Perform stability analysis for the following discretizations of the heat equation. You can assume a cubic domain. In other words, calculate the 1-step growth rates for:

5. 2D and 3D FTCS (10 pts)

2D FTCS:

$$\frac{u_{a,b}^{n+1} - u_{a,b}^{n}}{\Delta t} = \frac{\alpha_1}{\Delta x^2}\left(U_{a-1,b}^n - 2U_{a,b}^n + U_{a+1,b}^n\right) + \frac{\alpha_2}{\Delta y^2}\left(U_{a,b-1}^n - 2U_{a,b}^n + U_{a,b+1}^n\right)$$

Let $\quad U_{a,b}^n = G^n e^{ika\Delta x} e^{ikb\Delta y}$

$$\frac{(G^{n+1} - G^n)e^{ika\Delta x}e^{ikb\Delta y}}{\Delta t} = \frac{\alpha_1 G^n e^{ikb\Delta y}}{\Delta x^2}\left(e^{ik(a-1)\Delta x} - 2e^{ika\Delta x} + e^{ik(a+1)\Delta x}\right)$$
$$+ \frac{\alpha_2 G^n e^{ika\Delta x}}{\Delta y^2}\left(e^{ik(b-1)\Delta y} - 2e^{ikb\Delta y} + e^{ik(b+1)\Delta y}\right)$$

$$\frac{G^{n+1}}{G^n} = \left[1 + \frac{\alpha_1 \Delta t}{\Delta x^2}(e^{ik\Delta x} - 2 + e^{-ik\Delta x}) + \frac{\alpha_2 \Delta t}{\Delta y^2}(e^{ik\Delta y} - 2 + e^{-ik\Delta y})\right]$$
$$= \left[1 + 2\frac{\alpha_1 \Delta t}{\Delta x^2}(\cos(k\Delta x) - 1) + 2\frac{\alpha_2 \Delta t}{\Delta y^2}(\cos(k\Delta y) - 1)\right]$$

Let $\alpha_1 = \alpha_2, \Delta x = \Delta y$, so $\frac{\alpha_1 \Delta t}{\Delta x^2} = C_1$ and $\frac{\alpha_2 \Delta t}{\Delta y^2} = C_2$

Worst case: cos = -1

$$1 - 8C \le 1$$
$$-2 \le -8C \le 0$$
$$1 \ge 4C \ge 0$$
$$C \le \frac{1}{4}$$

3D FTCS:

$$\frac{u_{a,b,c}^{n+1} - u_{a,b,c}^{n}}{\Delta t} = \frac{\alpha_1}{\Delta x^2}\left(U_{a-1,b,c}^{n} - 2U_{a,b,c}^{n} + U_{a+1,b,c}^{n}\right) + \frac{\alpha_2}{\Delta y^2}\left(U_{a,b-1,c}^{n} - 2U_{a,b,c}^{n} + U_{a,b+1,c}^{n}\right) + \frac{\alpha_3}{\Delta y^3}\left(U_{a,b,c-1}^{n} - 2U_{a,b,c}^{n} + U_{a,b,+1c}^{n}\right)$$

Let $\quad U_{a,b,c}^{n} = G^n e^{ika\Delta x} e^{ikb\Delta y} e^{ikc\Delta z}$

$$\frac{(G^{n+1} - G^n) e^{ika\Delta x} e^{ikb\Delta y} e^{ikc\Delta z}}{\Delta t} = \frac{\alpha_1 G^n e^{ikb\Delta y} e^{ikc\Delta y}}{\Delta x^2}\left(e^{ik(a-1)\Delta x} - 2e^{ika\Delta x} + e^{ik(a+1)\Delta x}\right) + \frac{\alpha_2 G^n e^{ika\Delta x} e^{ikc\Delta y}}{\Delta y^2}\left(e^{ik(b-1)\Delta y} - 2e^{ikb\Delta y} + e^{ik(b+1)\Delta y}\right) + \frac{\alpha_3 G^n e^{ika\Delta x} e^{ikb\Delta y}}{\Delta z^3}\left(e^{ik(c-1)\Delta y} - 2e^{ikc\Delta y} + e^{ik(c+1)\Delta y}\right)$$

$$\frac{G^{n+1}}{G^n} = \left[1 + \frac{\alpha_1 \Delta t}{\Delta x^2}(e^{ik\Delta x} - 2 + e^{-ik\Delta x}) + \frac{\alpha_2 \Delta t}{\Delta y^2}(e^{ik\Delta y} - 2 + e^{-ik\Delta y}) + \frac{\alpha_3 \Delta t}{\Delta z^2}(e^{ik\Delta z} - 2 + e^{-ik\Delta z})\right]$$
$$= \left[1 + 2\frac{\alpha_1 \Delta t}{\Delta x^2}(\cos(k\Delta x) - 1) + 2\frac{\alpha_2 \Delta t}{\Delta y^2}(\cos(k\Delta y) - 1) + 2\frac{\alpha_3 \Delta t}{\Delta z^2}(\cos(k\Delta z) - 1)\right]$$

