Exam 2023 March

Problems

Problem 1: Lab Leak - Graph Traversal

Lab Leak

Problem ID: lableak

Oh no! A pathogen has escaped from your lab by infecting one of the scientists. Let's just hope it doesn't reach you.

Thanks to excellent surveillance technology, you have information of all human contacts in the city. In particular, you know for each pair of individuals u and v if a pathogen could be transmitted from u to v (and vice versa) due to human contact. Let's agree that individual 0 is the infected scientist, and $n-1$ represents you.

Input

On the first line of input, the number n of individuals and the number m of connections. Both numbers are integers with $n \geq 2$ and $m \geq 0$; we assume the individuals are numbered $0, \ldots, n-1$. Then follow m lines each containing two integers u and v with $0 \leq u < v < n$, meaning that the pathogen would be transmitted from u to v and from v to u.

Output

Print 'oh, no!' if the pathogen can be transmitted from 0 to $n-1$. Otherwise print 'phew!'

Sample Input 1	Sample Output 1
5 4 0 1 0 2 1 2 3 4	phew!

Sample Input 2	Sample Output 2
5 4 0 1 1 2 2 3 3 4	oh, no!

Problem 2: Mette - Greedy

Mette
Problem ID: mette

Your are the leader of the largest political party of a small nation and are tasked with forming a stable government. A government is stable if it has a majority of seats in parliament. (To be precise, a government consists of a subset of parties, and its number of seats is the sum of the seats of the individual parties.)

To make your life as a government leader easy, you want to include as few parties as possible.

For example, in sample input 1, there are 175 seats in total. The 3 parties 'Moderaterne', 'Socialdemokratiet', and 'Venstre' together have $16 + 50 + 23 + = 89$ seats, which is strictly more than $\frac{175}{2} = 87.5$. In this election, there is no way to form a government with only 2 parties that have at least 88 seats in total, so '3' is the minimum number of parties needed.

For simplicity, you can assume that the total number of seats is odd, and that there is exactly one largest party (namely, yours).

Input

On the first line, the number p of parties, a nonzero positive integer. For $1 \leq i \leq p$, the ith of the of the following p lines contains the name of the ith party and its number s_i of seats. You can assume that s_i is a nonnegative integer.

Output

The smallest number of parties needed to form a government.

Sample Input 1

```
14
Alternativet 6
Danmarksdemokraterne 14
Dansk Folkeparti 5
Enhedslisten 9
Frie Grønne 0
Konservative 10
Kristendemokraterne 0
Liberal Alliance 14
Moderaterne 16
Nye Borgerlige 6
Radikale Venstre 7
Socialdemokratiet 50
Socialistisk Folkeparti 15
Venstre 23
```

Sample Output 1

```
3
```

Sample Input 2

```
1
The Only Party 101
```

Sample Output 2

```
1
```

Problem 3: Mirror - Dynamic Programming

Mirror
Problem ID: mirror

A string of uppercase letters is a *mirror* if it is the same backwards and forwards. For instance `OTTO` is a mirror, and so are `LEVEL` and `RACECAR`. By definition, one-letter strings such `A` or `L` are mirrors. On the other hand `MIRROR` is not a mirror, nor is `BANANA`.

What is the minimum number of characters that must be inserted into a given string to make it a mirror? For instance, `BANANA` is turned into a mirror by inserting a single `B` (at the end), and `LOVELY` can be turned into `YLOEVEOLY` by inserting 3 letters.

Input

A string S of n uppercase letters from the English alphabet, `A` to `Z`.

Output

The minimum number of letters that must be inserted into S so that it becomes a mirror.

Sample Input 1	Sample Output 1
LOVELY	3

Sample Input 2	Sample Output 2
LEVEL	0

Sample Input 3	Sample Output 3
BANANA	1

Problem 4: Pipeline - Flow

Pipeline
Problem ID: pipeline

To your annoyance, the enemy is sending fossil fuel through a vast system of pipes to one of your vassal states. You have decided to put an end to this by destroying some of the pipes. Each such operation takes resources and jeopardises your international standing, so you want to destroy as few pipes as possible.

Input

On the first line, the integers n and m. We have $1 < n \leq m$. There are n states; the enemy is state 0 and the vassal state is $n-1$. Then follow m lines each containing two integer u and v with $0 \leq u < v < n$, meaning that there is a pipe between state u and state v that allows fossil fuel to flow in either direction.

