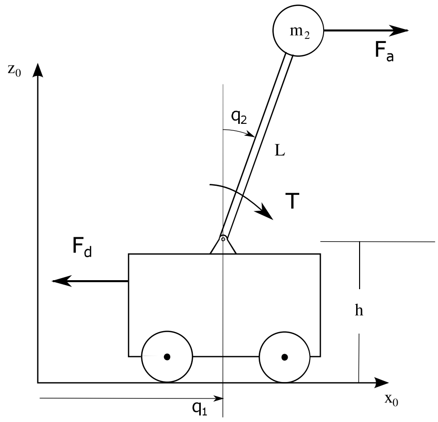
TTK4130 Modelling and simulation

Gutta på MS

May 18, 2024



Wagon with inverted pendulum

Department of Engineering Cybernetics

Contents

1 Kinematics

1.1 Vector notation

There are two ways of expressing vectors:

$$\mathbf{r}^i = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{1a}$$

$$\vec{r}_{a/b} = a\vec{i}_i + b\vec{j}_i + c\vec{k}_i \tag{1b}$$

Where i is the frame of reference, and subscript a/b denotes from point b to point a. If the vectors are velocities, subscript a/b denotes the velocity of point a relative to point b. It is important to be consistent in the notation. Never do arithmetic operations on vectors expressed in different frames. This means:

$$\xcancel{\vec{r}^a} \tag{2a}$$

$$\xcancel{\mathbf{v}^i + \mathbf{u}^a} \tag{2b}$$

$$\xcancel{\mathbf{v}^i \times \mathbf{u}^a} \tag{2c}$$

1.2 Skew matrix notation

If you have $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^\top$ and $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^\top$:

$$\mathbf{u}^\times = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \tag{3a}$$

$$\mathbf{u}^\times \mathbf{v} = \mathbf{u} \times \mathbf{v} \tag{3b}$$

$$\mathbf{u}^\times \mathbf{u} = 0 \tag{3c}$$

$$\left(\mathbf{u}^\times\right)^\times = -\mathbf{u}^\times \tag{3d}$$

$$\det\left(\mathbf{u}^\times\right) = 0 \tag{3e}$$

The cross product of two vectors can be calculated by finding the determinant of this matrix:

$$\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \tag{4}$$

1.3 Rotation matrices

The rotation matrices are defined on each axis as:

$$\mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{5a}$$

$$\mathbf{R}_y(\phi) = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix} \tag{5b}$$

$$\mathbf{R}_z(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5c}$$

Now lets say frame a relative to frame i is rotated by θ about the x-axis, ϕ about the y-axis, and ψ about the z-axis. The rotation matrix from frame a to frame i is then:

$$\mathbf{R}_i^a = \mathbf{R}_z(\psi)\mathbf{R}_y(\phi)\mathbf{R}_x(\theta) \tag{6}$$

Which means that $\mathbf{R}_i^a$ is called the *rotation matrix* from frame a to frame i.

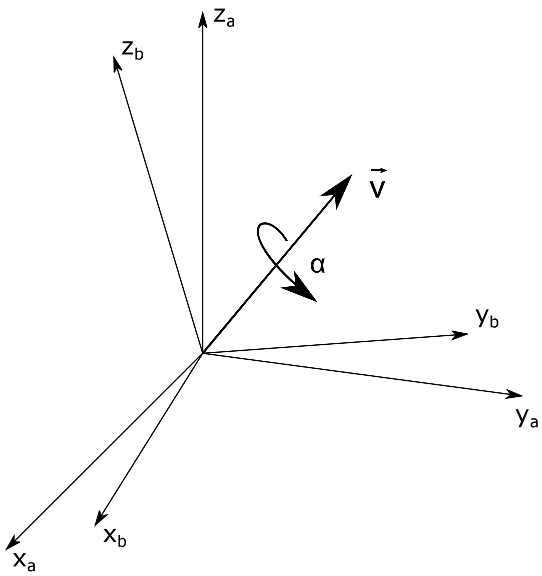


Figure 1: Rotation about arbitrary axis

Now, if we have a rotation about an arbitrary axis $\mathbf{v}$, we can use the equation:

$$\mathbf{R}_{\alpha,\vec{v}} = \cos(\alpha)\mathbf{I} + \mathbf{v}^{\times}\sin(\alpha) + \mathbf{v}\mathbf{v}^{\top}(1 - \cos(\alpha)) = \mathbf{R}_b^a(\alpha) \tag{7}$$

1.3.1 Properties of rotation matrices

A couple of maybe interesting properties of rotation matrices:

$$\mathbf{v}^b = \mathbf{R}_a^b \mathbf{v}^a \tag{8a}$$

$$\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b \tag{8b}$$

$$\mathbf{R}_a^b \mathbf{R}_b^a = \mathbf{I} \tag{8c}$$

$$\mathbf{R}_a^b = (\mathbf{R}_b^a)^{-1} = (\mathbf{R}_b^a)^\top \tag{8d}$$

$$\dot{\mathbf{R}}_b^a = (\omega_{ab}^a)^\times \mathbf{R}_b^a = \mathbf{R}_b^a \left(\omega_{ab}^b\right)^\times \tag{8e}$$

$$(\vec{\omega}_{ab}^a)^\times = \dot{\mathbf{R}}_b^a (\mathbf{R}_b^a)^\top \tag{8f}$$

$$\vec{\omega}_{ad} = \vec{\omega}_{ab} + \vec{\omega}_{bc} + \vec{\omega}_{cd} \tag{8g}$$

Where the vector $\vec{\omega}_{ab}^a$ is the angular velocity vector from frame b relative to frame b with respect to frame a.