Let $\alpha_1 = \alpha_2 = \alpha_3$, $\Delta x = \Delta y = \Delta z$, so $\frac{\alpha_1 \Delta t}{\Delta x^2} = C_1$ and $\frac{\alpha_2 \Delta t}{\Delta y^2} = C_2$ and $\frac{\alpha_3 \Delta t}{\Delta z^2} = C_3$

Worst case: cos = -1

$$1 - 12C \le 1$$
$$-2 \le -12C \le 0$$
$$1 \ge 6C \ge 0$$
$$C \le \frac{1}{6}$$

6. 1D, 2D, and 3D BECS (10 pts)

1D BECS:

$$\frac{u_a^{n+1}-u_a^n}{\Delta t}=\frac{\alpha_1}{\Delta x^2}\left(U_{a-1,b}^n-2U_a^n+U_{a+1}^n\right)$$

$$\frac{u_{a,b}^{n+1}-u_{a,b}^n}{\Delta t}=\frac{\alpha_1}{\Delta x^2}\left(U_{a-1,b}^n-2U_{a,b}^n+U_{a+1,b}^n\right)+\frac{\alpha_2}{\Delta y^2}\left(U_{a,b-1}^n-2U_{a,b}^n+U_{a,b+1}^n\right)$$

Let $U_{a,b}^n=G^n e^{ika\Delta x}$

$$\frac{(G^{n+1}-G^n)e^{ika\Delta x}}{\Delta t}=\frac{\alpha_1 G^n}{\Delta x^2}\left(e^{ik(a-1)\Delta x}-2e^{ika\Delta x}+e^{ik(a+1)\Delta x}\right)$$

$$\frac{G^{n+1}}{G^n}=\frac{1}{1-\frac{\alpha_1\Delta t}{\Delta x^2}(e^{ik\Delta x}-2+e^{-ik\Delta x})}$$

$$=\frac{1}{1-2\frac{\alpha_1\Delta t}{\Delta x^2}(\cos(k\Delta x)-1)}$$

Let $\frac{\alpha_1\Delta t}{\Delta x^2}=C$

Cos lies between -1 and 1

$1-2C(\cos(k\Delta x)-1)\geq 1$
$2C(\cos(k\Delta x)-1)\leq 0$
$C\geq 0$
$\frac{G^{n+1}}{G^n}$ will always be between 0 and 1

1D BECS is unconditionally stable

2D BECS:

$$\frac{u_{a,b}^{n+1}-u_{a,b}^n}{\Delta t}=\frac{\alpha_1}{\Delta x^2}\left(U_{a-1,b}^n-2U_{a,b}^n+U_{a+1,b}^n\right)+\frac{\alpha_2}{\Delta y^2}\left(U_{a,b-1}^n-2U_{a,b}^n+U_{a,b+1}^n\right)$$

Let $U_{a,b}^n=G^n e^{ika\Delta x}e^{ikb\Delta y}$

$$\frac{(G^{n+1}-G^n)e^{ika\Delta x}e^{ikb\Delta y}}{\Delta t}=\frac{\alpha_1 G^n e^{ikb\Delta y}}{\Delta x^2}\left(e^{ik(a-1)\Delta x}-2e^{ika\Delta x}+e^{ik(a+1)\Delta x}\right)$$
$$+\frac{\alpha_2 G^n e^{ika\Delta x}}{\Delta y^2}\left(e^{ik(b-1)\Delta y}-2e^{ikb\Delta y}+e^{ik(b+1)\Delta y}\right)$$

$$\frac{G^{n+1}}{G^n}=\frac{1}{1-\frac{\alpha_1\Delta t}{\Delta x^2}(e^{ik\Delta x}-2+e^{-ik\Delta x})-\frac{\alpha_2\Delta t}{\Delta y^2}(e^{ik\Delta y}-2+e^{-ik\Delta y})}$$

$$=\frac{1}{1-2\frac{\alpha_1\Delta t}{\Delta x^2}(\cos(k\Delta x)-1)-2\frac{\alpha_2\Delta t}{\Delta y^2}(\cos(k\Delta y)-1)}$$

