

# Lecture 6: Parametric curves and surfaces, implicit surfaces

DHBW, Computer Graphics

Lovro Bosnar

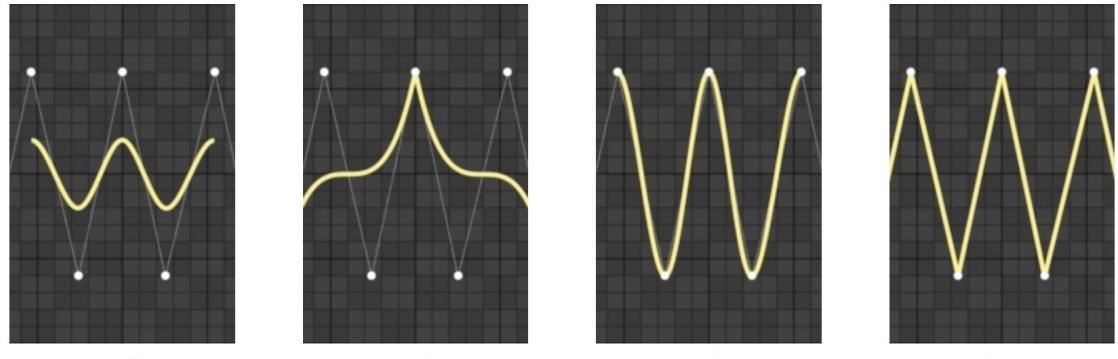
8.2.2023.

# Syllabus

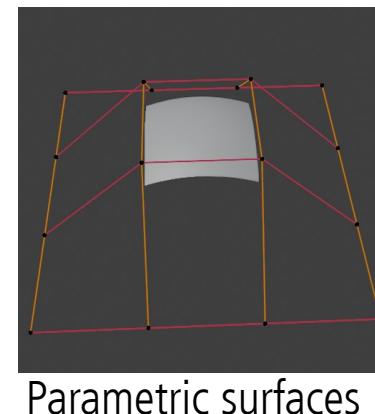
- 3D scene
    - Objects
      - Shape
      - Material
    - Lights
    - Cameras
  - Rendering
  - Image and display
- 
- A blue arrow points from the 'Shape' bullet point under '3D scene' to a rounded rectangular callout box. The callout box contains three bullet points: 'Parametric curves', 'Parametric surfaces', and 'Implicit surfaces'. The 'Shape' bullet point is enclosed in a small blue rectangular box.

# Recap: shape representations

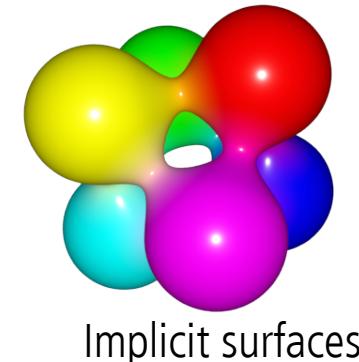
- Points
  - Point clouds
  - Particles and Particle systems
- Lines
  - Parametric curves
- Surfaces:
  - Polygonal mesh
  - Subdivision surfaces
  - **Parametric surfaces**
  - **Implicit surfaces**
- Volumetric objects/solids
  - Voxels
  - Space partitioning data-structures



Parametric curves



Parametric surfaces



Implicit surfaces

# Recap: shape representations

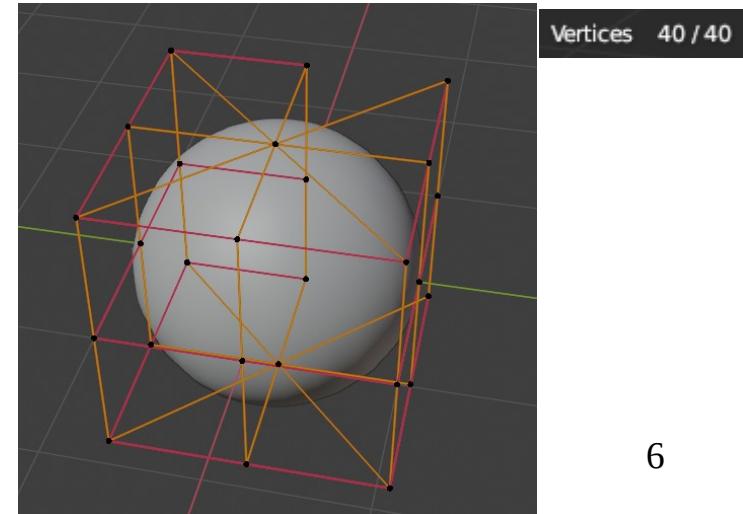
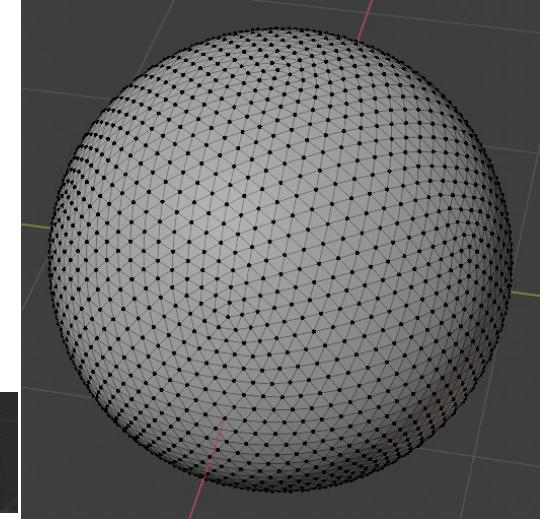
- Mesh surface representation: the most used and widespread
  - **Triangle:** basic atomic rendering primitive for GPU graphics pipelines and most ray-tracers
- Objects made in modeling systems can have many underlying geometric descriptions.
  - Easier and efficient modeling of shapes for the user
  - All higher-level geometrical descriptions are evaluated as set of triangles and then used

# Parametric surfaces vs Mesh

Parametric surfaces have advantages over meshes in certain scenarios:

- More **compact representation** than meshes:
  - Described with small number of control points and parametric equation
  - Less memory requirements
  - Less transformation operations
- **Scalable geometric primitives**
  - Geometry can be generated on the fly by evaluating the equations
- They can represent **smoother and more continuous primitives** than polygons
  - Convenient for representing organic and curved objects
- Certain modeling tasks can be efficiently performed
  - Example: less operations are needed for animation and collision

Vertices	2,562 / 2,562
Edges	7,680 / 7,680
Faces	5,120 / 5,120



# Applications



Paul de Casteljau



1959



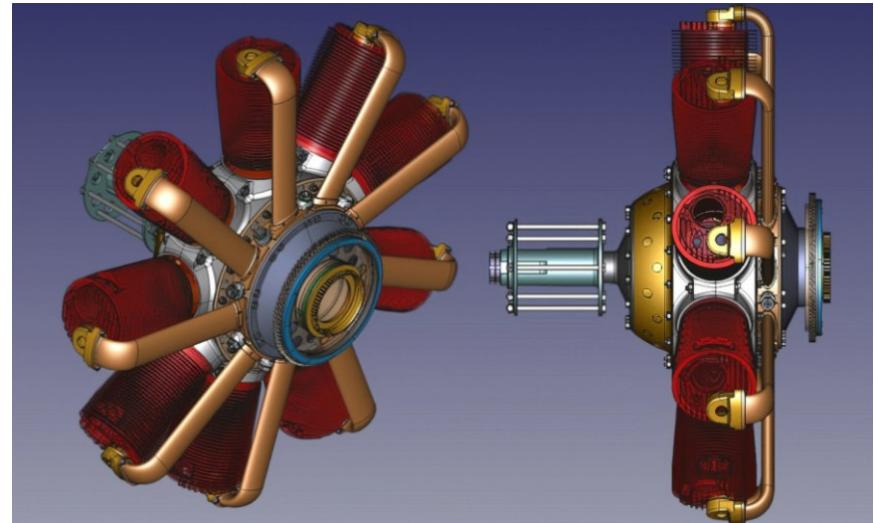
Pierre Bézier



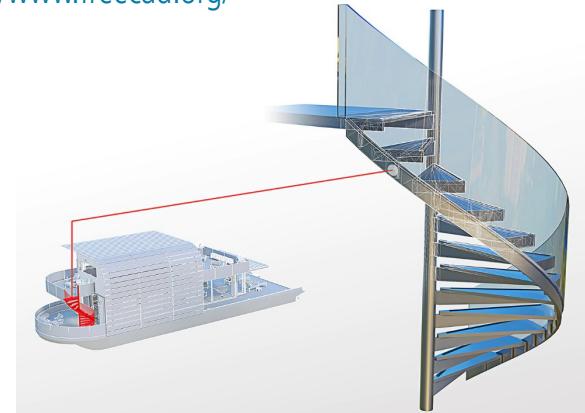
1962

[https://en.wikipedia.org/wiki/Paul\\_de\\_Casteljau](https://en.wikipedia.org/wiki/Paul_de_Casteljau)

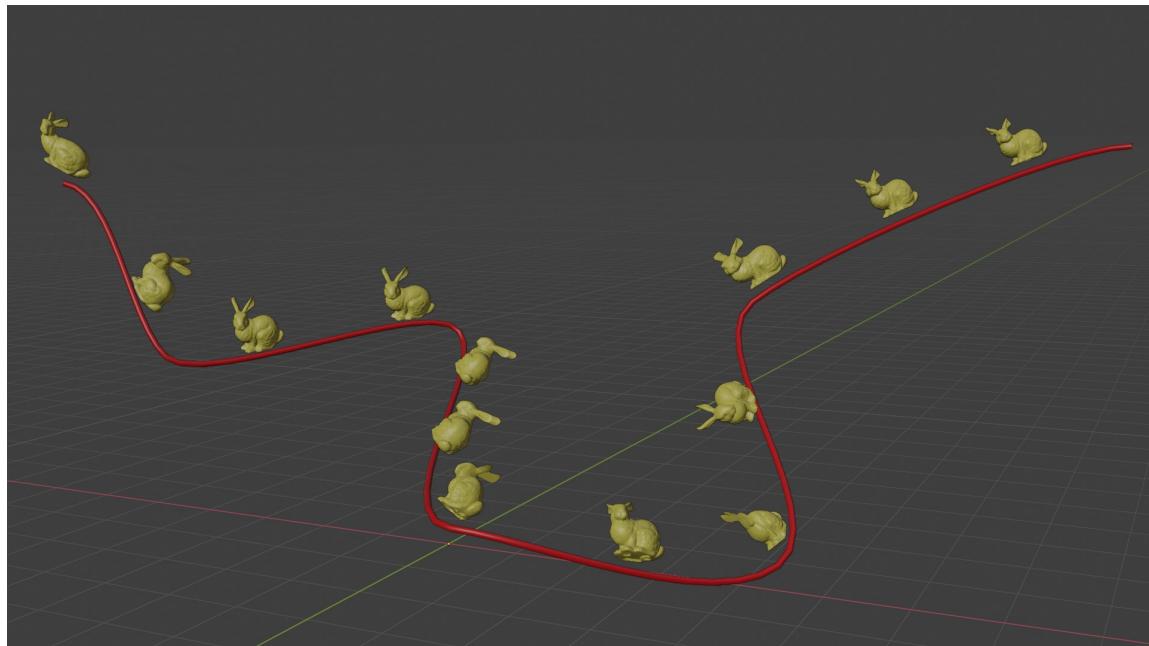
[https://en.wikipedia.org/wiki/Pierre\\_B%C3%A9zier](https://en.wikipedia.org/wiki/Pierre_B%C3%A9zier)



Manufacturing product design using parametric surfaces  
<https://www.freecad.org/>



Architecture and parametric surfaces:  
<https://www.autodesk.com/products/autocad/include-toolsets/autocad-architecture>



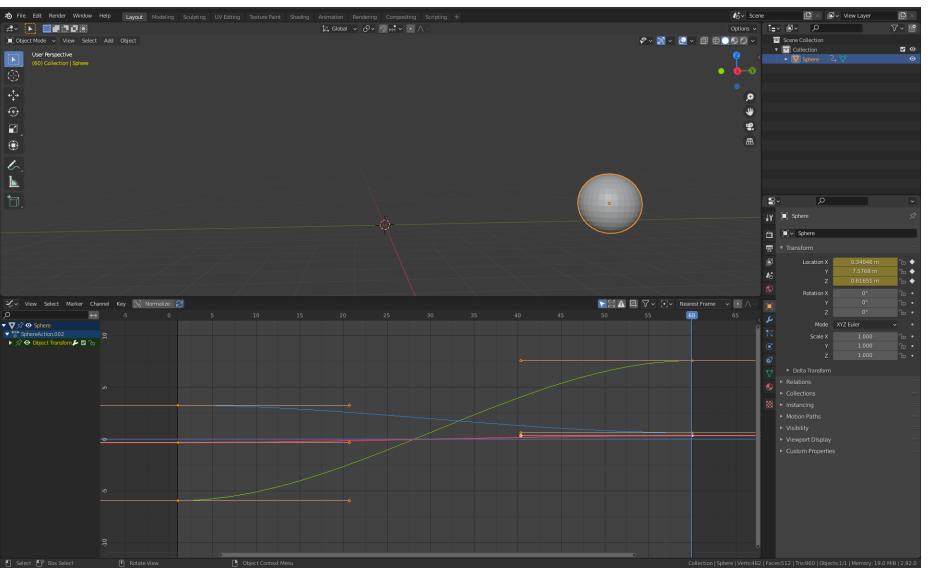
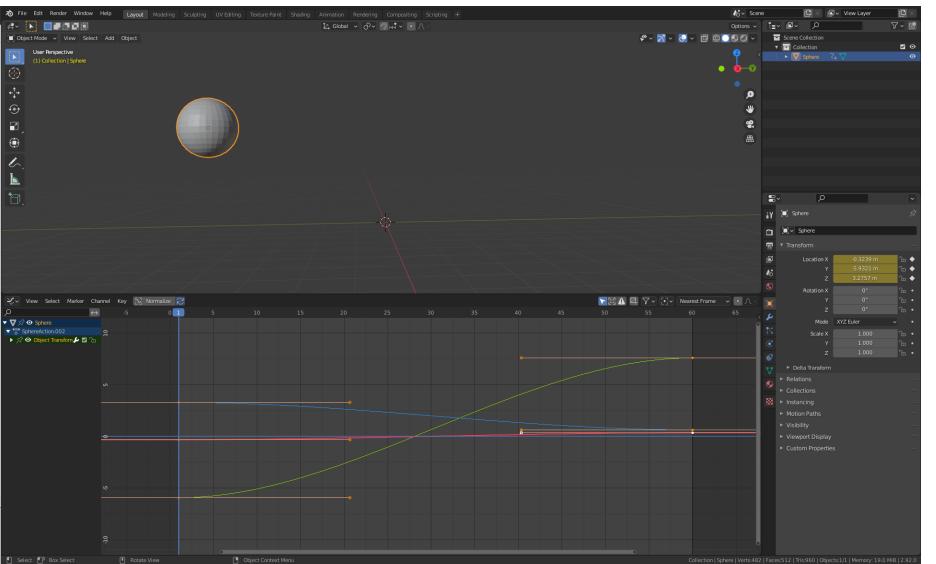
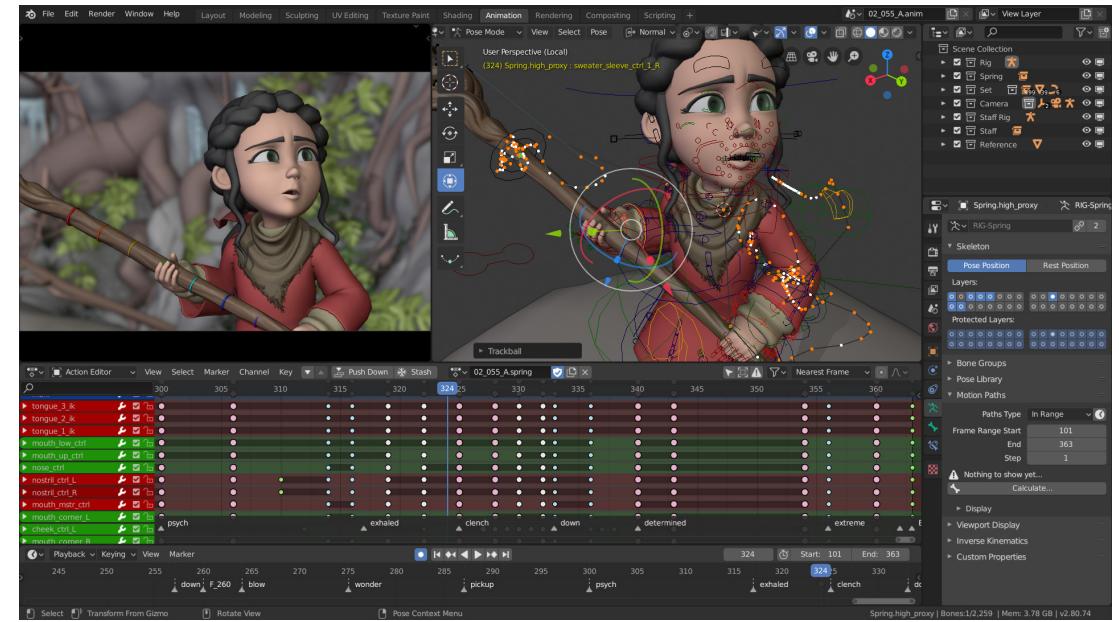
Animation paths using curves:

[https://docs.blender.org/manual/en/latest/animation/constraints/relationship/follow\\_path.html](https://docs.blender.org/manual/en/latest/animation/constraints/relationship/follow_path.html)

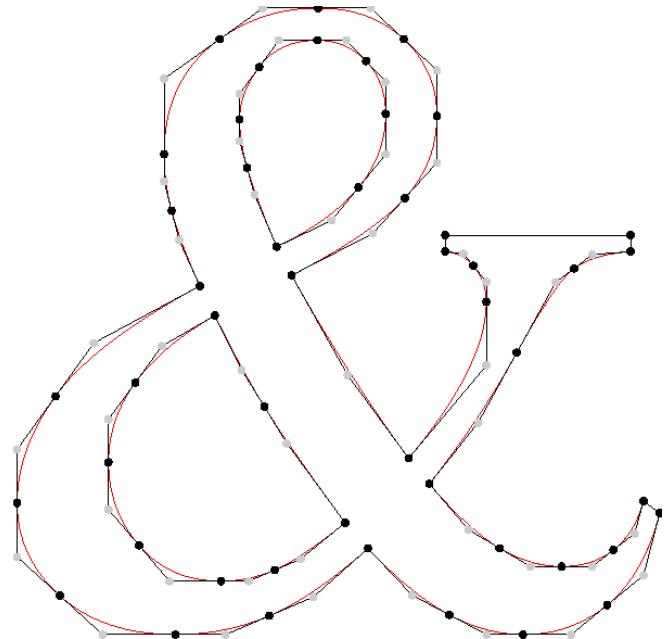


Rendering hair using curves:

<https://developer.nvidia.com/gpugems/gpugems2/part-iii-high-quality-rendering/chapter-23-hair-animation-and-rendering-nalu-demo>



Interpolation between key-frames  
(transforms) using curves:  
[https://docs.blender.org/manual/en/latest/editors/graph\\_editor/introduction.html](https://docs.blender.org/manual/en/latest/editors/graph_editor/introduction.html)



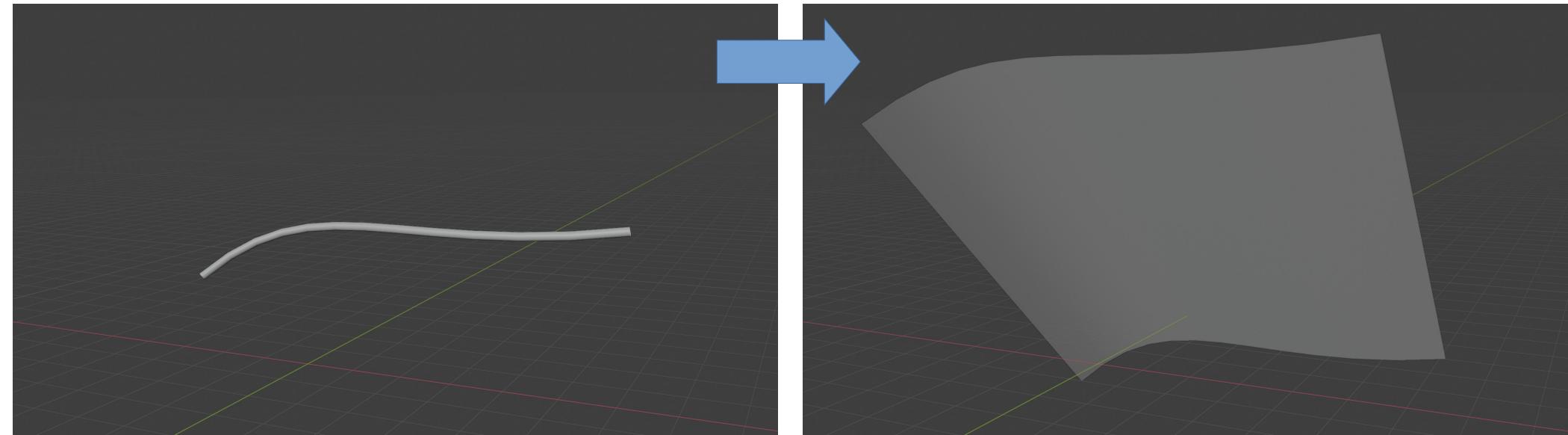
Font design (using Bezier curves):  
<https://font-bakers.github.io/knead/user-guide/>