You can assume that the entire system of pipes is connected; in particular there is at least one sequence of pipes connecting state 0 to state $n-1$.

Output

A single integer: the number of pipes you must destroy so that the enemy can not send any fossil fuel to the vassal state.

Sample Input 1

```
6 8
0 1
0 2
1 2
3 5
4 5
3 4
1 3
2 4
```

Sample Output 1

```
2
```

Sample Input 2

```
6 7
0 1
0 2
1 2
3 5
4 5
3 4
1 3
```

Sample Output 2

```
1
```

Problem 5: Polynomials - NP-Hard

Polynomials
Problem ID: polynomials

You are given m many polynomials in n variables $x_1, \ldots, x_n$, such as

$$(x_1)^2 - 1 \quad \text{and} \quad x_1 + x_2 + x_3 + 3\,.$$

(Recall that a polynomial is a sum of products of variables and constants.) You want to determine if you can make all polynomials equal to 0 by setting the variables $x_1, \ldots, x_n$ to integer values.

In the above example, this is possible by setting $x_1 = -1$, $x_2 = +5$ and $x_3 = -2$, because

$$(-1)^2 - 1 = 0 \text{ and } (-1) + (+5) + (-2) + 3 = 0\,.$$

There are many other solutions.

In general, a solution may not possible; here's a simple example of $m = 2$ polynomials in just a single variable:

$$x_1 - 1 \quad \text{and} \quad x_1 + 1\,.$$

No matter which value we choose for x_1, at least one of the resulting expressions will be nonzero.

Input

One the first line, the number n of variables and m of polynomials.

We assume that the variables are called $x_1, \ldots, x_n$ and that $1 \leq n$ and $0 \leq m$. Then follow m lines, each containing a polynomial. Each polynomial is given as an expression in a modern programming language, where the variables are written using brackets, like `x[35]`, multiplication is written using the asterisk $\star$.

Output

Output a sequence of n integers, the values of $x_1, \ldots, x_n$ so that all polynomials evaluate to 0.

If no valid solution exists, write "impossible".

Sample Input 1	Sample Output 1

`3 2`
`x[1] * x[1] - 1`
`x[1] + x[2] + x[3] + 3`
	`-1 5 -2`

Sample Input 2	Sample Output 2

`1 2`
`x[1] - 1`
`x[1] + 1`
	`impossible`

1. Greedy

One of the problems on pages 3–7 can be solved by a simple greedy algorithm.

1.a (1 pt.)

Which one?

1.a - Answer

"Mette" can be solved greedily.

1.b (2 pt.)

Describe the algorithm, for example, by writing it in pseudocode. (Ignore parsing the input.) You probably want to process the input in some order; be sure to make it clear which order this is (increasing or decreasing order of start time, alphabetic, colour, age, size, x-coordinate, distance, number of neighbours, scariness, etc.). In other words, don't just write "sort the input."

1.b - Answer

```
def mette(partiesIndividuals){

  // O(n log n)
  partiesIndividuals.sortMaxPeople() // Sort in descending order

  // O(n)
  majority = partiesIndividuals.Sum() / 2

  totalIndividuals = 0
  numberOfParties = 0

  // O(n)
  foreach (individuals in partiesIndividuals){
    if (majority < totalIndividuals) {
      return numberOfParties
    }

    totalIndividuals += individuals
    numberOfParties++
  }

  return "not enough individuals"
}
```

This code would ignore the name of the party and only take the seats into account.
The seats would be sorted from the largest to the smallest.
Then we would calculate the majority of the seats.
Then we would iterate over the sorted list and add the seats to the total until we reach the majority.

1.c (1 pt.)

State the running time of your algorithm in terms of the input parameters. (It must be polynomial in the input size.)

1.c - Answer

We have the following running times in the algorithm (where n is the number of parties):

- Sorting: $O(n \log n)$
- Summing: $O(n)$
- Iterating: $O(n)$

This can be reduced to $O(n \log n)$.

2. Graph Traversal

One of the problems on pages 3–7 can be efficiently solved using (possibly several applications of) standard graph traversal methods (such as breadth-first search, depth-first search, shortest paths, connected components, spanning trees, etc.), and without using more advanced design paradigms such as dynamic programming or network flows.

