Lecture 6: Parametric curves and surfaces, implicit surfaces

DHBW, Computer Graphics

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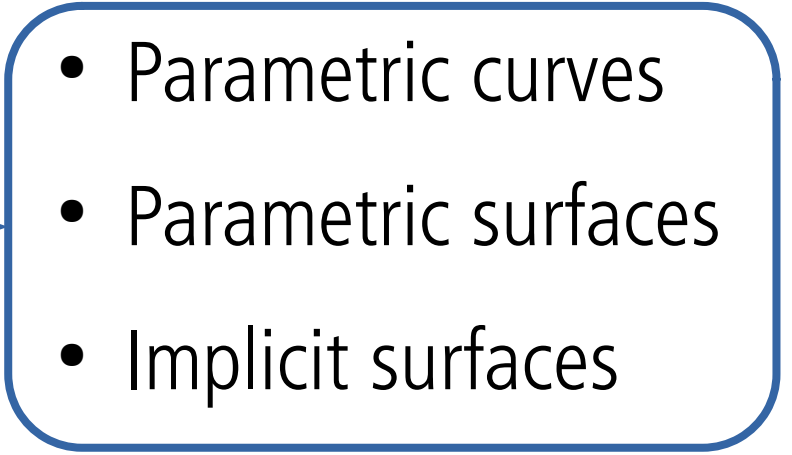
8.2.2023.

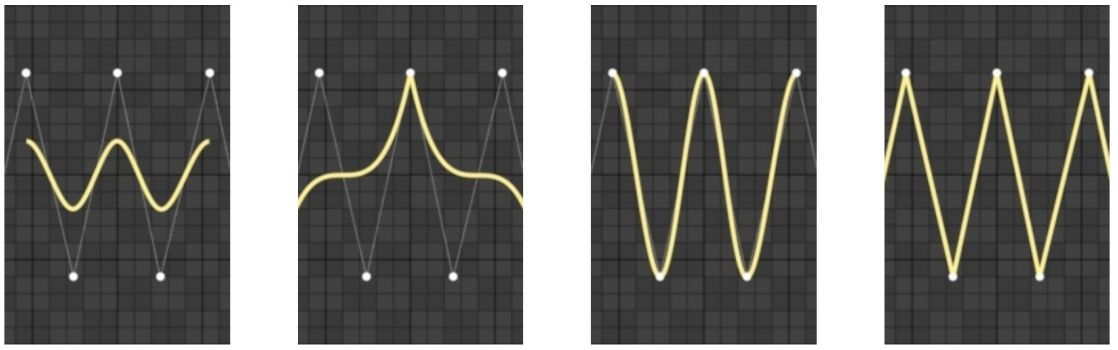
Syllabus

- 3D scene
  - Objects
    - Shape
    - Material
  - Lights
  - Cameras
- Rendering
- Image and display

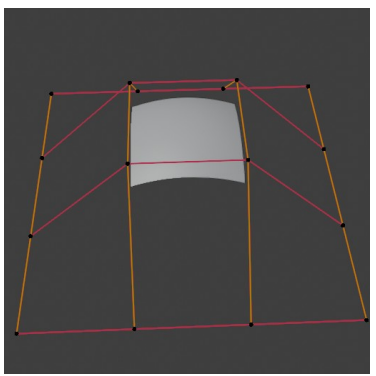


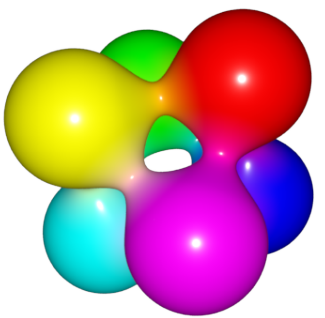
Recap: shape representations

- Points
  - Point clouds
  - Particles and Particle systems
- **Lines**
  - **Parametric curves**
- **Surfaces:**
  - Polygonal mesh
  - Subdivision surfaces
  - **Parametric surfaces**
  - **Implicit surfaces**
- Volumetric objects/solids
  - Voxels
  - Space partitioning data-structures



Parametric curves


Parametric surfaces


Implicit surfaces

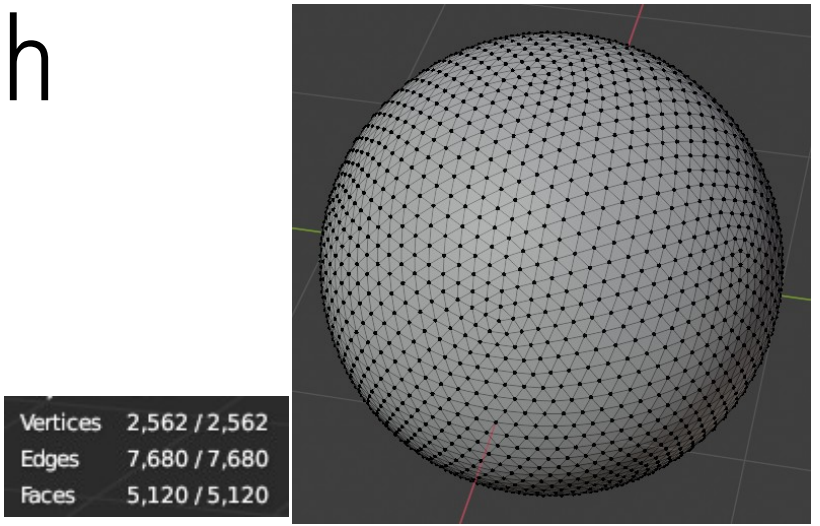
Recap: shape representations

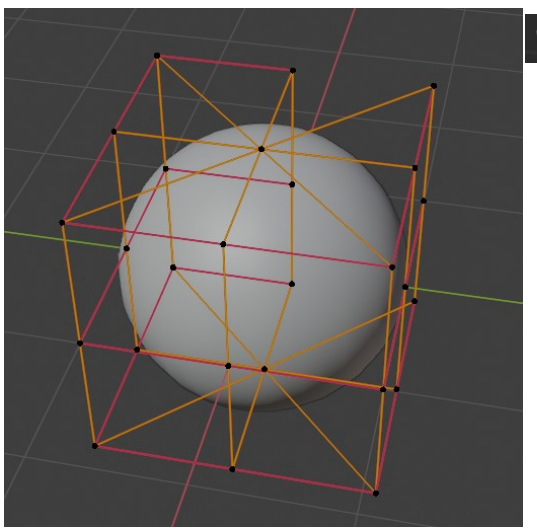
- Mesh surface representation: the most used and widespread
  - **Triangle:** basic atomic rendering primitive for GPU graphics pipelines and most ray-tracers
- Objects made in modeling systems can have many underlying geometric descriptions.
  - Easier and efficient modeling of shapes for the user
  - All higher-level geometrical descriptions are evaluated as set of triangles and then used

Parametric surfaces vs Mesh

Parametric surfaces have advantages over meshes in certain scenarios:

- More **compact representation** than meshes:
  - Less memory requirements
  - Less transformation operations
- **Scalable geometric primitives**
  - Geometry can be generated on the fly by evaluating the equations
- They can represent **smoother and more continuous primitives** than polygons
  - Convenient for representing organic and curved objects
- Certain modeling tasks can be efficiently performed, e.g., animation and collision





Applications

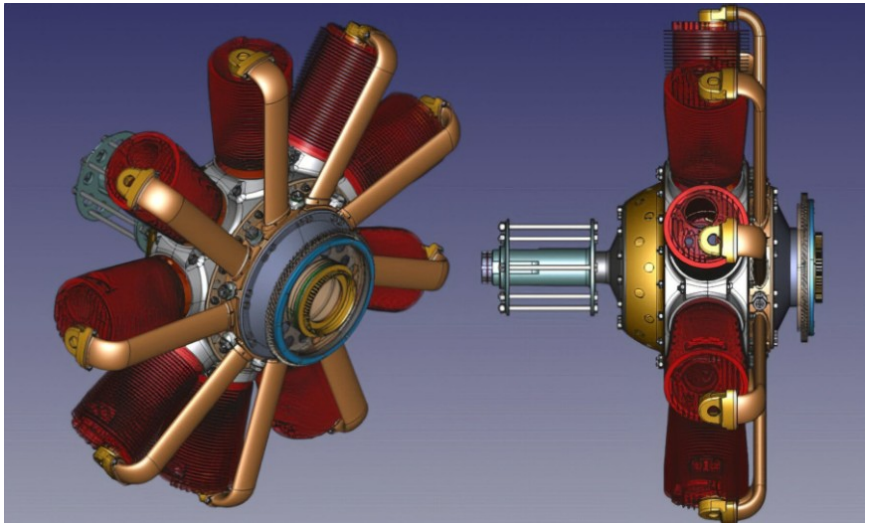
Paul de Casteljau

1959

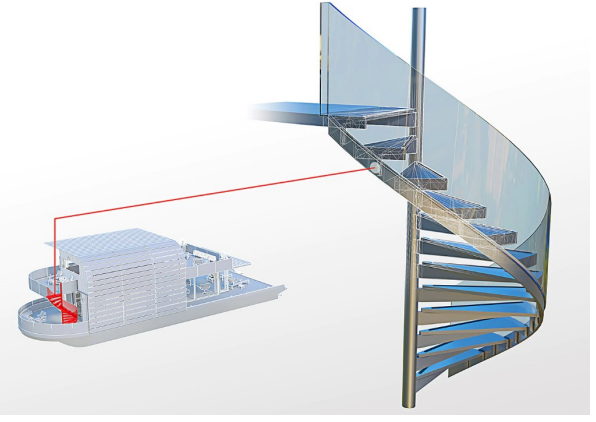
Pierre Bézier

1962

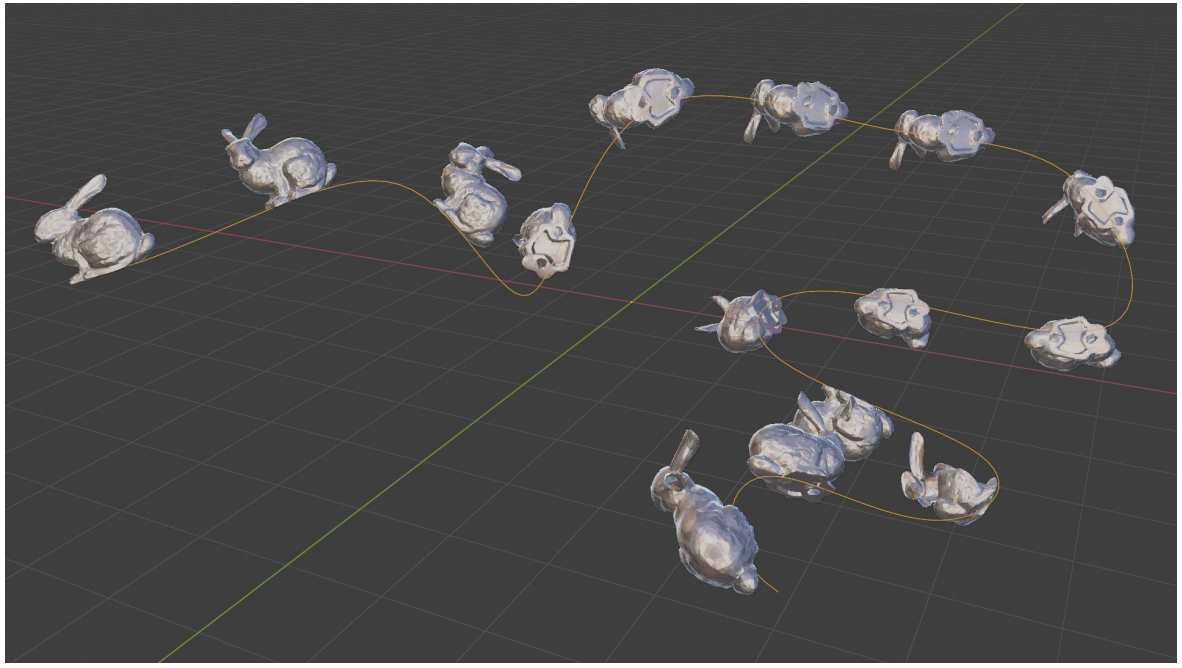
https://en.wikipedia.org/wiki/Paul_de_Casteljau
https://en.wikipedia.org/wiki/Pierre_B%C3%A9zier



Manufacturing product design: https://www.freecad.org/

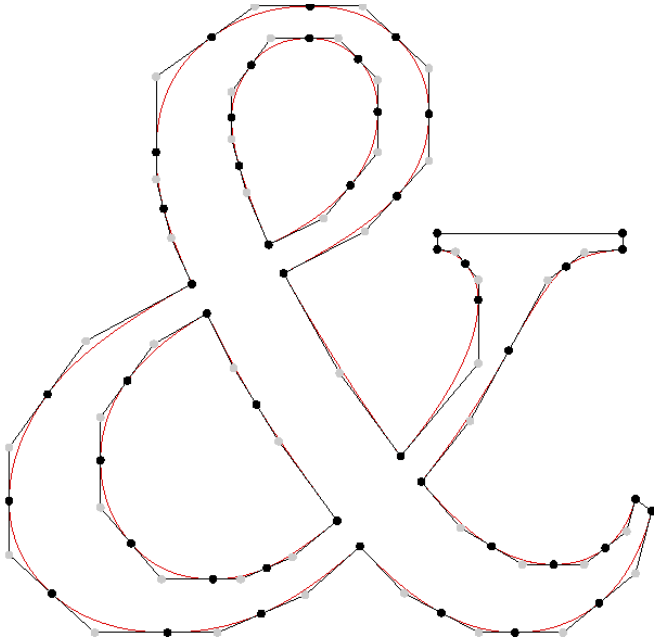


Arhitecture:
https://www.autodesk.com/products/autocad/included-toolsets/autocad-architecture

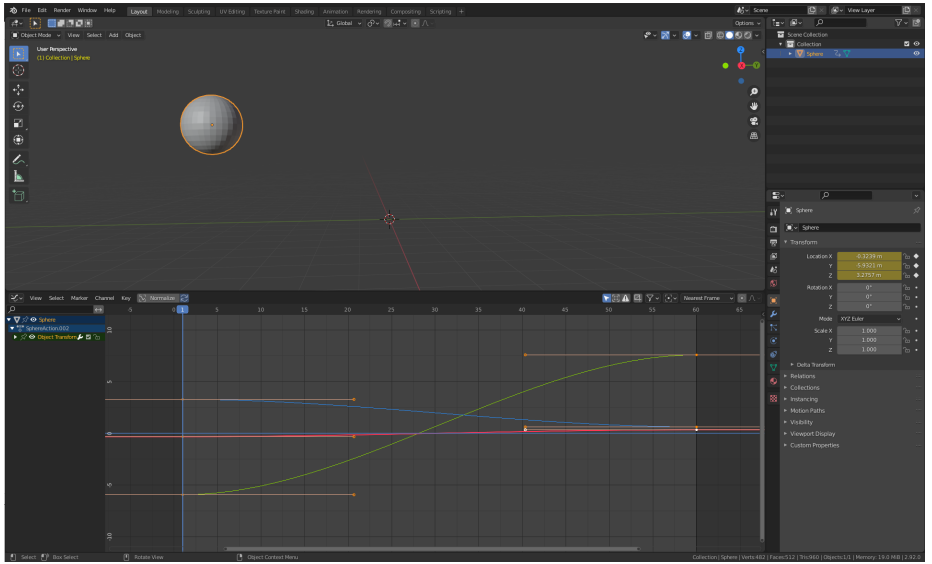


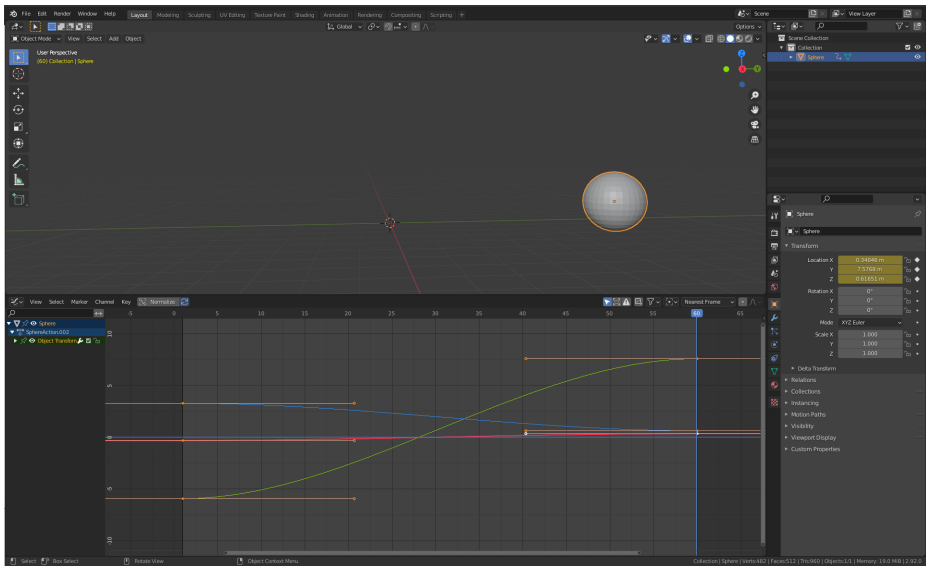
Animation paths:
https://docs.blender.org/manual/en/latest/animation/constraints/relationship/follow_path.html

Rendering hair:
https://developer.nvidia.com/gpugems/gpugems2/part-iii-high-quality-rendering/chapter-23-hair-animation-and-rendering-nalu-demo



Font design:
https://font-bakers.github.io/knead/user-guide/

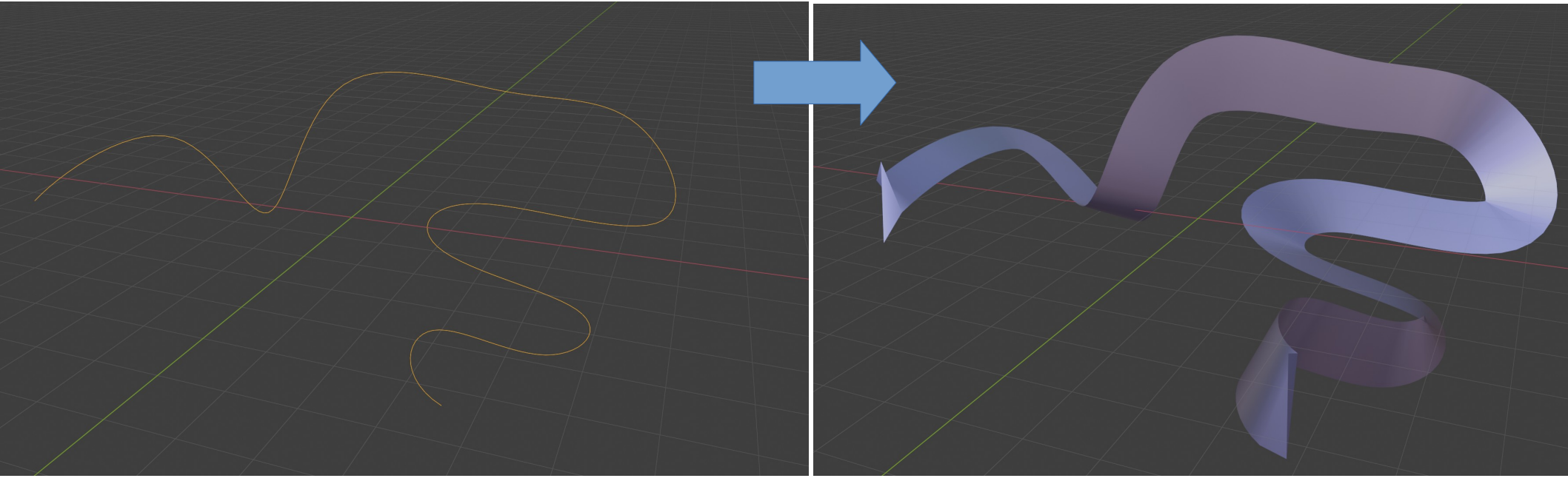




Interpolation between keyframes (transforms):
https://docs.blender.org/manual/en/latest/editors/graph_editor/introduction.html