Character modeling using Bezier patches

# Parametric curves and surfaces

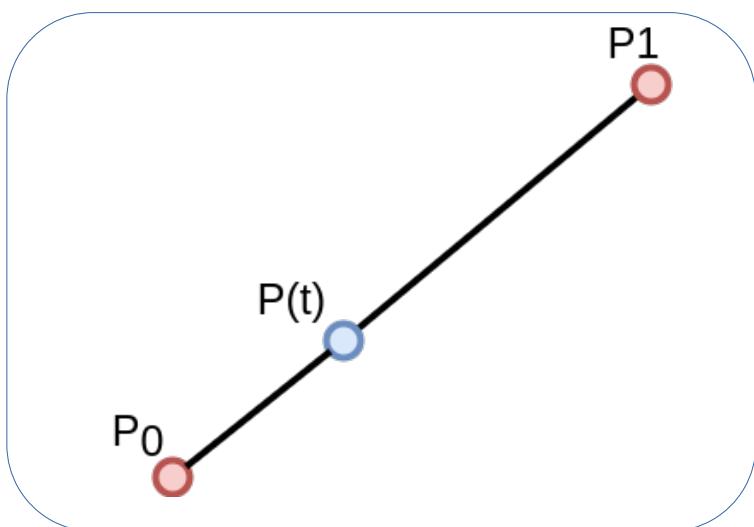
- Parametric surfaces are extension of parametric curves



# Parametric curves

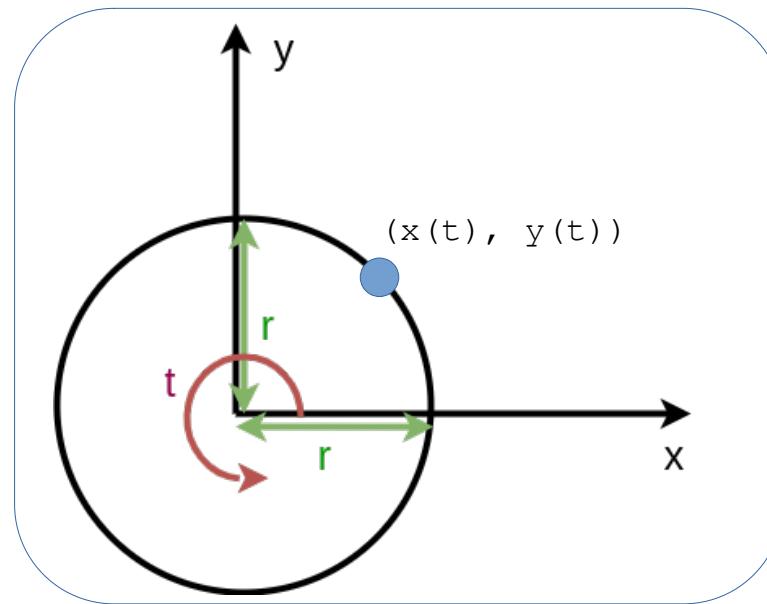
# Parametric curves

- Described with a formula as a function of parameter  $t$ :  $p(t)$ ,  $t$  in  $[a, b]$ 
  - Generated points are continuous



Example: line segment

$$P(t) = (1-t) P_0 + t P_1$$



Example: circle

$$x(t) = r \cos(2 * \pi * t), t \text{ in } [0, 1]$$

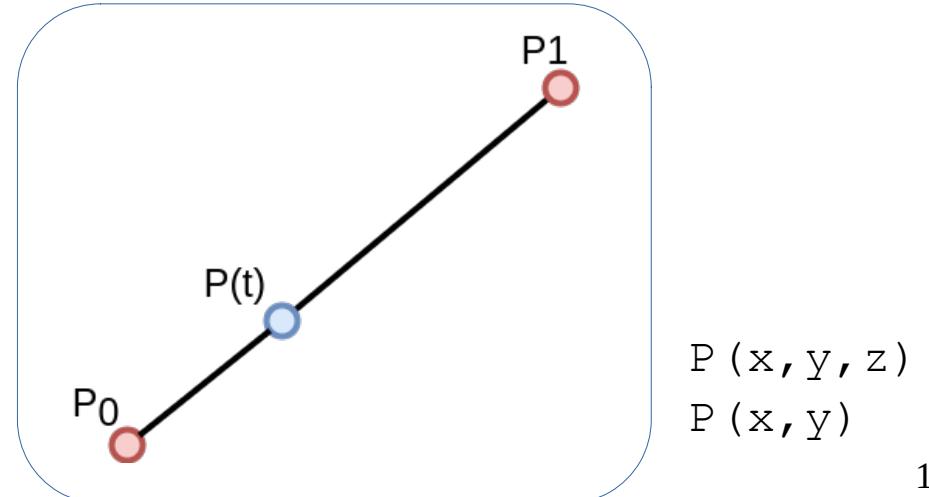
$$y(t) = r \sin(2 * \pi * t), t \text{ in } [0, 1]$$

# Parametric curves

- **Bezier curve**
- Hermite curve
- Catmull-Rom spline
- B-Splines

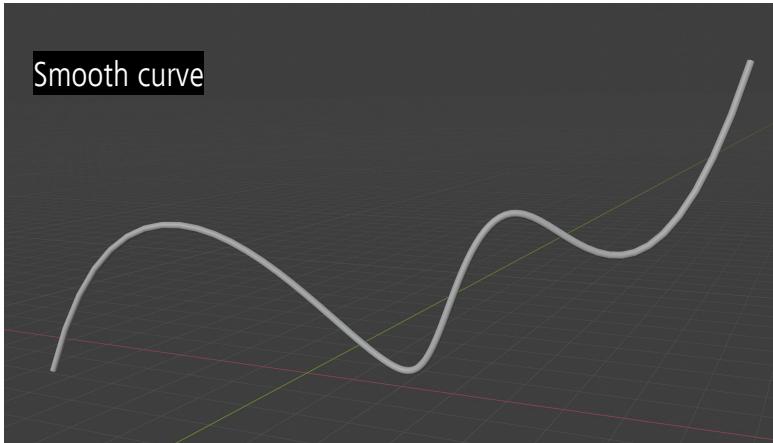
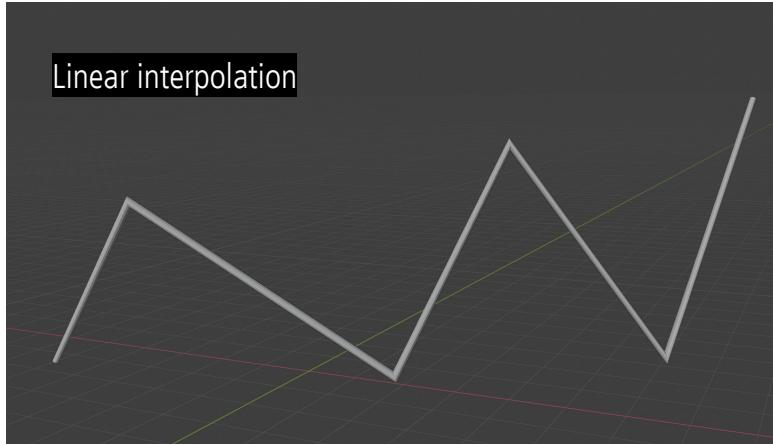
# Bezier curves: linear interpolation

- Linear interpolation between two control points  $P_0$  and  $P_1$  traces out straight line.
  - $P(t) = (1-t) P_0 + t P_1$
  - Short: `lerp( $P_0$ ,  $P_1$ ,  $t$ )`
  - For  $0 < t < 1$ , generated points are on straight line between  $P_0$  and  $P_1$ . Otherwise outside.
  - $P(0) = P_0$  and  $P(1) = P_1$



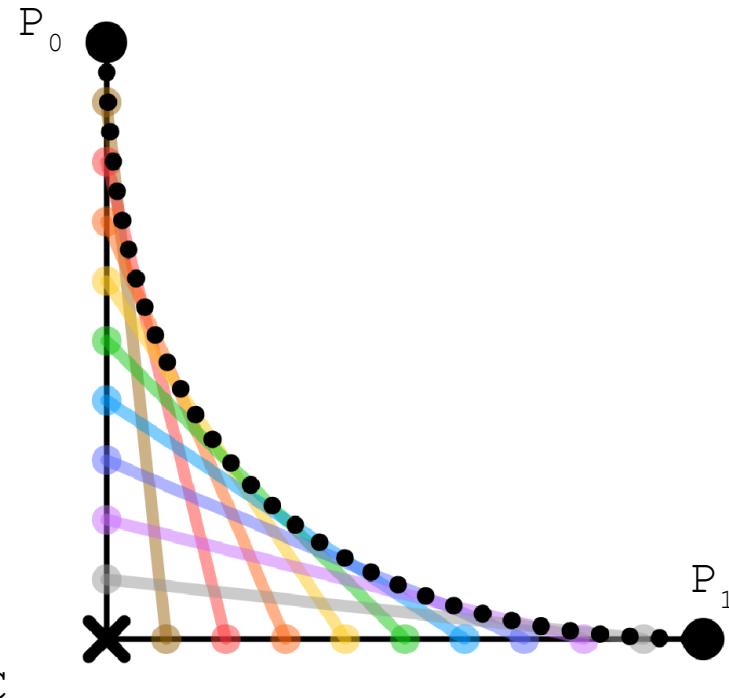
# Bezier curves: linear interpolation

- Linear interpolation between multiple points gives us straight segments with **sudden (discontinuous) changes at points**.



# Bezier curves: repeated interpolation

- Instead of linear interpolation between two points:  $P_0$  and  $P_1$  – **end points**, add another point  $C$  – **control point** and use it for **repeated linear interpolation**



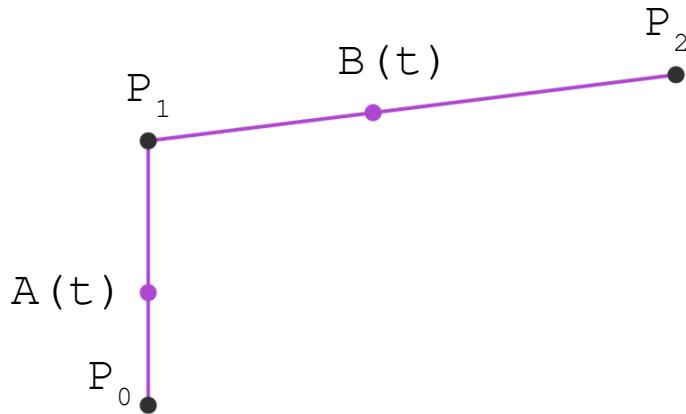
\* Independently discovered by Paul de Casteljau and Pierre Bezier for use in French car industry.

# Bezier curves: repeated interpolation

Linear interpolation



$$A(t) = \text{lerp}(P_0, P_1, t)$$

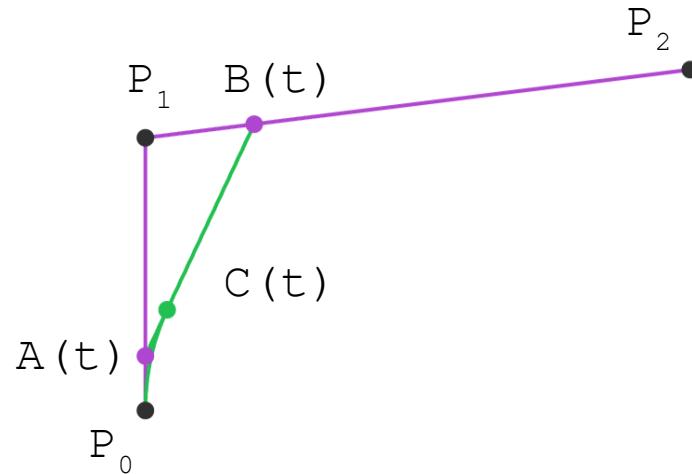


$$A(t) = \text{lerp}(P_0, P_1, t)$$

$$B(t) = \text{lerp}(P_1, P_2, t)$$

# Quadratic Bezier curve

- Repeated interpolation:
  - $A(t)$  : linear interpolation between  $P_0$  and  $P_1$
  - $B(t)$  : linear interpolation between  $P_0$  and  $P_1$
  - $C(t)$  : linear interpolation between  $A(t)$  and  $B(t)$



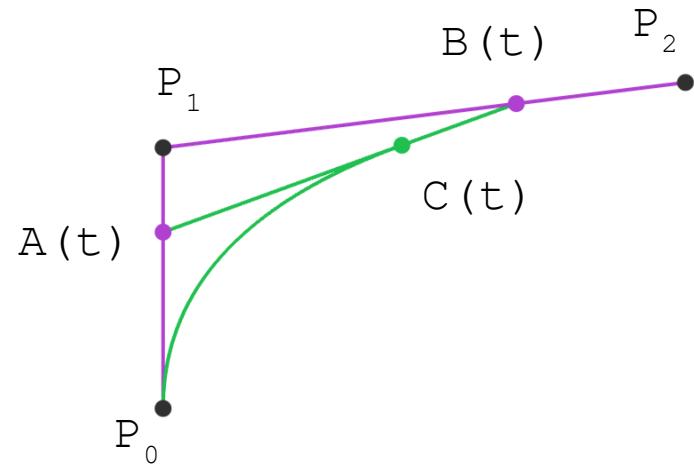
$$A(t) = \text{lerp}(P_0, P_1, t)$$

$$B(t) = \text{lerp}(P_1, P_2, t)$$

$$C(t) = \text{lerp}(A(t), B(t), t)$$

# Quadratic Bezier curve

- Repeated interpolation:
  - $A(t)$ : linear interpolation between  $P_0$  and  $P_1$
  - $B(t)$ : linear interpolation between  $P_0$  and  $P_2$
  - $C(t)$  : linear interpolation between  $A(t)$  and  $B(t)$



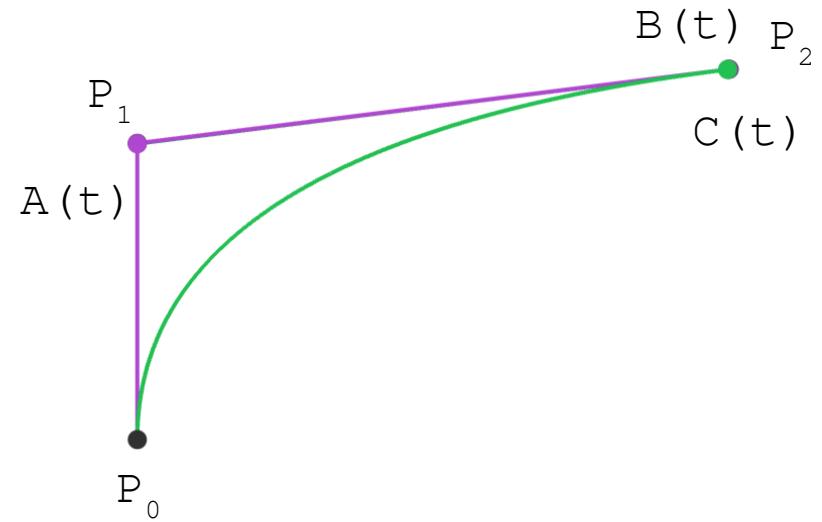
$$A(t) = \text{lerp}(P_0, P_1, t)$$

$$B(t) = \text{lerp}(P_1, P_2, t)$$

$$C(t) = \text{lerp}(A(t), B(t), t)$$

# Quadratic Bezier curve

- Repeated interpolation:
  - $A(t)$ : linear interpolation between  $P_0$  and  $P_1$
  - $B(t)$ : linear interpolation between  $P_0$  and  $P_2$
  - $C(t)$  : linear interpolation between  $A(t)$  and  $B(t)$

