Differential algebraic equations consists of both differential equations and algebraic equations where some of the states does not show up in their time-differentiated state, or there are not enough equations to explicitly define all the time-differentiated states. A differential equation is a DAE if the jacobian is rank-deficient. i.e.:

$$\begin{bmatrix} \frac{\partial F}{\partial \dot{\mathbf{x}}} & \frac{\partial F}{\partial \dot{\mathbf{z}}} \end{bmatrix} \text{ is rank deficient, where } \mathbf{z} \text{ are states that does not appear as time-differentiated.}$$

Be wary that if the jacobian beeing rank deficient are dependent on the values of **u**, **x** or **z**, the equation can switch between beeing an ODE and a DAE.

Fully-implicit DAEs occur when at least one of the states not appears in it's time-differentiated state, leaving at least one of the columns in the jacobian as 0. The states without their time differentiated state are often replaced with a z:

$$F(\dot{x}, x, z, u) = 0$$

Semi-explicit DAEs splits the differential and algebraic equations:

$$\begin{aligned} \dot{x} &= f(x, z, u) \\ 0 &= g(x, z, u) \end{aligned}$$

Linear DAEs are when the underlying functions are linear:

$$E\dot{x} = Ax + Bu$$

Here E is rank deficitent and not invertible, else it would have been an ODE.

Converting between forms

Fully-implicit to semi-explicit

$$\begin{bmatrix} \dot{x} = v \\ 0 = F(v, x, z, u) \end{bmatrix}$$

Semi-explicit to fully implicit

$$\begin{bmatrix} \dot{x} - f(x, z, u) \\ g(x, z, u) \end{bmatrix} = 0$$

The differential index of a DAE is the number of times the operator $\frac{d}{dt}$ must be applied to the equations (+ possibly an arbitrary amount of algebraic manipulations) in order to convert the DAE into an ODE. It's worth noting that when doing this we may end up with $\dot{u}$ in our system, this means that u needs to be differentiable in this case. An **easy** DAE is always of index 1. Remember that $\dot{\mathbf{g}}(\mathbf{x}, \mathbf{z}) = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{z}}\dot{\mathbf{z}}$.

A DAE is **easy** (index 1) if the Jacobian $\frac{\partial \mathbf{g}}{\partial \mathbf{z}}$ is full rank everywhere (**g** can be used to explicitly find **z**). Otherwise it's **hard**.

Models from **constrained lagrange mechanics** with position-dependent constraints of the form $c(q) = 0$ yield index-3 DAEs, and two time differentiations of the constraints yield an index-1 DAE:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}, \qquad \ddot{c}(q, \dot{q}, \ddot{q}) = 0$$

$$\begin{bmatrix} I & 0 & 0 \\ 0 & W(q) & 0 \\ 0 & 0 & M(q) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \\ z \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{q}} \\ Q - \frac{\partial}{\partial \mathbf{q}}(W(q)\dot{\mathbf{q}})\dot{\mathbf{q}} + \nabla_q T - \nabla_q V - \nabla_q c(q) z \\ -\frac{\partial}{\partial \mathbf{q}}\left(\frac{\partial c}{\partial \mathbf{q}}\dot{\mathbf{q}}\right)\dot{\mathbf{q}} \end{bmatrix}$$