2.a (1 pt.)

Which one?

2.a - Answer

"Lab Leak" can be solved with a normal graph traversal algorithm, such as DFS or BFS.

2.b (1 pt.)

Explain how you model the problem as a graph problem—what are the vertices, what are the edges, how many are there in terms of the parameters of the problem statement, are they directed, weighted, etc. Draw the graph(s) corresponding to the sample input(s).

2.b - Answer

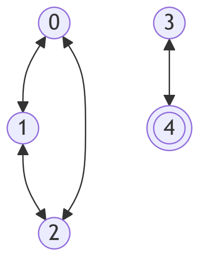
These are the denotations:

- The humans will be the vertices v
- The human connections will be the edges e

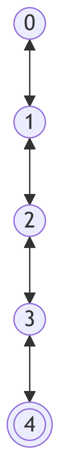
There will always be n many vertices and m many edges (the two first integers of the input).

All edges are undirected and unweighted.

Sample input 1 can be displayed as:



Sample input 2 can be displayed as:



2.c (2 pt.)

Describe your algorithm. Be explicit about arguments such as start vertices, stopping conditions, etc. As much as you can, make use of known algorithms. (For instance, don't re-invent a well-known algorithm. Instead, write something like "I will use Blabla's algorithm [KT, p. 342] to find a blabla in the blabla.")

2.c - Answer

> Note: Union find can also be used (and has a time complexity of $O(n \log n)$).

We will use a normal DFS algorithm to solve this problem.
This will have to start from v 0 and traverse the graph until we reach v $n-1$ (which is us - can also be seen as the terminal node) or until we have visited all vertices.
If the algorithm reaches the terminal node then we know that the virus has a possible path from the start (the infected) to the end (us) - thus we should print "oh, no!".
If we have visited all vertices and not reached the terminal node then we know that there is a disconnect between the infected and us - thus we should print "phew!".

2.d (1 pt.)

State the running time of your algorithm in terms of the parameters of the input.

2.d - Answer

We know that the running time of a normal DFS algorithm is $O(V + E)$ where V is the number of vertices and E is the number of edges.

3. Dynamic Programming

One of the problems on pages 3–7 is solved by dynamic programming.

3.a (1 pt.)

Which one?

3.a - Answer

"Mirror" can be solved with dynamic programming.

3.b (4 pt.)

Following the book's notation, let $\mathrm{OPT}(\dots)$ denote the value of a partial solution.
(Maybe you need more than one parameter, like $\mathrm{OPT}(i, v)$. Who knows?
Anyway, tell me what the parameters are—vertices, lengths, etc. and what their range is.
Use words like "where $i \in \{1, \dots, k^2\}$ denotes the length of BLABLA" or "where $v \in R$ is a red vertex.")
Give a recurrence relation for OPT, including relevant boundary conditions and base cases.
Which values of OPT are used to answer the problem?

3.b - Answer

$\mathrm{OPT}(i, j)$ denotes the lowest number of mismatches in $s[i : j]$ for $1 \leq i \leq j \leq n$ (or for $i, j \in \{1, \dots, n\}$).

$$\mathrm{OPT}(i, j) = \begin{cases} 0 & \text{if } i \geq j \\ \mathrm{OPT}(i+1, j-1) & \text{if } s_i = s_j \text{ (alternatively } s[i] = s[j]) \\ 1 + min \begin{cases} \mathrm{OPT}(i+1, j) \\ \mathrm{OPT}(i, j-1) \end{cases} & \text{otherwise} \end{cases}$$

Thus the solution to the problem is $\mathrm{OPT}(1, n)$.

3.c (1 pt.)

State the running time and space of the resulting algorithm in terms of the input parameters.

3.c - Answer

As we are building a 2D table and have to calculate all values we will both have a time and space complexity of:

$$O(n^2)$$

4. Flow

One of the problems on pages 3–7 is easily solved by a reduction to network flow.

4.a (1 pt.)

Which one?

4.a - Answer

"Pipeline" can be reduced to network flow where min-cut is utilized.

4.b (3 pt.)