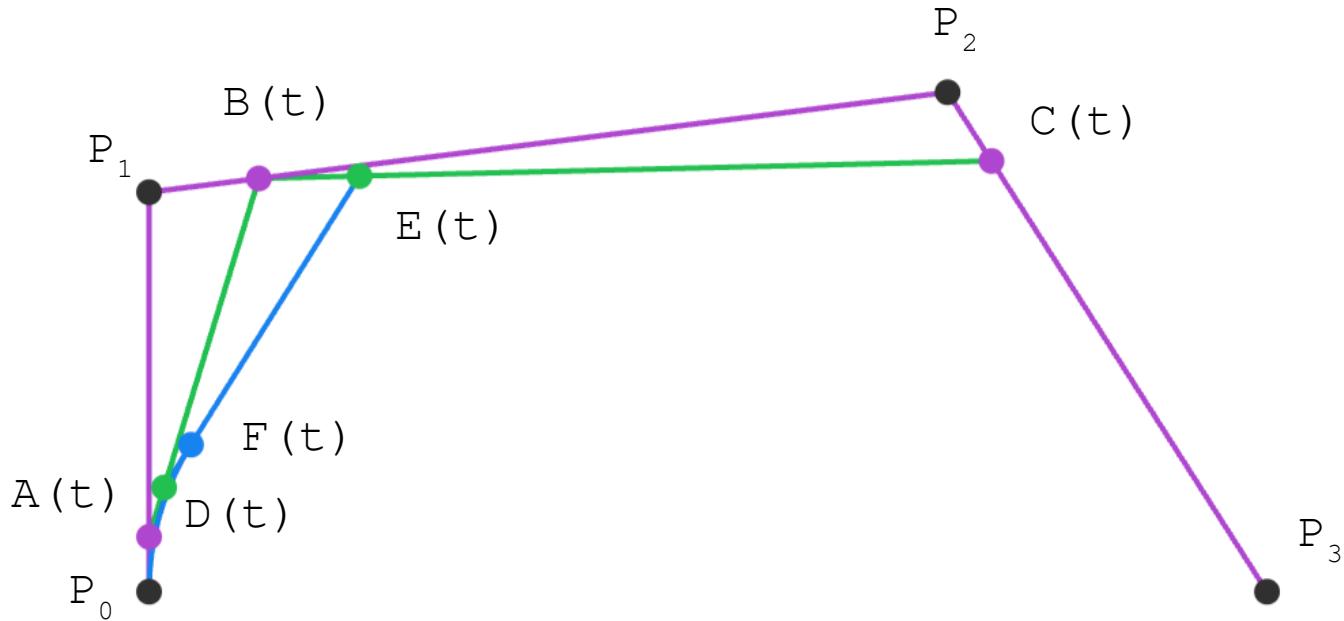


$$A(t) = \text{lerp}(P_0, P_1, t)$$

$$B(t) = \text{lerp}(P_1, P_2, t)$$

$$C(t) = \text{lerp}(A(t), B(t), t)$$

# Cubic Bezier curve



$$A(t) = \text{lerp}(P_0, P_1, t)$$

$$B(t) = \text{lerp}(P_1, P_2, t)$$

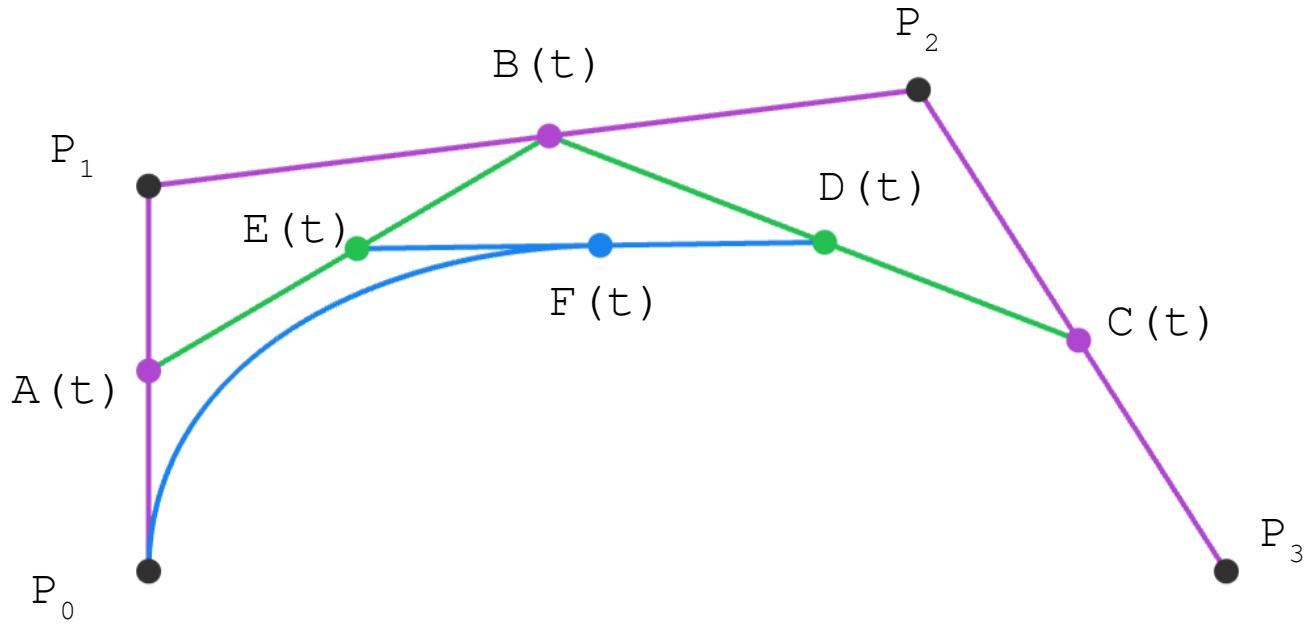
$$C(t) = \text{lerp}(P_2, P_3, t)$$

$$D(t) = \text{lerp}(A(t), B(t), t)$$

$$E(t) = \text{lerp}(B(t), C(t), t)$$

$$F(t) = \text{lerp}(D(t), E(t), t)$$

# Cubic Bezier curve



$$A(t) = \text{lerp}(P_0, P_1, t)$$

$$B(t) = \text{lerp}(P_1, P_2, t)$$

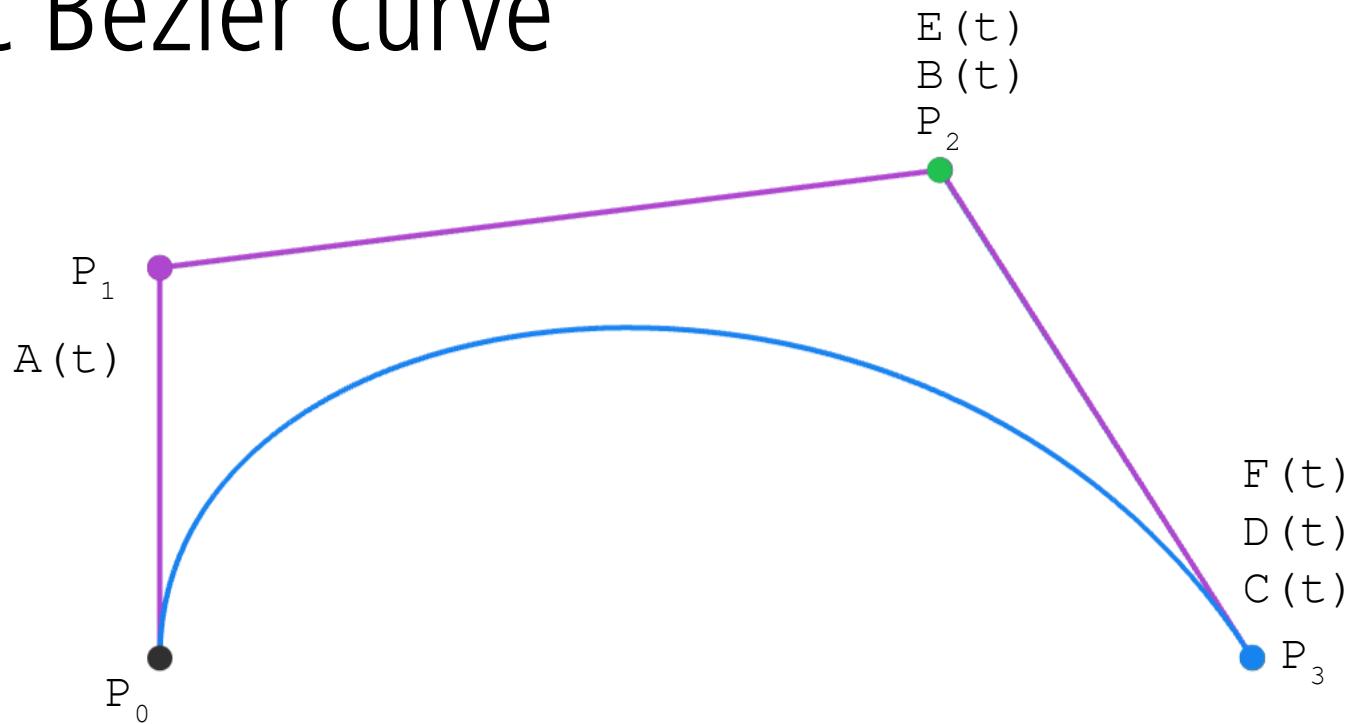
$$C(t) = \text{lerp}(P_2, P_3, t)$$

$$D(t) = \text{lerp}(A(t), B(t), t)$$

$$E(t) = \text{lerp}(B(t), C(t), t)$$

$$F(t) = \text{lerp}(D(t), E(t), t)$$

# Cubic Bezier curve



$$A(t) = \text{lerp}(P_0, P_1, t)$$

$$B(t) = \text{lerp}(P_1, P_2, t)$$

$$C(t) = \text{lerp}(P_2, P_3, t)$$

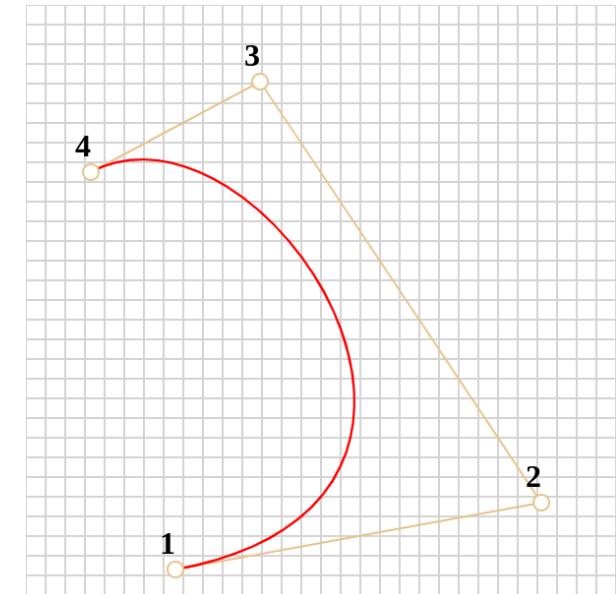
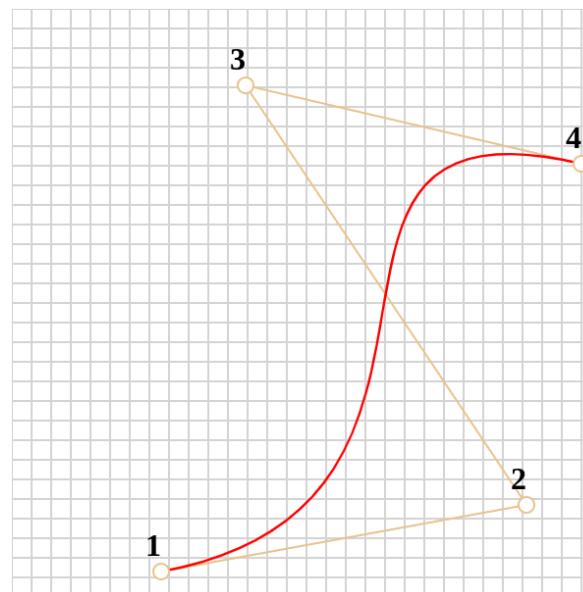
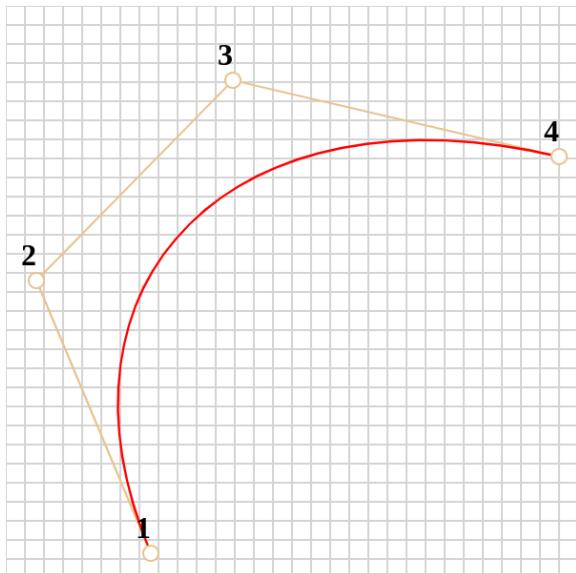
$$D(t) = \text{lerp}(A(t), B(t), t)$$

$$E(t) = \text{lerp}(B(t), C(t), t)$$

$$F(t) = \text{lerp}(D(t), E(t), t)$$

# Control points are for the user

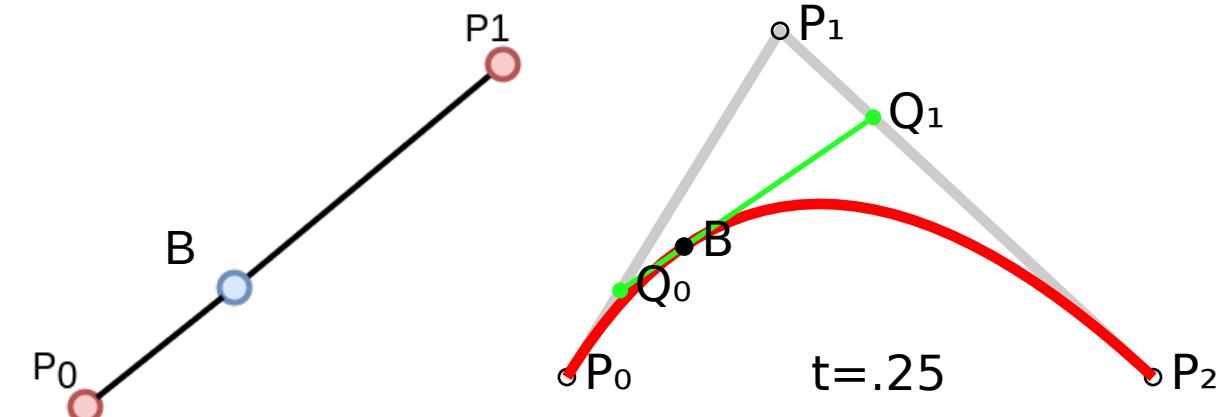
The point of curves in computer graphics is to interpolate between two or more points and that the shape of curve is modifiable via control points by the user to their liking.



# Bezier curves

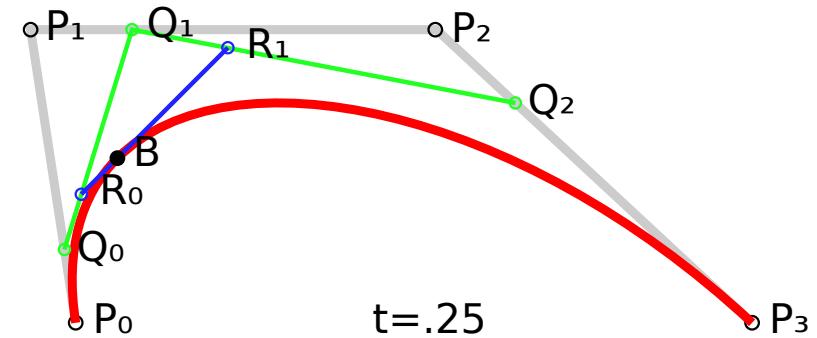
- Degree of curve is  $n$ , if  $n + 1$  control points are used.
  - More control points → more degrees of freedom
- Most often, cubic Bezier curve is used
  - Higher degree Bezier curves are expensive to evaluate and harder to control

$n = 1 \rightarrow 1^{\text{st}}$  degree, linear Bezier



$n = 2 \rightarrow 2^{\text{nd}}$  degree, quadratic Bezier

$n = 3 \rightarrow 3^{\text{rd}}$  degree, cubic Bezier



# Bezier curves: de Casteljau algorithm

- Repeated or recursive linear interpolation is often referred as **de Casteljau algorithm**.

Linear Bezier:

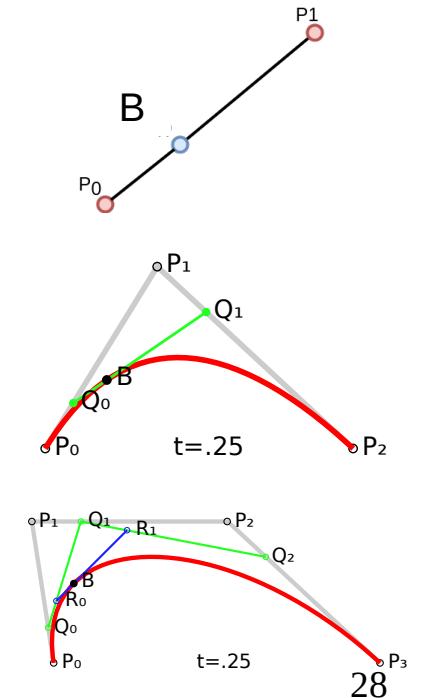
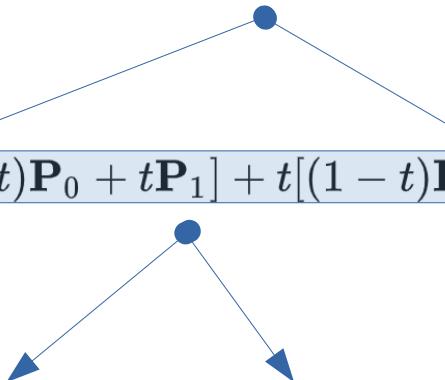
$$\mathbf{B}(t) = \mathbf{P}_0 + t(\mathbf{P}_1 - \mathbf{P}_0) = (1-t)\mathbf{P}_0 + t\mathbf{P}_1, \quad 0 \leq t \leq 1$$

Quadratic Bezier:

$$\mathbf{B}(t) = (1-t)[(1-t)\mathbf{P}_0 + t\mathbf{P}_1] + t[(1-t)\mathbf{P}_1 + t\mathbf{P}_2], \quad 0 \leq t \leq 1,$$

Cubic Bezier:

$$\mathbf{B}(t) = (1-t)\mathbf{B}_{\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2}(t) + t\mathbf{B}_{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3}(t), \quad 0 \leq t \leq 1.$$

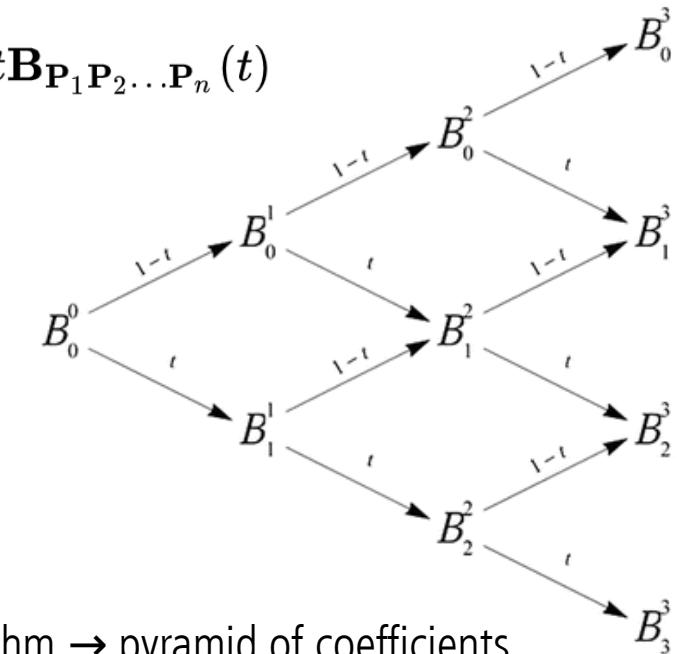


# De Casteljau algorithm: recursive formula\*

- Recursive formula for arbitrary Bezier curve degree:

$\mathbf{B}_{\mathbf{P}_0}(t) = \mathbf{P}_0$ , and

$$\mathbf{B}(t) = \mathbf{B}_{\mathbf{P}_0 \mathbf{P}_1 \dots \mathbf{P}_n}(t) = (1-t)\mathbf{B}_{\mathbf{P}_0 \mathbf{P}_1 \dots \mathbf{P}_{n-1}}(t) + t\mathbf{B}_{\mathbf{P}_1 \mathbf{P}_2 \dots \mathbf{P}_n}(t)$$



De Casteljau algorithm → pyramid of coefficients

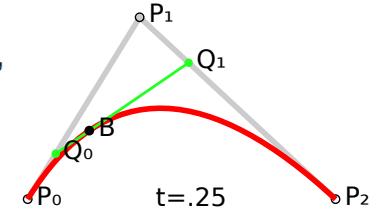
# Bezier curve: Bernstein form

- Quadratic Bezier:  $\mathbf{B}(t) = (1 - t)[(1 - t)\mathbf{P}_0 + t\mathbf{P}_1] + t[(1 - t)\mathbf{P}_1 + t\mathbf{P}_2]$ ,  $0 \leq t \leq 1$ ,
- Quadratic Bezier re-arranged  $\rightarrow$  algebraic description

$$\mathbf{B}(t) = (1 - t)^2 \mathbf{P}_0 + 2(1 - t)t \mathbf{P}_1 + t^2 \mathbf{P}_2, \quad 0 \leq t \leq 1$$

- Every Bezier curve can be described with algebraic fromula**
  - repeated interpolation is not needed.
- Generalized algebraic description: **Bernstein form**:

$$\begin{aligned}\mathbf{B}(t) &= \sum_{i=0}^n \binom{n}{i} (1 - t)^{n-i} t^i \mathbf{P}_i \\ &= (1 - t)^n \mathbf{P}_0 + \binom{n}{1} (1 - t)^{n-1} t \mathbf{P}_1 + \cdots + \binom{n}{n-1} (1 - t)^1 t^{n-1} \mathbf{P}_{n-1} + t^n \mathbf{P}_n, \quad 0 \leq t \leq 1\end{aligned}$$

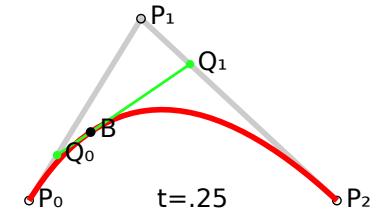


# Bezier curve: Bernstein form

- Example:  $n = 2$  (quadratic)

Bernstein  
form

$$B(t) = \sum_{i=0}^2 \binom{2}{i} t^i (1-t)^{(2-i)} P_i$$



$$B(t) = \binom{2}{0} t^0 (1-t)^2 P_0 + \binom{2}{1} t^1 (1-t)^1 P_1 + \binom{2}{2} t^2 (1-t)^0 P_2, \quad 0 \leq t \leq 1$$

$$\mathbf{B}(t) = (1-t)^2 \mathbf{P}_0 + 2(1-t)t \mathbf{P}_1 + t^2 \mathbf{P}_2, \quad 0 \leq t \leq 1$$

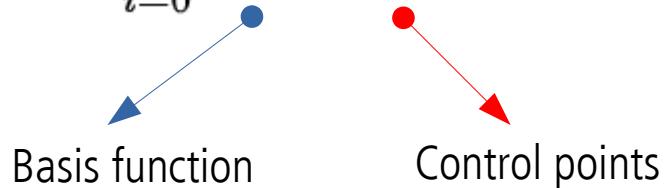
"Repeated  
interpolation"

$$\mathbf{B}(t) = (1-t)[(1-t)\mathbf{P}_0 + t\mathbf{P}_1] + t[(1-t)\mathbf{P}_1 + t\mathbf{P}_2], \quad 0 \leq t \leq 1,$$

# Bezier curve: Bernstein form

- Bernstein form:

$$\mathbf{B}(t) = \sum_{i=0}^n b_{i,n}(t) \mathbf{P}_i, \quad 0 \leq t \leq 1$$



- Bernstein polynomials aka Bezier basis functions:

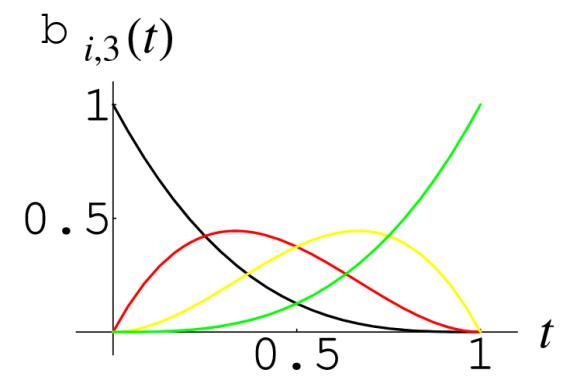
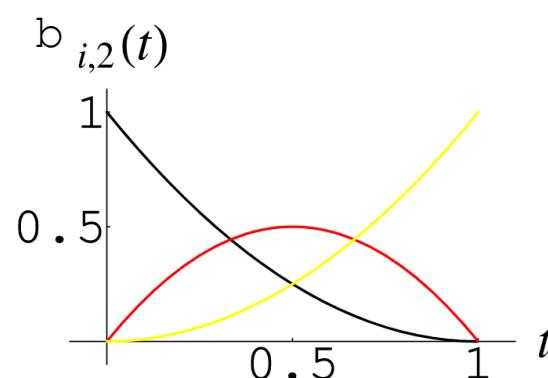
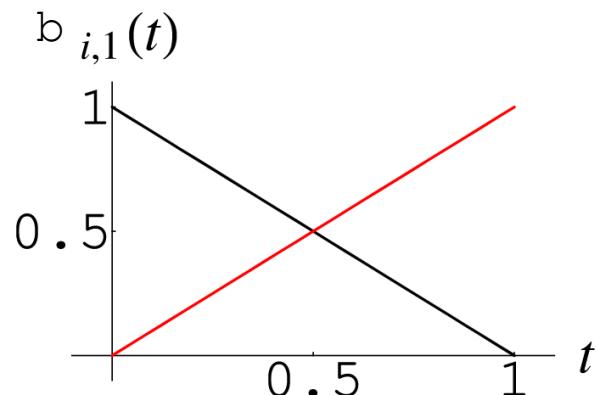
$$b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i = 0, \dots, n$$

# Basis functions

- Defines influence (weighting) of each control point as parameter  $t$  changes.
  - Example: when  $t$  increases, blending weight for  $P_0$  decreases and blending weight for  $P_1$  increases, etc.

$$\mathbf{B}(t) = \sum_{i=0}^n b_{i,n}(t) \mathbf{P}_i, \quad 0 \leq t \leq 1$$

$$b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i = 0, \dots, n$$



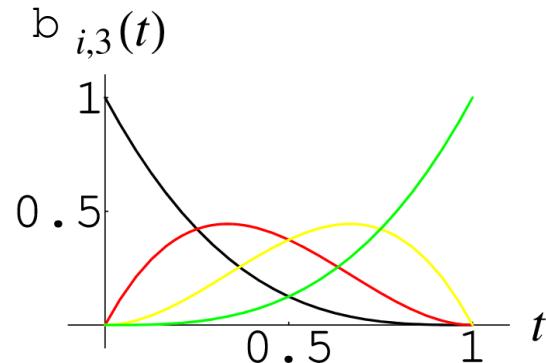
Bernstein polynomials aka blending functions: linear, quadratic, cubic

# Bernstein polynomials

- Bernstein polynomials (Bezier basis function) defines properties of a Bezier curve:

$$b_{i,n}(t) \in [0, 1] \text{ when } t \in [0, 1]$$

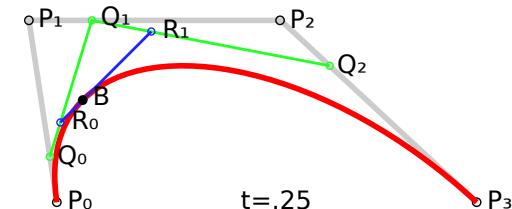
"Polynomials are in  $[0, 1]$  when  $t$  in  $[0, 1]$ "



$$\sum_{i=0}^n b_{i,n}(t) = 1$$

"Curve will stay close to the control points  $P_i$ "

- Whole Bezier curve will be located in convex hull of control points
  - useful for computing bounding area or volume of curve.