Parametric curves and surfaces

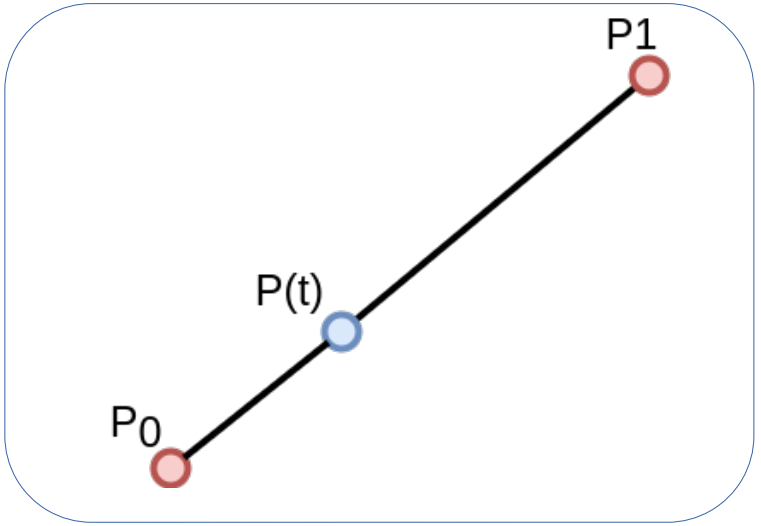
- Parametric surfaces are extension of parametric curves



Parametric curves

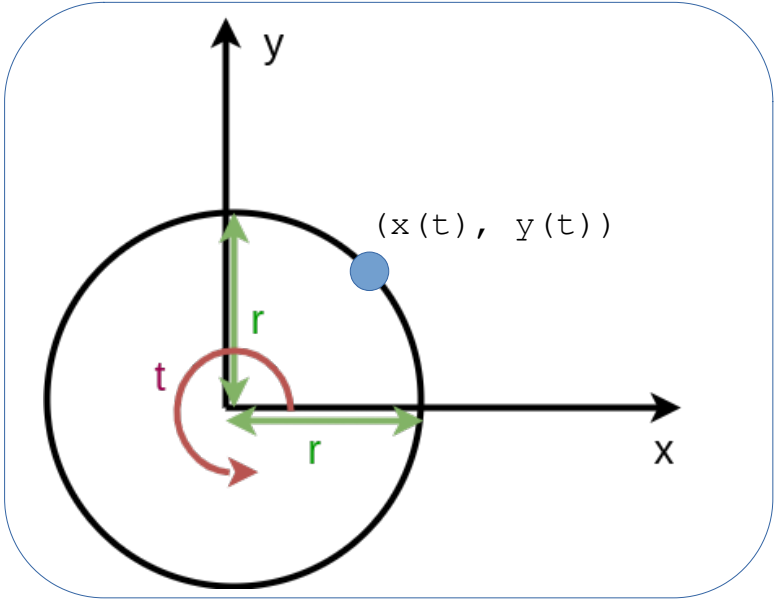
Parametric curves

- **Described with a formula as a function of parameter `t`**: `p(t), t in [a,b]`
  - Generated points are continuous



Example: **line segment**

$P(t) = (1-t)\ P_0 + t\ P_1$



Example: **circle**

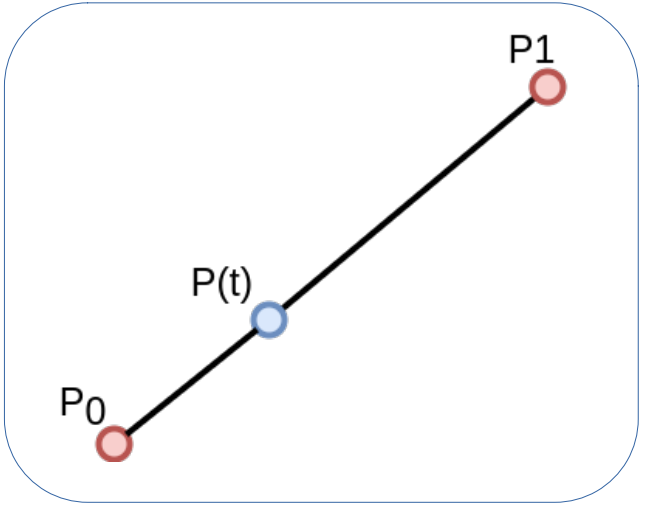
```
x(t) = r cos(2 * PI * t), t in [0,1]
y(t) = r sin(2 * PI * t), t in [0,1]
```

Arbitrary curves

- **Bezier curve**
- Hermite curve
- Catmull-Rom spline
- B-Splines

Bezier curves: linear interpolation

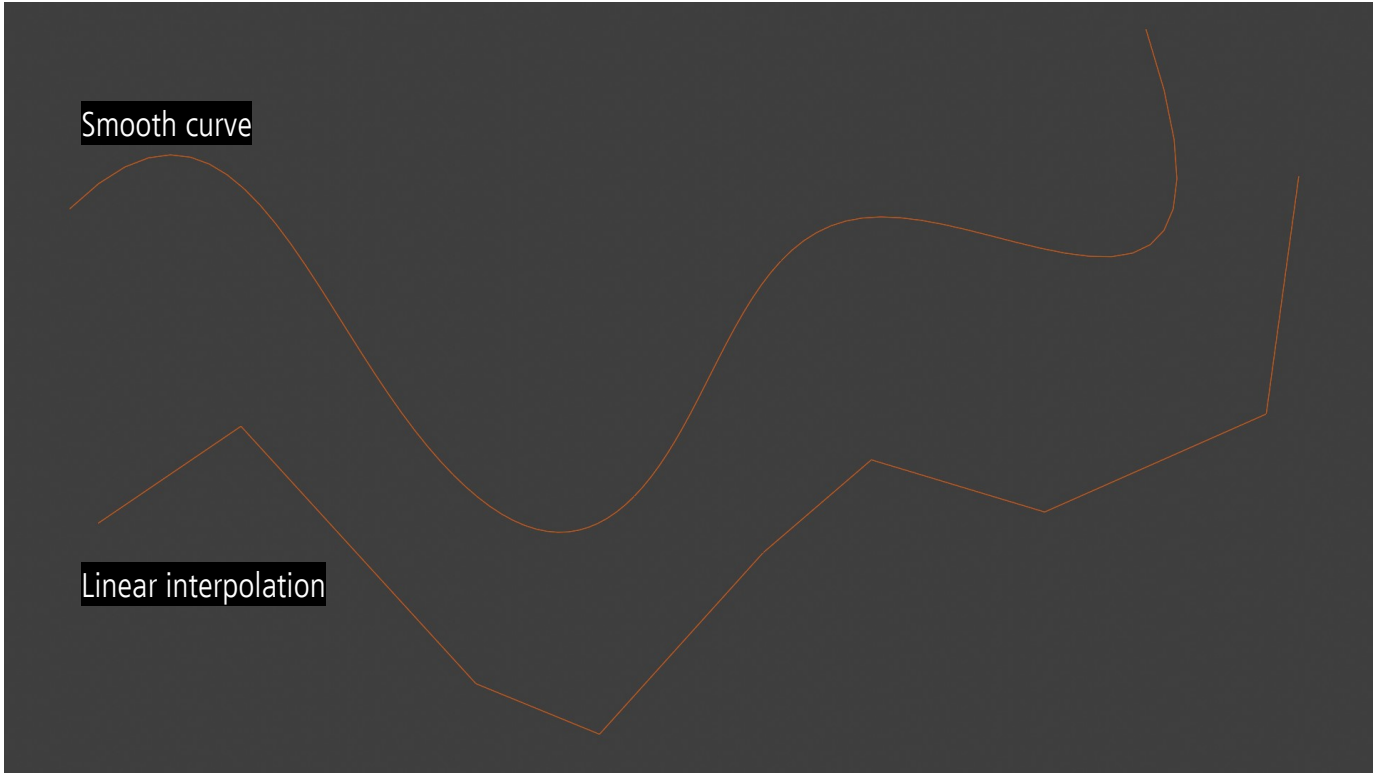
- **Linear interpolation** between two **control points** P_0 and P_1 traces out straight line.
  - $P(t) = (1-t)\ P_0 + t\ P_1$
  - Short: $\mathrm{lerp}(P_0, P_1, t)$
  - For $0 < t < 1$, generated points are on straight line between P_0 and P_1. Otherwise outside.
  - $P(0) = P_0$ and $P(1) = P_1$



$P(x,y,z)$
$P(x,y)$

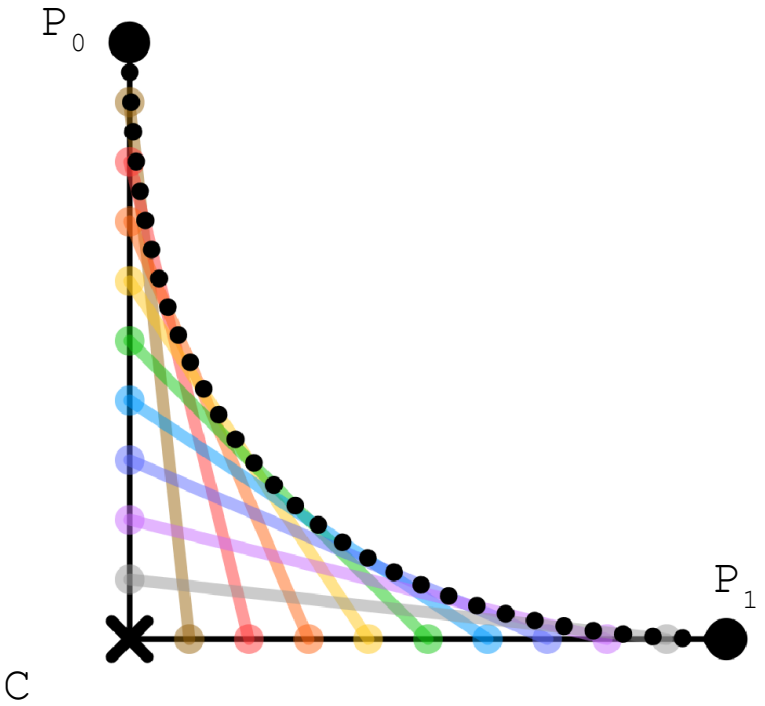
Bezier curves: linear interpolation

- Linear interpolation between multiple points gives us straight segments with sudden (discontinuous) changes at joints between.



Bezier curves: repeated interpolation

- Instead of linear interpolation between two points: P_0 and P_1 – **end points**, add another point C – **control point** and use it for **repeated interpolation**

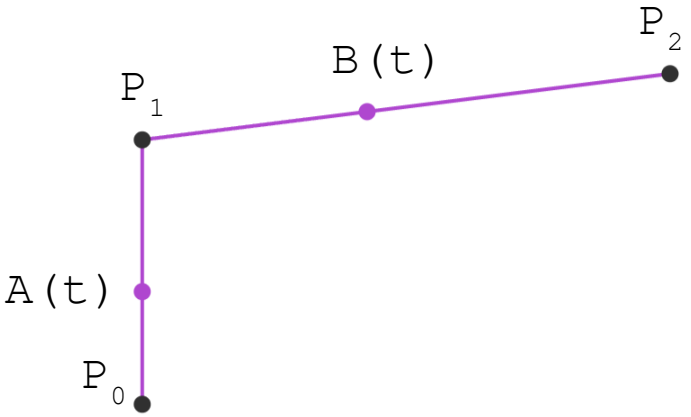


* Independently discovered by Paul de Casteljau and Pierre Bezier for use in French car industry.

Bezier curves: repeated interpolation

Linear interpolation

$A(t) = \text{lerp}(P_0, P_1, t)$

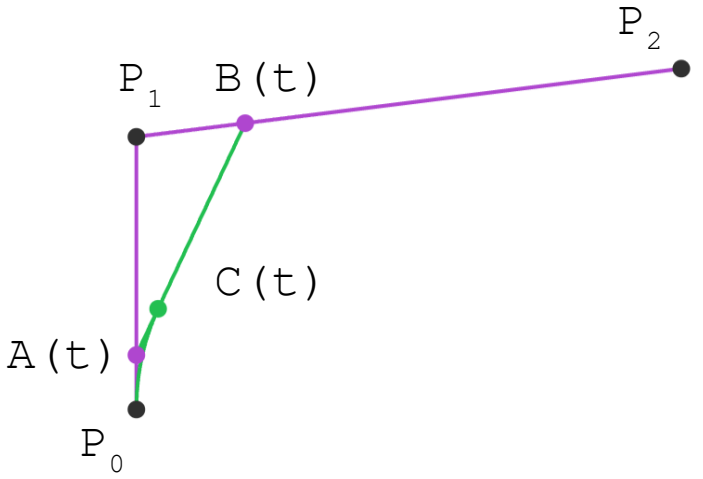


$A(t) = \text{lerp}(P_0, P_1, t)$

$B(t) = \text{lerp}(P_1, P_2, t)$

Quadratic Bezier curve

- Repeated interpolation:
  - A(t): linear interpolation between P_0 and P_1
  - B(t): linear interpolation between P_0 and P_1
  - `C(t):` linear interpolation between `A(t)` and `B(t)`



$$A(t) = \text{lerp}(P_0, P_1, t)$$
$$B(t) = \text{lerp}(P_1, P_2, t)$$
$$C(t) = \text{lerp}(A(t), B(t), t)$$

Quadratic Bezier curve

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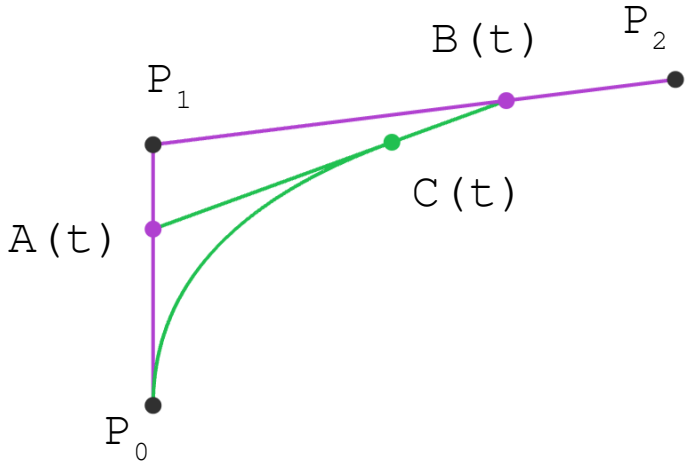

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Quadratic Bezier curve

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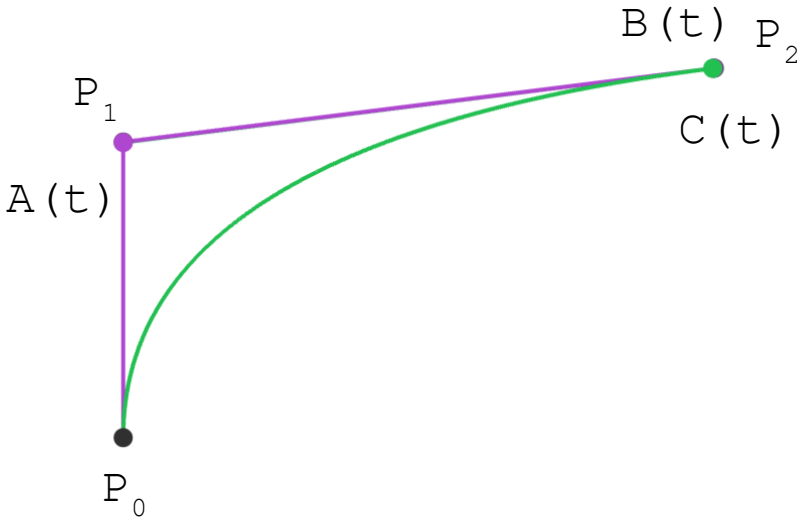
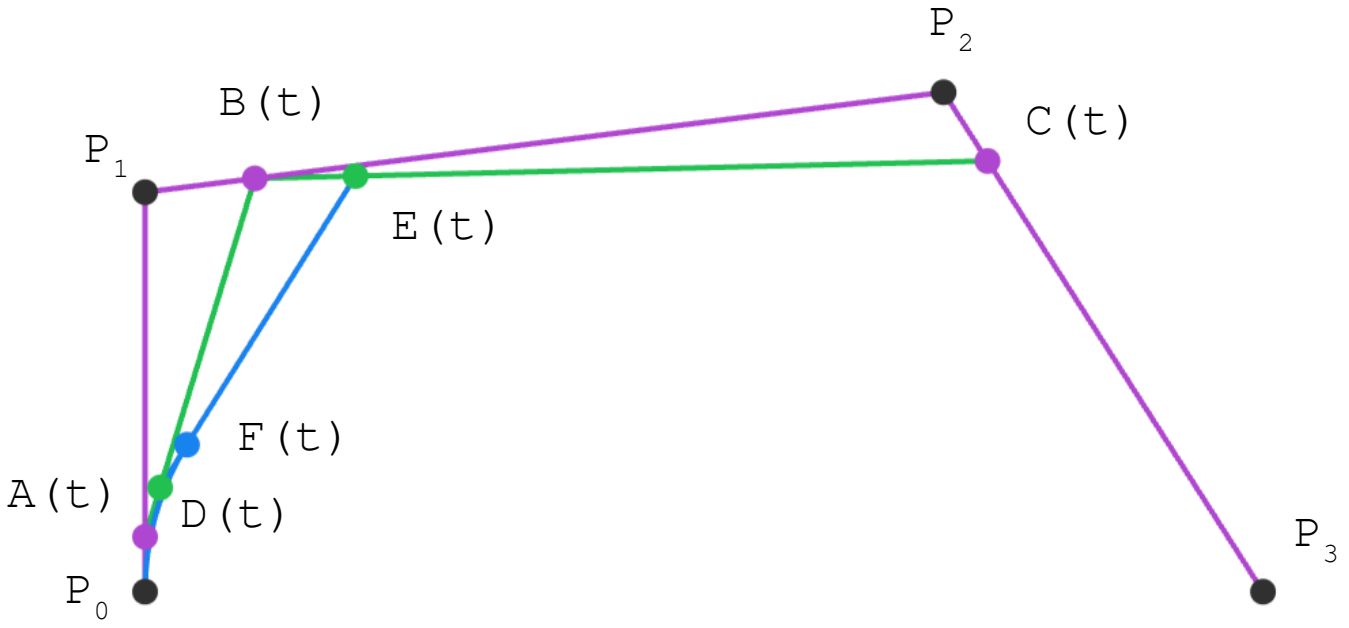


$$A(t) = \text{lerp}(P_0, P_1, t)$$
$$B(t) = \text{lerp}(P_1, P_2, t)$$
$$C(t) = \text{lerp}(A(t), B(t), t)$$

