General RK-Methods and Butcher Tableaus for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{t})$

$$K_1 = f\left(x_k + \Delta t \sum_{j=1}^{s} a_{1j} K_j, t_k + c_1 \Delta t\right)$$

$$\vdots$$

$$K_i = f\left(x_k + \Delta t \sum_{j=1}^{s} a_{ij} K_j, t_k + c_i \Delta t\right)$$

$$\vdots$$

$$K_s = f\left(x_k + \Delta t \sum_{j=1}^{s} a_{sj} K_j, t_k + c_s \Delta t\right)$$

$$x_{k+1} = x_k + \Delta t \sum_{i=1}^{s} b_i K_i$$

$$\begin{array}{c|ccc} c_1 & a_{11} & \dots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \dots & a_{ss} \\ \hline & b_1 & \dots & b_s \end{array}$$

$$c_i = \textstyle\sum_{j=1}^{s} a_{i,j} \text{ and } \sum_{j=1}^{s} b_j = 1$$
$$0 \leq c_i \leq 1$$

For Explicit Runge-Kutta methods, $\mathbf{A}$ is strictly lower triangular (all elements above the sub-diagonal are 0).

Collocation methods

$$a_{ji} = L_i(\tau_j), \quad b_i = L_i(1), \quad c_j = \tau_j$$

$$L_i(\tau) = \int_0^{\tau} \ell_i(\xi) d\xi \qquad \ell_i(\tau) = \prod_{j \neq i} \frac{\tau - \tau_j}{\tau_i - \tau_j} \qquad \ell_i(\tau_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Gauss-Legendre can be used to find the grid points $\tau_1, \dots, \tau_s$ effectively, by finding the roots of:

$$P_s(\tau) = \frac{1}{s!} \frac{d^s}{d\tau^s} [(\tau^2 - \tau)^s] \tag{1}$$

We then use the formulas from the collocation methods to find the butcher tableaus. Gauss-Legendre methods are always order $o = 2s$, never L-stable, but always A-stable.

Some common ERK-methods

Explicit Euler RK1

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

Mid-point RK2

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array}$$

Ralston's RK2

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}$$

Heun's RK2

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

RK4 ("the RK method")

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$