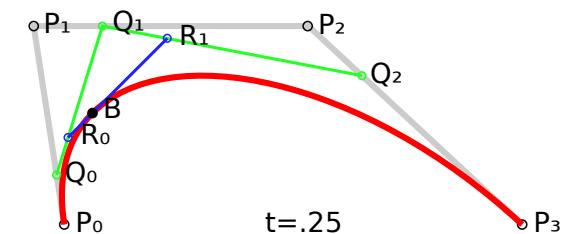


# Bezier curve: matrix representation

- In practice, for efficient calculation, Bezier curve is represented in a **matrix form**
- Example: **cubic Bezier curve**

$$\mathbf{B}(t) = (1 - t)^3 \mathbf{P}_0 + 3(1 - t)^2 t \mathbf{P}_1 + 3(1 - t)t^2 \mathbf{P}_2 + t^3 \mathbf{P}_3, \quad 0 \leq t \leq 1$$

$$B(t) = (1 \ t \ t^2 \ t^3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ 1 & 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$



# Bezier curves: good properties

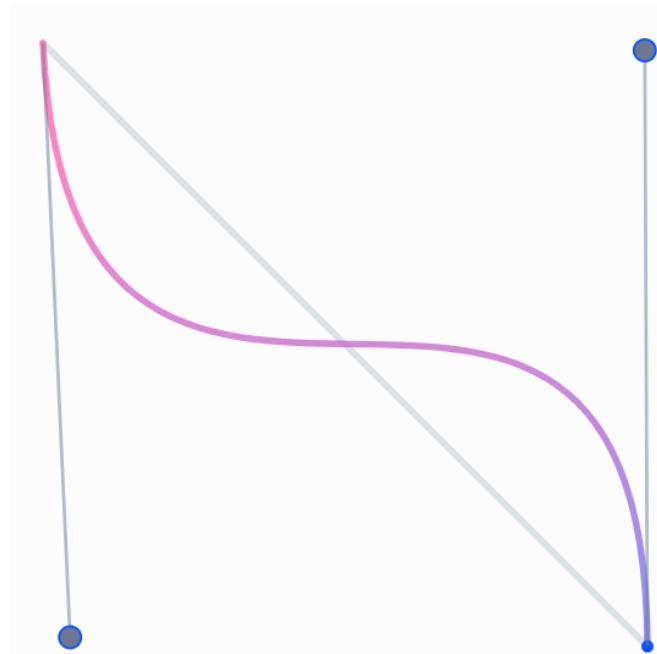
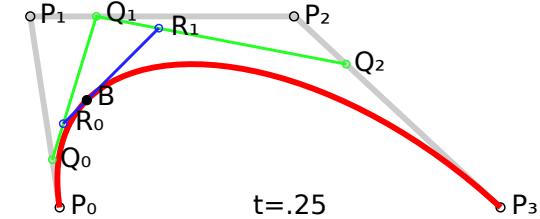
- **Intuitive theory**, good for understanding curves: repeated interpolation
- **Compact form**: Bernstein form and matrix representation
- **Tangent vector at curve point** can be calculated easily (derivation of polynomial)
- **Easy manipulation of large number of points**
  - Large number of points can be generated from parametric description
  - If rotation of those points is needed, then curve (few control points) are rotated and then points are generated

# Bezier curves: problems

- **Non-interpolating:** do not pass through all control points (except endpoints)
- Not many degrees of freedom:
  - Only position of control points can be chosen freely
  - Not every curve can be described with Bezier curve (e.g., simple circle can not be described with one or collection of Bezier curves)
  - Alternative is rational Bezier curve\*
- Degree increases with number of control points
  - Hard to control and complex computation
  - Bernstein polynomials do not interpolate well

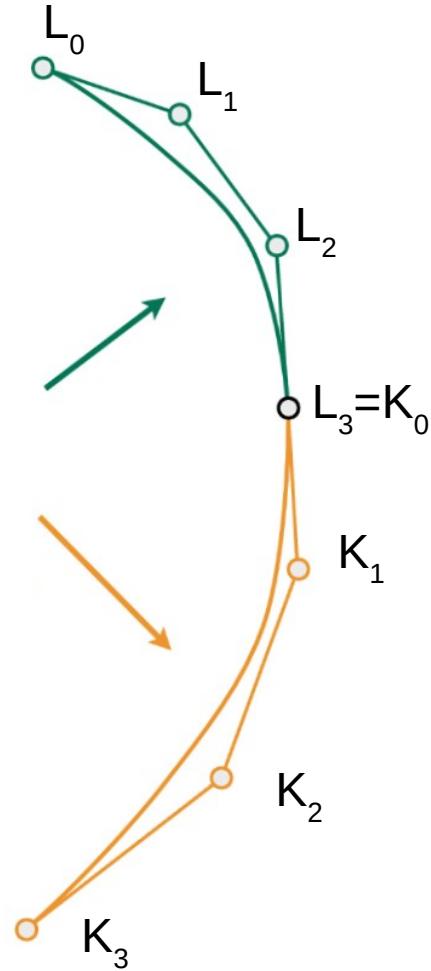
# Joining cubic Bezier curves

- In practice, cubic Bezier curves are concatenated to form larger spline
  - Cubic curves are lowest degree curves that can describe complex form, e.g., S-shaped curve called **inflection**
  - Lower computation complexity
  - Easy to manipulate with
  - Resulting composite curve will pass through the set of points



# Joining cubic Bezier curves

- Example: two cubic Bezier curves (4 control points):
  - $L_i$ ,  $i = 0, 1, 2, 3$
  - $K_i$ ,  $i = 0, 1, 2, 3$
- To join the curves we can set  $L_3 = K_0$ 
  - **Joint** - point where curves are joined
- This way, many cubic Bezier curves are chained into composite bezier curve

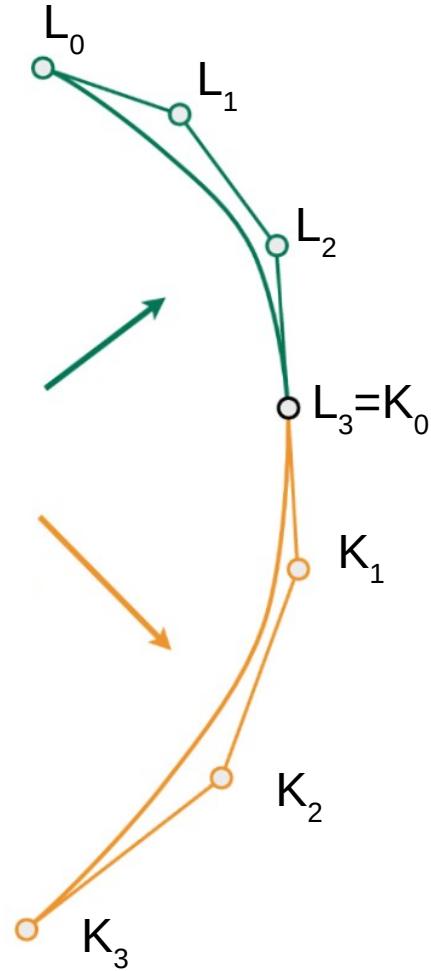


# Joining cubic Bezier curves

- Composite curve is called **piecewise Bezier curve,  $p(t)$** 
  - First segment is  $p(t)$ , where  $t$  in  $[0, 1]$
  - Second segment is  $p(t')$ , where  $t' = (t - t_1) / (t_2 - t_1)$ ,  $t$  in  $[0, 1]$
- Two curves connected just using  $L_3 = K_0$  will not be smooth at joint.
  - Improved smoothness can be achieved by **joining tangentially**: tangent of first and second curve at joint must be parallel\*:

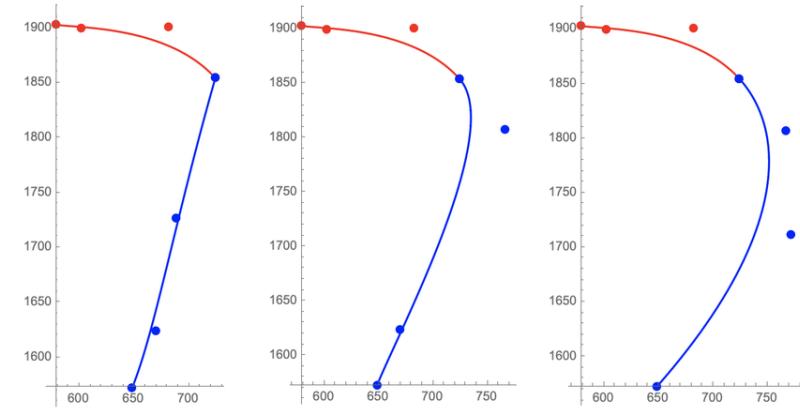
$$(K_1 - K_0) = c(L_3 - L_2), c > 0$$

$$c = \frac{t_2 - t_1}{t_1 - t_0}$$



# Joined curves continuity

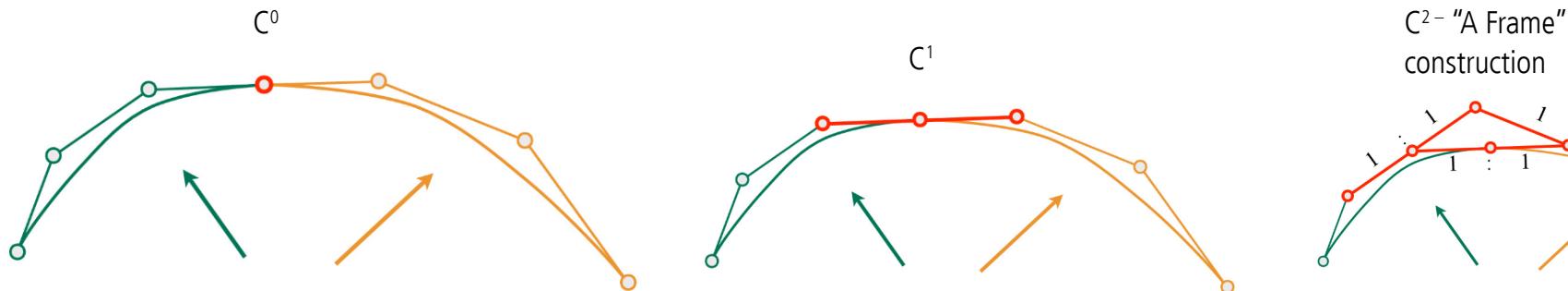
- Continuity of joined curves:
  - $C^0$  – **positional continuity** - segments should joint at the same point
  - $C^1$  – **velocity continuity** - derivation of any point (also joints) must be continuous
  - $C^2$  - **acceleration continuity** - first and second derivatives are continuous functions



$C^0$

$C^1$

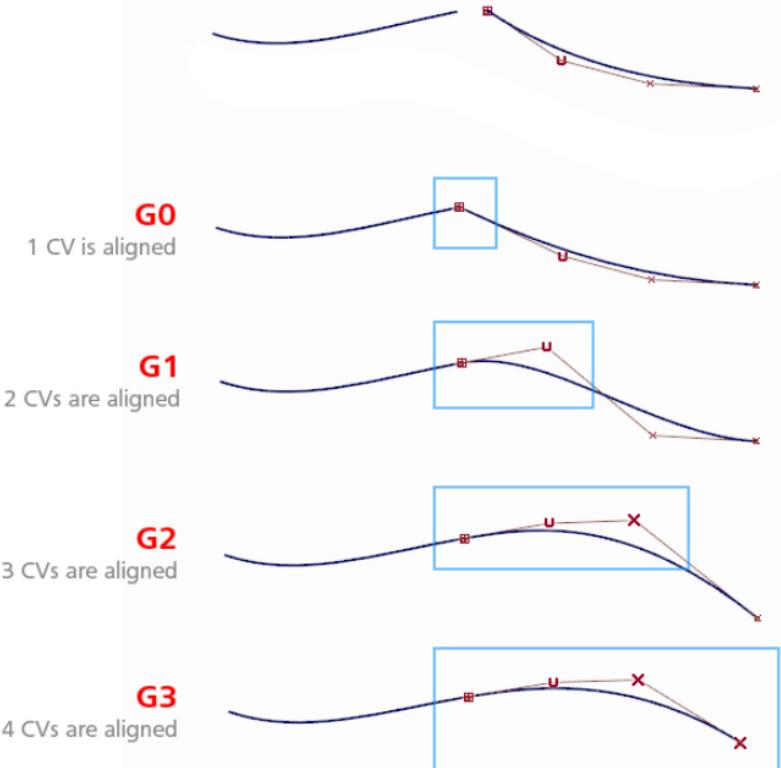
$C^2$



41

# Joined curves continuity

- Geometrical continuity:
  - $G^0$  – **positional continuity**: holds when the end points of two curves coincide
  - $G^1$  – **tangent continuity** - tangent vectors from curve segments that meet at joint should be parallel and have same direction – no sharp edges
  - $G^2$  – **curvature continuity** - tangent vectors from curve segments that meet at joint should be of same length and rate of length change

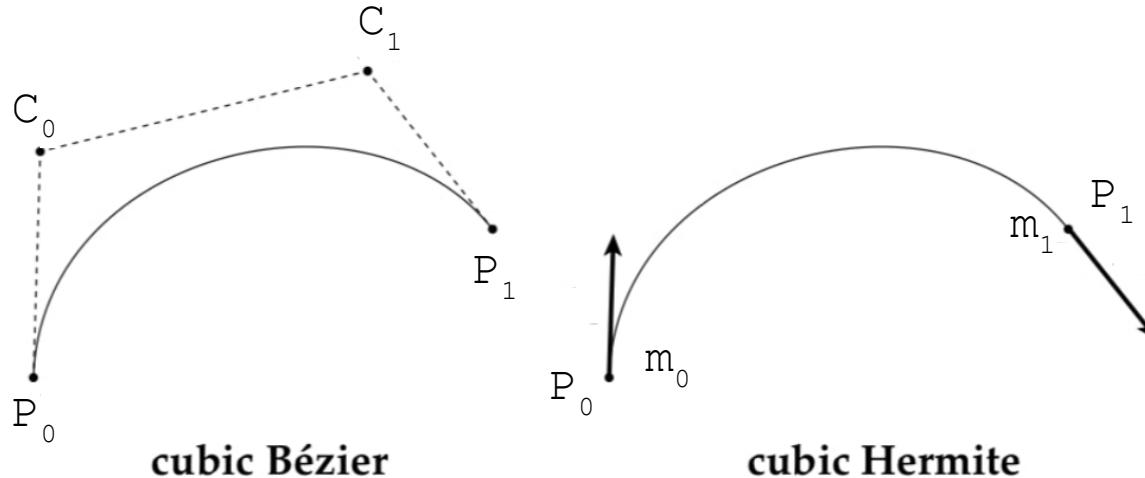


# Parametric curves

- Bezier curve
  - Intuitive theory: repeated interpolation de Casteljau algorithm
  - Formalization: Bernstein and matrix form
  - Concatenating cubic curves into composite curves
  - Curve continuity
- **Hermite curve**
  - In practice, **cubic Hermite curve** is used for simpler authoring
- Catmull-Rom spline
- B-Splines

# Cubic Hermite curve

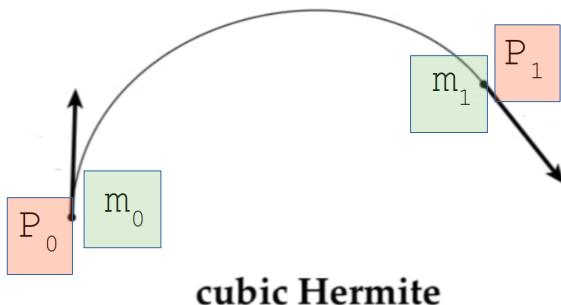
- Cubic Hermite interpolation curve:
  - Interpolates points  $P_0$  and  $P_1$  – **starting and ending control points**
  - Interpolation is **controlled with tangent vectors**:  $m_0$  at  $P_0$  and  $m_1$  at  $P_1$ 
    - Tangent vector magnitude and direction influence the curve shape



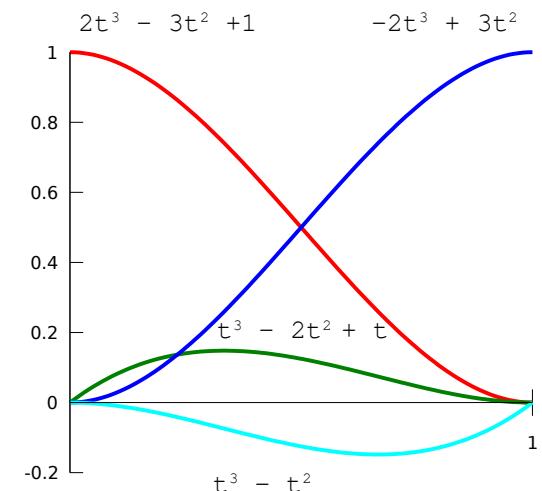
# Cubic Hermite curve

- Cubic Hermite curve  $p(t)$ ,  $t$  in  $[0, 1]$ :

$$p(t) = (2t^3 - 3t^2 + 1)p_0 + (t^3 - 2t^2 + t)m_0 + (-2t^3 + 3t^2)p_1 + (t^3 - t^2)m_1$$



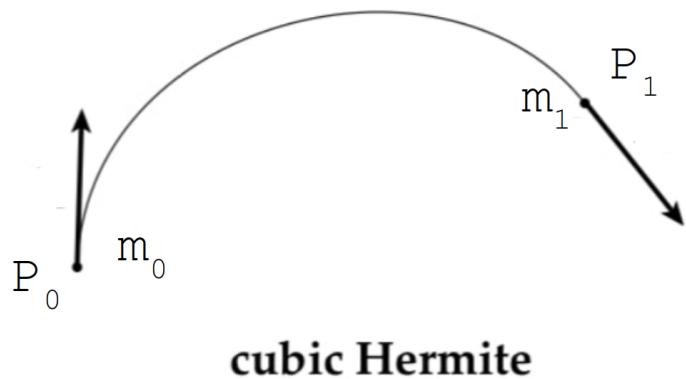
- Similarly as discussed with Bezier: **Bernstein form** with **Hermite basis function** exists
  - Control points polynomial are in  $[0, 1]$  for  $t$  in  $[0, 1]$ . This is not the case for tangent polynomials



# Cubic Hermite curve

- In practice, cubic Hermite interpolation curve is represented with **matrix form** for efficient computation.

$$\mathbf{p}(t) = (2t^3 - 3t^2 + 1)\mathbf{p}_0 + (t^3 - 2t^2 + t)\mathbf{m}_0 + (-2t^3 + 3t^2)\mathbf{p}_1 + (t^3 - t^2)\mathbf{m}_1$$

