z(s)] \end{aligned} \tag{9}$$

All important linear perturbations of an ID is contained in the above equation. Particularly, closed-orbit distortions focusing and *transversal coupling*. At this point we neglect all displacements induced by the ID field, that is, we assume that $x(s) = x_0 + x'_0 s$ and $z(s) = z_0 + z'_0 s$. If the field components are expanded in linear order in the initial condition parameters the angles x' and z' are then approximately

$$\begin{aligned}
\Delta x'(y) &= -\alpha \left\{ I_z^{(0)}(y) + (x_0\partial_x + z_0\partial_z) I_z^{(1)}(x,y,z)\Big|_0 + (x'_0\partial_x + z'_0\partial_z) I_z^{(2)}(x,y,z)\Big|_0 \right\} \\
\Delta z'(y) &= +\alpha \left\{ I_x^{(0)}(y) + (x_0\partial_x + z_0\partial_z) I_x^{(1)}(x,y,z)\Big|_0 + (x'_0\partial_x + z'_0\partial_z) I_x^{(2)}(x,y,z)\Big|_0 \right\} - \frac{1}{f_z} z_0
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
I_{x,z}^{(1)}(x,y,z) &\equiv \int_{-\infty}^{y} ds\, b_{x,z}(x,s,z) \\
I_{x,z}^{(2)}(x,y,z) &\equiv \int_{-\infty}^{y} ds\, s\, b_{x,z}(x,s,z)
\end{aligned} \tag{11}$$

and $I^{(0)}(y) = I^{(1)}(0,y,0)$. The effective focalization length f_z has been introduced in order to tale into account vertical focusing. Integrals $I^{(2)}$ are, apart from a change of sign, just the second integrals of the field components. This can be readily checked by integration by parts. But since we are neglecting displacements due to the ID field, we will also drop terms that contain $I^{(2)}$ in order to maintain consistency. We will also drop terms with $I^{(0)}$ since they only induce closed orbit distortion and no focusing or coupling.

The map of the ID is finally given by evaluating the expression above at $y \to \infty$:

$$\begin{pmatrix} x_f \\ x'_f \\ z_f \\ z'_f \end{pmatrix} = \begin{pmatrix} 1 & \ell & 0 & 0 \\ X_{21} & 1 & m_{21} & 0 \\ 0 & 0 & c & d \\ e & f & g & h \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ z_0 \\ z'_0 \end{pmatrix} \tag{12}$$

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