Machine Learning

Contents

1 Regression

Important: Gradient descent and predictions are different. When finding optimal parameters, a dataset will be used, while a single data point will be used in prediction, outputting a single prediction value.

1.1 Linear regression

1.1.1 Squared error cost function

Measures how well line fits training data
m = num of training examples
$\vec{x}$ holds training example x values (length m)
$\vec{y}$ holds training example y values (length m)
$\hat{y}^{(i)} = w\vec{x}^{(i)} + b$

$$J(w, b) = \frac{1}{2m} \sum_{i=1}^{m} (\hat{y}^{(i)} - \vec{y}_i)^2$$

$\frac{1}{m}$ finds average error for larger data sets, $\frac{1}{2m}$ makes later calculations neater

1.1.2 Gradient descent

Find w, b for minimum of cost function $J(w, b)$

1. Start with some w, b (commonly $0, 0$)
2. Look around starting point and find direction that will move the point furthest downwards for a small step size

α = learning rate
Must simultaneously update w and b

$$w_1 = w_0 - \alpha \frac{\partial}{\partial w} J(w_0, b_0)$$
$$b_1 = b_0 - \alpha \frac{\partial}{\partial b} J(w_0, b_0)$$
$$\frac{\partial}{\partial w} J(w, b) = \frac{1}{m} \sum_{i=1}^{m} (\hat{y}^{(i)} - \vec{y}_i)\vec{x}^{(i)}$$
$$\frac{\partial}{\partial b} J(w, b) = \frac{1}{m} \sum_{i=1}^{m} (\hat{y}^{(i)} - \vec{y}_i)$$

1.2 Multiple linear regression

n_f = number of features
m = number of data points
$\vec{w}$ = vector of weights (length n_f)
X is a list of x vectors which hold n_f features (size $m \times n_f$)
Sum of predictions of all features is the prediction of multiple linear reg

$$f_{\vec{w},b}(\vec{x}) = \vec{w} \cdot \vec{x} + b$$

Gradient descent

$$\vec{w}_j = \vec{w}_j - \alpha \frac{\partial}{\partial \vec{w}_j} J(\vec{w}, b)$$
$$b = b - \alpha \frac{\partial}{\partial b} J(\vec{w}, b)$$

Cost function and its partial derivatives

$$J(\vec{w}, b) = \frac{1}{2m} \sum_{i=1}^{m} (f_{\vec{w},b}(X^{(i)}) - \vec{y}_i)^2$$
$$\frac{\partial}{\partial \vec{w}_j} J(\vec{w}, b) = \frac{1}{m} \sum_{i=1}^{m} (f_{\vec{w},b}(X^{(i)}) - \vec{y}_i)X_j^{(i)}$$
$$\frac{\partial}{\partial b} J(\vec{w}, b) = \frac{1}{m} \sum_{i=1}^{m} (f_{\vec{w},b}(X^{(i)}) - \vec{y}_i)$$

1.3 Logistic regression

Sigmoid function

$$g(z) = \frac{1}{1+e^{-z}}$$
$$z = f_{\vec{w},b}(\vec{x})$$
$$\hat{y}^{(i)} = g(f_{\vec{w},b}(X^{(i)}))$$

$\hat{y}^{(i)}$ can be interpreted as the "probability" that class is 1, $0 \leq \hat{y}^{(i)} \leq 1$
ex. $\hat{y}^{(i)} = 0.7$ means there is a 70% chance y is 1

Logistic regression requires a new cost function because $f_{\vec{w},b}(\vec{x})$ for logistic regression is non-convex, trapping gradient descend in local minima.