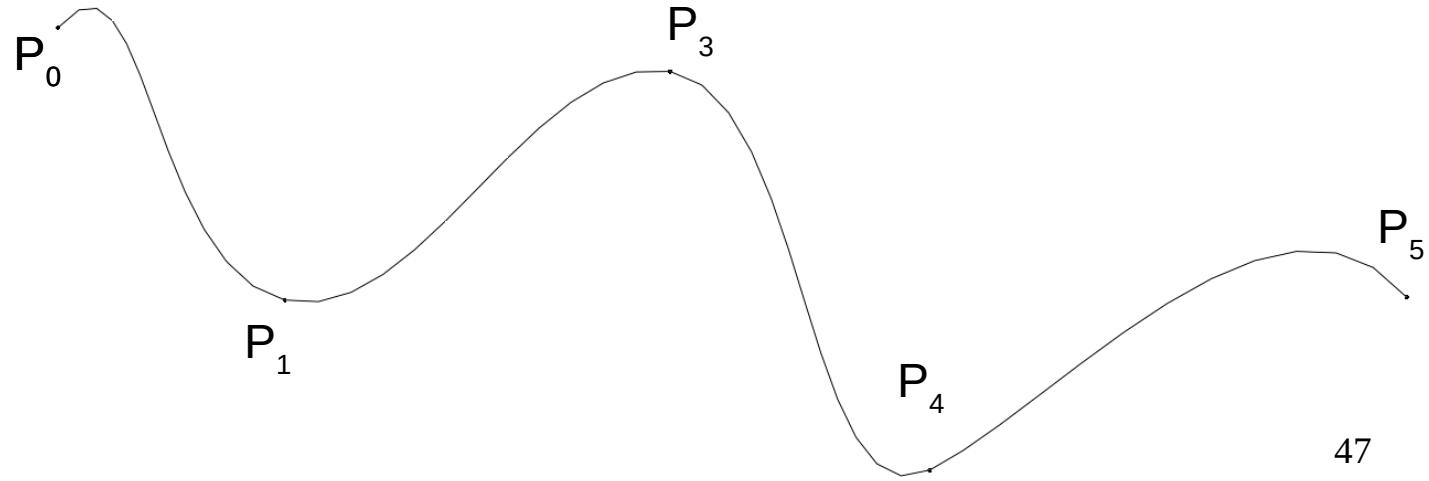


$$p(t) = (t^3 \ t^2 \ t \ 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ m_0 \\ m_1 \end{pmatrix}$$

# Splines: assembly of curves

- When interpolating more than two points, several Hermite curves can be connected together (similarly as done with Bezier).
- Connecting multiple points  $P_0 \dots P_n$ 
  - $n - 1$  cubic Hermite curves must be used
  - Resultant is **assembly of curves** called **spline**
  - Elements of assembly are called **segments** (e.g., Hermite segments)

How to calculate the tangent vectors in given points?  
Solution: Catmull-Rom spline

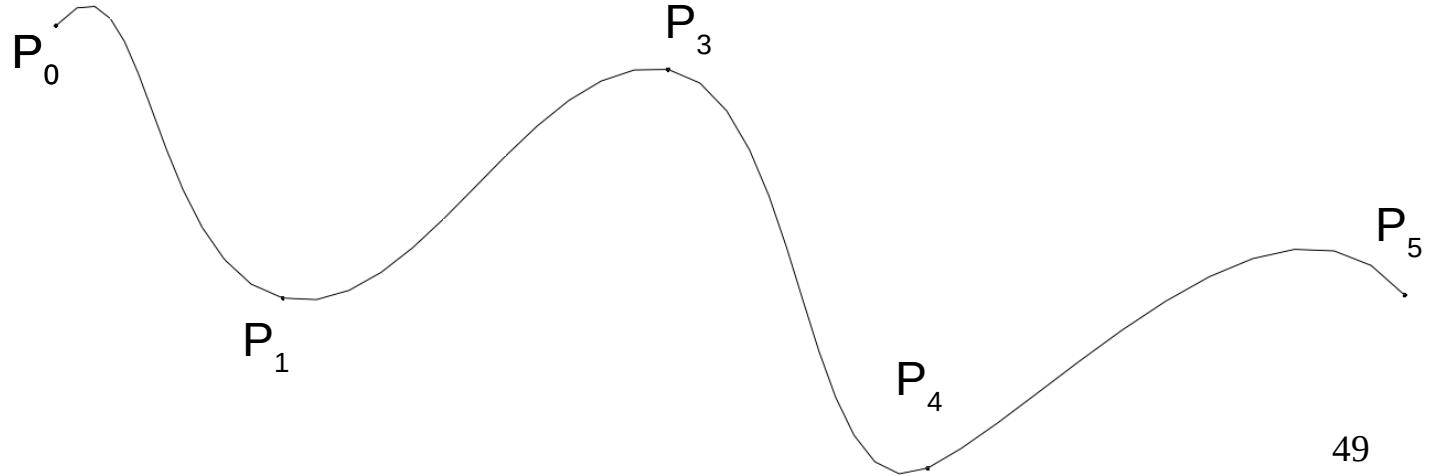


# Parametric curves

- Bezier curve
- Hermite curve
- **Catmull-Rom spline**
  - Use cubic Hermite curve to build a spline
- B-Splines

# Catmull-Rom spline

- Cubic Hermite interpolation curve tangents can be hard to control
- Therefore, **tangents are calculated from control points  $P_0 \dots P_n$**  specified by the user
  - Given points  $P_0 \dots P_n$ , the goal is to calculate tangents needed for Hermite curves which would create spline: continuous differentiable curve passing through each point → **Catmull-Rom spline**



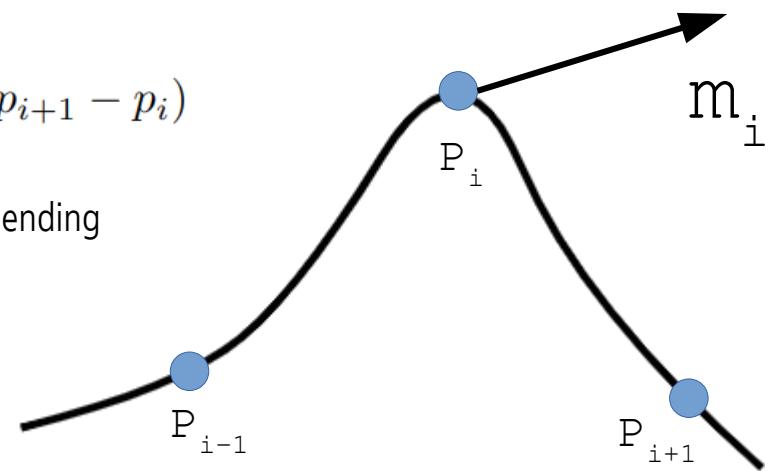
# Kochanek-Bartels method

- Method for computing tangents  $m_i$  at joints  $P_i$
- Assume that there is only **one tangent per control point**
  - Tangent  $m_i$  at  $P_i$  can be computed as combination of two chords:  $P_i - P_{i-1}$  and  $P_{i+1} - P_i$

$$m_i = \frac{(1-a)(1+b)}{2}(p_i - p_{i-1}) + \frac{(1-a)(1-b)}{2}(p_{i+1} - p_i)$$

Tension parameter -  $a$ : length of tangent. Higher values → sharper curve bending

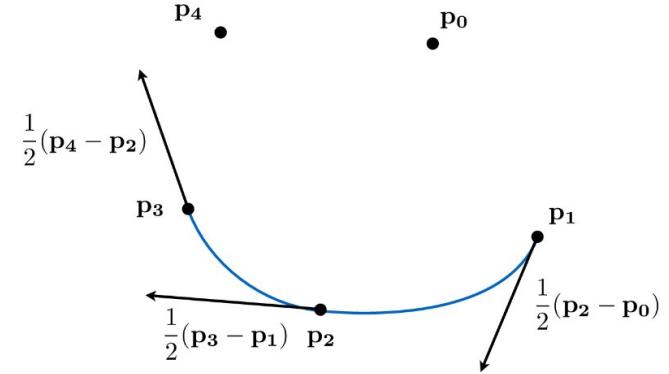
Bias parameter -  $b$ : direction of tangent



# Catmull-Rom spline

- Catmull-Rom spline is a special case where tension and bias parameters of Kochanek-Bartels method are set to 0.
- Tangents  $m_i$  in  $P_0 \dots P_n$  are calculated using:

$$m_i = \frac{1}{2}(p_i - p_{i-1}) + \frac{1}{2}(p_{i+1} - p_i)$$



<https://cs184.eecs.berkeley.edu/sp22/lecture/7-50/intro-to-geometry-splines-and-be>

Properties:

- Finding points on Catmull-Rom spline is fast and efficient
- Interpolating: passing through all control points but doesn't stay inside convex hull of points

# Parametric curves

- Bezier curve
- Hermite curve
- Catmull-Rom spline
- B-Splines
  - Next to Catmull-Rom spline, B-splines are widely used

# B-spline

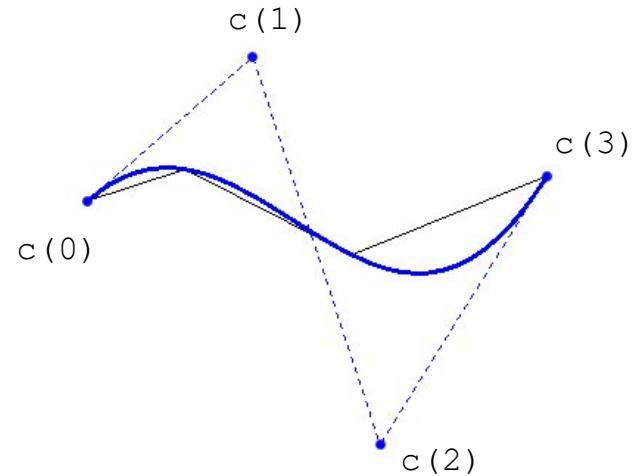
- Types:
  - Uniform: uniformly space control points
  - Non-Uniform
- Parameterized function of  $\mathbf{x}$ . (Similarly to Bezier curve)

$$s_n(x) = \sum_{k \in \mathbb{Z}} c(k) \beta_n(x - k).$$

$n-1$  – degree,  $c(k)$  – control points,  $\beta_n$  - basis function

- Cubic B-spline is most widely used.
- Basis function of uniform cubic B-spline:

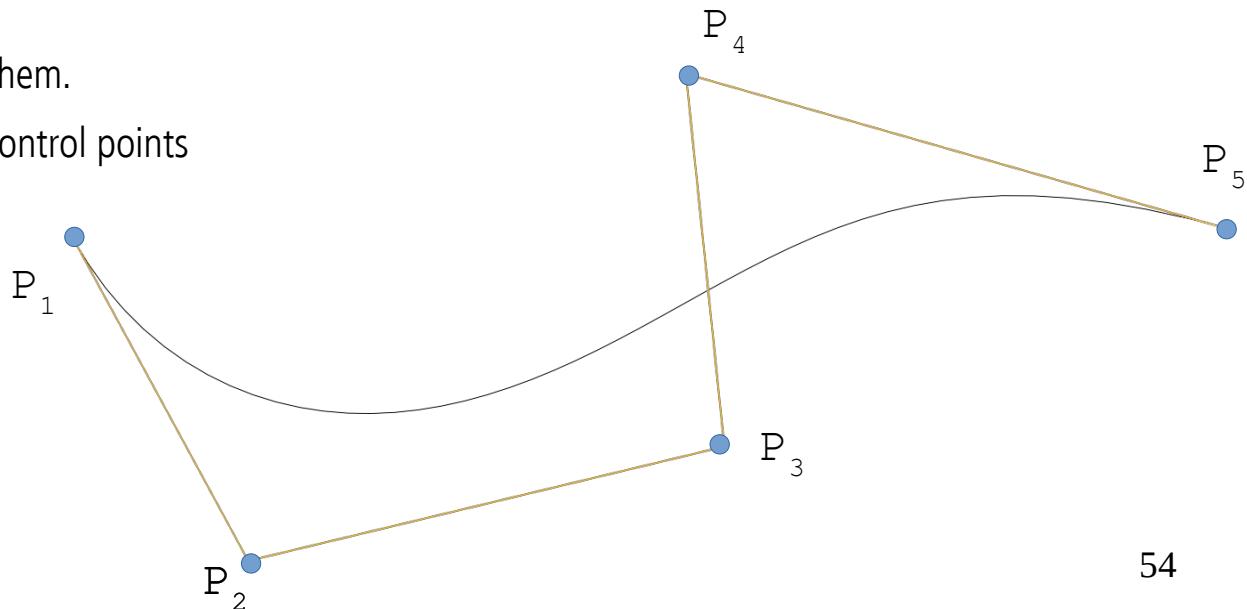
$$\beta_3(x) = \begin{cases} 0, & |x| \geq 2 \\ \frac{1}{6} \cdot (2 - |x|)^3, & 1 \leq |x| < 2 \\ \frac{2}{3} - \frac{1}{2}|x|^2 \cdot (2 - |x|), & |x| < 1 \end{cases}$$



Cubic B-spline: non-interpolating  
<https://docs.bentley.com/LiveContent/web/ProStructures%20Help-v6/en/SECurvesPlaceBsplineCurve.html>

# Uniform cubic B-spline

- Multiple uniform cubic B-splines curves can be joined together
  - Cubic B-spline is  $C^2$  continuous
  - Curve composed of multiple B-splines will be  $C^2$  continuous
- Cubic B-spline is **non-interpolating**
  - Passing near control points, not through them.
  - No guarantee that it is interpolating the control points



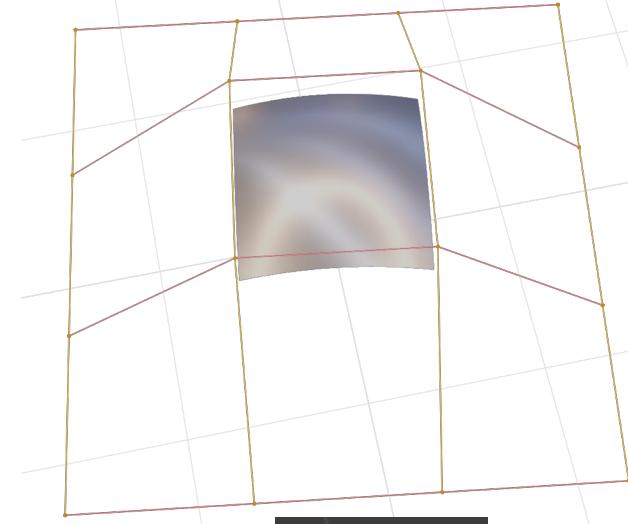
# Limitations and generalizations

- Representing circle is not possible
  - Solution, a generalization is to introduce additional coordinate to B-spline → **rational B-spline**
- Uniformly spaced control points:
  - Example: animation when positions of object are known at  $t = 1, 2, 3$  and  $10$ .
  - Further generalization: **Non-uniform rational B-splines (NURBS)**
    - Very often used in CAD tools

# Parametric surfaces

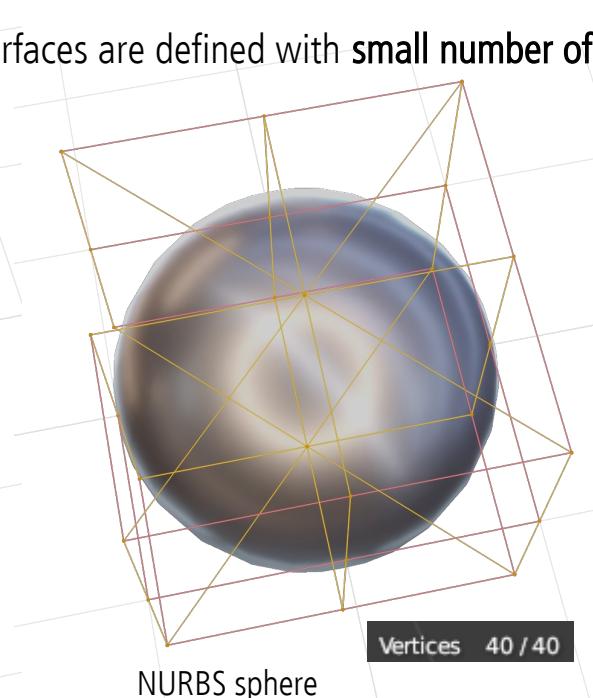
# Parametric surfaces

- Natural extension of parametric curves are parametric surfaces
  - Similarly as triangle or polygon is extension of a line segment
- Very useful for modeling **curved surfaces**
- Similarly as parametric curves, parametric surfaces are defined with **small number of control points and parametric equation**



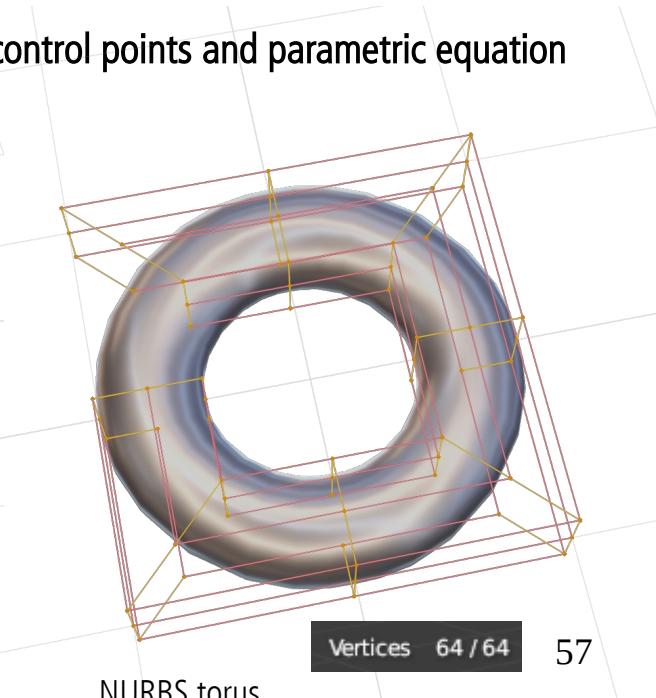
NURBS surface

Vertices 16 / 16



NURBS sphere

Vertices 40 / 40



NURBS torus

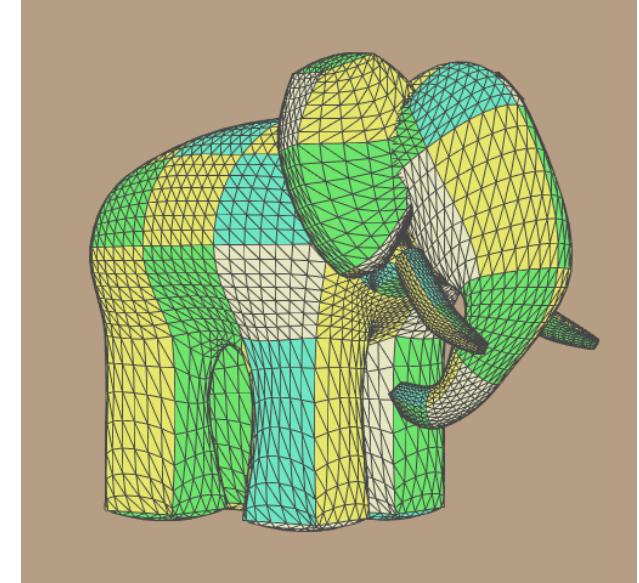
Vertices 64 / 64

# Parametric surfaces

- Common types
  - Bezier surfaces
  - B-spline surfaces



NURBS Renderman: <https://rmanwiki.pixar.com/display/REN24/NURBS>

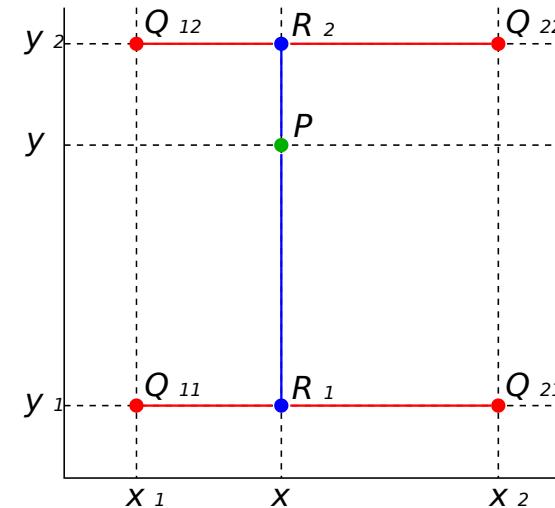


Ed Catmull's "Gumbo" model, composed from Bezier patches  
[https://en.wikipedia.org/wiki/B%C3%A9zier\\_surface](https://en.wikipedia.org/wiki/B%C3%A9zier_surface)

# Bezier surface: bilinear interpolation

- Bezier curve is extended so it has **two parameters (u,v)** which define surface
- As linear Bezier curve is defined with linear interpolation, **planar Bezier patch** is defined with **bilinear interpolation**.