Cubic Bezier curve



$A(t) = \text{lerp}(P_0, P_1, t)$

$B(t) = \text{lerp}(P_1, P_2, t)$

$C(t) = \text{lerp}(P_2, P_3, t)$

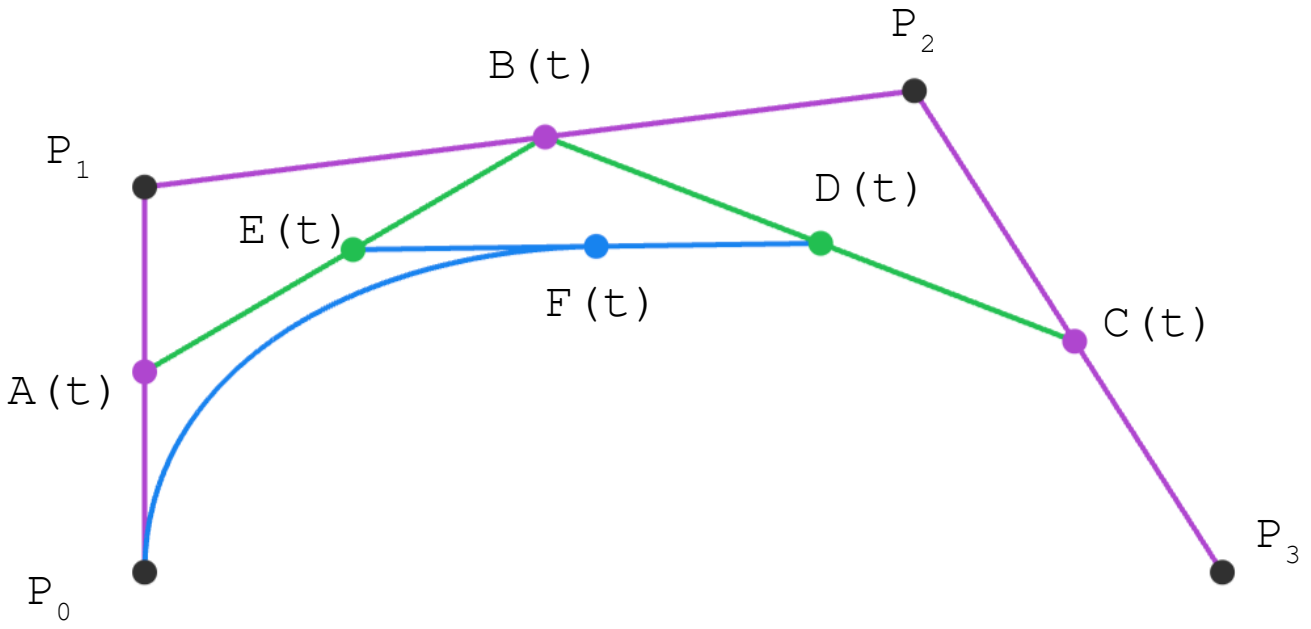
$D(t) = \text{lerp}(A(t), B(t), t)$

$E(t) = \text{lerp}(B(t), C(t), t)$

$C(t) = \text{lerp}(D(t), E(t), t)$

$C(t) = \text{lerp}(D(t), E(t), t)$

Cubic Bezier curve

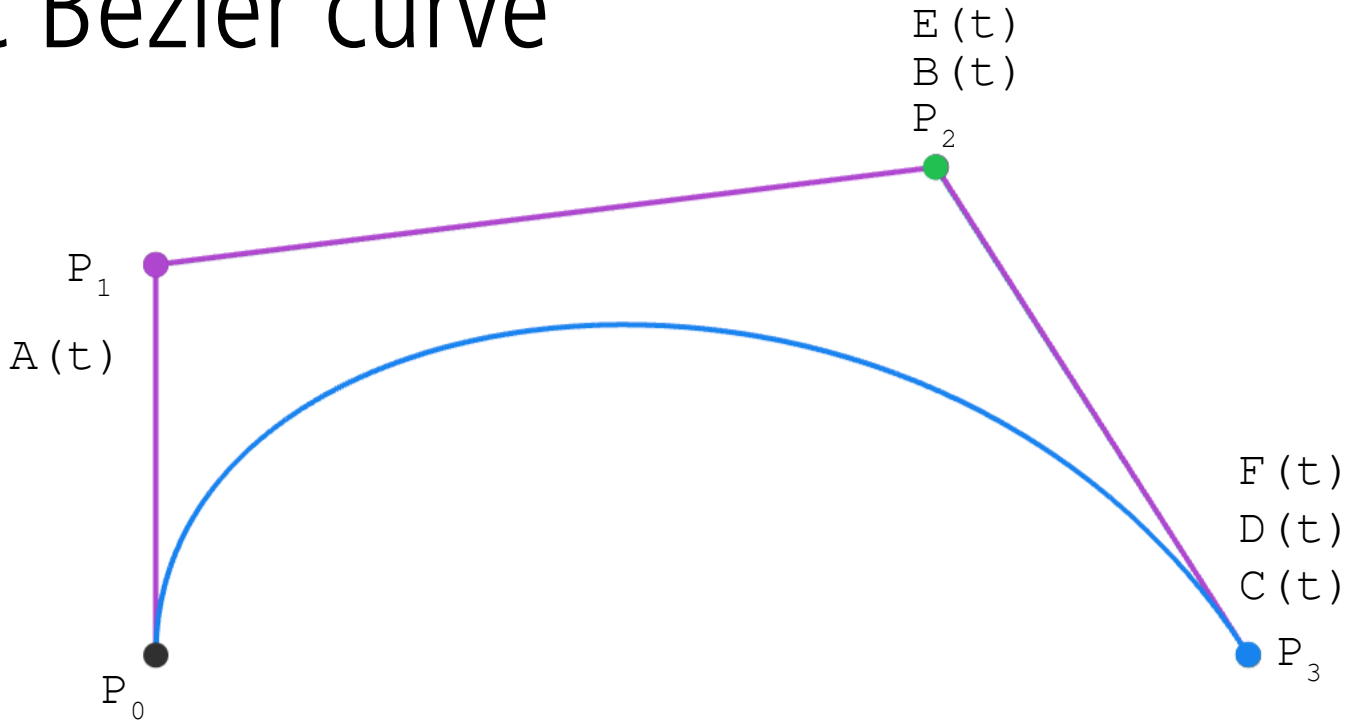


$A(t) = \text{lerp}(P_0, P_1, t)$
$B(t) = \text{lerp}(P_1, P_2, t)$
$C(t) = \text{lerp}(P_2, P_3, t)$

$D(t) = \text{lerp}(A(t), B(t), t)$
$E(t) = \text{lerp}(B(t), C(t), t)$
$C(t) = \text{lerp}(D(t), E(t), t)$

$C(t) = \text{lerp}(D(t), E(t), t)$

Cubic Bezier curve



$A(t) = \text{lerp}(P_0, P_1, t)$
$B(t) = \text{lerp}(P_1, P_2, t)$
$C(t) = \text{lerp}(P_2, P_3, t)$

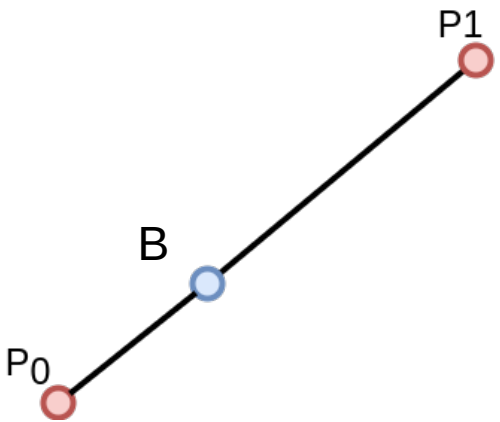
$D(t) = \text{lerp}(A(t), B(t), t)$
$E(t) = \text{lerp}(B(t), C(t), t)$
$C(t) = \text{lerp}(D(t), E(t), t)$

$C(t) = \text{lerp}(D(t), E(t), t)$

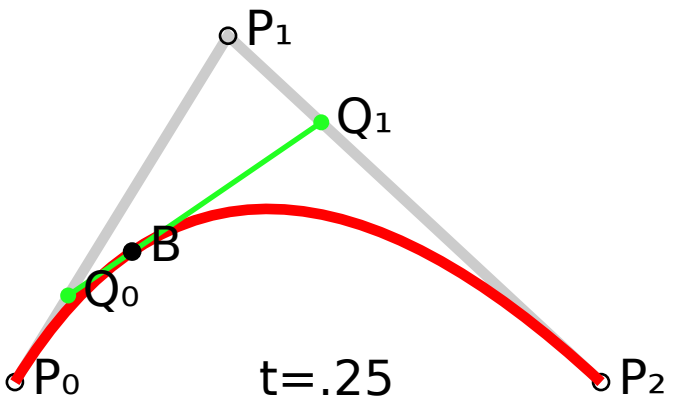
Bezier curves

- **Degree of curve** is n, if n +1 control points are used.
  - More control points → more degrees of freedom
- **Most often, cubic Bezier curve is used**
  - Higher degree Bezier curves are expensive to evaluate

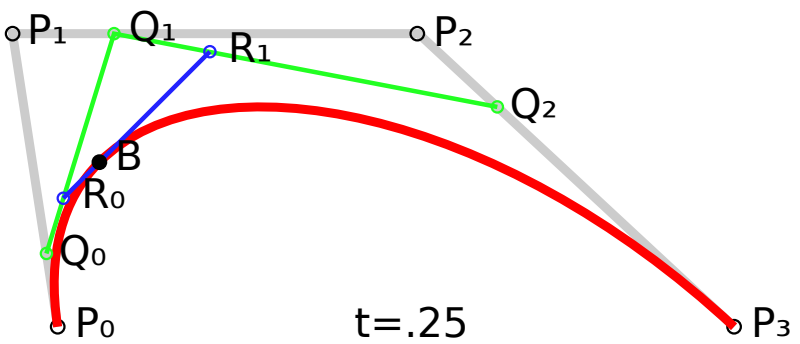
n = 1 → 1st degree, **linear Bezier**



n = 2 → 2nd degree, **quadratic Bezier**



n = 3 → 3rd degree, **cubic Bezier**



https://acko.net/files/fullfrontal/fullfrontal/wdcode/online.html
https://www.youtube.com/watch?v=jvPPXbo87ds&ab_channel=FreyaHolm%C3%A9r

Bezier curves: de Casteljau algorithm

- Repeated or recursive linear interpolation is often referred as **de Casteljau algorithm**.

Linear Bezier:

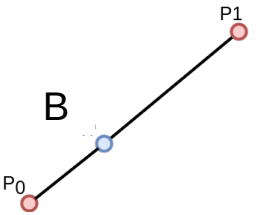
$$\mathbf{B}(t) = \mathbf{P}_0 + t(\mathbf{P}_1 - \mathbf{P}_0) = (1-t)\mathbf{P}_0 + t\mathbf{P}_1, \quad 0 \le t \le 1$$

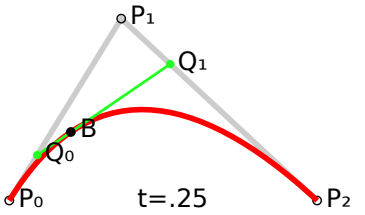
Quadratic Bezier:

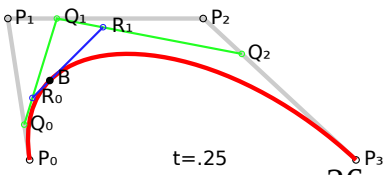
$$\mathbf{B}(t) = (1-t)[(1-t)\mathbf{P}_0 + t\mathbf{P}_1] + t[(1-t)\mathbf{P}_1 + t\mathbf{P}_2], \quad 0 \le t \le 1,$$

Cubic Bezier:

$$\mathbf{B}(t) = (1-t)\mathbf{B}_{\mathbf{P}_0,\mathbf{P}_1,\mathbf{P}_2}(t) + t\mathbf{B}_{\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3}(t), \quad 0 \le t \le 1.$$





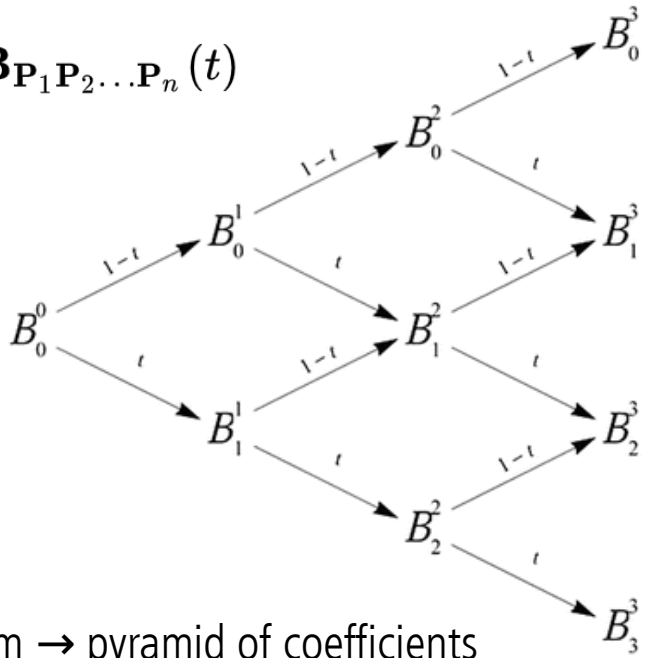


De Casteljau algorithm: recursive formula*

- Recursive formula for arbitrary Bezier curve degree:

$$\mathbf{B}_{\mathbf{P}_0}(t) = \mathbf{P}_0, \text{ and}$$

$$\mathbf{B}(t) = \mathbf{B}_{\mathbf{P}_0\mathbf{P}_1\ldots\mathbf{P}_n}(t) = (1-t)\mathbf{B}_{\mathbf{P}_0\mathbf{P}_1\ldots\mathbf{P}_{n-1}}(t) + t\mathbf{B}_{\mathbf{P}_1\mathbf{P}_2\ldots\mathbf{P}_n}(t)$$

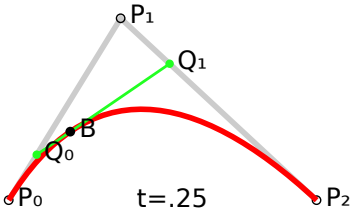


De Casteljau algorithm → pyramid of coefficients

Bezier curve: Bernstein form

- Quadratic Bezier: $\mathbf{B}(t) = (1-t)[(1-t)\mathbf{P}_0 + t\mathbf{P}_1] + t[(1-t)\mathbf{P}_1 + t\mathbf{P}_2],\ 0 \le t \le 1,$
- Quadratic Bezier re-arranged → algebraic description

$$\mathbf{B}(t) = (1-t)^2\mathbf{P}_0 + 2(1-t)t\mathbf{P}_1 + t^2\mathbf{P}_2,\ 0 \le t \le 1$$



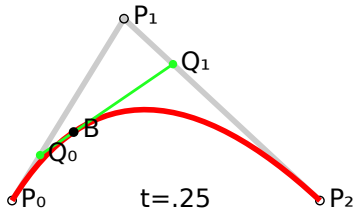
- **Every Bezier curve can be described with algebraic fromula**
  - repeated interpolation is not needed.
- Generalized algebraic description: **Bernstein form**:

$$\mathbf{B}(t) = \sum_{i=0}^{n} \binom{n}{i}(1-t)^{n-i}t^i\mathbf{P}_i$$

$$= (1-t)^n\mathbf{P}_0 + \binom{n}{1}(1-t)^{n-1}t\mathbf{P}_1 + \cdots + \binom{n}{n-1}(1-t)t^{n-1}\mathbf{P}_{n-1} + t^n\mathbf{P}_n, \qquad 0 \leqslant t \leqslant 1$$

Bezier curve: Bernstein form

- Example: n = 2 (quadratic)



$$B(t) = \sum_{i=0}^{2} \binom{2}{i} t^i (1-t)^{(2-i)} P_i$$

$$B(t) = \binom{2}{0} t^0 (1-t)^2 P_0 + \binom{2}{1} t^1 (1-t)^1 P_1 + \binom{2}{2} t^2 (1-t)^0 P_2, 0 \le t \le 1$$

$$\mathbf{B}(t) = (1-t)^2 \mathbf{P}_0 + 2(1-t)t\mathbf{P}_1 + t^2 \mathbf{P}_2, \ 0 \le t \le 1$$

$$\mathbf{B}(t) = (1-t)[(1-t)\mathbf{P}_0 + t\mathbf{P}_1] + t[(1-t)\mathbf{P}_1 + t\mathbf{P}_2], \ 0 \le t \le 1,$$

Bezier curve: Bernstein form

- Bernstein form:

$$\mathbf{B}(t) = \sum_{i=0}^{n} b_{i,n}(t)\mathbf{P}_i, \quad 0 \le t \le 1$$

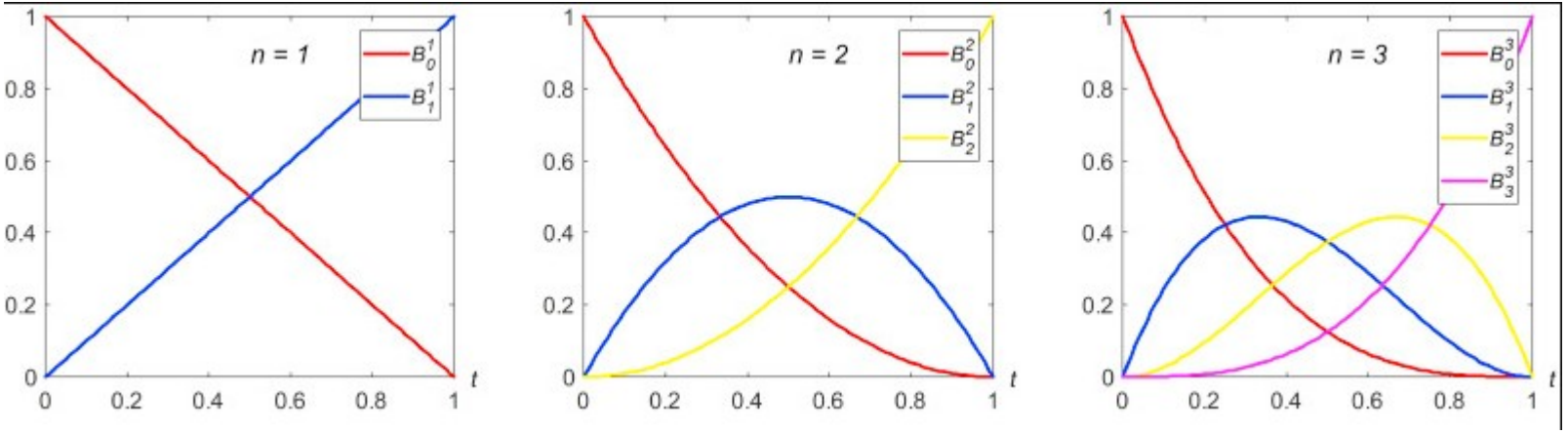
- **Bernstein polynomials** aka **Bezier basis functions**:

$$b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i = 0, \ldots, n$$

Basis functions

- Defines influence (weighting) of each control points as parameter `t` changes.
  - Example: when `t` increases, blending weight form P_0 decreases and blending weight for P_1 increases, etc.

$$\mathbf{B}(t) = \sum_{i=0}^{n} b_{i,n}(t)\mathbf{P}_i, \quad 0 \le t \le 1 \qquad b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i = 0, \ldots, n$$



Bernstein polynomials aka blending functions

https://www.sciencedirect.com/science/article/abs/pii/S0997753817306381

Bernstein polynomials

- **Bernstein polynomials (Bezier basis function)** defines properties of a Bezier curve:

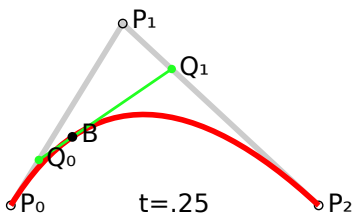
$$b_{i,n}(t) \in [0,1] \text{ when } t \in [0,1]$$

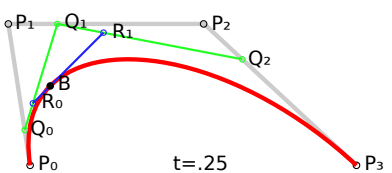
"Polynomials are in `[0,1]` when `t` in `[0,1]`"

$$\sum_{i=0}^{n} b_{i,n}(t) = 1$$

"Curve will stay close to the control points P_i"

- **Whole Bezier curve will be located in convex hull of control points**
  - useful for computing bounding area or volume of curve.