Let $\alpha_1 = \alpha_2, \Delta x = \Delta y$, so $\frac{\alpha_1 \Delta t}{\Delta x^2} = C_1 = \frac{\alpha_2 \Delta t}{\Delta y^2} = C_2 = C$

Cos lies between -1 and 1

$1 - 4C(\cos(k\Delta x) - 4) \geq 1$
$4C(\cos(k\Delta x) - 4) \leq 0$
$C \geq 0$
$\frac{G^{n+1}}{G^n}$ will always be between 0 and 1

2D BECS is unconditionally stable

3D BECS:

$$\frac{u_{a,b,c}^{n+1} - u_{a,b,c}^{n}}{\Delta t} = \frac{\alpha_1}{\Delta x^2}\left(U_{a-1,b,c}^{n} - 2U_{a,b,c}^{n} + U_{a+1,b,c}^{n}\right) + \frac{\alpha_2}{\Delta y^2}\left(U_{a,b-1,c}^{n} - 2U_{a,b,c}^{n} + U_{a,b+1,c}^{n}\right) + \frac{\alpha_3}{\Delta z^2}\left(U_{a,b,c-1}^{n} - 2U_{a,b,c}^{n} + U_{a,b,c+1}^{n}\right)$$

Let $\quad U_{a,b,c}^{n} = G^n e^{ika\Delta x} e^{ikb\Delta y} e^{ikc\Delta y}$

$$\frac{(G^{n+1} - G^n) e^{ika\Delta x} e^{ikb\Delta y} e^{ikc\Delta y}}{\Delta t} = \frac{\alpha_1 G^n e^{ikb\Delta y} e^{ikc\Delta y}}{\Delta x^2}\left(e^{ik(a-1)\Delta x} - 2e^{ika\Delta x} + e^{ik(a+1)\Delta x}\right) + \frac{\alpha_2 G^n e^{ika\Delta x}\ e^{ikc\Delta y}}{\Delta y^2}\left(e^{ik(b-1)\Delta y} - 2e^{ikb\Delta y} + e^{ik(b+1)\Delta y}\right) + \frac{\alpha_3 G^n e^{ika\Delta x}\ e^{ikb\Delta y}}{\Delta z^2}\left(e^{ik(c-1)\Delta y} - 2e^{ikc\Delta y} + e^{ik(c+1)\Delta y}\right)$$

$$\frac{G^{n+1}}{G^n} = \frac{1}{1 - \frac{\alpha_1 \Delta t}{\Delta x^2}(e^{ik\Delta x} - 2 + e^{-ik\Delta x}) - \frac{\alpha_2 \Delta t}{\Delta y^2}(e^{ik\Delta y} - 2 + e^{-ik\Delta y}) - \frac{\alpha_3 \Delta t}{\Delta z^2}(e^{ik\Delta y} - 2 + e^{-ik\Delta y})}$$

$$= \frac{1}{1 - 2\frac{\alpha_1 \Delta t}{\Delta x^2}(\cos(k\Delta x) - 1) - 2\frac{\alpha_2 \Delta t}{\Delta y^2}(\cos(k\Delta y) - 1) - 2\frac{\alpha_3 \Delta t}{\Delta z^2}(\cos(k\Delta z) - 1)}$$

Let $\alpha_1 = \alpha_2 = a_3, \Delta x = \Delta y = \Delta z$, so $\frac{\alpha_1 \Delta t}{\Delta x^2} = C_1 = \frac{\alpha_2 \Delta t}{\Delta y^2} = C_2 = \frac{\alpha_3 \Delta t}{\Delta z^2} = C_3 = C$

Cos lies between -1 and 1

$1 - 2C(\cos(k\Delta x) - 6) \geq 1$
$2C(\cos(k\Delta x) - 6) \leq 0$
$C \geq 0$
$\frac{G^{n+1}}{G^n}$ will always be between 0 and 1

3D BECS is unconditionally stable

7. 1D, 2D, and 3D Crank-Nicolson (10 pts)

1D CN:

$$\frac{u_a^{n+1} - u_a^n}{\Delta t} = \frac{\alpha_1}{\Delta x^2}\left[(U_{a-1}^{n+1} - 2U_a^{n+1} + U_{a+1}^{n+1}) + \left(U_{a-1,b}^n - 2U_{a,b}^n + U_{a+1,b}^n\right)\right]$$

