TTK4130 Modelling and simulation

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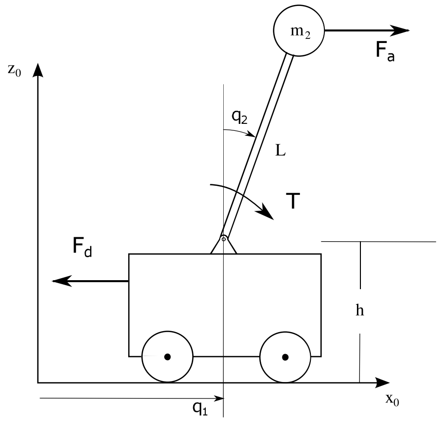

Wagon with inverted pendulum

Department of Engineering Cybernetics

Contents

1 Kinematics

1.1 Vector notation

There are two ways of expressing vectors:

$$\mathbf{r}^i = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{1a}$$

$$\vec{r}_{a/b} = a\vec{i}_i + b\vec{j}_i + c\vec{k}_i \tag{1b}$$

Where i is the frame of reference, and subscript a/b denotes from point b to point a. If the vectors are velocities, subscript a/b denotes the velocity of point a relative to point b. It is important to be consistent in the notation. Never do arithmetic operations on vectors expressed in different frames. This means:

$$\xcancel{\vec{r}^{a}} \tag{2a}$$

$$\xcancel{\mathbf{v}^i + \mathbf{u}^a} \tag{2b}$$

$$\xcancel{\mathbf{v}^i \times \mathbf{u}^a} \tag{2c}$$

1.2 Skew matrix notation

If you have $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^\top$ and $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^\top$:

$$\mathbf{u}^\times = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \tag{3a}$$

$$\mathbf{u}^\times \mathbf{v} = \mathbf{u} \times \mathbf{v} \tag{3b}$$

$$\mathbf{u}^\times \mathbf{u} = 0 \tag{3c}$$

$$\left(\mathbf{u}^\times\right)^\times = -\mathbf{u}^\times \tag{3d}$$

$$\det\left(\mathbf{u}^\times\right) = 0 \tag{3e}$$

The cross product of two vectors can be calculated by finding the determinant of this matrix:

$$\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \tag{4}$$

1.3 Rotation matrices

The rotation matrices are defined on each axis as:

$$\mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{5a}$$

$$\mathbf{R}_y(\phi) = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix} \tag{5b}$$

$$\mathbf{R}_z(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5c}$$

Now lets say frame a relative to frame i is rotated by θ about the x-axis, ϕ about the y-axis, and ψ about the z-axis. The rotation matrix from frame a to frame i is then:

$$\mathbf{R}_i^a = \mathbf{R}_z(\psi)\mathbf{R}_y(\phi)\mathbf{R}_x(\theta) \tag{6}$$

Which means that $\mathbf{R}_i^a$ is called the *rotation matrix* from frame a to frame i.

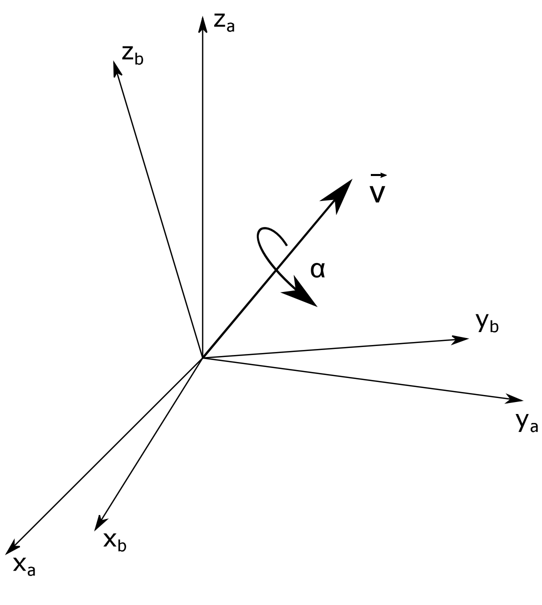

Figure 1: Rotation about arbitrary axis

Now, if we have a rotation about an arbitrary axis $\mathbf{v}$, we can use the equation:

$$\mathbf{R}_{\alpha,\vec{v}} = \cos(\alpha)\mathbf{I} + \mathbf{v}^{\times}\sin(\alpha) + \mathbf{v}\mathbf{v}^{\top}(1-\cos(\alpha)) = \mathbf{R}_b^a(\alpha) \tag{7}$$

1.3.1 Properties of rotation matrices

A couple of maybe interesting properties of rotation matrices:

$$\mathbf{v}^b = \mathbf{R}_a^b \mathbf{v}^a \tag{8a}$$

$$\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b \tag{8b}$$

$$\mathbf{R}_a^b \mathbf{R}_b^a = \mathbf{I} \tag{8c}$$

$$\mathbf{R}_a^b = (\mathbf{R}_b^a)^{-1} = (\mathbf{R}_b^a)^\top \tag{8d}$$

$$\dot{\mathbf{R}}_b^a = (\omega_{ab}^a)^\times \mathbf{R}_b^a = \mathbf{R}_b^a \left(\omega_{ab}^b\right)^\times \tag{8e}$$

$$(\vec{\omega}_{ab}^a)^\times = \dot{\mathbf{R}}_b^a \, (\mathbf{R}_b^a)^\top \tag{8f}$$

$$\vec{\omega}_{ad} = \vec{\omega}_{ab} + \vec{\omega}_{bc} + \vec{\omega}_{cd} \tag{8g}$$

Where the vector $\vec{\omega}_{ab}^a$ is the angular velocity vector from frame b relative to frame b with respect to frame a.

1.4 Other stuff that might be usefull

Linear momentum does not depend upn its point of reference:

$$\vec{p} = m\vec{v} \tag{9a}$$

$$\dot{\vec{p}} = m\vec{a} = \vec{F} \tag{9b}$$

Angular momentum of point p with respect t origin o, where $\vec{r}_{p/o}$ is the position of p and $\vec{p}$ is the linear momentum. (Angular momentum depends upon its point of reference):

$$\vec{h}_{p/o} = \vec{r}_{p/o} \times \vec{p} \tag{10a}$$

$$\dot{\vec{h}}_{p/o} = \vec{r}_{p/o} \times \dot{\vec{p}} = \vec{T} \tag{10b}$$