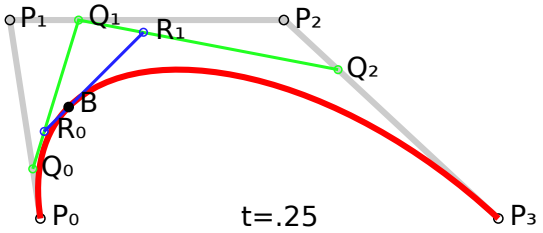


Bezier curve: matrix representation

- In practice, for efficient calculation, Bezier curve is represented in a **matrix form**
- Example: **cubic Bezier curve**

$$\mathbf{B}(t) = (1-t)^3\mathbf{P}_0 + 3(1-t)^2 t\mathbf{P}_1 + 3(1-t)t^2\mathbf{P}_2 + t^3\mathbf{P}_3,\ 0 \le t \le 1$$

$$B(t) = (1\ t\ t^2\ t^3\ ) \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ 1 & 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$



Check matrix

https://blog.demofox.org/2016/03/05/matrix-form-of-bezier-curves/

Bezier curves: good properties

- **Intuitive theory**, good for understanding curves: repeated interpolation
- **Compact form**: Bernstein form and matrix representation
- **Tangent vector at curve point** can be calculated easily (derivation of polynomial)
- **Easy manipulation of large number of points**
  - Large number of points can be generated from parametric description
  - If rotation of those points is needed, then curve (few control points) are rotated and then points are generated

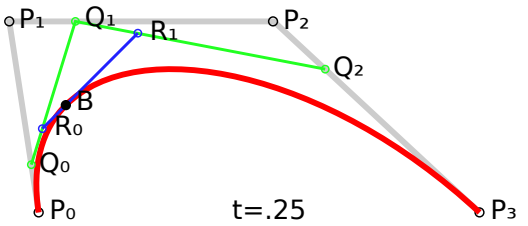
Experiments and analysis: https://www.youtube.com/watch?v=aVwxzDHniEw&ab_channel=FreyaHolm%C3%A9r

Bezier curves: problems

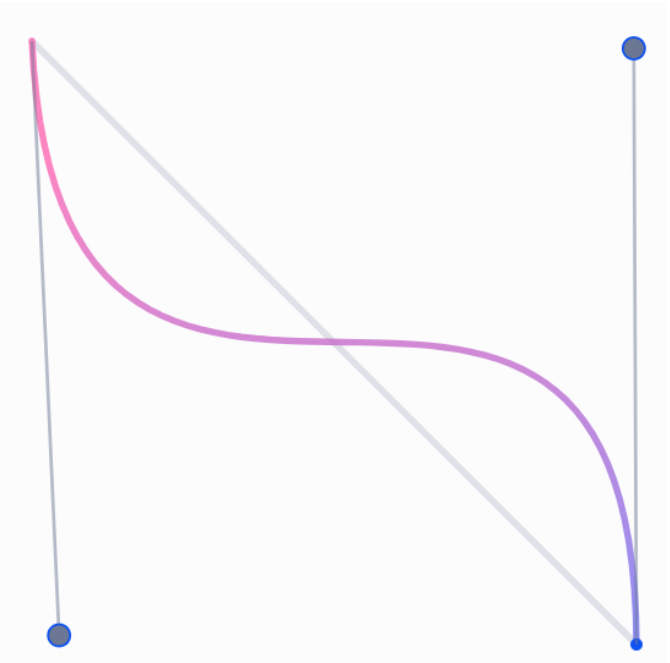
- **Non-interpolating:** do not pass through all control points (except endpoints)
- Not many degrees of freedom:
  - **Only position of control points can be chosen freely**
  - Not every curve can be described with Bezier curve (e.g., simple circle can not be described with one or collection of Bezier curves)
  - Alternative is rational Bezier curve*
- Degree increases with number of control points
  - Hard to control and complex computation
  - Bernstein polynomials do not interpolate well

* Hoschek, J. and Lasser, D. (1993) Fundamentals of Computer Aided Geometric Design. A K Peters.

Joining cubic Bezier curves



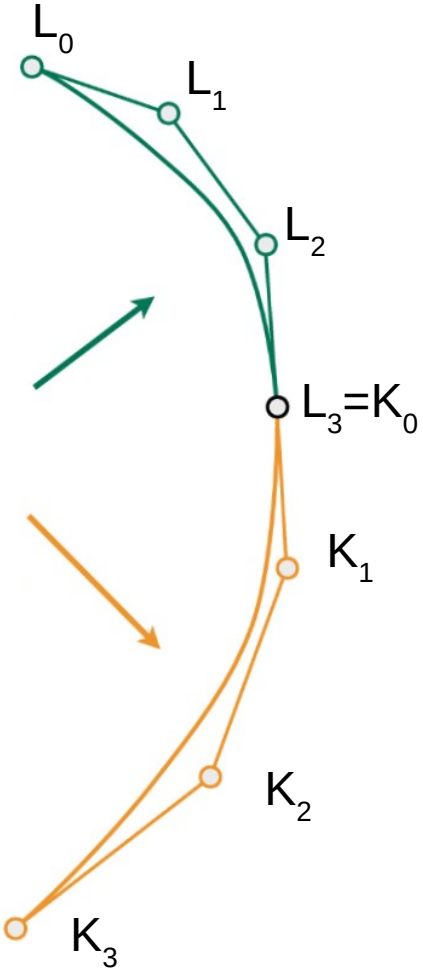
- **In practice, cubic Bezier curves are concatenated to form larger spline**
  - Cubic curves are lowest degree curves that can describe complex form, e.g., S-shaped curve called inflection
  - Lower computation complexity
  - Resulting spline will pass through the set of points



Cubic Bezier with inflection: https://blog.maximeheckel.com/posts/cubic-bezier-from-math-to-motion/

Joining cubic Bezier curves

- Example: **two cubic Bezier curves** (4 control points):
  - L_i, i = 0, 1, 2, 3
  - K_i, i = 0, 1, 2, 3
- To join the curves we can set $L_3 = K_0$
  - **Joint** - point where curves are joined



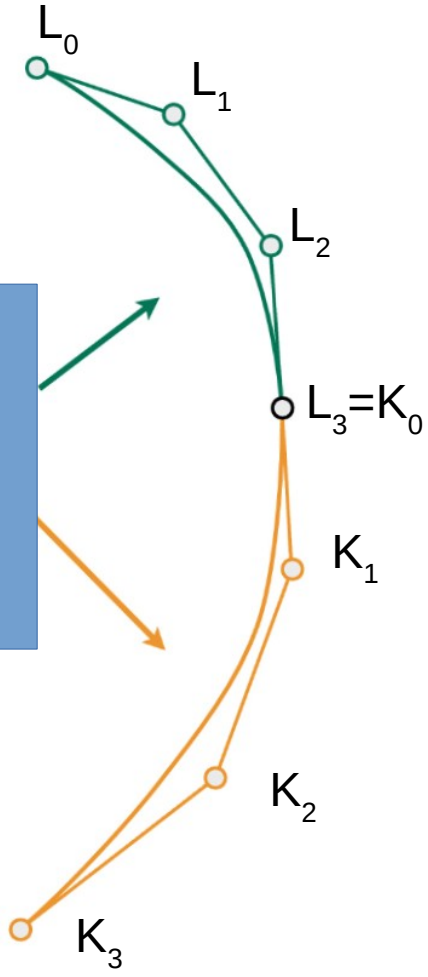
https://cs184.eecs.berkeley.edu/sp22/lecture/7-77/intro-to-geometry-splines-and-be

Joining cubic Bezier curves

- Composite curve formed from several curve pieces is called **piecewise Bezier curve, p(t)**
  - Now `t` is in `[t1, t2]`
  - First curve is `p(t')`, where `t'` in `[0,1]`
  - Second curve is `p(t')`, where `t' = (t - t1)`
- Two curves connected just using $L_3 = K_0$ will n
  joint.
- Improved smoothness is achieved using tangent constraint:

  $(K_1 - K_0) = c\ (L_3 - L_2),\ c > 0$*

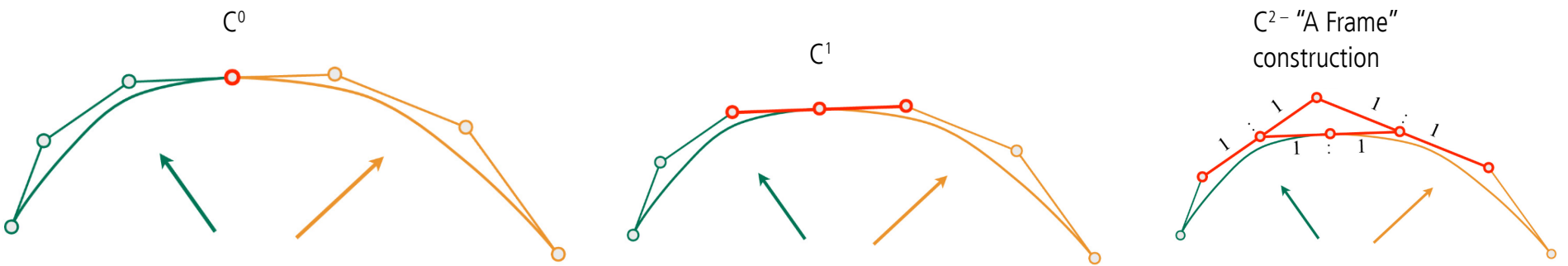
re-read



* Curves and Surfaces in Computer Aided Geometric Design, Farin, Gerald

Joined curves continuity

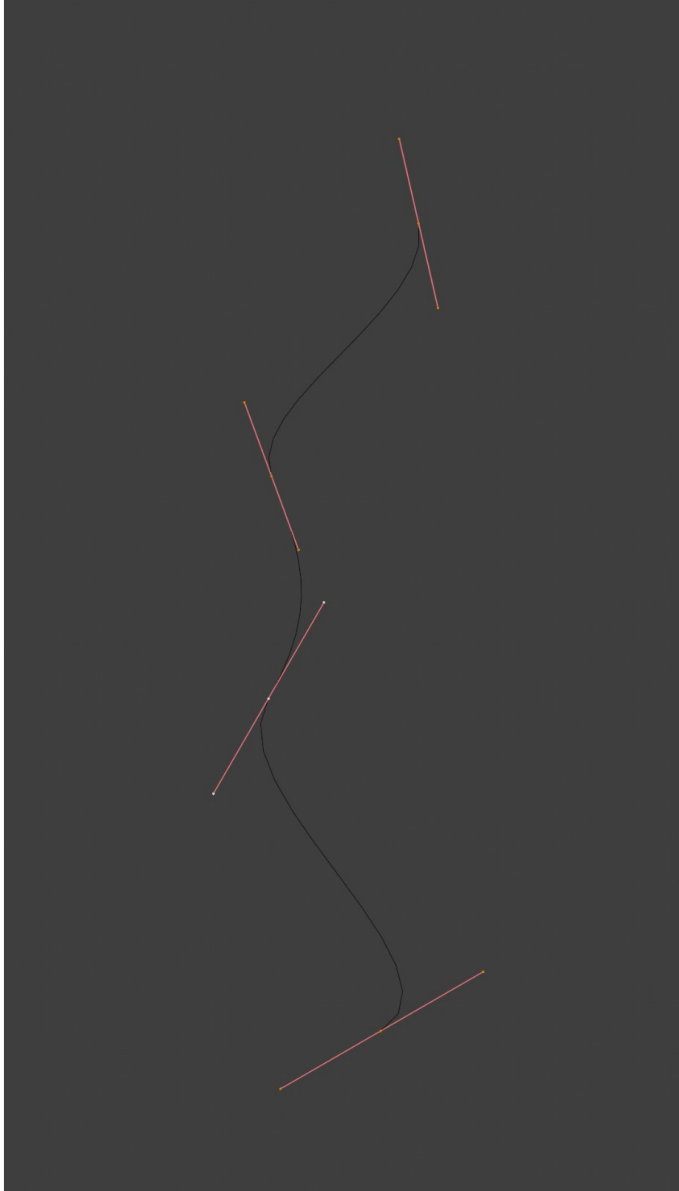
- This way, many cubic Bezier curves are chained into piecewise cubic Bezier line
- Continuity of joined curves:
  - C^0 – **positional continuity** - segments should joint at the same point
  - C^1 – **velocity continuity** - derivation of any point (including joints) must be continuous
  - C^2 - **acceleration continuity** - first and second derivatives are continuous functions



https://cs184.eecs.berkeley.edu/sp22/lecture/7/intro-to-geometry-splines-and-be

Joined curves continuity

- **Geometrical continuity**:
  - G^0 – **positional continuity**: holds when the end points of two curves coincide
  - G^1 – **tangent continuity** - tangent vectors from curve segments that meet at joint should be parallel and have same direction – no sharp edges. Continuous edges make splines look natural – often sufficient measure
  - G^2 – **curvature continuity** - tangent vectors from curve segments that meet at joint should be of same length and rate of length change – perfectly smooth surface – two joined surfaces appear as one

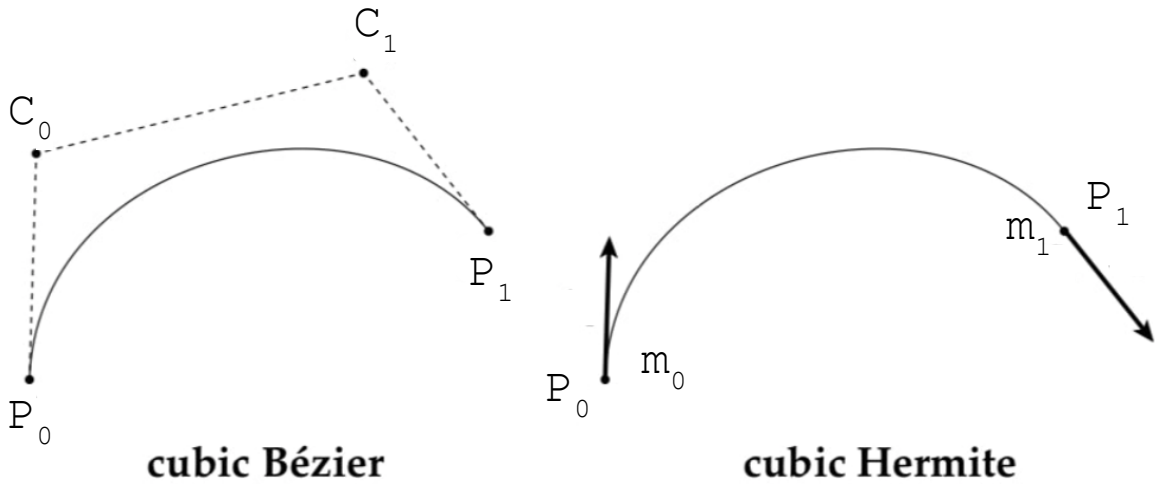


Arbitrary curves

- Bezier curve
  - Intuitive theory: repeated interpolation
  - Formalization: Bernsten and matrix form
  - Concatenating cubic curves into splines
  - Curve continuity
- **Hermite curve**
  - In practice, **cubic Hermite curve** is used for simpler authoring
- Catmull-Rom spline
- B-Splines

Cubic Hermite curve

- **Cubic Hermite interpolation curve:**
  - Interpolates points P_0 and P_1 – **starting and ending control points**
  - Interpolation is **controlled with tangent vectors**: m_0 at P_0 and m_1 at P_1
    - Tangent vector magnitude and direction influence the curve shape

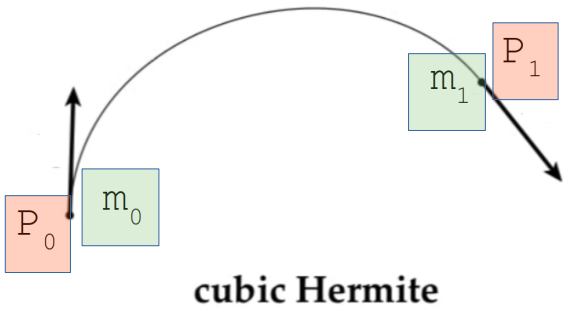


http://15462.courses.cs.cmu.edu/fall2020/article/10

Cubic Hermite curve

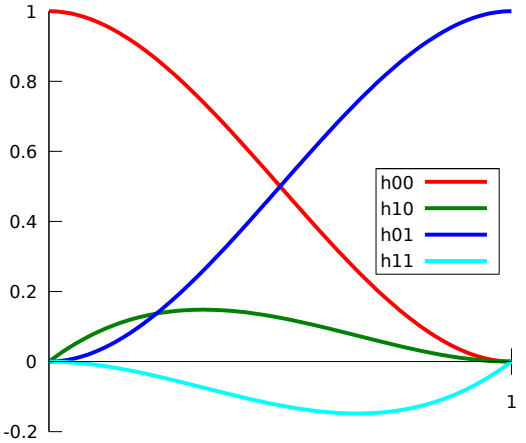
- Cubic Hermite curve `p(t), t in [0,1]`:

$$\boldsymbol{p}(t) = (2t^3 - 3t^2 + 1)\boldsymbol{p}_0 + (t^3 - 2t^2 + t)\boldsymbol{m}_0 + (-2t^3 + 3t^2)\boldsymbol{p}_1 + (t^3 - t^2)\boldsymbol{m}_1$$



cubic Hermite

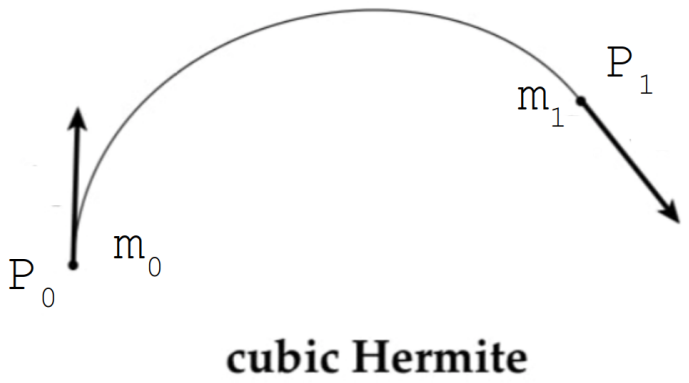
- Similarly as discussed with Bezier: **Bernstein form** with **Hermite basis function** exists



https://en.wikipedia.org/wiki/Cubic_Hermite_spline

Cubic Hermite curve

- In practice, cubic Hermite interpolation curve is represented with matrix for efficient computation.