Let $U_{a,b,c}^n = G^n e^{ika\Delta x}$

$$\frac{(G^{n+1} - G^n)e^{ika\Delta x}}{\Delta t} = \frac{\alpha_1(G^{n+1} + G^n)}{\Delta x^2}\left(e^{ik(a-1)\Delta x} - 2e^{ika\Delta x} + e^{ik(a+1)\Delta x}\right)$$

$$\frac{G^{n+1}}{G^n} = \frac{1 + \frac{\alpha_1 \Delta t}{\Delta x^2}(e^{ik\Delta x} - 2 + e^{-ik\Delta x})}{1 - \frac{\alpha_1 \Delta t}{\Delta x^2}(e^{ik\Delta x} - 2 + e^{-ik\Delta x})}$$

$$= \frac{1 + 2\frac{\alpha_1 \Delta t}{\Delta x^2}(\cos(k\Delta x) - 1)}{1 - 2\frac{\alpha_1 \Delta t}{\Delta x^2}(\cos(k\Delta x) - 1)}$$

Let $\frac{\alpha_1 \Delta t}{\Delta x^2} = C_1 = C$ and let $\cos(k\Delta x)$ be between -1 and 1

$\frac{1 - 2C}{1 + 2C} \leq 1$

$C \geq 0$

$\frac{G^{n+1}}{G^n}$ will always be less than 1

1D CN is unconditionally stable

2D CN:

$$\frac{u_{a,b}^{n+1} - u_{a,b}^n}{\Delta t} = \frac{\alpha_1}{\Delta x^2}\left[\left(U_{a-1,b}^{n+1} - 2U_{a,b}^{n+1} + U_{a+1,b}^{n+1}\right) + \left(U_{a-1,b}^n - 2U_{a,b}^n + U_{a+1,b}^n\right)\right] + \frac{\alpha_2}{\Delta y^2}\left[\left(U_{a,b-1}^{n+1} - 2U_{a,b}^{n+1} + U_{a,b+1}^{n+1}\right) + \left(U_{a,b-1}^n - 2U_{a,b}^n + U_{a,b+1}^n\right)\right]$$

Let $U_{a,b,c}^n = G^n e^{ika\Delta x} e^{ikb\Delta y}$

$$\frac{(G^{n+1} - G^n)e^{ika\Delta x}e^{ikb\Delta y}}{\Delta t} = \frac{\alpha_1(G^{n+1} + G^n)e^{ikb\Delta y}}{\Delta x^2}\left(e^{ik(a-1)\Delta x} - 2e^{ika\Delta x} + e^{ik(a+1)\Delta x}\right) + \frac{\alpha_2(G^{n+1} + G^n)e^{ika\Delta x}}{\Delta y^2}\left(e^{ik(b-1)\Delta y} - 2e^{ikb\Delta y} + e^{ik(b+1)\Delta y}\right)$$

$$\frac{G^{n+1}}{G^n} = \frac{1 + \frac{\alpha_1 \Delta t}{\Delta x^2}(e^{ik\Delta x} - 2 + e^{-ik\Delta x}) + \frac{\alpha_2 \Delta t}{\Delta y^2}(e^{ik\Delta y} - 2 + e^{-ik\Delta y})}{1 - \frac{\alpha_1 \Delta t}{\Delta x^2}(e^{ik\Delta x} - 2 + e^{-ik\Delta x}) - \frac{\alpha_2 \Delta t}{\Delta y^2}(e^{ik\Delta y} - 2 + e^{-ik\Delta y})}$$

$$= \frac{1 + 2\frac{\alpha_1 \Delta t}{\Delta x^2}(\cos(k\Delta x) - 1) + 2\frac{\alpha_2 \Delta t}{\Delta y^2}(\cos(k\Delta y) - 1)}{1 - 2\frac{\alpha_1 \Delta t}{\Delta x^2}(\cos(k\Delta x) - 1) - 2\frac{\alpha_2 \Delta t}{\Delta y^2}(\cos(k\Delta y) - 1))}$$

Let $\alpha_1 = \alpha_2, \Delta x = \Delta y$, so $\frac{\alpha_1 \Delta t}{\Delta x^2} = C_1 = \frac{\alpha_2 \Delta t}{\Delta y^2} = C_2 = C$ and let $\cos(k\Delta x)$ be between -1 and 1

$\frac{1-4C}{1+4C} \leq 1$
$C \geq 0$
$\frac{G^{n+1}}{G^n}$ will always be less than 1
2D CN is unconditionally stable

