Lipschitz continuity for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$: A function is Lipschitz continous if there are bounds on the first derivative of the function. This means that the function never increases more than a linear term with the constant L as the lipschitz constant.

$$||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|| \leq L||\mathbf{x} - \mathbf{y}||, \ \forall \ \mathbf{x}, \mathbf{y}$$

Theorems from lecture notes

Theorem 1: Consider the ODE $\dot{x} = f(x)$ where f is continuous. If f is also Lipschitz continuous, then the solution of $\dot{x} = f(x)$ exists and is unique for all t.

Theorem 2: Consider the ODE $\dot{x} = f(x)$. Then if f is continuously differentiable (i.e. the Jacobian $\frac{\partial f}{\partial x}$ exists and is continuous), then the solution to the ODE exists and is unique on some time interval.

Note: These two theorems are sufficient but not necessary to guarantee the existence of a solution. I.e there might be a solution to the ODE even though it doesn't fulfil these two theorems.

Theorem 8 (Tikhonov): Consider the ordinary differential equation (ODE):

$$\begin{aligned} \dot{\mathrm{x}} &= \mathrm{f}(\mathrm{x}, \mathrm{z}) \\ \epsilon\dot{\mathrm{z}} &= \mathrm{g}(\mathrm{x}, \mathrm{z}) \end{aligned}$$

where $0 < \epsilon << 1$ is very small. Let us label $\mathrm{x}_\epsilon(t)$, $\mathrm{z}_\epsilon(t)$ the solution to the ODE. Suppose that the dynamics $\dot{\mathrm{z}} = \mathrm{g}(\mathrm{x}, \mathrm{z})$ are stable $\forall$ x and that the matrix $\frac{\partial \mathrm{g}}{\partial \mathrm{z}}$ is full rank (i.e. invertible) everywhere. Then

$$\lim_{\epsilon \to 0} \mathrm{x}_\epsilon(t), \mathrm{z}_\epsilon(t) = \mathrm{x}_0(t), \mathrm{z}_0(t)$$

where $\mathrm{x}_0(t), \mathrm{z}_0(t)$ is the solution of

$$\begin{aligned} \dot{\mathrm{x}} &= \mathrm{f}(\mathrm{x}, \mathrm{z}) \\ 0 &= \mathrm{g}(\mathrm{x}, \mathrm{z}) \end{aligned}$$

A **stiff system** does not have a precise definition, but a few takes:

- Different variables have different time scales (rapid vs. slow part of solution)
- Negative but highly different eigenvalues (large stiffness ratio)
- Explicit methods work poorly (limited by stability criterion)
- Stability requirement dominates the choice of the step size

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$$SO(3) = \left\{\mathbf{R} \mid \mathbf{R} \in R^{3\times 3}, \quad \mathbf{R}^T\mathbf{R} = \mathbf{I} \quad \text{and} \quad \det(\mathbf{R}) = 1\right\}$$

Implicit function theorem (IFT), simplified version: Let the function $\phi(x, y)$ be smooth, and consider a point $(\bar{x}, \bar{y})$ such that $\phi(\bar{x}, \bar{y}) = 0$. Suppose that the Jacobian

$$\left.\frac{\partial \phi(x, y)}{\partial x}\right|_{x=\bar{x}, y=\bar{y}}$$

is full rank. Then there exists an open set Y around the point $\bar{y}$ in which there exists a unique, smooth function $x(y)$ satisfying:

$$\phi(x(y), y) = 0, \quad \forall y \in Y$$

Moreover, the Jacobian of function $x(y)$ is given by

$$\frac{\partial x(y)}{\partial y} = -\left(\frac{\partial \phi(x, y)}{\partial x}\right)^{-1} \left.\frac{\partial \phi(x, y)}{\partial y}\right|_{x=x(y), y}$$