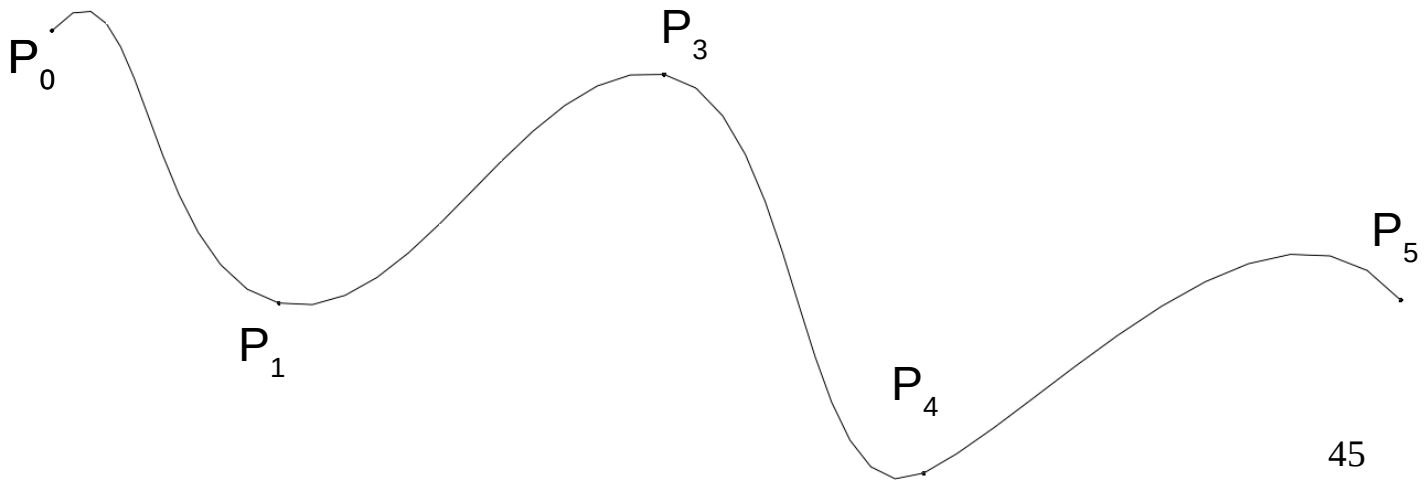
$$p(t) = (t^3\ t^2\ t\ 1\ ) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ m_0 \\ m_1 \end{pmatrix}$$

Check matrix

https://www.cubic.org/docs/hermite.htm

Splines: assembly of curves

- When interpolating more than two points, several Hermite curves can be connected together (similarly as done with Bezier).
- **Connecting multiple points $P_0 \ldots P_n$**
  - **$n - 1$ cubic Hermite curves** must be used
  - Resultant is **assembly of curves** called **spline**
  - Elements of assembly are called **segments** (e.g., Hermite sements)

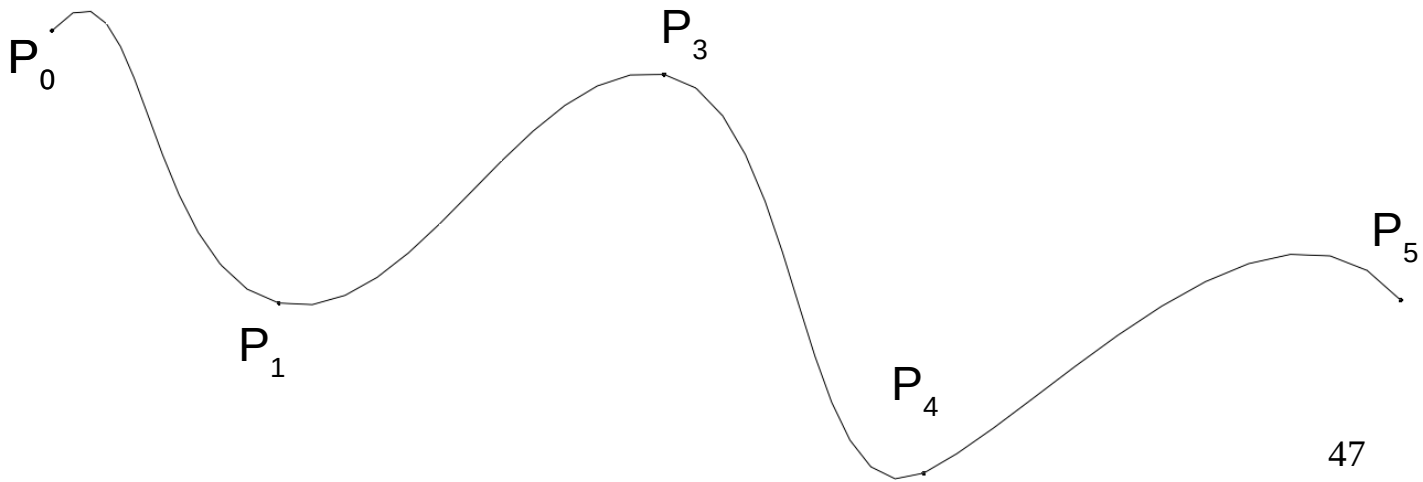


Arbitrary curves

- Bezier curve
- Hermite curve
- **Catmull-Rom spline**
- B-Splines

Catmull-Rom spline

- Cubic Hermite interpolation curve tangents can be hard to control
- Therefore, **tangents are calculated from control points** $P_0 \ldots P_n$ specified by the user
  - Given points $P_0 \ldots P_n$, the goal is to calculate tangents needed for Hermite curves which would create spline: continuous differentiable curve passing through each point → **Catmull-Rom spline**

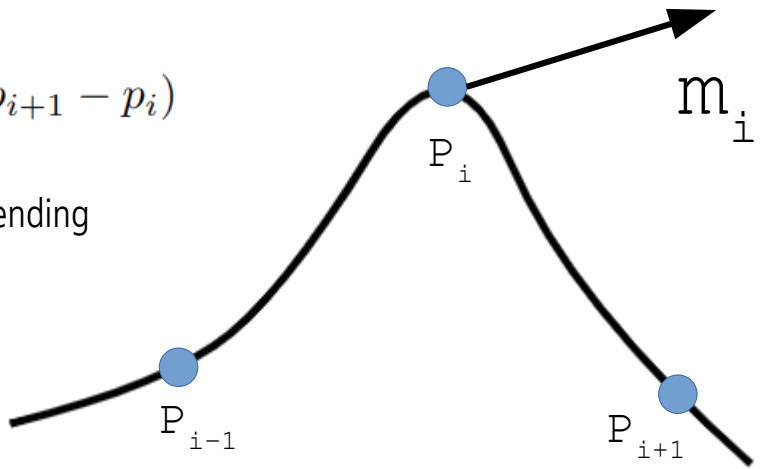


Kochanek-Bartels method

- Method for computing tangents m_i at joints P_i
- Assume that there is only **one tangent per control point**
  - Tangent m_i at P_i can be computed as combination of two chords: $P_i - P_{i-1}$ and $P_{i+1} - P_i$

$$m_i = \frac{(1-a)(1+b)}{2}(p_i - p_{i-1}) + \frac{(1-a)(1-b)}{2}(p_{i+1} - p_i)$$

Tension parameter - a: length of tangent. Higher values → sharper curve bending
Bias parameter - b: direction of tangent

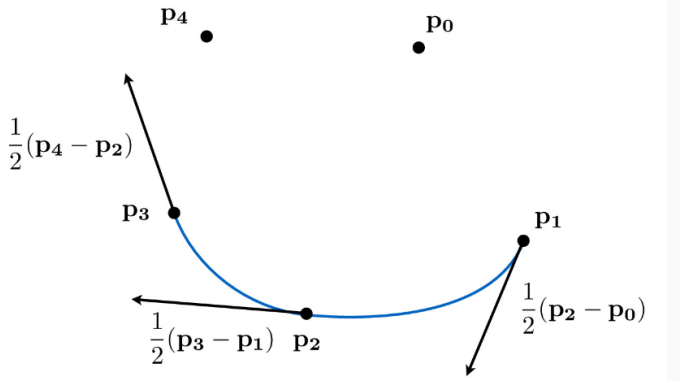


Kochanek-Bartels method can be extended with additional, continuity, parameter. For this, additional tangent at control points is introduced: https://www.cubic.org/docs/hermite.htm

Catmull-Rom spline

- **Catmull-Rom** spline is a special case where tension and bias parameters of **Kochanek-Bartels method** are set to 0.
- Tangents m_i in $P_0 \ldots P_n$ are calculated using:

$$m_i = \frac{1}{2}(p_i - p_{i-1}) + \frac{1}{2}(p_{i+1} - p_i)$$



https://cs184.eecs.berkeley.edu/sp22/lecture/7-50/intro-to-geometry-splines-and-be

Properties:

- Finding points on Catmull-Rom spline is fast and efficient
- **Interpolating**: passing through all control points but doesn't stay inside convex hull of points

Arbitrary curves

- Bezier curve
- Hermite curve
- Catmull-Rom spline
- **B-Splines**
  - Next to Catmull-Rom spline, B-splines are widely used

B-spline

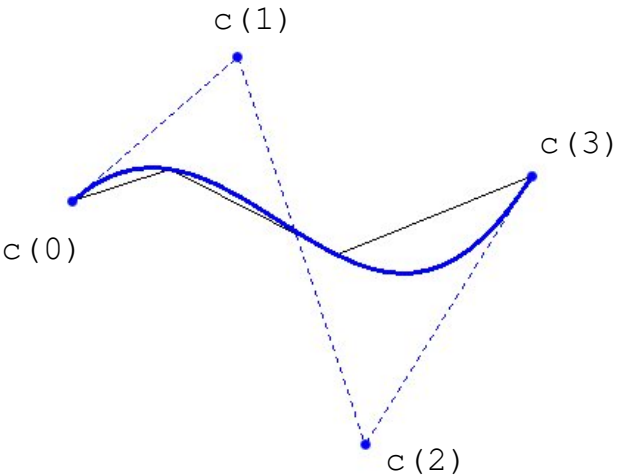
- Types:
  - **Uniform**: uniformly space control points
  - **Non-Uniform**
- **Parameterized function of $\mathbf{x}$.** (Similarly to Bezier curve)

$$s_n(x) = \sum_{k \in \mathbb{Z}} c(k)\beta_n(x-k).$$

n-1 – **degree**, c(k) – **control points**, β_n - **basis function**

- **Cubic B-spline is most widely used.**
- **Basis function of uniform cubic B-spline:**

$$\beta_3(x) = \begin{cases} 0, & |x| \geqq 2 \\ \frac{1}{6} \cdot (2-|x|)^3, & 1 \leqq |x| < 2 \\ \frac{2}{3} - \frac{1}{2}|x|^2 \cdot (2-|x|), & |x| < 1 \end{cases}$$



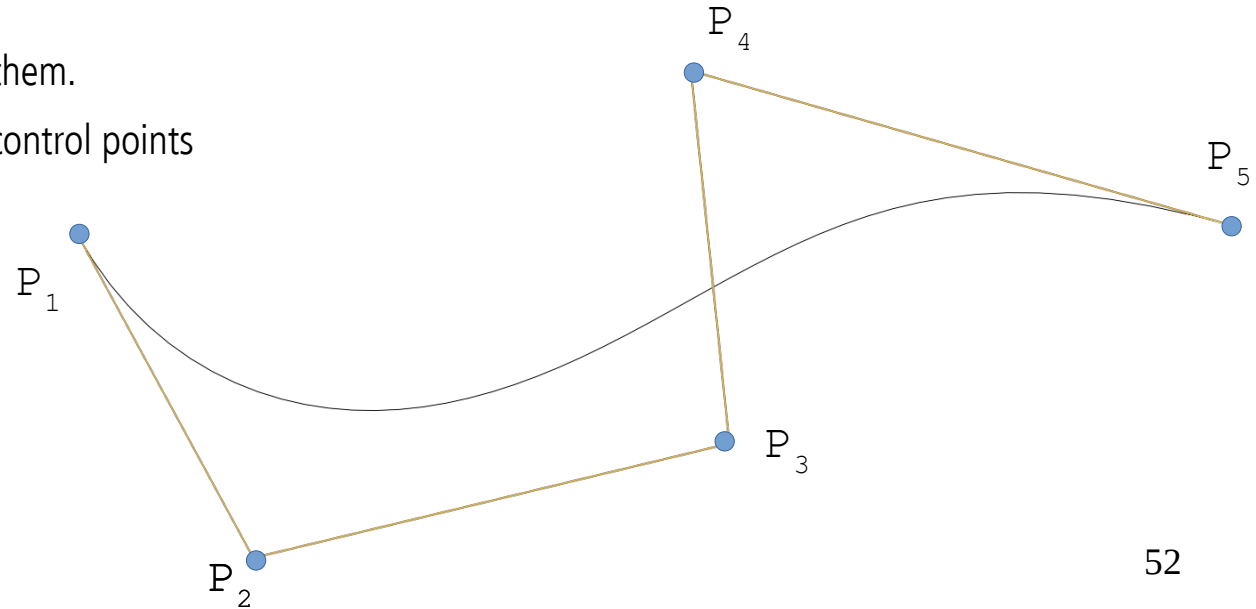
Cubic B-spline: non-interpolating

https://docs.bentley.com/LiveContent/web/ProStructures%20Help-v6/en/SECurvesPlaceBsplineCurve.html

https://www.researchgate.net/publication/220494113_Efficient_GPU-Based_Texture_Interpolation_using_Uniform_B-Splines

Uniform cubic B-spline

- **Multiple uniform cubic B-splines curves can be joined together**
  - Cubic B-spline is **C^2 continuous**
  - Curve composited of multiple B-splines will be C^2 continuous
- Cubic B-spline is **non-interpolating**
  - Passing near control points, not through them.
  - No guarantee that it is interpolating the control points



Limitations and generalizations

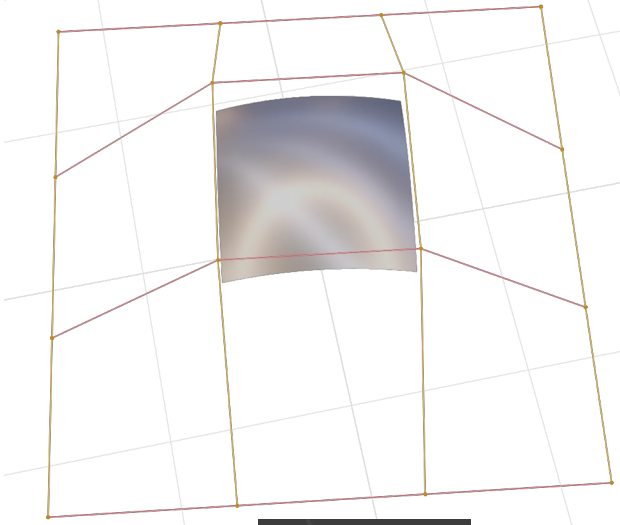
- **Representing circle is not possible**
  - Solution, a generalization is to introduce additional coordinate to B-spline → **rational B-spline**
- **Uniformly spaced control points**:
  - Example: animation when positions of object are known at `t = 1, 2, 3` and `10`.
  - Futher generalization: **Non-uniform rational B-splines (NURBS)**
    - Very often used in CAD tools

NURBS in Blender: https://docs.blender.org/manual/en/latest/modeling/curves/introduction.html

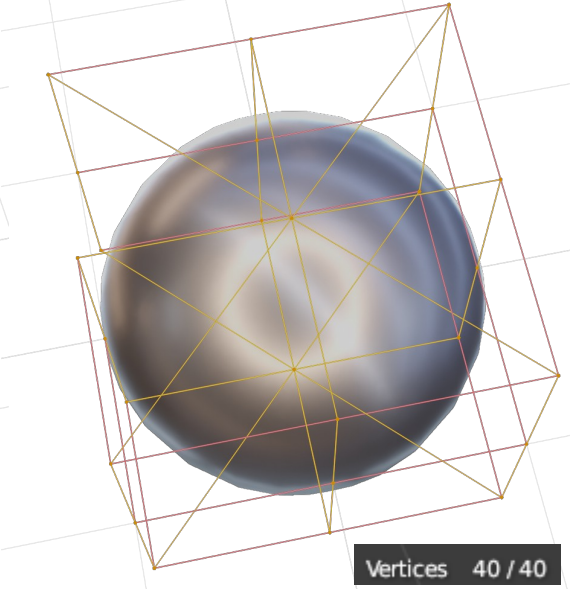
Parametric surfaces

Parametric surfaces

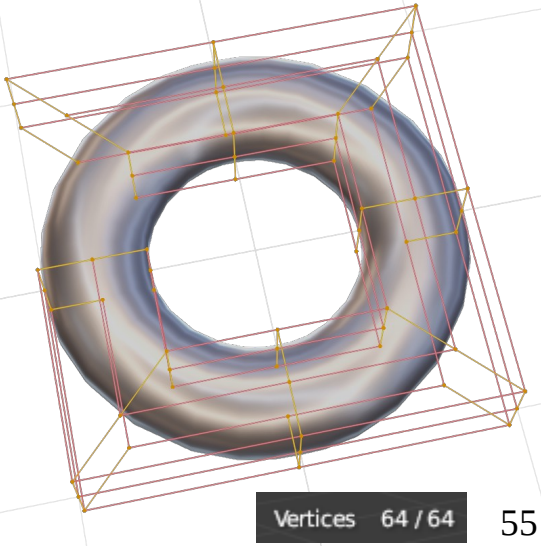
- **Natural extension of parametric curves are parametric surfaces**
  - Similarly as triangle or polygon is extension of a line segment
- Very useful for modeling **curved surfaces**
- Similarly as parametric curves, parametric surfaces are defined with **small number of control points and parametric equation**



NURBS surface



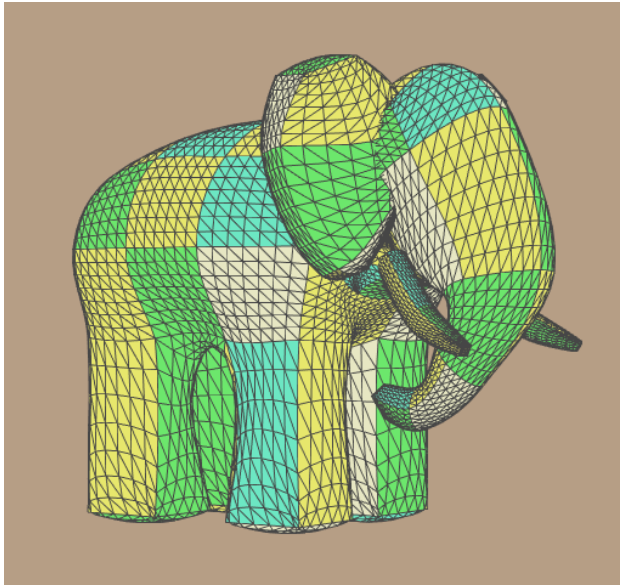
NURBS sphere



NURBS torus

Parametric surfaces

- Common types
  - **Bezier surfaces**
  - B-spline surfaces



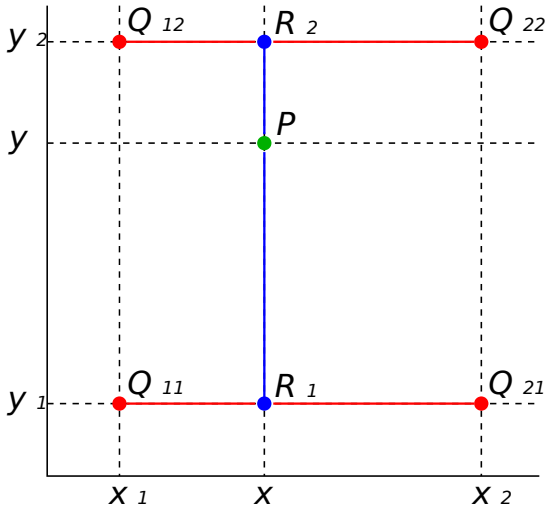
Ed Catmull's "Gumbo" model, composed from Bezier patches
https://en.wikipedia.org/wiki/B%C3%A9zier_surface

NURBS Renderman: https://rmanwiki.pixar.com/display/REN24/NURBS

Bezier surface: bilinear interpolation

- Bezier curve is extended so it has **two parameters (u,v) which define surface**
- As linear Bezier curve is defined with linear interpolation, **planar Bezier patch** is defined with **bilinear interpolation.**

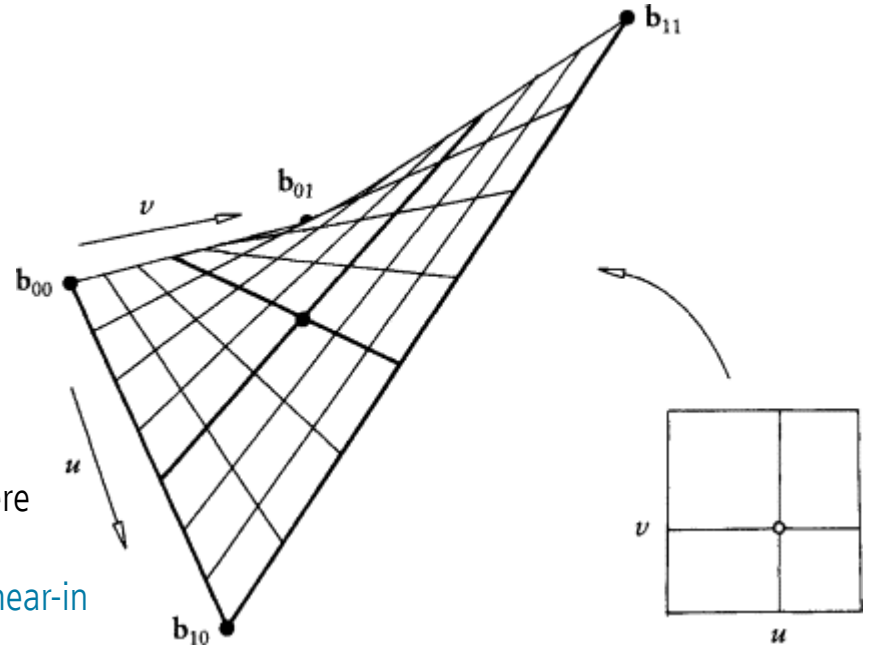
$$R_1(u) = \mathrm{lerp}(Q_{11}, Q_{12}, u)$$
$$R_2(u) = \mathrm{lerp}(Q_{12}, Q_{22}, u)$$
$$P(u,v) = \mathrm{lerp}(R_1(u), R_2(u), v)$$



