LMU München, Winter Term 2020/2021
Macroeconomics - Growth
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Macroeconomics - Growth, Problem Set 3: Endogenous Growth Models

Problem 1: The AK-Model

Consider a standard neo-classical growth model in continuous time. Assume the economy admits a representative household with preferences given by

$$\max_{[c(t),a(t)]_{t=0}^{\infty}} \int_0^{\infty} e^{-(\rho-n)t} \, \frac{c(t)^{1-\theta}-1}{1-\theta} \, dt \quad \{\rho, \theta\} > 0 \, .$$

Population grows at rate $n > 0$, capital depreciates at rate $\delta \in (0,1)$ and the law of motion for assets of the household is given by

$$\dot{a}(t) = (r(t) - n) \, a(t) + w(t) - c(t) \, .$$

1.1 Set up the Hamiltonian of your choice and derive all first order conditions. Write down the corresponding transversality condition and show that the Euler equation is given by

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\theta} \, .$$

1.2 For the remainder of Problem 1 assume that production is given by

$$Y(t) = AK(t) \, .$$

What are the crucial differences of this formulation when compared to the standard neo-classical growth model?

1.3 Simplify the law of motion and use the Euler equation to show that the optimal consumption path is given by

$$c(t) = c(0) \cdot e^{\frac{A-\delta-\rho}{\theta} t} \, .$$

1.4 Use the simplified law of motion, the solution for the optimal consumption path and the transversality condition to show that initial consumption and the path of capital per capita are given by

$$\begin{aligned} c(0) &= \frac{\rho - (1-\theta)(A-\delta) - \theta n}{\theta} k(0) \, , \\ k(t) &= k(0) \cdot e^{\frac{A-\delta-\rho}{\theta} t} \, . \end{aligned}$$

What additional assumptions are necessary to ensure positive growth?

Problem 2: Endogenous Growth with Capital Mobility

Consider a neo-classical growth model in continuous time. Assume the economy admits a representative household with preferences given by

$$\max_{[c(t),a(t)]_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{c(t)^{1-\theta}-1}{1-\theta} \, dt \, , \quad \{\rho, \theta\} > 0 \, .$$

Production takes place as follows: one sector produces the consumption good with production technology

$$C(t) = A_C K_C(t)^{\alpha} L(t)^{1-\alpha} \quad \alpha \in (0,1) \text{ and } A_C > 0 \, ,$$

where $K_C(t)$ is the total capital stock used in the production of the consumption good at time t and $L(t)$ the number of workers, with $L(0) > 0$.

A second sector produces investment goods with the production technology

$$I(t) = A_I K_I(t) \quad A_I > 0 \, ,$$

where $K_I(t)$ is the total capital stock used in the production of the investment good. Markets are perfectly competitive and capital is free to move between the production sectors; there is no population growth, i.e. $n = 0$. Capital depreciates at rate $\delta \in (0,1)$, with $K(0) > 0$ and its law of motion given by

$$\dot{K}(t) = I(t) - \delta K(t) \quad \text{and} \quad K_C(t) + K_I(t) = K(t) \, .$$

2.1 Denote the share of capital used in the investment sector with $x(t)$ and assume that the price of the consumption good is normalised to one, i.e. $p_C(t) \equiv 1$. Show that the price of the investment good is given by

$$p_I(t) = \alpha \cdot \frac{A_C}{A_I} \cdot \left(\frac{L}{(1-x(t))\, K(t)} \right)^{1-\alpha} \, .$$

2.2 The solution to the individual optimization problem gives the Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{r_C(t) - \rho}{\theta} \, ,$$

where $r_C(t)$ is the net return to capital in the consumption goods sector and given by

$$r_C(t) = A_I - \delta + \frac{\dot{p}_I(t)}{p_I(t)} - \frac{\dot{p}_C(t)}{p_C(t)} \, .$$

Use this information in combination with the equilibrium conditions

$$\begin{aligned} \frac{\dot{c}(t)}{c(t)} =& \alpha \left[\frac{\dot{K}(t)}{K(t)} - \frac{\dot{x}(t)}{1-x(t)} \right] \\ p_I(t) =& \alpha \cdot \frac{A_C}{A_I} \cdot \left(\frac{L}{(1-x(t))\, K(t)} \right)^{1-\alpha} \, . \end{aligned}$$

to show that the consumption path in this economy is given by

$$c(t) = c(0) \cdot e^{\frac{\alpha(A_I - \delta - \rho)}{1-\alpha(1-\theta)} t} \, .$$

2.3 Suppose that $x(t) \equiv \bar{x}$ is constant over time. Use the previous results and the law of motion for capital to show that under this assumption the path of capital in the economy and the solution for $\bar{x}$ are given by

$$
\begin{aligned}
K(t) =& K(0) \cdot e^{\frac{A_I - \delta - \rho}{1-\alpha(1-\theta)} \cdot t} \\
\bar{x} =& \frac{1}{A_I} \cdot \left(\frac{A_I - \delta - \rho}{1 - \alpha(1-\theta)} + \delta \right) .
\end{aligned}
$$

2.4 What additional assumption is needed in order for this economy to have a balanced growth path? Does it admit a steady state? If so, under what condition(s), if not, why not?