Cost function

$$J(\vec{w},b) = \frac{1}{m}\sum_{i=1}^{m} L(\hat{y}^{(i)}, \vec{y}_i)$$
$$L(\hat{y}^{(i)}, \vec{y}_i) = \begin{cases} -\log(\hat{y}^{(i)}) & \text{if } \vec{y}_i = 1 \\ -\log(1-\hat{y}^{(i)}) & \text{if } \vec{y}_i = 0 \end{cases}$$

Simplified form

$$L(\hat{y}^{(i)}, \vec{y}_i) = -\vec{y}_i \log(\hat{y}^{(i)}) - (1-\vec{y}_i)\log(1-\hat{y}^{(i)})$$

The loss function will decrease as $\hat{y}^{(i)}$ approaches $\vec{y}_i$ on a graph of L vs f.
$\frac{\partial J(\vec{w},b)}{\partial \vec{w}_j}$ and $\frac{\partial J(\vec{w},b)}{\partial b}$ are the same as in linear regression, just the definition of f has changed.

1.4 Softmax regression

Generalization of logistic regression, y can have more than two possible values.

The most probable value of y is the value that when given to L yields the largest loss.

Calculate z_i with $\vec{x}$ only consisting of data points that have label i. In implementation, set all y values of data points with label equal to i to 1, and 0 for everything else.

n_f = num features
n_y = number of possible y outputs
W is a matrix of dimensions $n_y \times n_f$.
$\vec{b}$, $\vec{z}$, $\vec{a}$ are vectors of length n_y.

$$1 \leq i \leq n_y$$
$$\vec{z}_i = W^{(i)} \cdot \vec{x} + \vec{b}_i$$
$$\vec{a}_i = \frac{e^{\vec{z}_i}}{\sum_{k=1}^{n_y} e^{\vec{z}_k}}$$

$$L(\vec{a}, y) = \begin{cases} -\log \vec{a}_1 \text{ if } y = 1 \\ -\log \vec{a}_2 \text{ if } y = 2 \\ \vdots \\ -\log \vec{a}_n \text{ if } y = n \end{cases} \tag{1}$$

1.5 Feature scaling: z-score normalization

After z-score normalization, all features will have a mean of 0 and a standard deviation of 1
n_f = num features
$\vec{\mu}_j$ = mean of all values for feature j (length n_f)
$\vec{\sigma}_j$ = standard deviation of feature j (length n_f)

$$X_j^{(i)} = \frac{X_j^{(i)} - \vec{\mu}_j}{\vec{\sigma}_j}$$
$$\vec{\mu}_j = \frac{1}{m}\sum_{i=0}^{m-1} X_j^{(i)}$$
$$\vec{\sigma}_j^2 = \frac{1}{m}\sum_{i=0}^{m-1}(X_j^{(i)} - \vec{\mu}_j)^2$$

1.6 Over / underfitting

Underfit / high bias: does not fit training set well ($wx + b$ fit onto data points with $x + x^2$ shape)

Overfit / high variance: fits training set extremely well but does not generalize well ($w_1x + w_2x^2 + w_3x^3 + w_4x^4 + b$ fit onto training set of shape $x + x^2$ can have zero cost but predicts values outside the training set inaccurately)

Addressing overfitting

- Collect more data
- Select features ("Feature selection")
- Reduce size of parameters ("Regularization")

1.6.1 Regularization

Small values of $w_1, w_2, \cdots, w_n, b$ for simpler model, less likely to overfit

Given n_f features, there is no way to tell which features are important and which features should be penalized, so all features are penalized.

$$J_r(\vec{w}, b) = J(\vec{w}, b) + \frac{\lambda}{2m}\sum_{j=1}^{n_f} \vec{w}_j^2$$

Can include b by adding $\frac{\lambda}{2m}b^2$ to J_r but typically doesn't make a large difference.

The extra term in J_r is called the regularization term.

Effectively, $\lambda \propto \frac{1}{w}$. When trying to minimize cost, either the error term or the regularization term must decrease. The larger the lambda, the more the regularization term should decrease to minimize cost, decreasing w parameters.