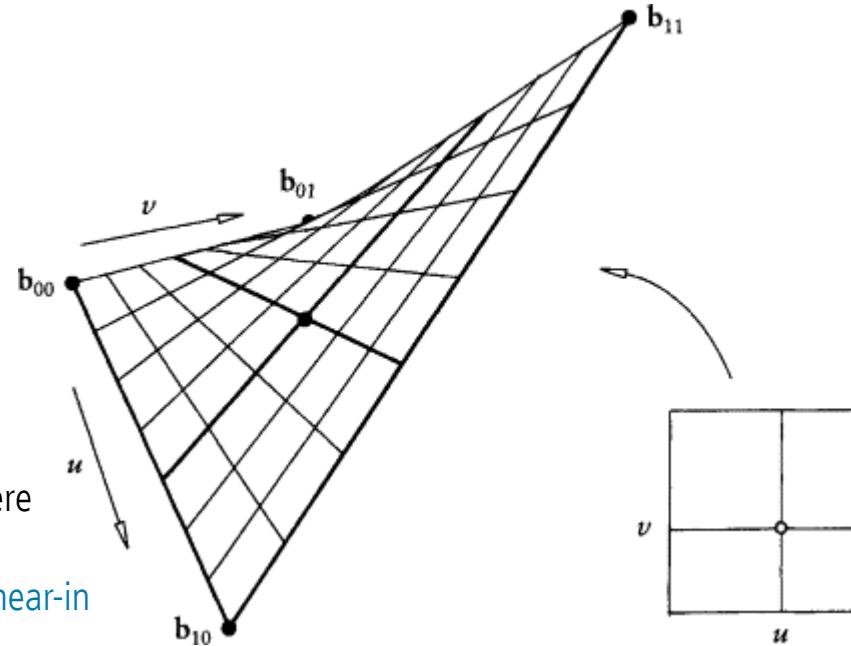
$$\begin{aligned} R_1(u) &= \text{lerp}(Q_{11}, Q_{12}, u) \\ R_2(u) &= \text{lerp}(Q_{12}, Q_{22}, u) \\ P(u, v) &= \text{lerp}(R_1(u), R_2(u), v) \end{aligned}$$



Bilinear interpolation

# Bezier surface: patch

- Planar Bezier patch  $p(u, v)$  is simplest, non-planar parametric surface
- It has **rectangular domain** and thus resulting surface is called a **patch**

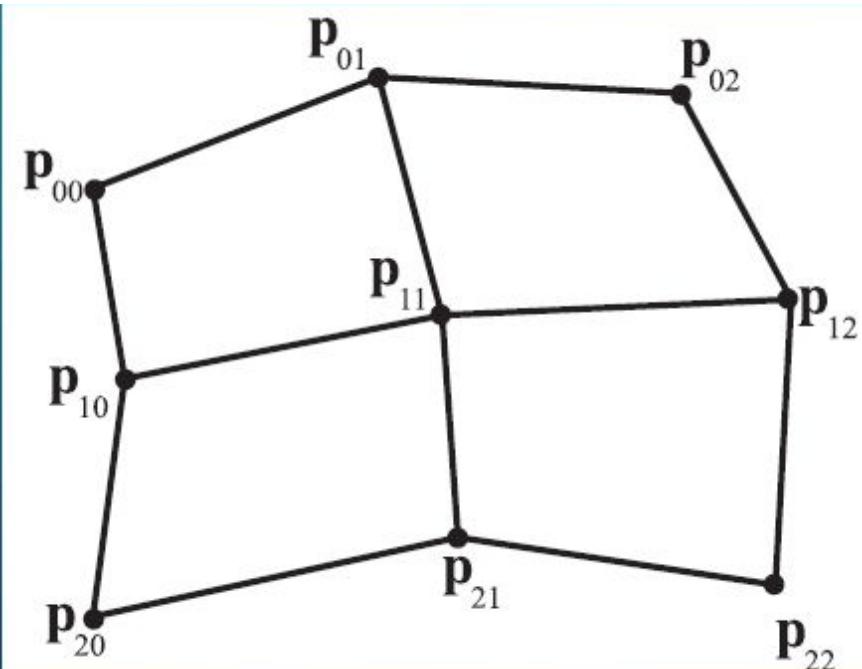


Bilinear interpolation defines planar Bezier patch  $p(u,v)$  where  $(u,v)$  are in  $[0,1]$ .

<https://www.sciencedirect.com/topics/computer-science/bilinear-interpolation>

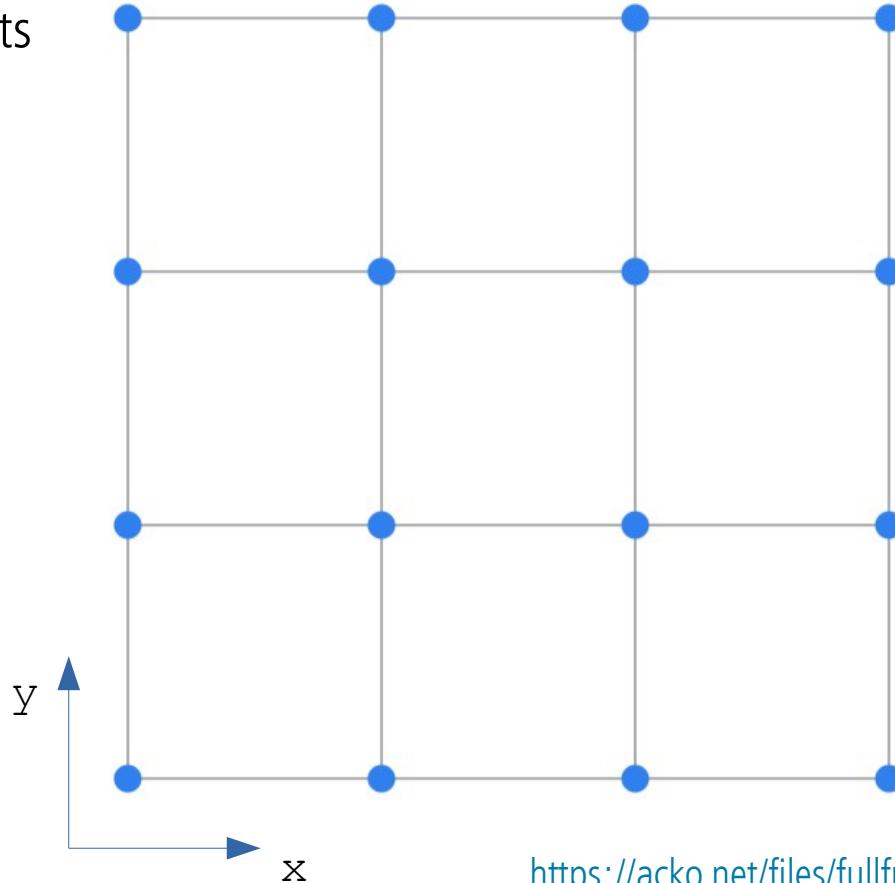
# Bezier surface

- Bezier surface is achieved by adding more points to bilinear interpolation (similarly as we added more points to linear interpolation to obtain Bezier curve)
- Repeated bilinear interpolation is extension of de Casteljau's algorithm to Bezier patches.
- Example: biquadratic Bezier surface
  - 9 control points



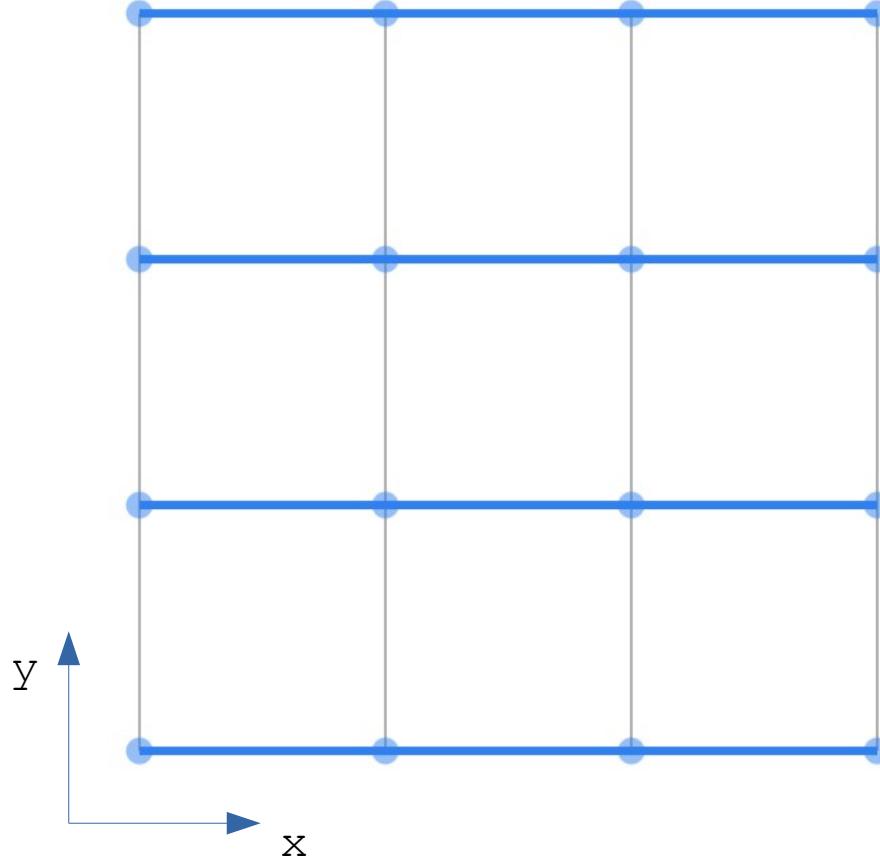
# Bezier surface: repeated bilinear interpolation

- Starting control points



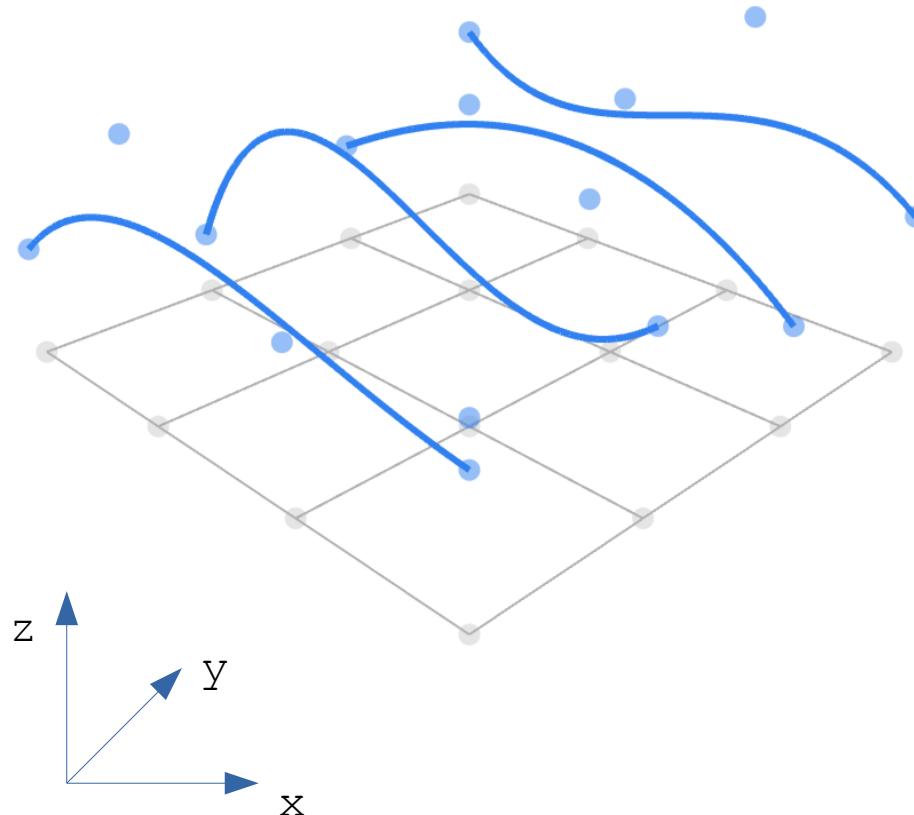
# Bezier surface: repeated bilinear interpolation

- Repeated linear interpolation



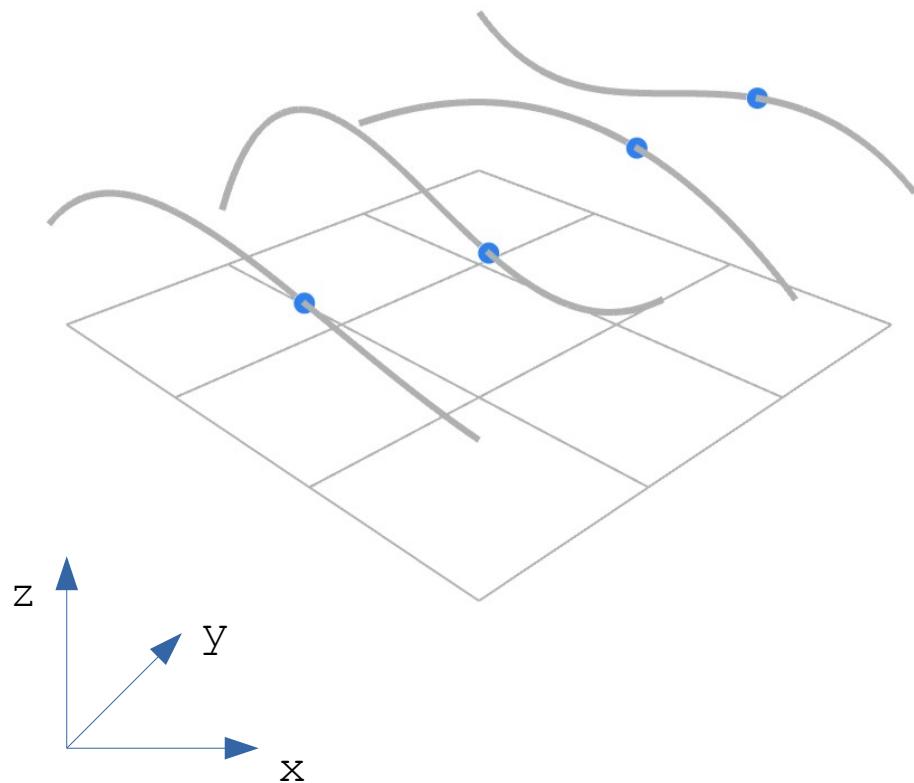
# Bezier surface: repeated bilinear interpolation

- Repeated linear interpolation
  - Bezier cubic curves



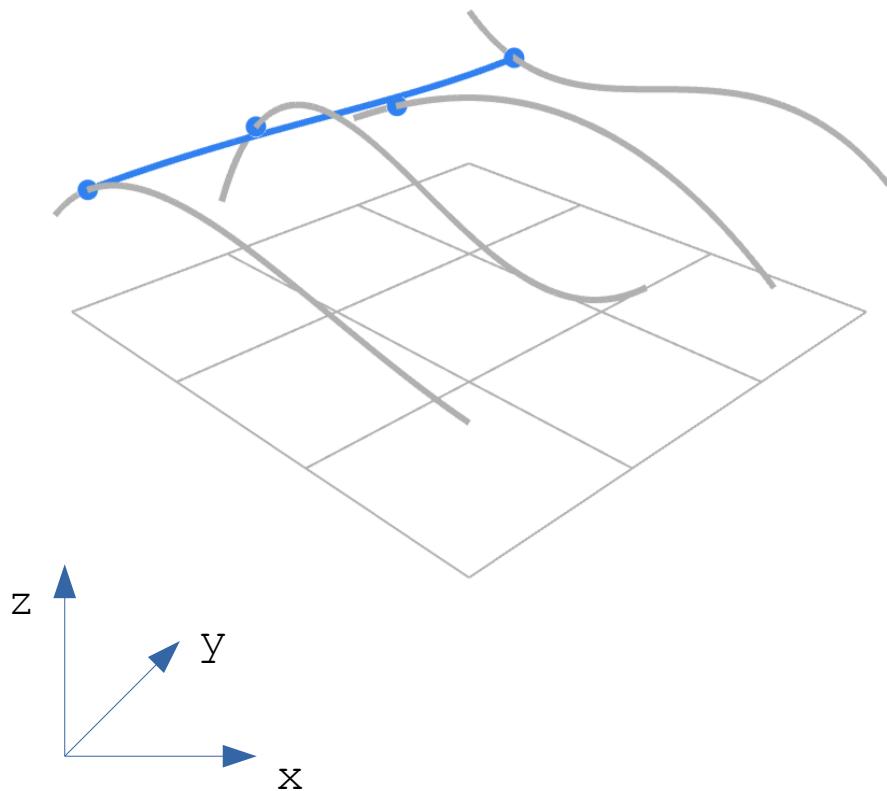
# Bezier surface: repeated bilinear interpolation

- Repeated linear interpolation



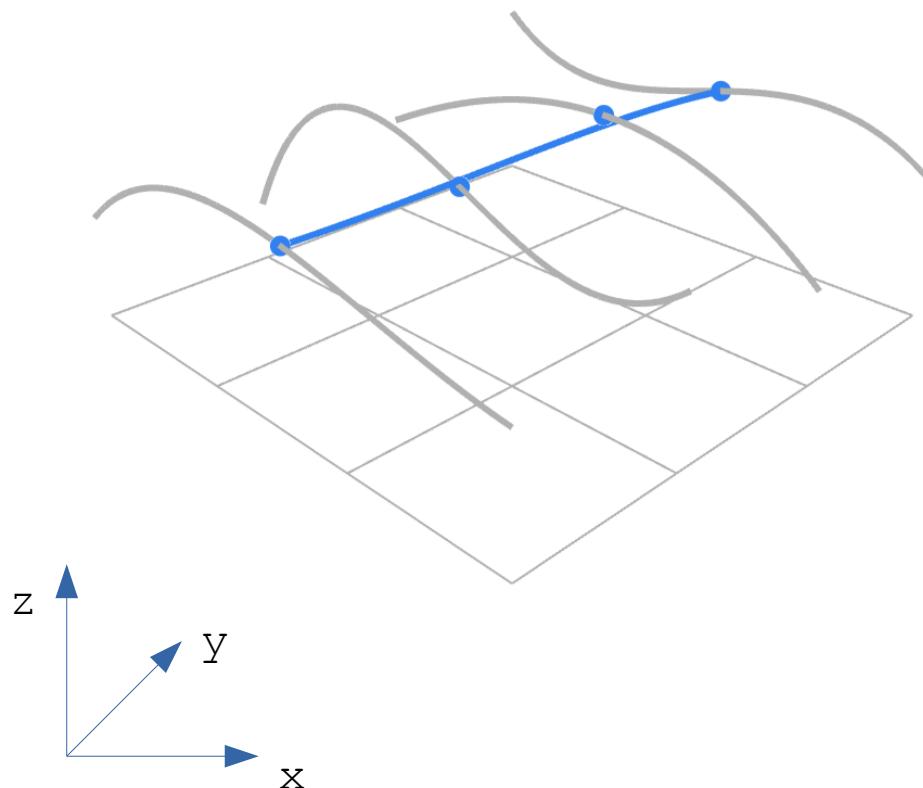
# Bezier surface: repeated bilinear interpolation

- Repeated bilinear interpolation



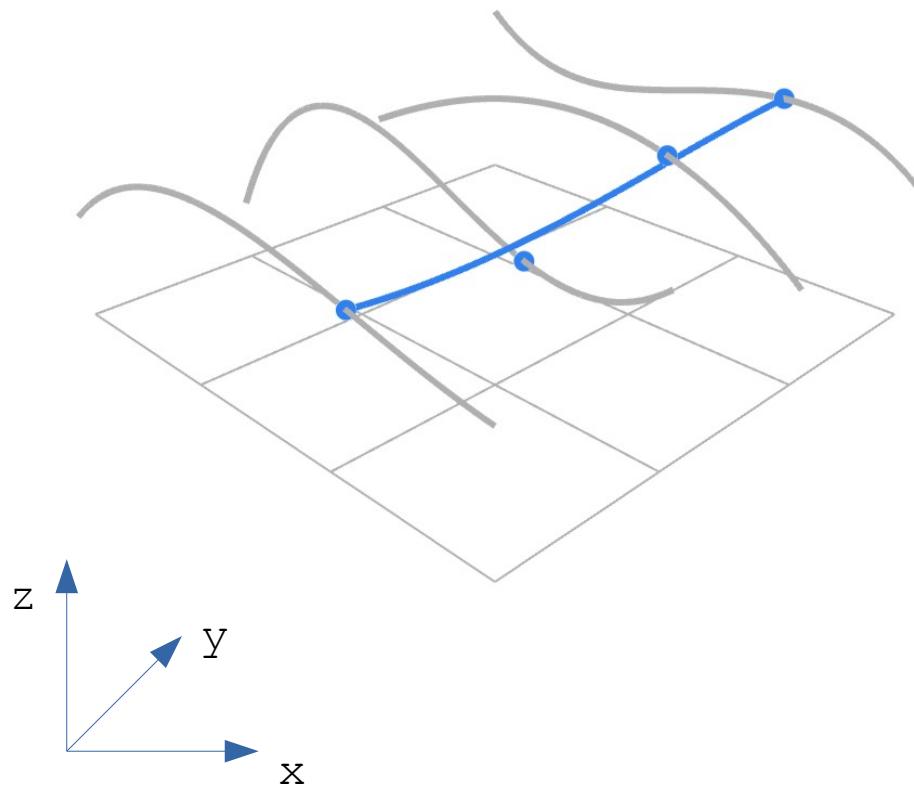
# Bezier surface: repeated bilinear interpolation