Problem 3: Directed Technical Change

Consider a neo-classical growth model in continuous time. There is no capital and static population (i.e. $n = 0$). The final output is produced according to a CES production function

$$Y(t) = \left[Y^L(t)^{\frac{\varepsilon-1}{\varepsilon}} + Y^H(t)^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}} \qquad \varepsilon > 1 ,$$

where $Y^L(t)$ and $Y^H(t)$ are quantities of goods produced with unskilled and skilled labor, subject to the following production functions

$$\begin{aligned} Y^L(t) &= L(t)^{1-\alpha} \int_0^{A_L(t)} \left(x_L^j(t) \right)^{\alpha} \, dj , \\ Y^H(t) &= H(t)^{1-\alpha} \int_0^{A_H(t)} \left(x_H^j(t) \right)^{\alpha} \, dj , \end{aligned}$$

where $L(t)$ and $H(t)$ are quantities of unskilled and skilled labor while $x_L^j(t)$ and $x_H^j(t)$ are intermediate goods (technologies) complementing unskilled and skilled labor, respectively. Final goods production is competitive, while intermediate goods production is subject to monopolistic competition with each producer owning the patent for a single intermediate variety.

3.1 Show that the demand for intermediate varieties and the profit of monopolists is given by

$$\begin{aligned} x_L^j(t) = L(t) \cdot \left[\frac{\alpha P_L(t)}{p_L^j(t)} \right]^{\frac{1}{1-\alpha}} \quad &\text{and} \quad x_H^j(t) = H(t) \cdot \left[\frac{\alpha P_H(t)}{p_H^j(t)} \right]^{\frac{1}{1-\alpha}} , \\ \pi_L(t) = (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}} \left[P_L(t) \right]^{\frac{1}{1-\alpha}} L(t) \quad &\text{and} \quad \pi_H(t) = (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}} \left[P_H(t) \right]^{\frac{1}{1-\alpha}} H(t) . \end{aligned}$$

3.2 Use the demand for intermediate varieties to show that the price ratio of skilled to unskilled goods and the skill premium of skilled to unskilled labor is given by

$$\begin{aligned} \frac{P_H(t)}{P_L(t)} &= \left[\frac{A_H(t)}{A_L(t)} \cdot \frac{H(t)}{L(t)} \right]^{-\frac{1-\alpha}{\sigma}} , \\ \frac{w_H(t)}{w_L(t)} &= \left[\frac{A_H(t)}{A_L(t)} \right]^{\frac{\sigma-1}{\sigma}} \left[\frac{H(t)}{L(t)} \right]^{\frac{-1}{\sigma}} , \end{aligned}$$

where $\sigma = 1 + (1-\alpha)(\varepsilon - 1)$. What are the implications of this result?

3.3 Assume that the development of a new intermediate good has a fixed cost μ and there is free entry. Show that this implies an equilibrium skill bias of

$$\frac{A_H(t)}{A_L(t)} = \left[\frac{H(t)}{L(t)} \right]^{\sigma-1} ,$$

and an equilibrium skill premium of

$$\frac{w_H(t)}{w_L(t)} = \left[\frac{H(t)}{L(t)} \right]^{\sigma-2} .$$

What are the implications of this result for inequality in the long-run?

Problem 4: A Numerical Implementation of the AK-Model

Consider the model from Problem 1. The paths of capital and consumption are given by

$$
\begin{aligned}
k(t) &= k(0) \cdot e^{\frac{A-\delta-\rho}{\theta}t} , \\
c(t) &= \frac{\rho - (1-\theta)(A-\delta) - \theta n}{\theta} k(0) \cdot e^{\frac{A-\delta-\rho}{\theta}t} .
\end{aligned}
$$

4.1 Assume parameter values of $A = 0.1$, $\delta = 0.05$, $n = 0.02$, $\rho = 0.04$, $\theta = 0.8$ and $K(0) = L(0) = 1$. Simulate the model for $t = 100$. Does the model admit a steady state?

4.2 Assume that from $t \geq 50$ onwards the government levies an unanticipated tax $\tau = 0.1$ on the net return to capital, such that after tax return is given by $(1-\tau)r(t)$.

Simulate the paths of capital and consumption for economies with and without taxation. Comment on the result.