1.4 Other stuff that might be usefull

Linear momentum does not depend upn its point of reference:

$$\vec{p} = m\vec{v} \tag{9a}$$

$$\dot{\vec{p}} = m\vec{a} = \vec{F} \tag{9b}$$

Angular momentum of point p with respect t origin o, where $\vec{r}_{p/o}$ is the position of p and $\vec{p}$ is the linear momentum. (Angular momentum depends upon its point of reference):

$$\vec{h}_{p/o} = \vec{r}_{p/o} \times \vec{p} \tag{10a}$$

$$\dot{\vec{h}}_{p/o} = \vec{r}_{p/o} \times \dot{\vec{p}} = \vec{T} \tag{10b}$$

2 Lagrange modelling

2.1 Generalized coordinates

The generalized coordinates are $\mathbf{q} \in \mathbb{R}^n$, where $n \geq$ DOF. DOF is degrees of freedom, and is the number of independent coordinates needed to describe the configuration of a system. The generalized coordinates are not unique, and can be chosen in many ways. The choice of generalized coordinates is important, as it can simplify the equations of motion. The generalized velocities $\dot{\mathbf{q}}$ are the time derivatives of the generalized coordinates. The generalized accelerations $\ddot{\mathbf{q}}$ are the time derivatives of the generalized velocities. The generalized forces $\mathbf{Q}$ are the forces that act on the system.

2.2 Definition of the lagrangian $\mathcal{L}$

$$\mathcal{L} = T - V - \mathbf{z}^\top \mathbf{c} \tag{11a}$$

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q}) - \mathbf{z}^\top \mathbf{c}(\mathbf{q}) \tag{11b}$$

Where T is the kinetic energy, V is the potential energy, $\mathbf{q}$ is the generalized coordinates, $\dot{\mathbf{q}}$ is the generalized velocities, $z \in \mathbb{R}^n$ is the lagrangian multiplier and n is the number of constraints. It is also valid to use $+\mathbf{z}^\top \mathbf{c}$. Equation (11b)

$$T = \frac{1}{2} m \dot{\mathbf{p}}^\top \dot{\mathbf{p}} + \frac{1}{2} I \dot{\theta}^2 \tag{12a}$$

$$\dot{\mathbf{p}} = \frac{\partial \mathbf{p}}{\partial \mathbf{q}} \dot{\mathbf{q}} \tag{12b}$$

$$T = \frac{1}{2} m v^\top v + \frac{1}{2} I \omega^\top \omega \tag{12c}$$

$$V = \underbrace{mgh}_{\text{gravity}} + \underbrace{\frac{1}{2} k x^2}_{\text{potential energy in spring}} \tag{12d}$$

$$T = \frac{1}{2} \dot{\mathbf{q}}^\top \mathbf{W} \dot{\mathbf{q}} \tag{12e}$$

$$\mathbf{W} = m \left(\frac{\partial \mathbf{p}}{\partial \mathbf{q}} \right)^\top \frac{\partial \mathbf{p}}{\partial \mathbf{q}} + \beta^\top \mathbf{J} \beta \tag{12f}$$

Where m is the mass, I is the inertia, W is the mass inertia matrix, p is the position vector, θ is the angle, v is the velocity, ω is the angular velocity, h is the height, k is the spring constant, and x is the displacement.
If the object is a point mass, we don't use the inertia term $\frac{1}{2} I \dot{\theta}^2$.
As for equation (12f), $\omega = \beta(\mathbf{q}) \dot{\mathbf{q}}$ and $\mathbf{J}$ is the inertia matrix.

2.3 System dynamics

With this lagrangian, we can describe the system dynamics. Let $\mathbf{Q}$ be the generalized forces, then the equations of motion are:

$$\mathbf{Q} = \frac{\partial \mathbf{p}}{\partial \mathbf{q}}^{\top} \mathbf{F} \tag{13a}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{\partial \mathbf{c}}{\partial \mathbf{q}} \mathbf{z} = \mathbf{Q} \tag{13b}$$

$$\frac{d}{dt} \mathbf{W} \dot{\mathbf{q}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q} \tag{13c}$$

$$\mathbf{W} \ddot{\mathbf{q}} + \dot{\mathbf{W}} \dot{\mathbf{q}} - \frac{\partial T}{\partial \mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}} = \mathbf{Q} \tag{13d}$$

Lastly, we have the $\boldsymbol{GOAT}$ matrix, which you get by solving $\frac{d^2}{dt^2} \mathbf{c}(\mathbf{q}) = 0$ and using the equation above:

$$\begin{bmatrix} \mathbf{W} & \frac{\partial \mathbf{c}}{\partial \mathbf{q}} \\ \left(\frac{\partial \mathbf{c}}{\partial \mathbf{q}}\right)^{\top} & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} + \frac{\partial T}{\partial \mathbf{q}} - \dot{\mathbf{W}} \dot{\mathbf{q}} - \frac{\partial V}{\partial \mathbf{q}} \\ -\frac{\partial}{\partial \mathbf{q}} \left(\frac{\partial \mathbf{c}}{\partial \mathbf{q}} \dot{\mathbf{q}}\right) \dot{\mathbf{q}} \end{bmatrix} \tag{14}$$