Explain the reduction. Start by drawing the graph corresponding to Sample Input 1. Be ridiculously precise about which nodes and arcs there are, how many there are (in terms of size measures of the original problem), how the nodes are connected and directed, and what the capacities are. Describe the reduction in general (use words like "every node corresponding to a giraffe is connected to every node corresponding to a letter by an

undirected arc of capacity the length of the neck").
What does a maximum flow mean in terms of the original problem, and what size does it have in terms of the original parameters?

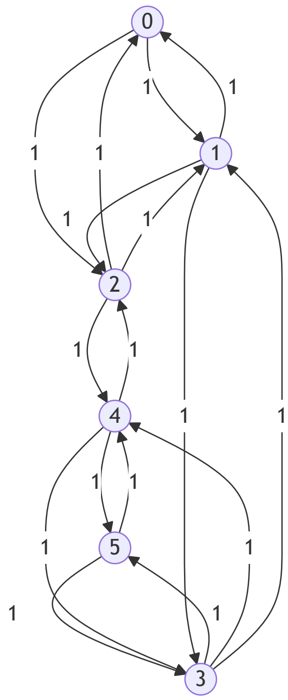
4.b - Answer

We have the following denotations:

- $|V| = n$
- $|E| = 2m$

```
foreach (u,v) in States:
  (u, v) in E, cap 1
  (v, u) in E, cap 1
```

Sample input 1 can be displayed as:



The source node is the first state, 0, and the sink node is the last state, $n-1$.
We will have to try to force as much flow as possible from the source to the sink, which will be bounded by m.

As we first have to find the max-flow we can use Ford-Fulkerson.
Specifically we can use the Edmonds-Karp algorithm.
Edmonds-Karp runs in $O(VE^2)$.

4.c (1 pt.)

State the running time of the resulting algorithm, be precise about which flow algorithm you use.
(Use words like "Using Krampfmeier–Strumpfnudel's algorithm ((5.47) in the textbook), the total running time will be $O(r^{17}\log^3\epsilon + \log^2 k)$, where r is the number of froontzes and k denotes the maximal weight of a giraffe.")

5. NP-hard

One of the problems on pages 3–7 is NP-hard.

5.a (1 pt.)

Which problem is it? (Let's call it P_1.)

5.a - Answer

"Polynomials" is NP-hard.

5.b (1 pt.)

The easiest way to show that P_1 is NP-hard is to consider another well-known NP-hard problem (called P_2). Which?

5.b - Answer

We choose the 3-SAT problem as P_2.
Thus $P_2 = \text{3-SAT}$.

5.c (0 pt.)

Do you now need to prove $P_1 \leq_p P_2$ or $P_2 \leq_p P_1$?

5.c - Answer

We have to prove that "3-SAT" P_2 can be reduced to "Polynomials" P_1.
Thus:

$$P_2 \leq_p P_1$$

5.d (3 pt.)

Describe the reduction. Do this both in general and for a small but complete example. In particular, be ridiculously precise about what instance is **given**, and what instance is **constructed** by the reduction, the parameters of the instance you produce (for example, number of vertices, edges, sets, colors) in terms of the parameters of the original instance, what the solution of the transformed instance means in terms of the original instance, etc. For the love of all that is Good and Holy, please start your reduction with words like “Given an instance to BLABLA, we will construct an instance of BLABLA as follows.”

5.d - Answer

> Note:
> Remember $x * y * z = 0 \text{ if } 0 \in \{x, y, z\}$

Given a "3-SAT" instance $X = \{x_1, x_2, \ldots, x_n\}, C_1, C_2, \ldots, C_m$ (where x_i is a boolean and the C_s are a conjunctions of disjunctions).

Construct instance to "Polynomials" n, m, M (where M is a list of expressions) as follows:

- $n = |X|,\ m = |C|$

```
M := for each C_i = (x_1, x_2, x_3)
construct an expression
f(x_1)*f(x_2)*f(x_3)
f(x_i) = x[i]   if x_i is false
(1-x[i])        if x_i is true
```

Assume S is an answer to "Polynomials"
If $S = \text{impossible}$ the answer to "3-SAT" is "no"
Otherwise the answer is "yes".

Example $x = \{x_1, x_2, x_3\}, C_1 = (\bar{x_1} \lor x_2 \lor x_3), (x_1 \lor x_2 \lor \bar{x_3})$

$n = 3,\ m = 2$

$x[1] * (1 - x[2]) * (1 - x[3])$
$(1 - x[1]) * (1 - x[2]) * x[3]$