Regularized linear regression

$$J_r(\vec{w}, b) = \frac{1}{2m}\sum_{i=1}^{m}\left[(f_{\vec{w},b}(X^{(i)}) - \vec{y}_i)^2\right] + \frac{\lambda}{2m}\sum_{j=1}^{n_f} \vec{w}_j^2$$

For gradient descent, only $\frac{\partial J_r}{\partial \vec{w}_j}$ changes (b is not regularized):

$$\frac{\partial J_r}{\partial \vec{w}_j} = \frac{1}{m}\sum_{i=1}^{m}\left[(f_{\vec{w},b}(X^{(i)}) - \vec{y}_i)X_j^{(i)}\right] + \frac{\lambda}{m}\vec{w}_j$$

Regularized logistic regression

$$J_r(\vec{w}, b) = \frac{1}{m}\sum_{i=1}^{m} L(f_{\vec{w},b}(X^{(i)}), \vec{y}_i) + \frac{\lambda}{2m}\sum_{j=1}^{n_f} \vec{w}_j^2$$

For gradient descent, only $\frac{\partial J_r}{\partial w_j}$ changes (b is not regularized):

$$\frac{\partial J_r}{\partial \vec{w}_j} = \frac{1}{m}\sum_{i=1}^{m}\left[(f_{\vec{w},b}(X^{(i)}) - \vec{y}_i)X_j^{(i)}\right] + \frac{\lambda}{m}\vec{w}_j$$

2 Neural networks

n_ℓ = num layers excluding input
$n_n^{[\ell]}$ = n neurons in layer ℓ
n_f = num features
$\vec{W}$ is a vector (length n_ℓ) of matrices of size $n_n^{[\ell]} \times n_n^{[\ell-1]}$
$\vec{x}$ is a vector of outputs from each neuron in previous layer
$\vec{b}$ is a vector (length n_ℓ), holds a bias value for each layer
Z and A: vector (length n_ℓ) of vectors (length $n_n^{[\ell]}$)
g: activation function
$1 \leq i \leq n_\ell$
a (activation) = scalar output of a single neuron
Superscript $[i]$ is used to notate information relating to the ith layer in a neural network.

2.1 Choosing an activation function

sigmoid: $g(z) = \frac{1}{1+e^{-z}}$
tanh: $g(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}$
linear: $g(z) = z$
ReLU: $g(z) = \max(0, z)$
Leaky ReLU: $g(z) = \max(\epsilon z, z)$ where ϵ is a small nonzero positive value < 1

For output layer

Binary classification, y = 0 or 1: use sigmoid
Regression, $-\infty \leq y \leq \infty$: use linear activation function
Regression, y $\geq$ 0: use ReLU

For hidden layer

ReLU is most common

2.2 Forward propagation

Input $A^{[\ell-1]}$, output $A^{[\ell]}$, cache $Z^{[\ell]}$, $W^{[\ell]}$, $\vec{b}^{[\ell]}$

$$\begin{aligned} Z^{[\ell]} &= W^{[\ell]} A^{[\ell-1]} + \vec{b}^{[\ell]} \\ A^{[\ell]} &= g^{[\ell]}(Z^{[\ell]}) \end{aligned}$$

Up to $A^{[n_\ell]}$, in which case $\hat{y} = A_0^{[n_\ell]}$ assuming output layer has one unit

2.3 Back propagation

Input d$a^{[\ell]}$, output d$a^{[\ell-1]}$, d$W^{[\ell-1]}$, d$\vec{b}^{[\ell-1]}$

$$\begin{aligned} \mathrm{d}Z^{[\ell]} &= \mathrm{d}A^{[\ell]} \cdot \frac{\mathrm{d}g^{[\ell]}(Z^{[\ell]})}{\mathrm{d}Z^{[\ell]}} \\ \mathrm{d}W^{[\ell]} &= \frac{1}{m} \mathrm{d}Z^{[\ell]} \cdot A^{[\ell-1]T} \\ \mathrm{d}\vec{b}^{[\ell]} &= \frac{1}{m} \sum_i \mathrm{d}Z_i^{[\ell]} \\ \mathrm{d}A^{[\ell-1]} &= W^{[\ell]T} \cdot \mathrm{d}Z^{[\ell]} \end{aligned}$$