3D CN:

$$\frac{u_{a,b,c}^{n+1} - u_{a,b,c}^{n}}{\Delta t} = \frac{\alpha_1}{\Delta x^2}\left[\left(U_{a-1,b,c}^{n+1} - 2U_{a,b,c}^{n+1} + U_{a+1,b,c}^{n+1}\right) + \left(U_{a-1,b,c}^{n} - 2U_{a,b,c}^{n} + U_{a+1,b,c}^{n}\right)\right] + \frac{\alpha_2}{\Delta y^2}\left[\left(U_{a,b-1,c}^{n+1} - 2U_{a,b,c}^{n+1} + U_{a,b+1,c}^{n+1}\right) + \left(U_{a,b-1,c}^{n} - 2U_{a,b,c}^{n} + U_{a,b+1,c}^{n}\right)\right] + \frac{\alpha_3}{\Delta z^2}\left[\left(U_{a,b,c-1}^{n+1} - 2U_{a,b,c}^{n+1} + U_{a,b,c+1}^{n+1}\right) + \left(U_{a,b,c-1}^{n} - 2U_{a,b,c}^{n} + U_{a,b,c+1}^{n}\right)\right]$$

Let $U_{a,b,c}^n = G^n e^{ika\Delta x} e^{ikb\Delta y} e^{ikc\Delta y}$

$$\frac{(G^{n+1} - G^n) e^{ika\Delta x} e^{ikb\Delta y} e^{ikc\Delta y}}{\Delta t} = \frac{\alpha_1 (G^{n+1} + G^n) e^{ikb\Delta y} e^{ikc\Delta y}}{\Delta x^2}\left(e^{ik(a-1)\Delta x} - 2e^{ika\Delta x} + e^{ik(a+1)\Delta x}\right) + \frac{\alpha_2 (G^{n+1} + G^n) e^{ika\Delta x}\ e^{ikc\Delta y}}{\Delta y^2}\left(e^{ik(b-1)\Delta y} - 2e^{ikb\Delta y} + e^{ik(b+1)\Delta y}\right) + \frac{\alpha_3 (G^{n+1} + G^n) e^{ika\Delta x}\ e^{ikb\Delta y}}{\Delta z^2}\left(e^{ik(c-1)\Delta y} - 2e^{ikc\Delta y} + e^{ik(c+1)\Delta y}\right)$$

$$\frac{G^{n+1}}{G^n} = \frac{1 + \frac{\alpha_1 \Delta t}{\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right) + \frac{\alpha_2 \Delta t}{\Delta y^2}\left(e^{ik\Delta y} - 2 + e^{-ik\Delta y}\right) + \frac{\alpha_3 \Delta t}{\Delta z^2}\left(e^{ik\Delta y} - 2 + e^{-ik\Delta y}\right)}{1 - \frac{\alpha_1 \Delta t}{\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right) - \frac{\alpha_2 \Delta t}{\Delta y^2}\left(e^{ik\Delta y} - 2 + e^{-ik\Delta y}\right) - \frac{\alpha_3 \Delta t}{\Delta z^2}\left(e^{ik\Delta y} - 2 + e^{-ik\Delta y}\right)}$$

$$= \frac{1 + 2\frac{\alpha_1 \Delta t}{\Delta x^2}(\cos(k\Delta x) - 1) + 2\frac{\alpha_2 \Delta t}{\Delta y^2}(\cos(k\Delta y) - 1) + 2\frac{\alpha_3 \Delta t}{\Delta z^2}(\cos(k\Delta z) - 1)}{1 - 2\frac{\alpha_1 \Delta t}{\Delta x^2}(\cos(k\Delta x) - 1) - 2\frac{\alpha_2 \Delta t}{\Delta y^2}(\cos(k\Delta y) - 1) - 2\frac{\alpha_3 \Delta t}{\Delta z^2}(\cos(k\Delta z) - 1)}$$

Let $\alpha_1 = \alpha_2 = a_3, \Delta x = \Delta y = \Delta z$, so $\frac{\alpha_1 \Delta t}{\Delta x^2} = C_1 = \frac{\alpha_2 \Delta t}{\Delta y^2} = C_2 = \frac{\alpha_3 \Delta t}{\Delta z^2} = C_3 = C$ and let $\cos(k\Delta x)$ be between -1 and 1

$\frac{1-6C}{1+6C} \leq 1$
$C \geq 0$
$\frac{G^{n+1}}{G^n}$ will always be less than 1
3D CN is unconditionally stable