- Repeated bilinear interpolation



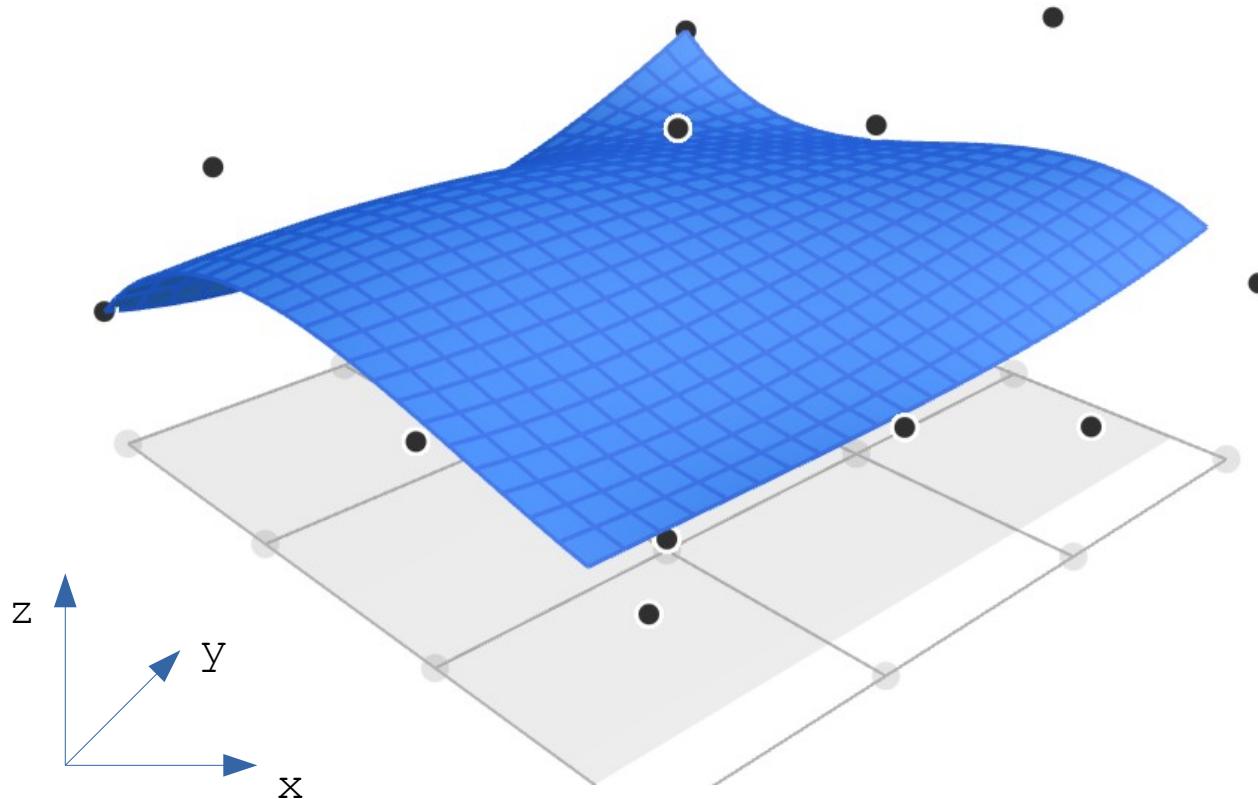
# Bezier surface: repeated bilinear interpolation

- Repeated bilinear interpolation



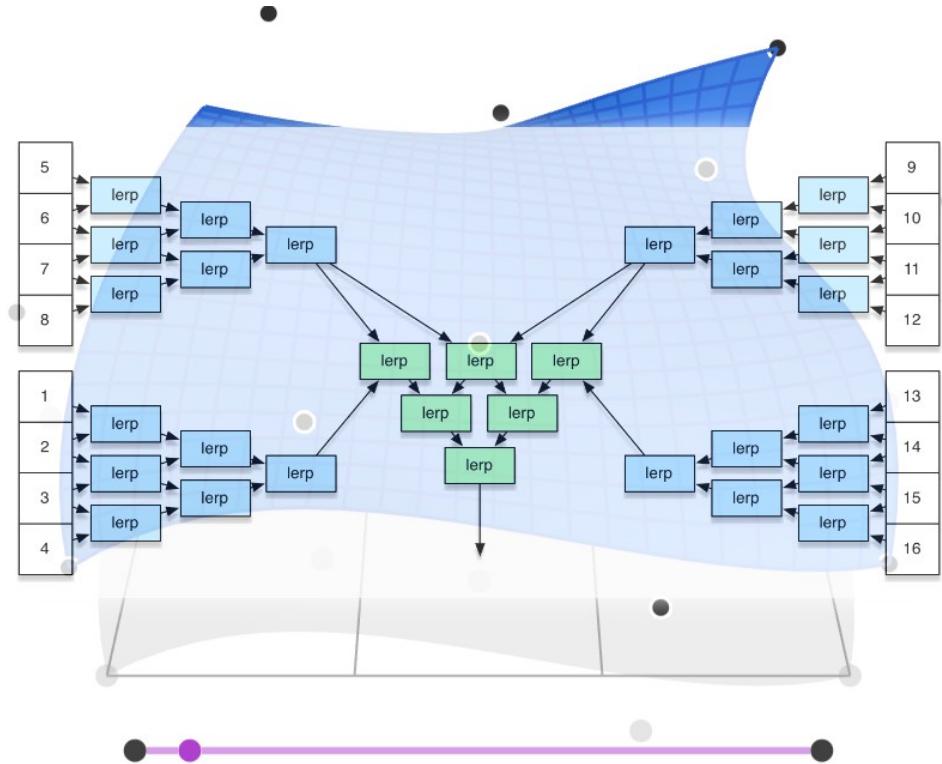
# Bezier surface: repeated bilinear interpolation

- Bicubic Bezier surface



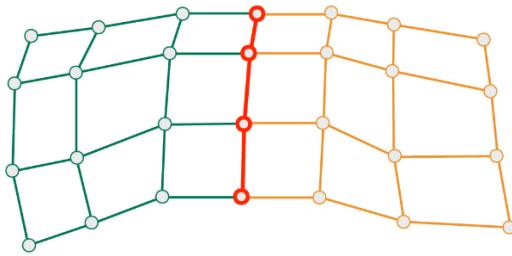
# Bezier surface

- Repeated bilinear interpolation is extension of **De Casteljau algorithm** to patches
- Point on Bezier Patch can be described in **Bernstein form** using Bernstein polynomials
- **Bicubic Bezier surface** defined with 16 control points
  - Building block for more complex surfaces

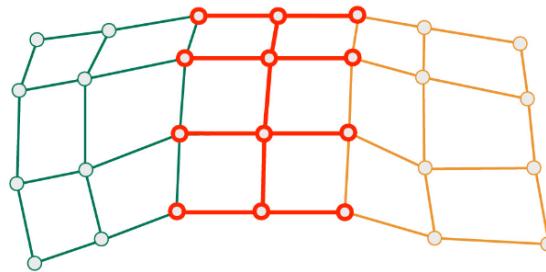


# Bezier surface: continuity

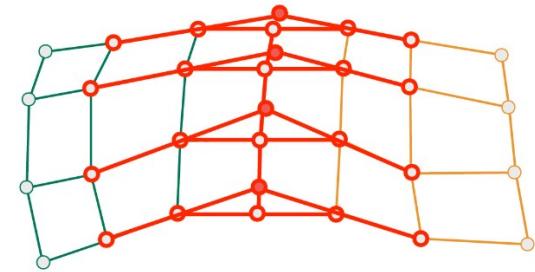
Surface continuity



$C^0$  – **positional continuity** –  
joined at the same point

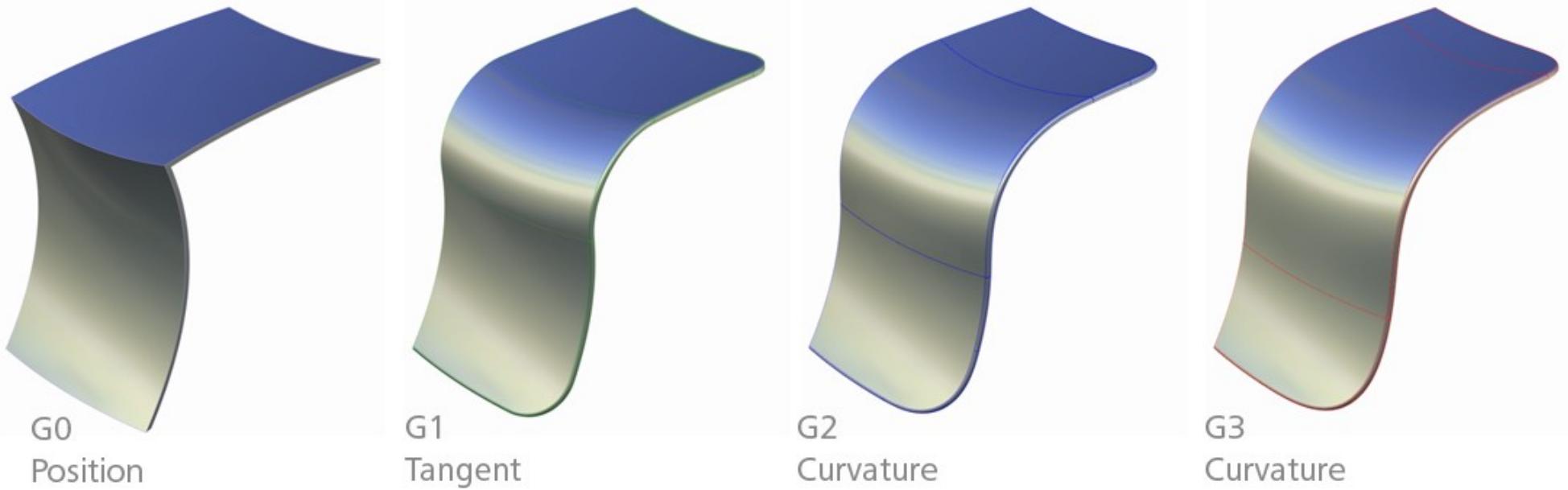


$C^1$  – **velocity continuity** -  
derivation of any point  
(including joints) must be  
continuous



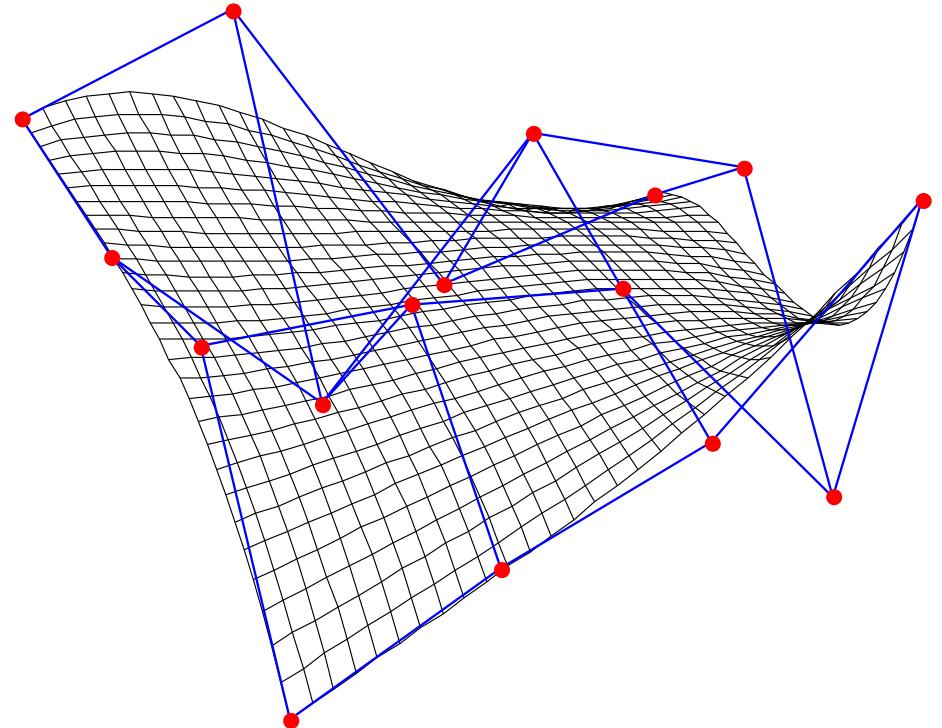
$C^2$  - **acceleration continuity** -  
first and second derivatives are  
continuous functions

# Bezier surface: continuity



# Bi-cubic Bezier surface properties

- **Non-interpolating:** passes through only corner control points
- **Boundary of the patch:** Bezier curve of degree  $n$
- **Tangents at border points:** described with Bezier curve at border points – two tangents: for  $u$  and  $v$  direction
  - Derivative is straightforward (derivation of polynomials)
- **Patch lies within convex hull of its control points**
- Arbitrary number of points on patch can be generated
  - Prior generation, transformation on patch can be done which is efficient than transformation on generated points
- Extension: **rational Bezier patches**
  - <https://theses.hal.science/tel-01064604/document/>

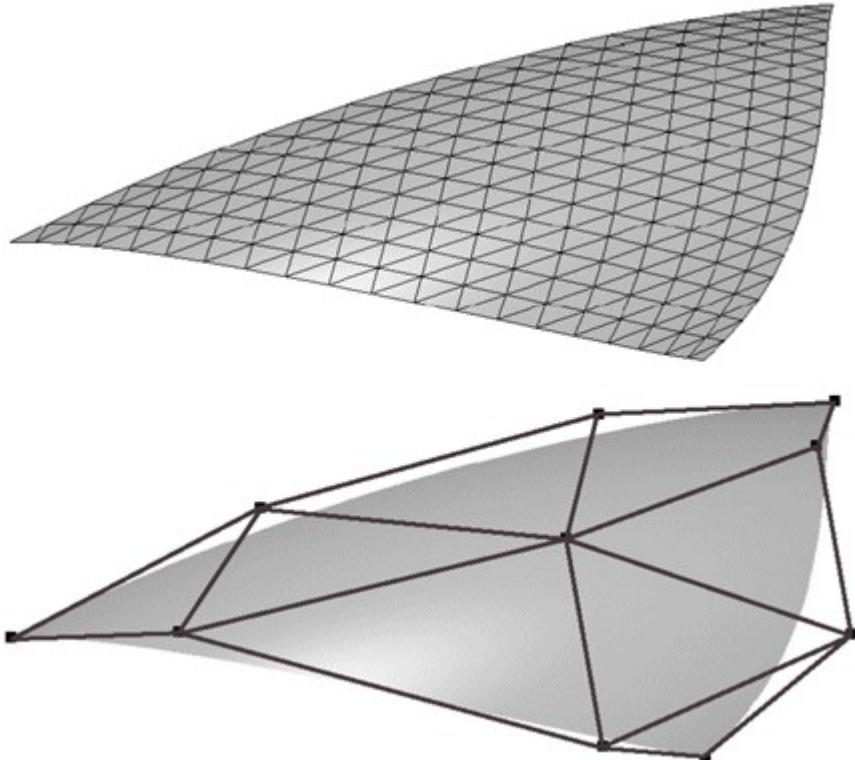


# Parametrized surfaces based on Bezier surface

- Bezier triangles
- Point-Normal (PN) Triangles
- Phong tessellation

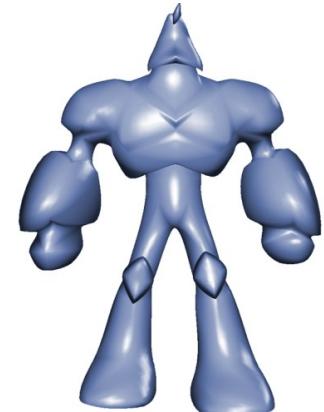
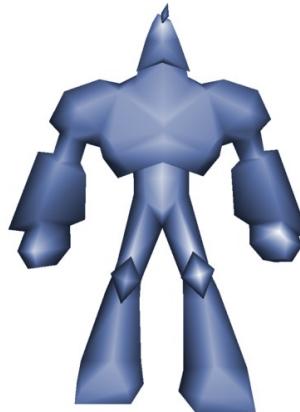
# Bezier triangles

- Control points are located in triangulated grid
- Based on **repeated interpolation**:
  - de Casteljau algorithm
  - Bernstein triangles
- Constructing complex object requires stitching Bezier triangles so that composite surface contains desired properties and look, e.g., continuity



# Point-Normal (PN) Triangles

- Given triangle mesh with normals at each vertex, the goal is to construct smoother surface using Bezier triangles
- Properties:
  - Improves mesh shading and silhouettes by creating **curved surface to replace each triangle**
  - Creases in PN triangles are hard to control
  - Continuity between Bezier triangles is  $C_0$  but looks acceptable for certain applications

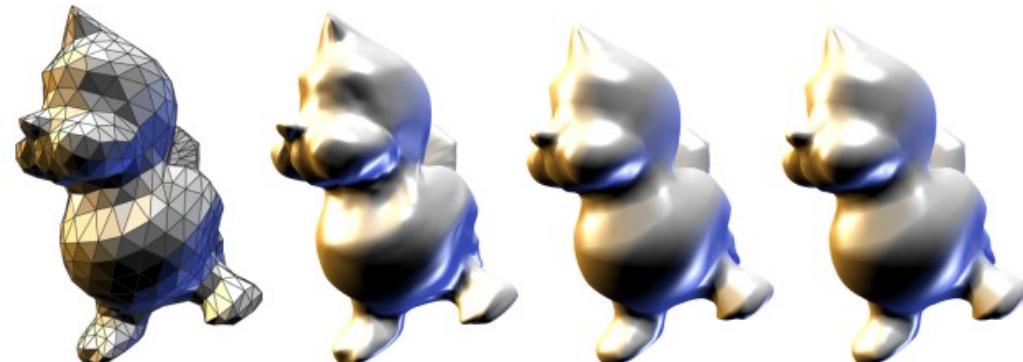


Comparison of original mesh and PN triangle counterpart  
<https://www.gamedeveloper.com/programming/b-zier-triangles-and-n-patches>  
<https://alex.vlachos.com/graphics/CurvedPNTriangles.pdf>

\* Game engines (e.g., unity and unreal) support those methods since triangle mesh is basic building primitive.

# Phong tessellation

- Similar as PN triangles, given the triangle points with normals, construct smoother surface
- Phong tessellation attempts to create geometric version of Phong shading normal using repeated interpolation resulting in Bezier triangles.