Bilinear interpolation

Bezier surface: patch

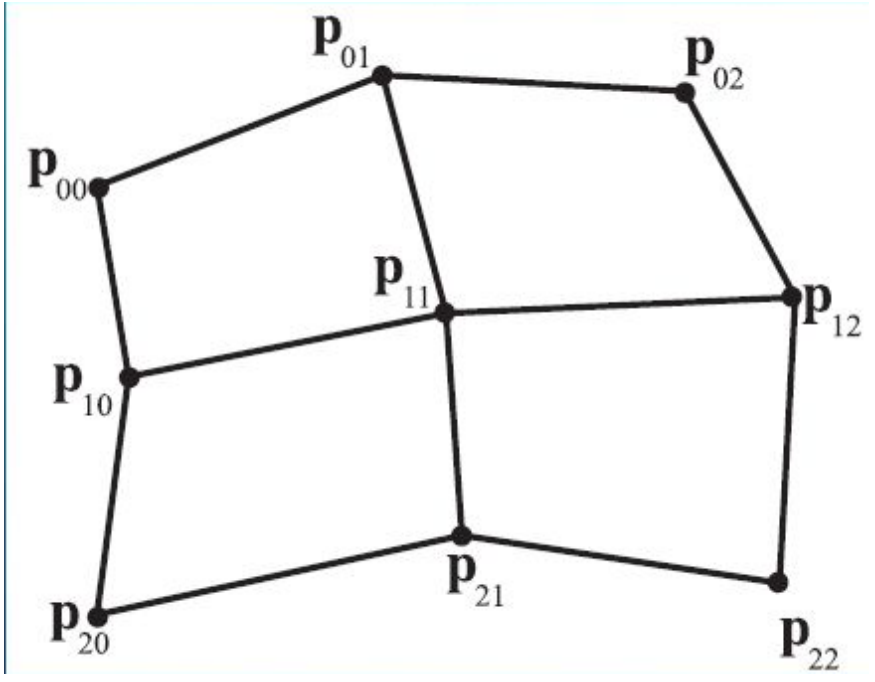
- Planar Bezier patch `p(u,v)` is simplest, non-planar parametric surface
- It has **rectangular domain** and thus resulting surface is called a **patch**



Bilinear interpolation defines planar Bezier patch p(u,v) where (u,v) are in [0,1].
https://www.sciencedirect.com/topics/computer-science/bilinear-interpolation

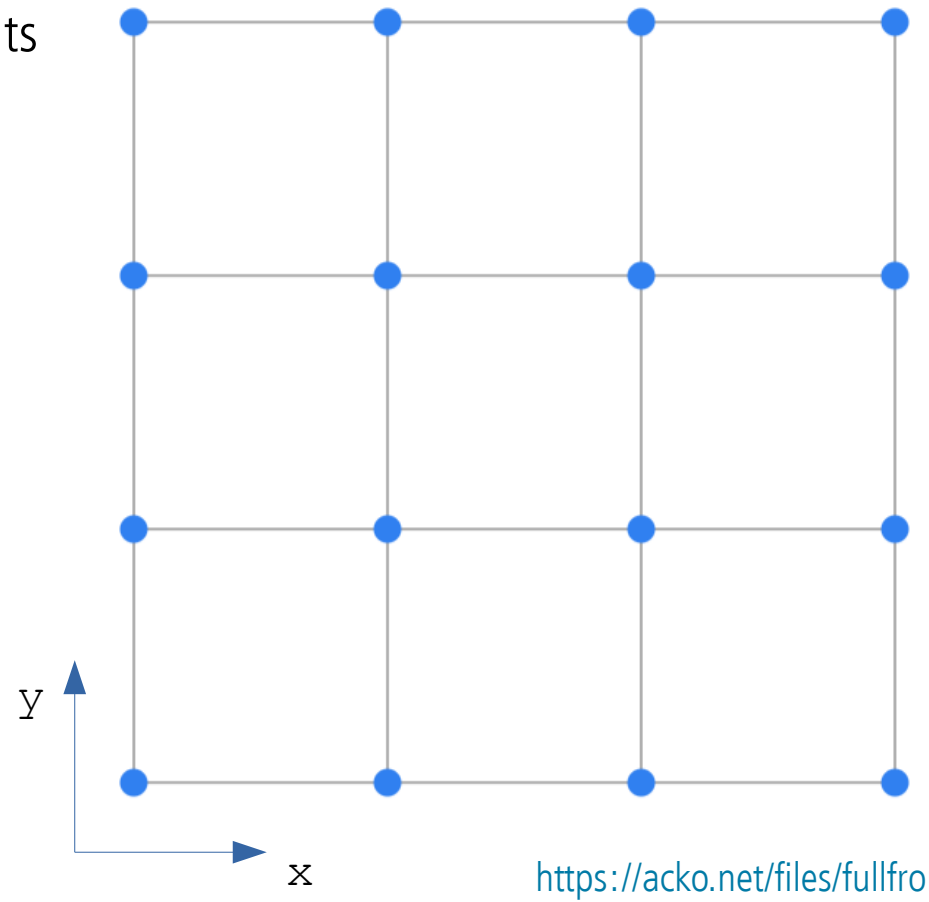
Bezier surface

- **Bezier surface** is achieved by adding more points to bilinear interpolation (similarly as we added more points to linear interpolation to obtain Bezier curve)
- **Repeated bilinear interpolation** is extension of de Casteljau's algorithm to patches.
- Example: **biquadratic Bezier surface**
  - 9 control points



Bezier surface: repeated bilinear interpolation

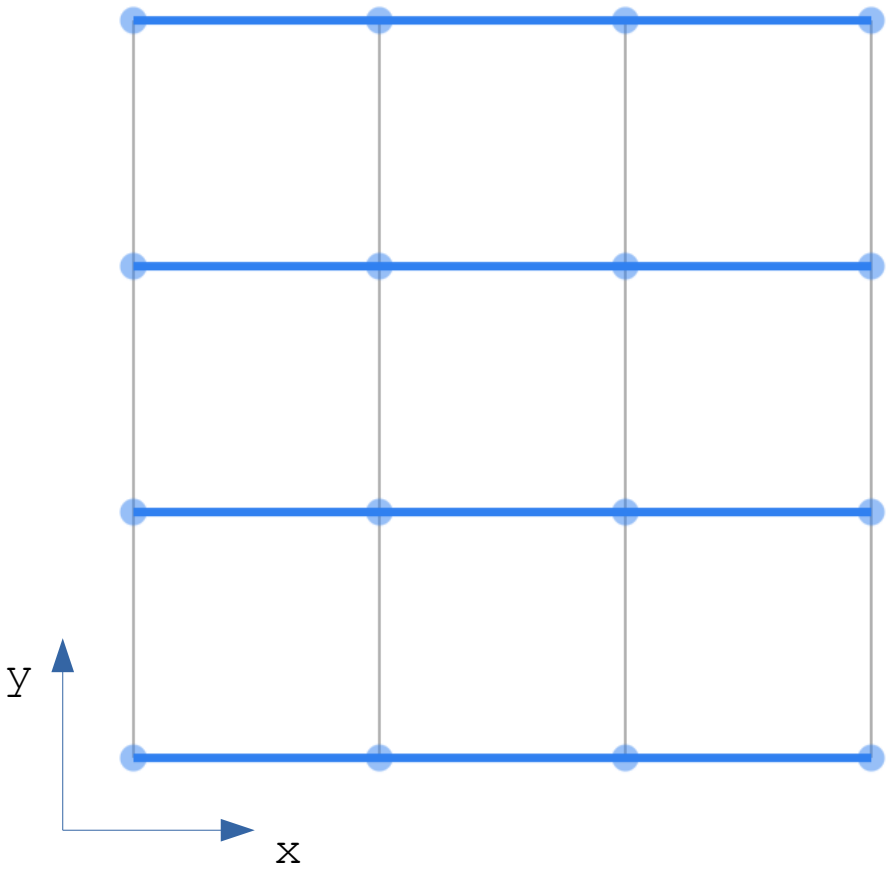
- Starting control points



https://acko.net/files/fullfrontal/fullfrontal/wdcode/online.html

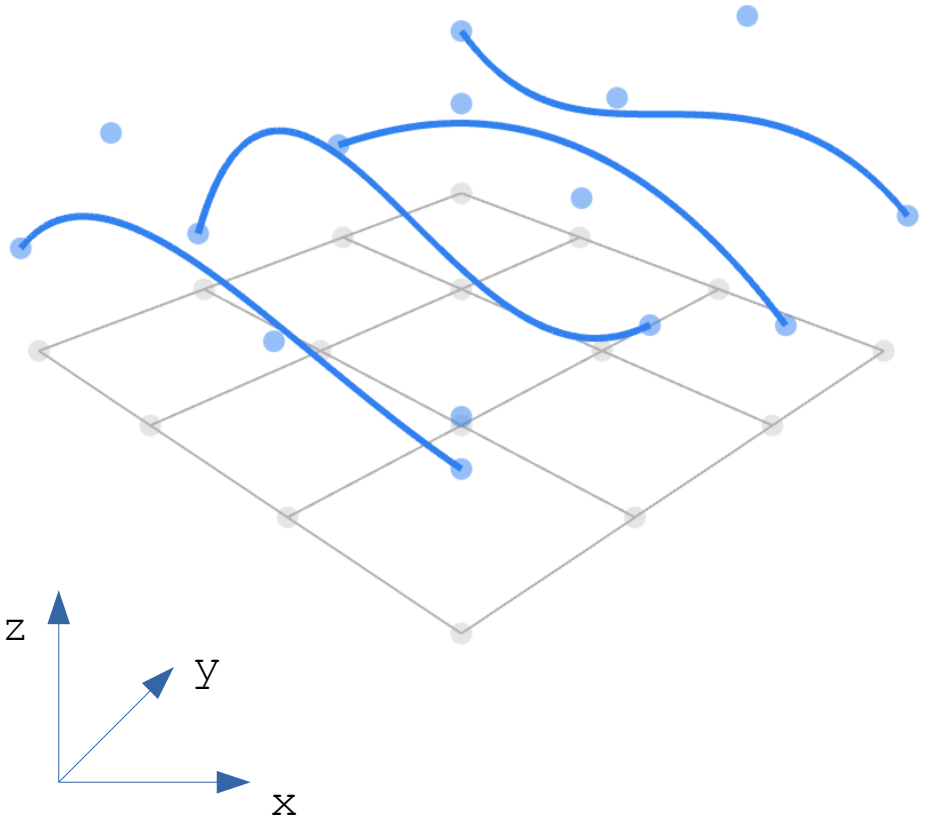
Bezier surface: repeated bilinear interpolation

- Linear interpolation



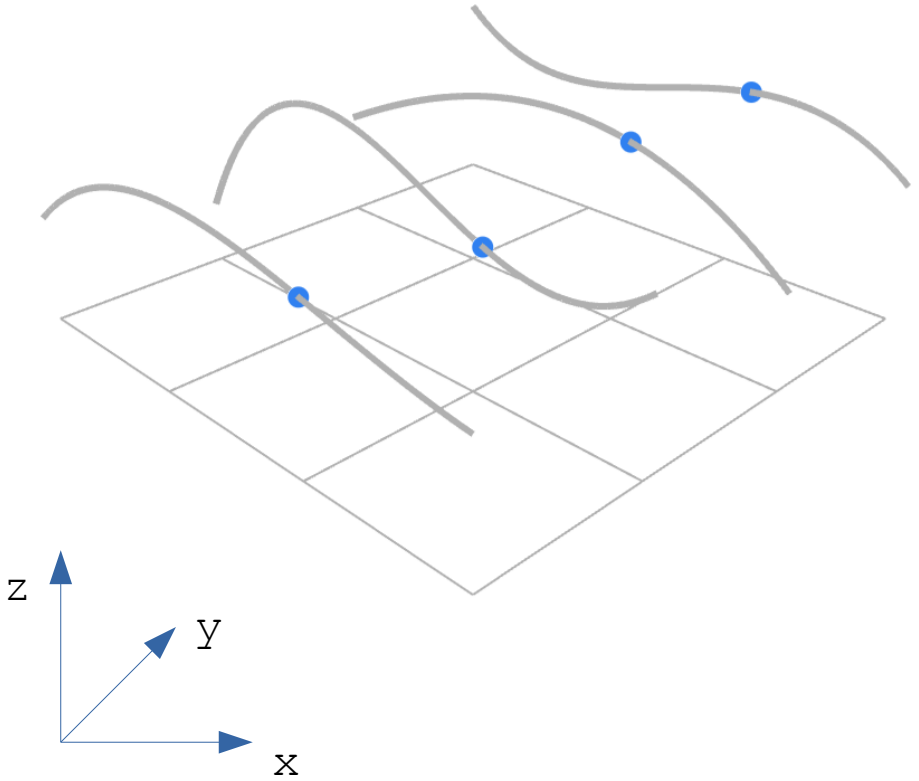
Bezier surface: repeated bilinear interpolation

- Repeated linear interpolation
  - Bezier cubic curves



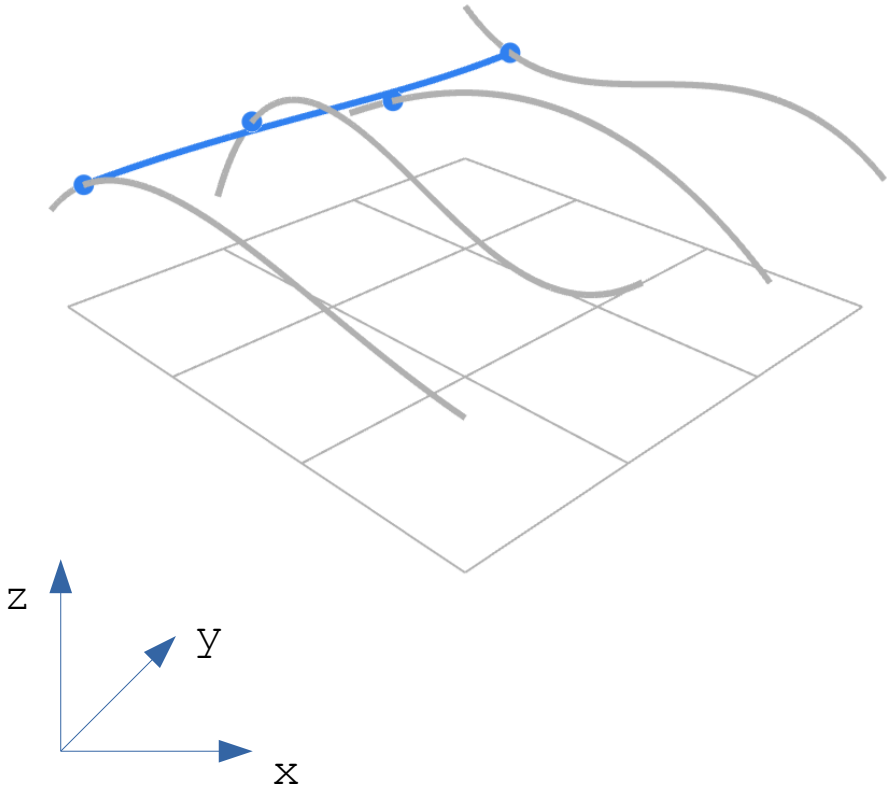
Bezier surface: repeated bilinear interpolation

- Repeated linear interpolation



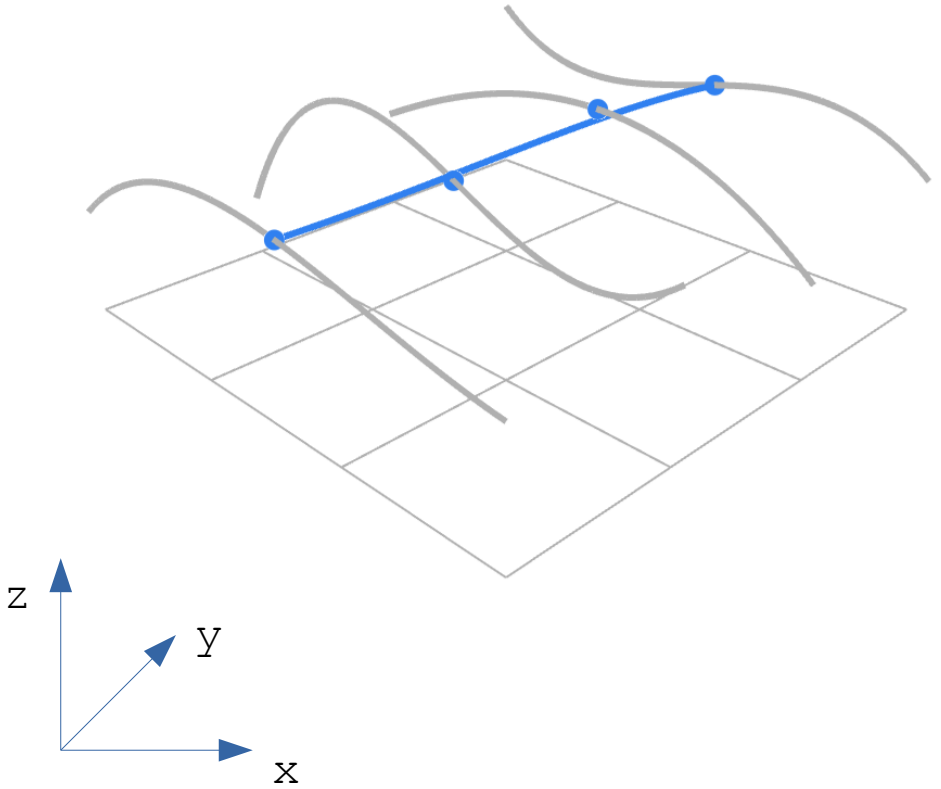
Bezier surface: repeated bilinear interpolation

- Repeated bilinear interpolation



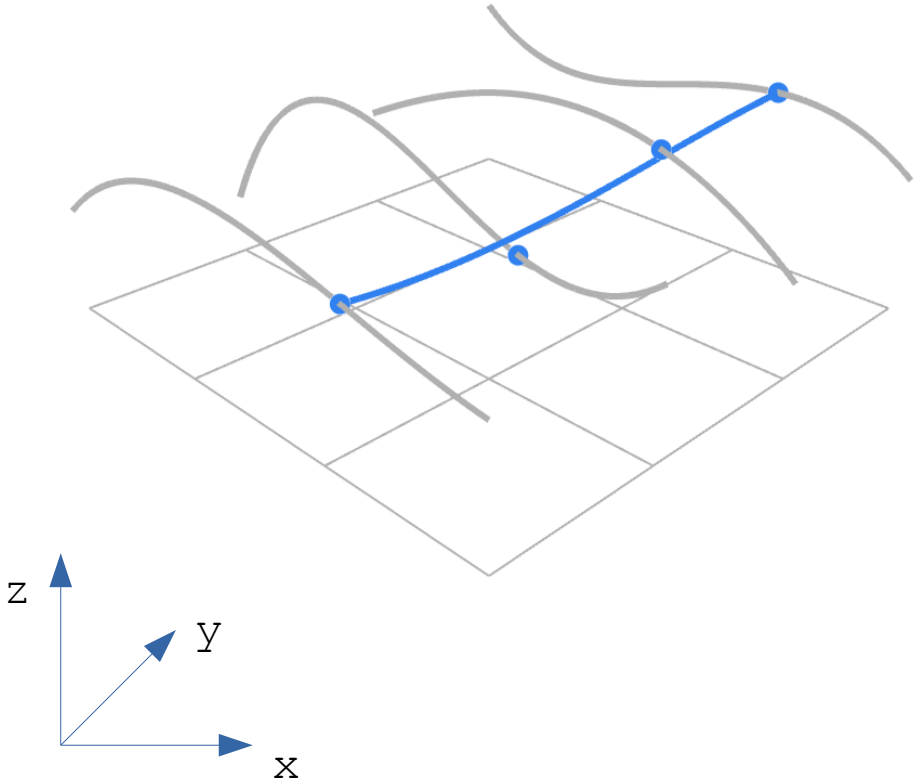
Bezier surface: repeated bilinear interpolation

