Some common IRK-methods (stability)

Implicit Euler (L)

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

Trapezoidal method (L)

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Gauss-Legendre Collocation (IRK4) (A)

$$\begin{array}{c|cc} \frac{1}{2} - \frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{2} + \frac{\sqrt{3}}{6} & \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Stability of RK-methods and the stability function for system $\dot{\mathbf{x}} = \lambda \mathbf{x}$

The region of stability is $||R(z)|| \leq 1$ where $z := \lambda \Delta t \in \mathbb{C}$, and the stability function is defined as:

$$R(z) = 1 + z\mathbf{b}^T(\mathbf{I} - z\mathbf{A})^{-1}\mathbf{1} = \frac{\det(\mathbf{I} - z(\mathbf{A} - \mathbf{1}\mathbf{b}^T))}{\det(\mathbf{I} - z\mathbf{A})}, \quad \text{where } \mathbf{A} \text{ and } \mathbf{b} \text{ are determined by the RK-method.}$$

A system is **A-stable** if $|R(\lambda h)| \leq 1$ for all λ with $\mathrm{Re}(\lambda) \leq 0$ (ERK-methods are never A-stable). An A-stable system is not dependent on the time step to be stable.

An RK method is **L-stable** if it is A-stable, and in addition $|R(i\omega h)| \to 0$ when $\omega \to \pm\infty$. It will dampen out oscillations in the system, so not necessarily the best for systems that oscillates on a set frequency, but will not create extra oscillations. If not L-stable, the simulation can suffer from aliasing.

Error control is important in RK-methods. We need a small Δt, but having it too small is computational expensive and unnecessary, and how small it needs to be can vary throughout the simulation, therefore we have **adaptive simulations** that change Δt during the simulation. This is done by simulating with two different methods, normally with different b-coefficients, if the difference is too big the time step is decreased.

The **order** of an ERK is never bigger than the number of steps. max $o = s$ for $s \leq 4$, then the order "stalls". An IRK can maximally achieve a order of $o = 2s$ for all s.

To **simulate DAEs** we can do a slight alteration to the formulas for $K_1, \dots, K_s$ to make them implicit:

$$\begin{bmatrix} F(K_1, z_1, x_k + \Delta t \sum_{j=1}^{s} a_{1j} K_j, u(t_k + c_1 \Delta t) \\ \vdots \\ F(K_i, z_i, x_k + \Delta t \sum_{j=1}^{s} (a_{ij} K_j), u(t_k + c_i \Delta t) \\ \vdots \\ F(K_s, z_s, x_k + \Delta t \sum_{j=1}^{s} (a_{sj} K_j), u(t_k + c_s \Delta t) \end{bmatrix} = 0$$

This will also work for **implicit ODEs** without the z. Note that this only works for DAEs with index 1.

Some taylor expansions

$$\dot{f}(t_k, x(t_k)) = \frac{\partial f}{\partial x} f + \frac{\partial f}{\partial t}$$

$$x(t_{k+1}) = x_k + \Delta t \cdot f(t_k, x_k) + \frac{\Delta t^2}{2} \cdot \dot{f}(t_k, x(t_k)) + O(\Delta t^3)$$

$$f(t_k + c\Delta t, x_k + a\Delta t f(t_k, x_k)) = f(t_k, x_k) + a\Delta t f(t_k, x_k) \frac{\partial f}{\partial x} + c\Delta t \frac{\partial f}{\partial t} + O(\Delta t^2)$$