Mesh, Butterfly subdivision, PN Triangles, Phong Tessellation

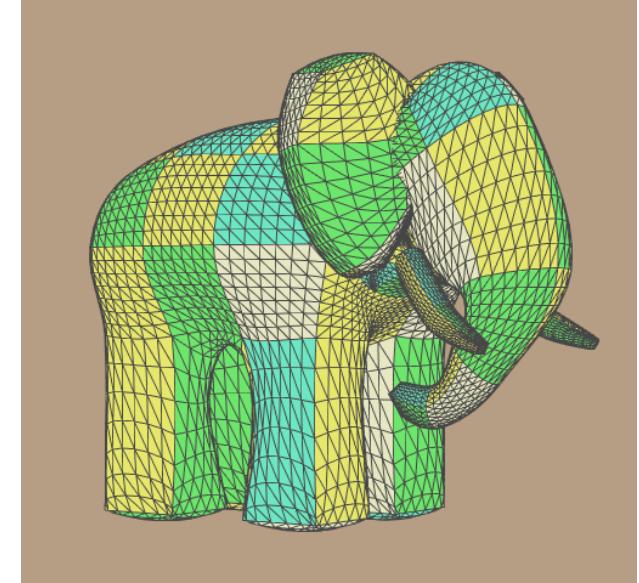
<https://perso.telecom-paristech.fr/boubek/papers/PhongTessellation/>

# Parametric surfaces

- Common types
  - Bezier surfaces
  - B-spline surfaces



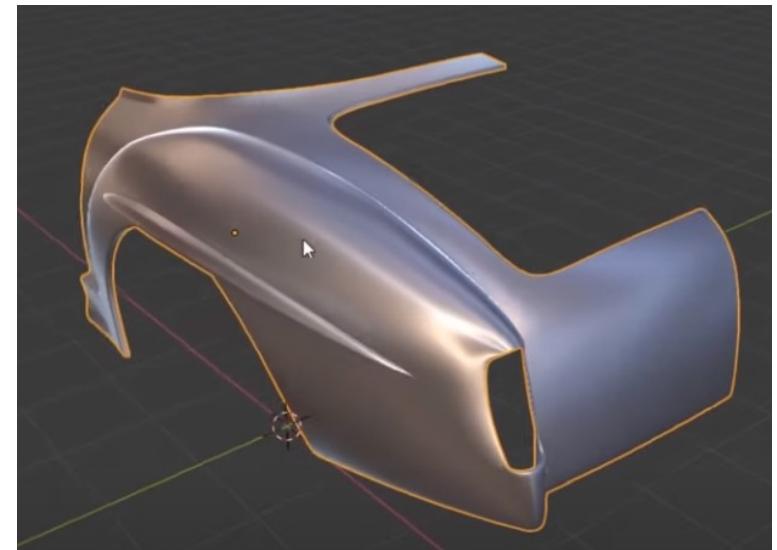
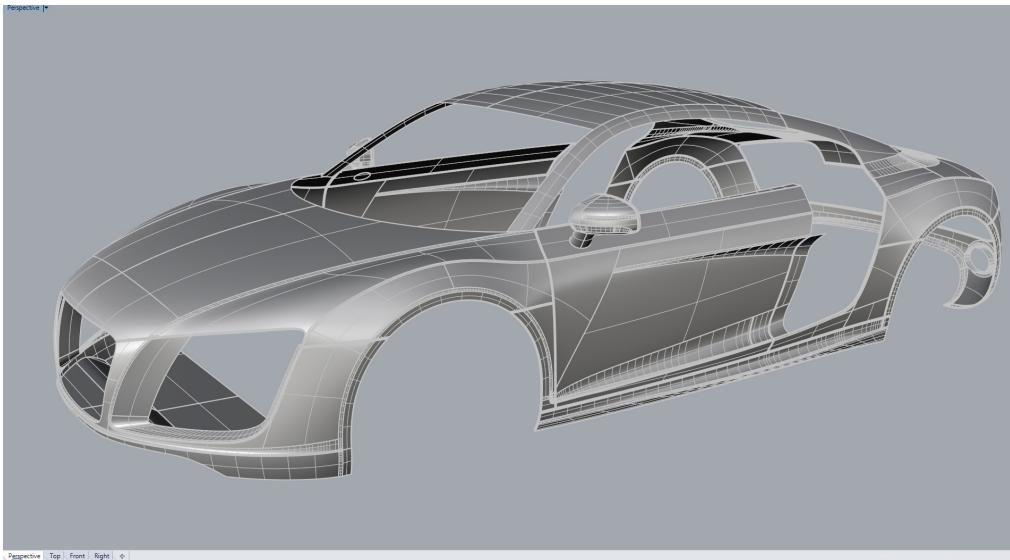
NURBS Renderman: <https://rmanwiki.pixar.com/display/REN24/NURBS>



Ed Catmull's "Gumbo" model, composed from Bezier patches  
[https://en.wikipedia.org/wiki/B%C3%A9zier\\_surface](https://en.wikipedia.org/wiki/B%C3%A9zier_surface)

# B-Spline surfaces

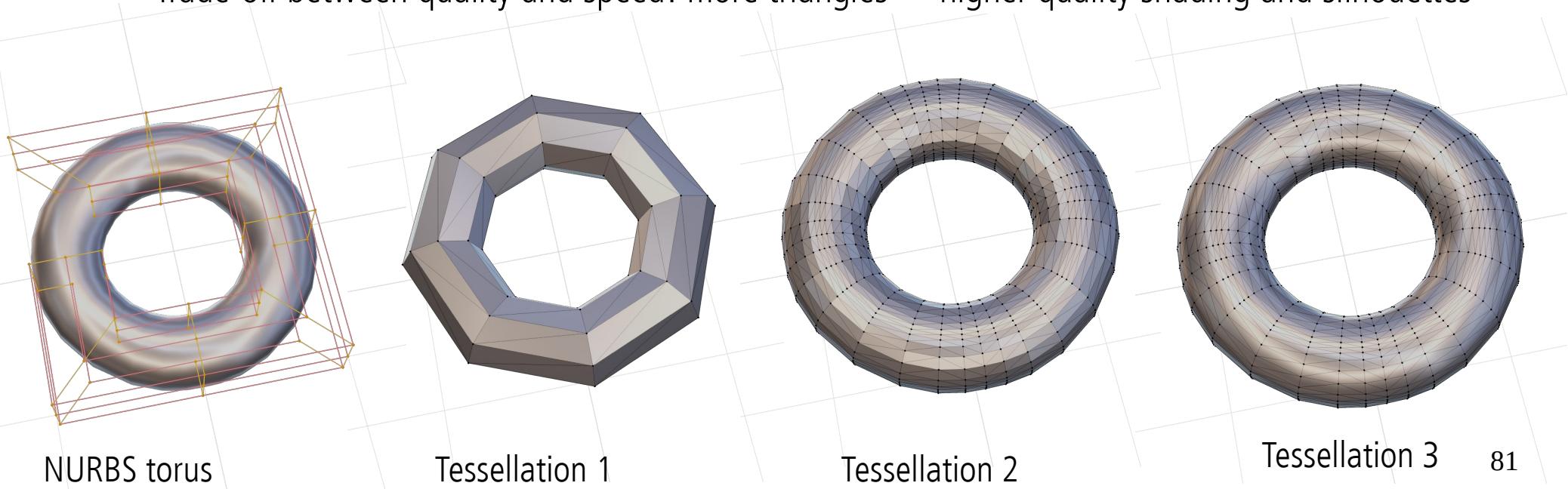
- B-Spline curves can be extended to **B-Spline surfaces** which are similar to Bezier surface
- Often **bicubic B-Spline surface is used to form composite surface**
  - Basis for Catmull-Clark subdivision surfaces
- **Non-uniform rational B-Spline surface (NURBS)** is often used in 3D modeling software



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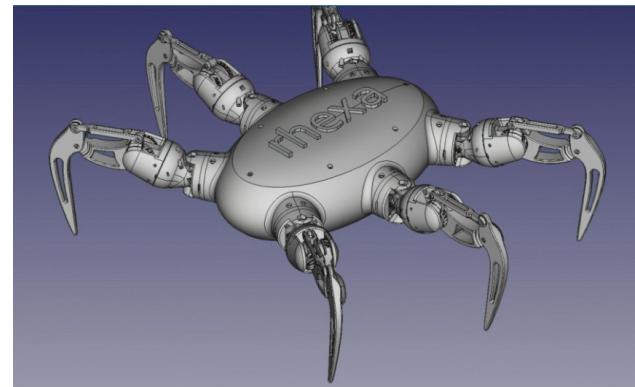
# Rendering of parametric surfaces

- Ray-parametric-surface intersection can be defined
- Model made with parametric surfaces is **tessellated** for efficient rendering process.
  - Trade-off between quality and speed: more triangles → higher quality shading and silhouettes



# Modeling and transferring parametric surfaces

- Parametric curves and surfaces are extensively used in CAD and modeling software:
  - <https://www.freecad.org/>
  - <https://docs.blender.org/manual/en/latest/modeling/surfaces/index.html>
  - <https://www.autodesk.com/collections/product-design-manufacturing/overview>
- Parametric surfaces are stored using:
  - **STEP file format:** <https://www.adobe.com/creativecloud/file-types/image/vector/step-file.html>
  - **IGES file format:** <https://www.adobe.com/creativecloud/file-types/image/vector/iges-file.html>



# Exploring parametric curves and surfaces

- Parametric curves and surfaces are highly used and researched method in computer graphics. We only covered foundations.
- NURBS in real-time: <https://www.gamedeveloper.com/programming/using-nurbs-surfaces-in-real-time-applications>
- Blender NURBS surfaces: <https://docs.blender.org/manual/en/latest/modeling/surfaces/introduction.html>
- Blender NURBS and Bezier curves: <https://docs.blender.org/manual/en/latest/modeling/curves/index.html>
- Tutorial on curves in Blender: <https://behreajj.medium.com/scripting-curves-in-blender-with-python-c487097efd13>
- Houdini: <https://www.sidefx.com/docs/houdini/nodes/sop/curve.html>
- Library for creating and manipulating NURBS surfaces and curves: <http://verbnurbs.com/>
- More examples: <https://www.realtimerendering.com/#curves>

# Implicit surfaces

# Implicit surfaces

- Described with a function. Instead of using parameters  $(u, v)$  surface is described with implicit function  $f(p)$ :

$f(p) = 0$ , if  $p$  on surface

$f(p) < 0$ , if  $p$  inside surface

$f(p) > 0$ , if  $p$  outside surface

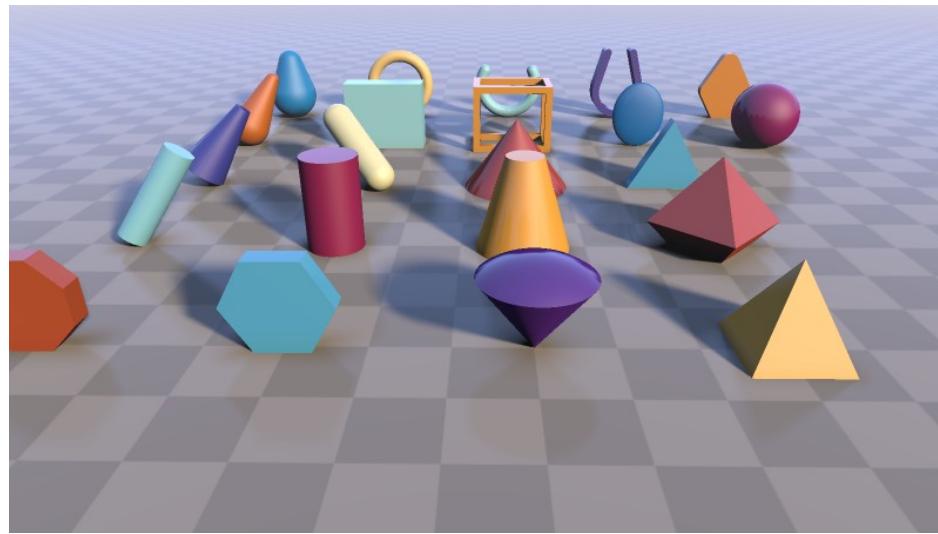
- Surface can not be computed directly
  - Distance is computed by evaluating point  $p(x, y, z)$  in space → **signed distance function**

```
float sdSphere( vec3 p, float s )
{
    return length(p)-s;
}
```

```
float sdTorus( vec3 p, vec2 t )
{
    vec2 q = vec2(length(p.xz)-t.x,p.y);
    return length(q)-t.y;
}
```

```
float sdRoundBox( vec3 p, vec3 b, float r )
{
    vec3 q = abs(p) - b;
    return length(max(q,0.0)) + min(max(q.x,max(q.y,q.z)),0.0) - r;
}
```

```
float sdRoundedCylinder( vec3 p, float ra, float rb, float h )
{
    vec2 d = vec2( length(p.xz)-2.0*ra+rb, abs(p.y) - h );
    return min(max(d.x,d.y),0.0) + length(max(d,0.0)) - rb;
}
```



# Implicit surfaces: normal

- Crucial information for shading is normal.
- Normal vector  $\mathbf{N}$  of implicit surface in point  $p$ 
  - Normalized gradient of function  $f$  in point  $p$ :  $\nabla f(p)$
  - Intuition: Gradient points in direction of steepest ascent in the function near  $p$  and thus is orthogonal to surface



## Analytical expression:

- Function must be differentiable

$$\nabla f(x, y, z) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

## Approximation:

- Central difference using small number  $\epsilon$

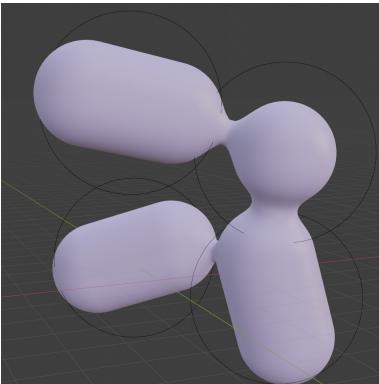
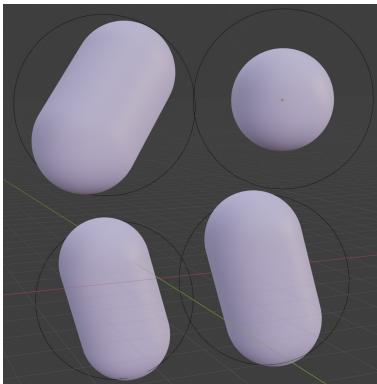
$$\nabla f_x = f(p + \epsilon e_x) - f(p - \epsilon e_x), e_x = (1, 0, 0)$$

$$\nabla f_y = f(p + \epsilon e_y) - f(p - \epsilon e_y), e_y = (0, 1, 0)$$

$$\nabla f_z = f(p + \epsilon e_z) - f(p - \epsilon e_z), e_z = (0, 0, 1)$$

# Implicit surfaces: modeling

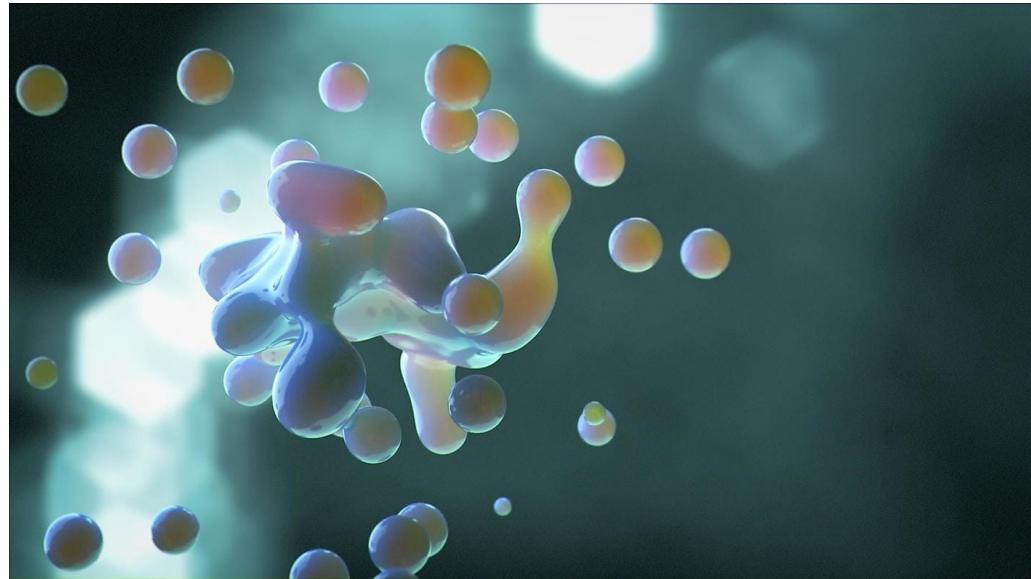
- Blending of implicit surfaces: blobby modeling, soft objects or metaballs
  - Blender: <https://docs.blender.org/manual/en/latest/modeling/metaballs/index.html>



- Example of blending: resulting distance  $d$  by blending between two distances  $d_1$  and  $d_2$  with a blend radius  $r_b$

$$d = (1 - h)d_2 + hd_1 + r_b h(1 - h)$$

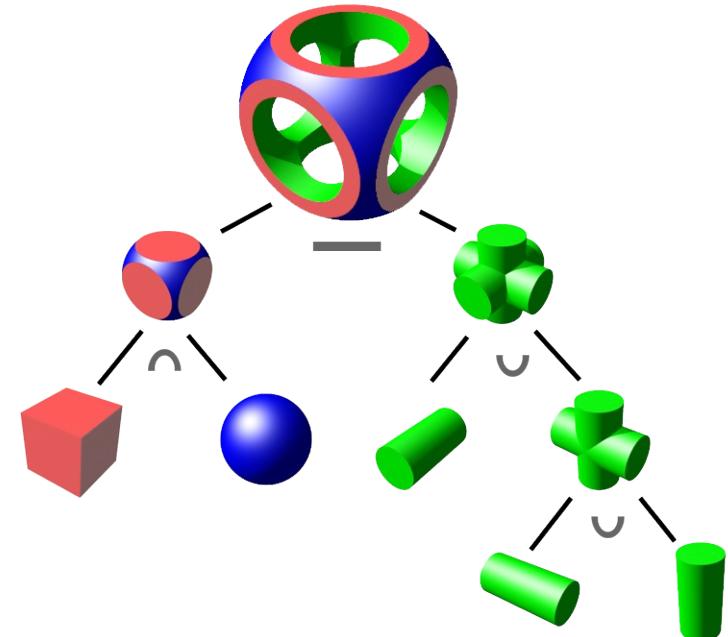
$$h = \min(\max(0.5 + 0.5(d_2 - d_1)/r_b, 0.0), 1.0)$$



<https://www.chaos.com/gallery/elmar-glaubauf-metaballs>

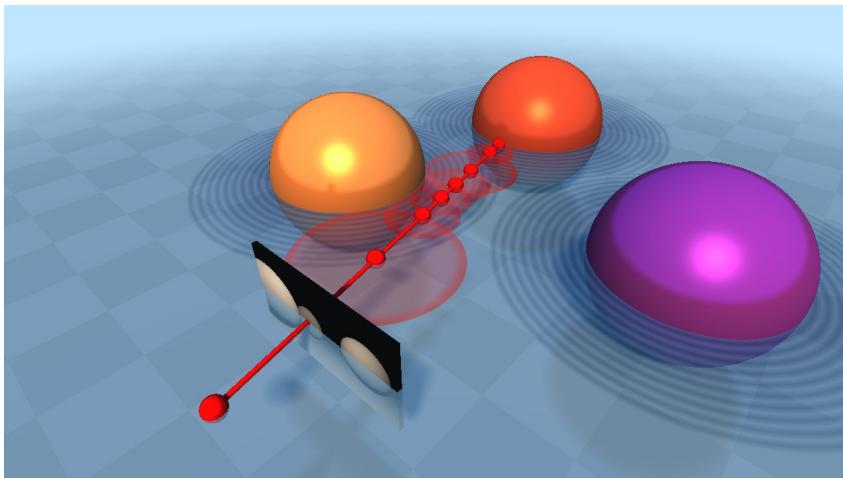
# Implicit surfaces: modeling

- **Constructive solid geometry:** set of operations –  
Boolean operations:
  - Intersection
  - Union
  - Difference
- Well defined for closed objects (volumetric bodies)
- Used to create complex shapes from simple, base shapes (e.g., cubes, spheres, etc.)



# Implicit surfaces: rendering

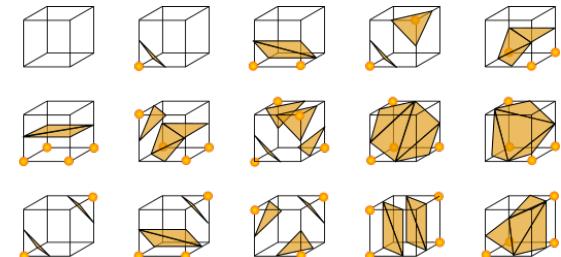
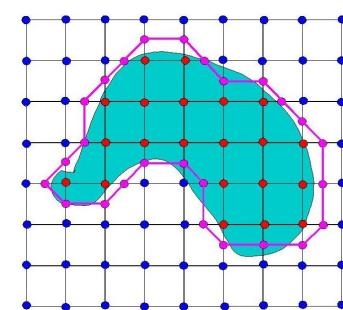
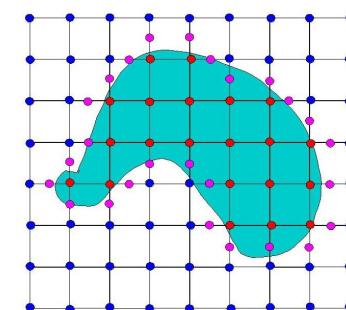
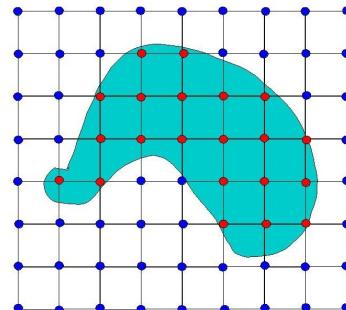
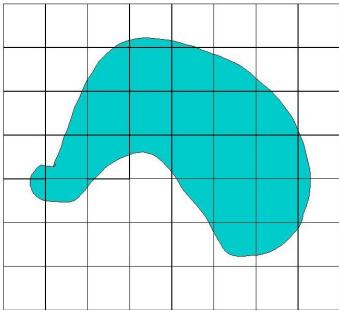
- Ray-marching
  - At first point  $p$  on the ray, shortest distance  $d$  to the surface is evaluated (sphere around the point)
  - Move in ray direction for distance  $d$
  - Repeat until surface is reached within some distance (small threshold) or max number of steps is reached (background hit)
- Effects: shadows, reflections, ambient occlusion, etc.



<https://iquilezles.org/articles/raymarchingdf/>

# Implicit surfaces: rendering

- Turn surface into triangles (triangulation) and then render
- **Marching cubes algorithm:** surface extraction, polygonization
  - First, surface is partitioned into adjacent cells at whose corners the implicit surface is evaluated
  - Negative values are considered outside surface, positive values inside surface
  - Within each cell, the intersection of cell edges with the implicit surface are connected to form one or more polygons



3D case: creating cells, labeling cell corners, calculating cell edge points, connecting edge points to form polygon

3D case

# Summary questions

- [https://github.com/lorentzo/IntroductionToComputerGraphics/tree/main/lectures/6\\_parametric\\_curves\\_surfaces](https://github.com/lorentzo/IntroductionToComputerGraphics/tree/main/lectures/6_parametric_curves_surfaces)

# Reading material

- <https://github.com/lorentzo/IntroductionToComputerGraphics>