- Repeated bilinear interpolation



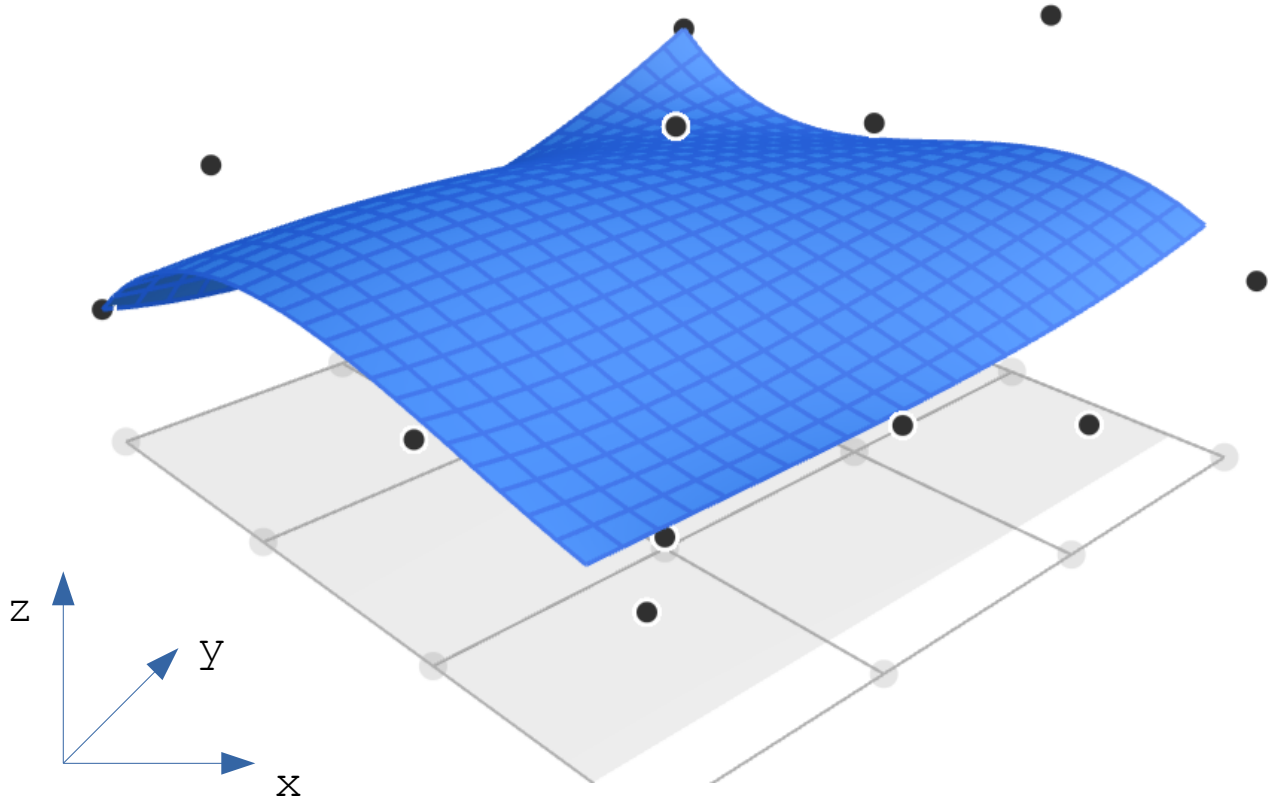
Bezier surface: repeated bilinear interpolation

- Repeated bilinear interpolation



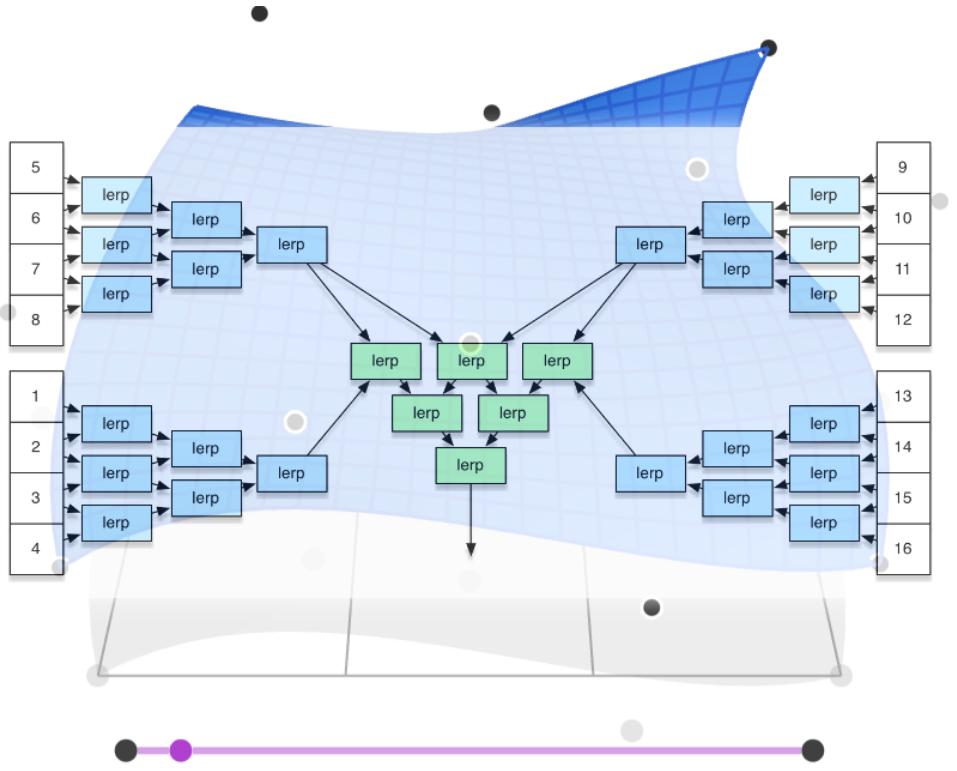
Bezier surface: repeated bilinear interpolation

- Bicubic Bezier surface



Bezier surface

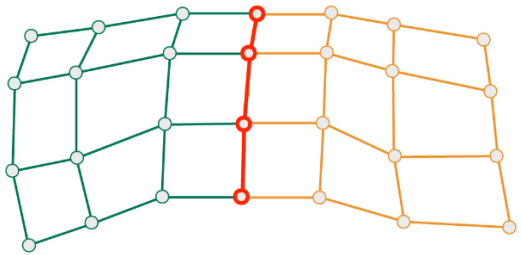
- Repeated bilinear interpolation is extension of **De Casteljau algorithm** to patches
- Point on Bezier Patch can be described in **Bernstein form** using Bernstein polynoms



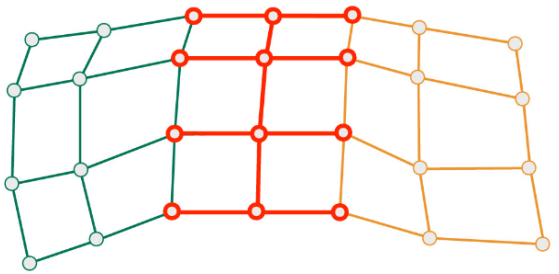
- Bicubic Bezier surface defined with 16 control points

Bezier surface: continuity

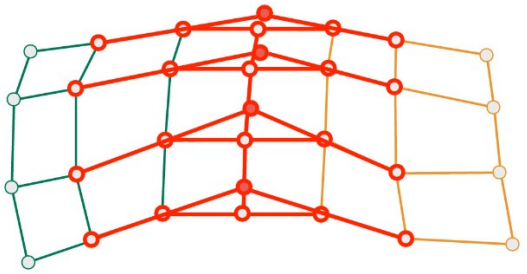
Surface continuity



C^0 – **positional continuity** – joined at the same point



C^1 – **velocity continuity** - derivation of any point (including joints) must be continuous

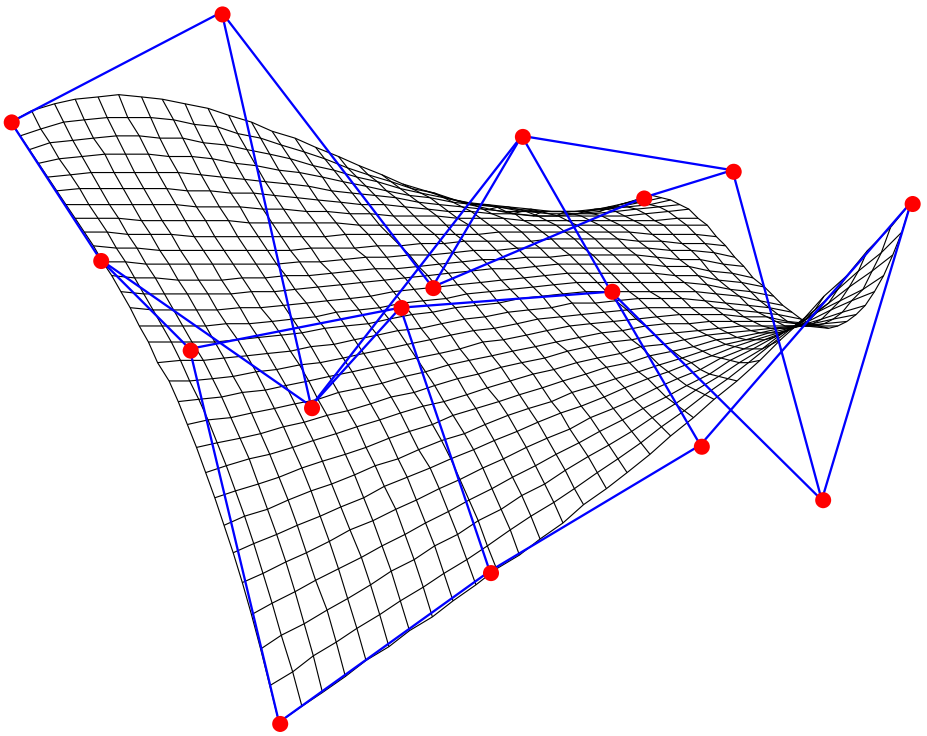


C^2 - **acceleration continuity** - first and second derivatives are continuous functions

https://cs184.eecs.berkeley.edu/sp22/lecture/7-96/intro-to-geometry-splines-and-be

Bezier surface properties

- **Non-interpolating**: passes through only corner control points
- **Boundary of the patch:** Bezier curve of degree `n`
- **Tangents at border points:** described with Bezier curve at border points – two tangents: for u and v direction
  - Derivative is straightforward (derivation of polynomials)
- **Patch lies within convex hull of its control points**
- Arbitrary number of points on patch can be generated
  - Prior generation, transformation on patch can be done which is efficient than transformation on generated points
- Extension: **rational Bezier patches**
  - https://theses.hal.science/tel-01064604/document/

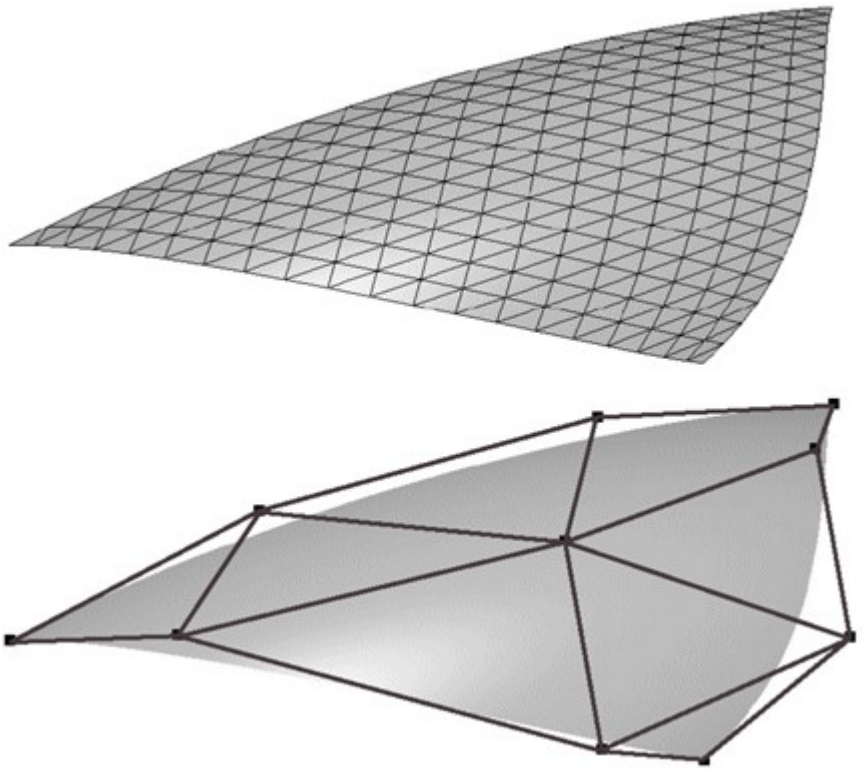


Parametrized surfaces based on Bezier surface

- Bezier triangles
- Point-Normal (PN) Triangles
- Phong tessellation

Bezier triangles

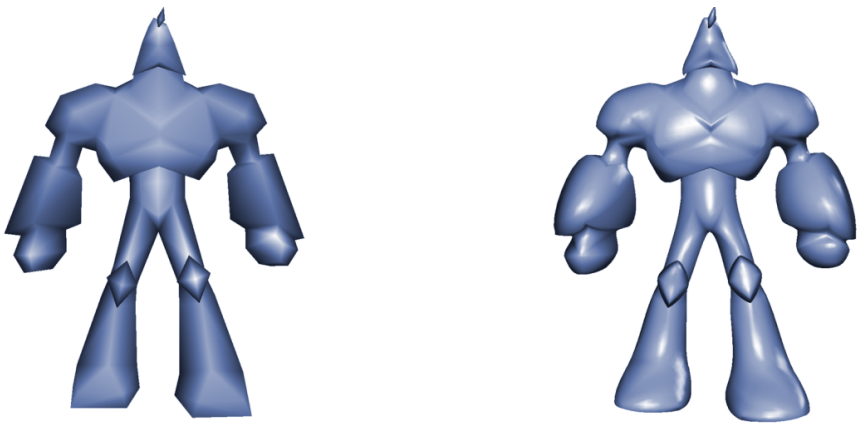
- **Control points are located in triangulated grid**
- Based on **repeated interpolation**:
  - de Casteljau algorithm
  - Bernstein triangles
- Constructing complex object requires stitching Bezier triangles so that composite surface contains desired properties and look, e.g., continuity



https://bezier.readthedocs.io/en/stable/python/reference/bezier.triangle.html
https://github.com/xtmacbook/notebook/blob/master/B%C3%A9zier/B%C3%A9zier%20triangle.md

Point-Normal (PN) Triangles

- **Given triangle mesh with normals at each vertex, the goal is to construct smoother surface using Bezier triangles**
- Properties:
  - Improves mesh shading and silhouettes by creating **curved surface to replace each triangle**
  - Creases in PN triangles are hard to control
  - Continuity between Bezier triangles is C_0 but looks acceptable for certain applications

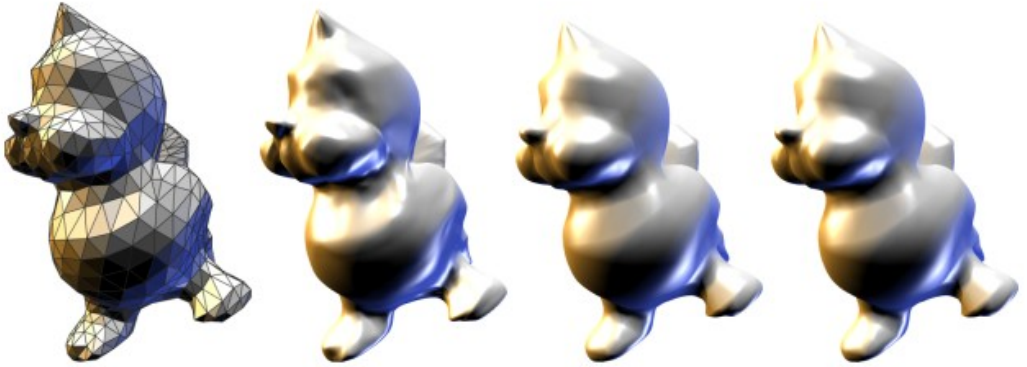


Comparison of original mesh and PN triangle counterpart
https://www.gamedeveloper.com/programming/b-zier-triangles-and-n-patches
https://alex.vlachos.com/graphics/CurvedPNTriangles.pdf

* Game engines (e.g., unity and unreal) support those methods since triangle mesh is basic building primitive.

Phong tessellation

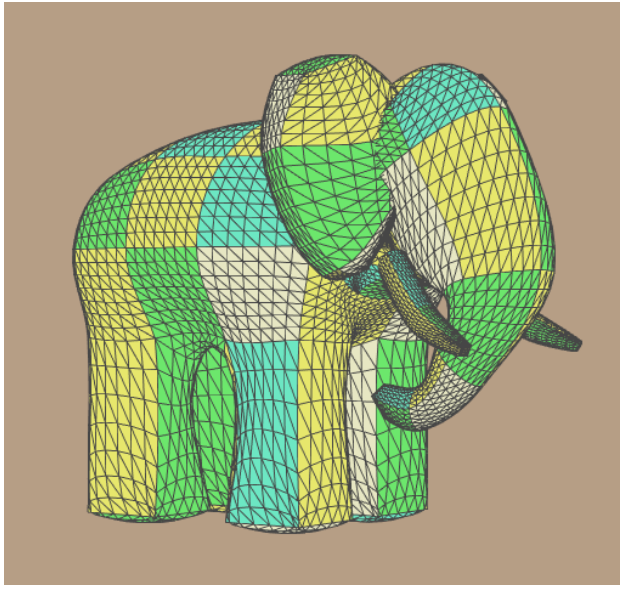
- Similar as PN triangles, **given the triangle points with normals, construct smoother surface**
- Phong tessallation **attempts to create geometric version of Phong shading normal** using repeated interpolation resulting in Bezier triangles.



Mesh, Butterfly subidivision, PN Triangles, Phong Tessellation
https://perso.telecom-paristech.fr/boubek/papers/PhongTessellation/

Parametric surfaces

- Common types
  - Bezier surfaces
  - **B-spline surfaces**

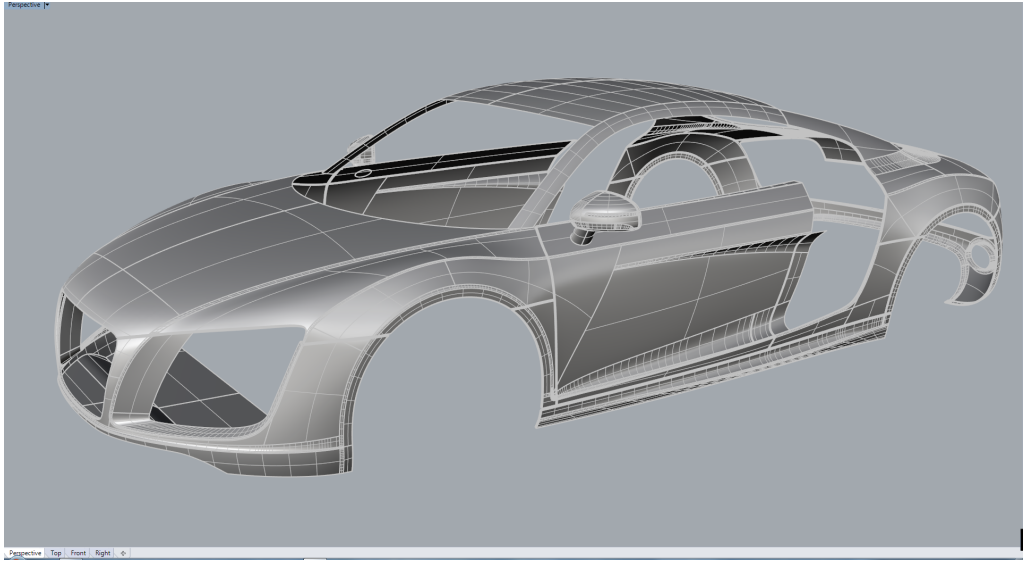


Ed Catmull's "Gumbo" model, composed from Bezier patches
https://en.wikipedia.org/wiki/B%C3%A9zier_surface

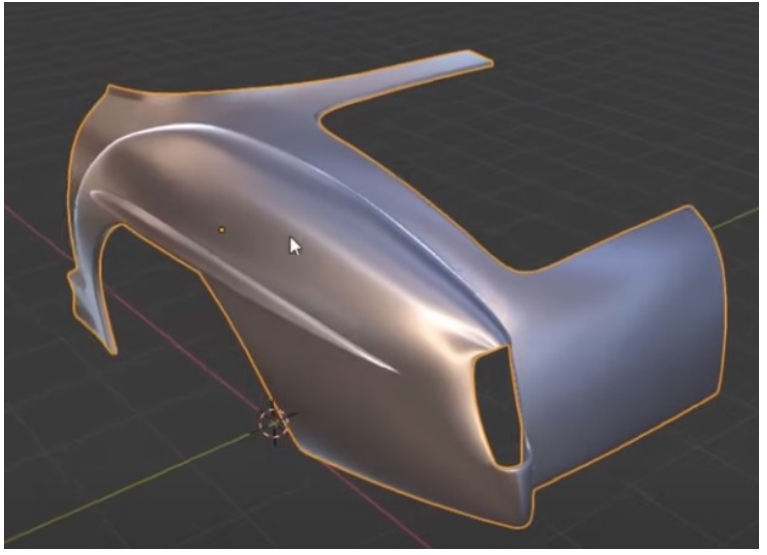
NURBS Renderman: https://rmanwiki.pixar.com/display/REN24/NURBS

B-Spline surfaces

- B-Spline curves can be extended to **B-Spline surfaces** which are similar to Bezier surface
- Often **bicubic B-Spline surface is used to form composite surface**
  - Basis for Catmull-Clark subidivison surfaces
- **Non-uniform rational B-Spline surface (NURBS)** is often used in 3D modeling software



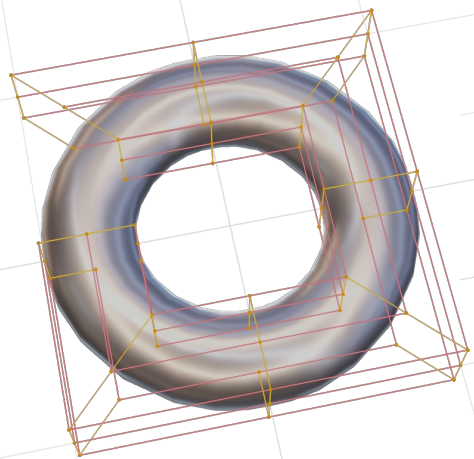
https://forums.cgsociety.org/t/audi-r8-v10-2010-nurbs-model-with-rhino-3d/1761843/3



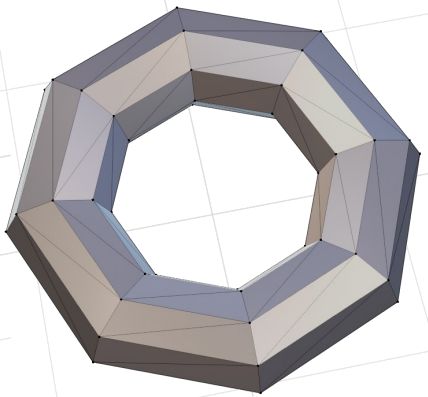
https://www.youtube.com/watch?v=1zNfh_g8jXM&ab_channel=BlenderSecrets

Rendering of parametric surfaces

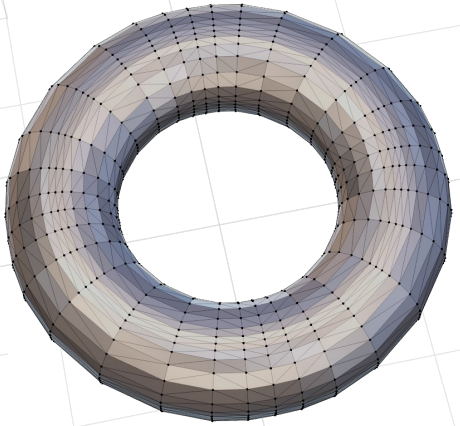
- **Ray-parametric-surface intersection** can be defined
- Model made with parametric surfaces is **tessellated** for efficient rendering process.
  - Trade-off between quality and speed: more triangles → higher quality shading and silhouettes



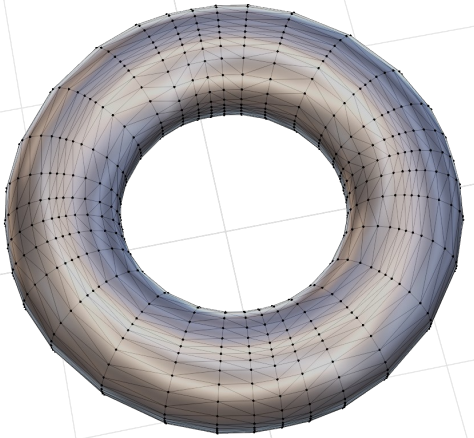
NURBS torus



Tessellation 1



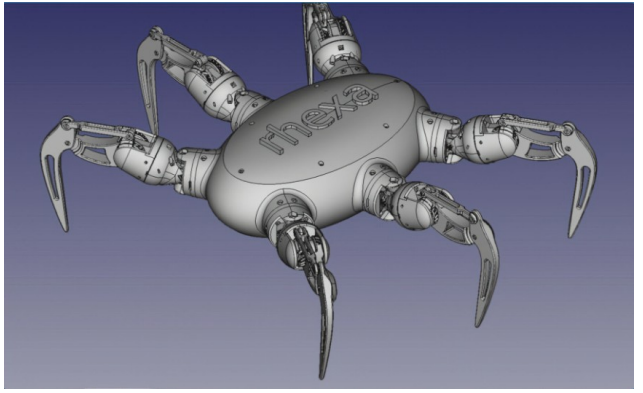
Tessellation 2



Tessellation 3

Modeling and transferring parametric surfaces

- Parametric curves and surfaces are extensively used in CAD software:
  - https://www.freecad.org/
  - https://www.autodesk.com/collections/product-design-manufacturing/overview
- Parametric surfaces are stored using:
  - **STEP file format:** https://www.adobe.com/creativecloud/file-types/image/vector/step-file.html
  - **IGES file format:** https://www.adobe.com/creativecloud/file-types/image/vector/iges-file.html



Exploring parametric curves and surfaces

- Parametric curves and surfaces are highly used and researched method in computer graphics. We only covered foundations.
- NURBS in real-time: https://www.gamedeveloper.com/programming/using-nurbs-surfaces-in-real-time-applications
- Blender NURBS surfaces: https://docs.blender.org/manual/en/latest/modeling/surfaces/introduction.html
- Blender NURBS and Bezier curves: https://docs.blender.org/manual/en/latest/modeling/curves/index.html
- Tutorial on curves in Blender: https://behreajj.medium.com/scripting-curves-in-blender-with-python-c487097efd13
- Houdini: https://www.sidefx.com/docs/houdini/nodes/sop/curve.html
- Library for creating and manipulating NURBS surfaces and curves: http://verbnurbs.com/
- More examples: https://www.realtimerendering.com/#curves

Implicit surfaces

Implicit surfaces

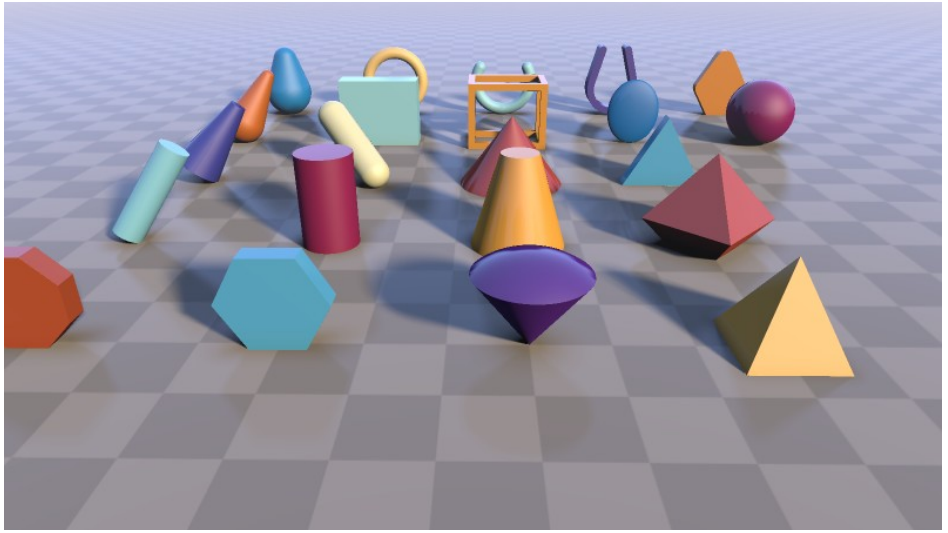
- Described with a function. Instead of using parameters `(u,v)` surface is described with implicit function `f(p)`:

```
f(p) = 0, if p on surface
f(p) < 0, if p inside surface
f(p) > 0, if p outside surface
```

- Surface can not be computed directly
  - Distance is computed by evaluating point `p(x,y,z)` in space → **signed distance function**

```
float sdSphere( vec3 p, float s )
{
  return length(p)-s;
}
```

```
float sdTorus( vec3 p, vec2 t )
{
  vec2 q = vec2(length(p.xz)-t.x,p.y);
  return length(q)-t.y;
}
```



```
float sdRoundBox( vec3 p, vec3 b, float r )
{
  vec3 q = abs(p) - b;
  return length(max(q,0.0)) + min(max(q.x,max(q.y,q.z)),0.0) - r;
}
```

```
float sdRoundedCylinder( vec3 p, float ra, float rb, float h )
{
  vec2 d = vec2( length(p.xz)-2.0*ra+rb, abs(p.y) - h );
  return min(max(d.x,d.y),0.0) + length(max(d,0.0)) - rb;
}
```

Implicit surfaces: normal

- **Crucial information for shading is normal.**
- Normal vector $\mathtt{N}$ of implicit surface in point $\mathtt{p}$
  - **Normalized gradient of function f in point p: ∇f(p)**
  - Intuition: Gradient points in direction of steepest ascent in the function near $\mathtt{p}$ and thus is orthogonal to surface

Analytical expression:

- Function must be differentiable

$$\nabla f(x,y,z) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$$

Approximation:

- Central difference using small number ε

$$\nabla f_x = f(p + \epsilon e_x) - f(p - \epsilon e_x), e_x = (1,0,0)$$

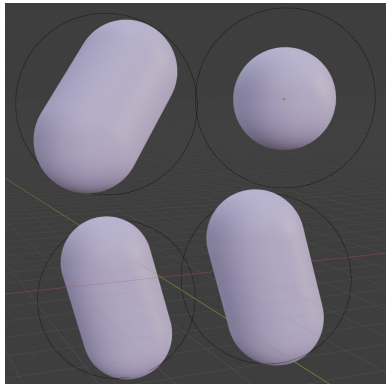
$$\nabla f_y = f(p + \epsilon e_y) - f(p - \epsilon e_y), e_y = (0,1,0)$$

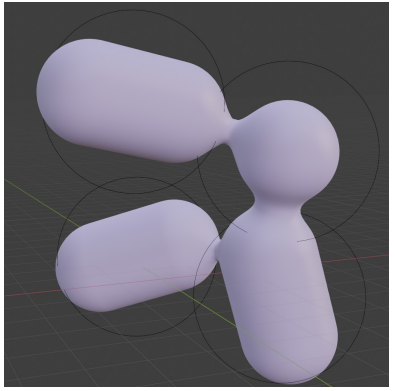
$$\nabla f_z = f(p + \epsilon e_z) - f(p - \epsilon e_z), e_z = (0,0,1)$$

https://iquilezles.org/articles/normalsSDF/

Implicit surfaces: modeling

- Blending of implicit surfaces: blobby modeling, soft objects or metaballs
  - Blender: https://docs.blender.org/manual/en/latest/modeling/metas/index.html

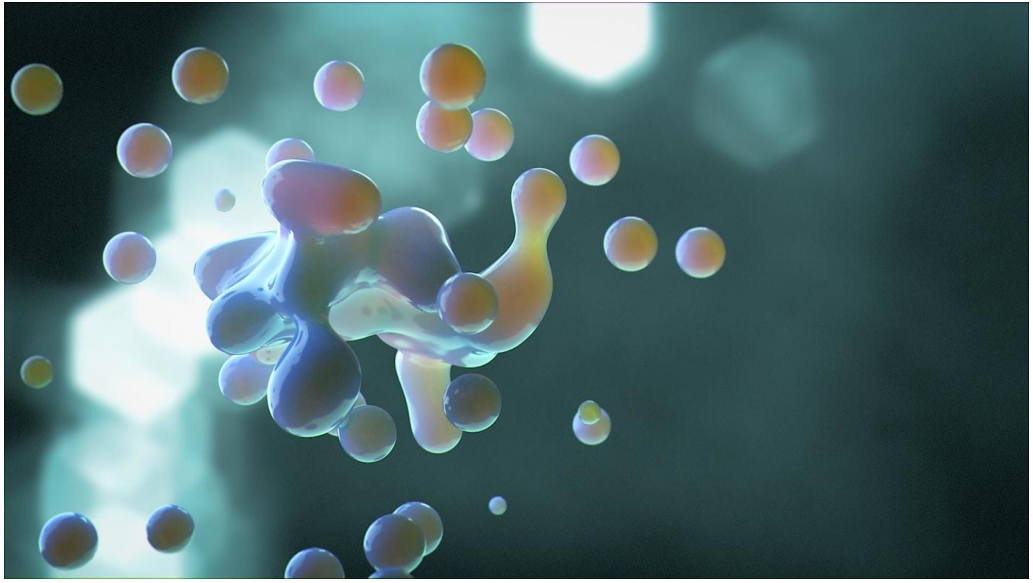




- Example of blending: resulting distance d by blending between two distances d_1 and d_2 with a blend radius r_b

$$d = (1-h)d_2 + hd_1 + r_b h(1-h)$$

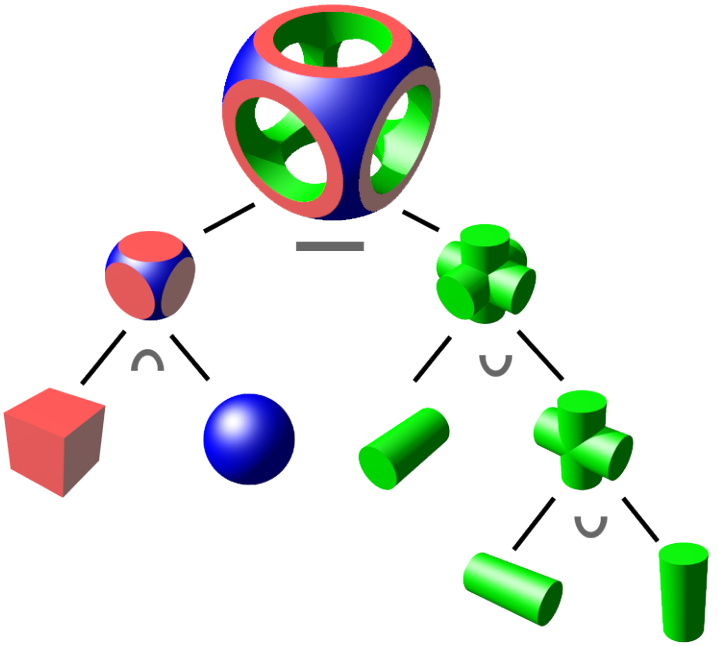
$$h = min(max(0.5 + 0.5(d_2 - d_1)/r_b, 0.0), 1.0)$$



https://www.chaos.com/gallery/elmar-glaubauf-metaballs

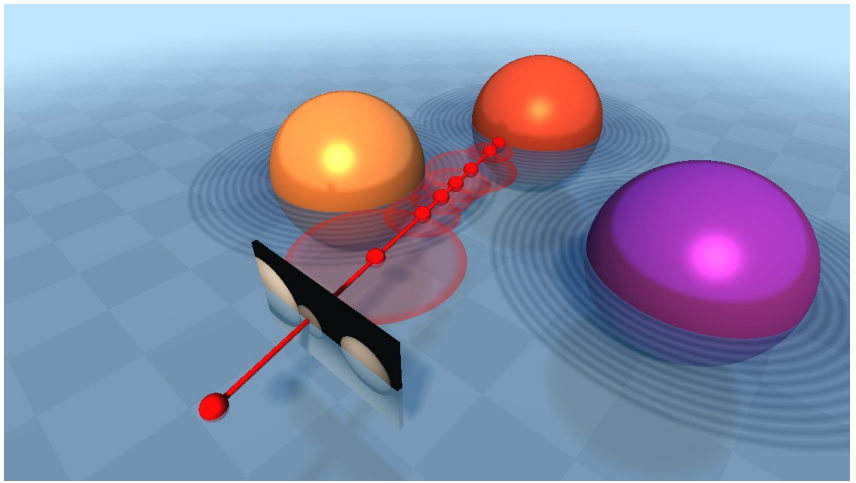
Implicit surfaces: modeling

- **Constructive solid geometry**: set of operations – **Boolean operations:**
  - Intersection
  - Union
  - Difference
- Well defined for closed objects (volumetric bodies)
- Used to create complex shapes from simple, base shapes (e.g., cubes, spheres, etc.)



Implicit surfaces: rendering

- **Ray-marching**
  - At first point `p` on the ray, shortest distance `d` to the surface is evaluated (sphere around the point)
  - Move in ray direction for distance `d`
  - Repeat until surface is reached within some distance (small threshold) or max number of steps is reached (background hit)
- Effects: shadows, reflections, ambient occlusion, etc.

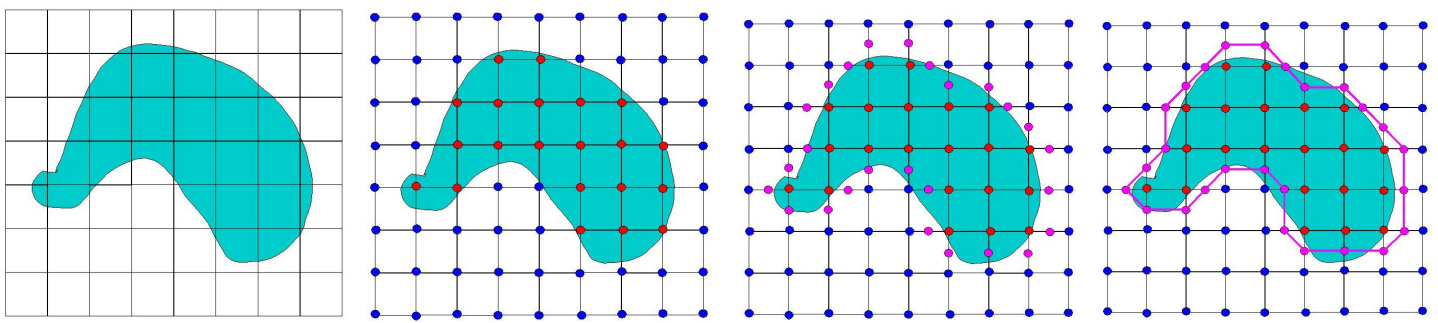


https://iquilezles.org/articles/raymarchingdf/

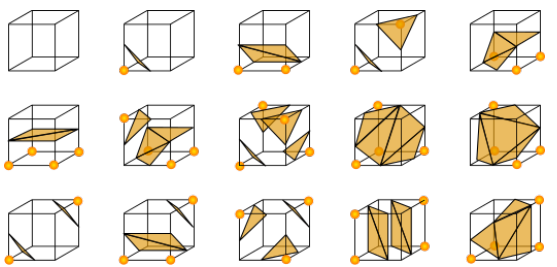
https://nanzhaodocumentations.wordpress.com/2021/01/15/creative-computing-term-1-assessment-my-shader-world-in-raymarching-%F0%9F%91%BE/
https://iquilezles.org/articles/nvscene2008/rwwtt.pdf

Implicit surfaces: rendering

- **Turn surface into triangles (triangulation) and then render**
- **Marching cubes algorithm:** surface extraction, polygonization
  - First, surface is partitioned into adjacent cells at whose corners the implicit surface is evaluated
  - Negative values are considered outside surface, positive values inside surface
  - Within each cell, the intersection of cell edges with the implicit surface are connected to form one or more polygons



3D case: creating cells, labeling cell corners, calculating cell edge points, connecting edge points to form polygon



3D case

https://people.eecs.berkeley.edu/~jrs/meshpapers/Bloomenthal.pdf
https://www.cs.carleton.edu/cs_comps/0405/shape/marching_cubes.html

Summary questions

- https://github.com/lorentzo/IntroductionToComputerGraphics/tree/main/lectures/6_parametric_curves_surfaces

Reading material

- https://github.com/lorentzo/IntroductionToComputerGraphics