Macroeconomics - Growth

Neo-Classical Growth

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Dynamic Optimization in Continuous Time

- So far: either exogenous savings (Solow) or finite lifetime with overlapping generations (intra-generational equilibrium in OLG).
- Now: endogenous savings, infinite horizon and intergenerational allocation.
- Solving these problems needs a bit of knowledge in continuous time dynamic optimization ("optimal control").
- Goal: get you acquainted with the tools and intuition behind them.

Dynamic Optimization in Continuous Time

Generic set-up:

- Households maximize their lifetime utility (in intensive form)

$$U = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^t u(c_t) \ ,$$

where

  - $u(c_t)$: instantaneous utility.
  - ρ: the rate of time preference ($\beta \equiv \frac{1}{1+\rho}$)
- The higher ρ, the less utility weight is given to future consumption.
- Households choose their consumption path to maximize their lifetime utility under the law of motion

$$k_{t+1} = k_t + g\left(c_t, k_t\right) \ .$$

  - Consumption c_t is a control variable (can be chosen).
  - The level of savings k_t is a state variable (describes available resources).

Optimization:

$$\max_{c_t, c_{t+1}, k_{t+1}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t u(c_t) \qquad \text{s.t.} \qquad k_{t+1} = k_t + g\left(c_t, k_t\right)$$

The Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty} \left\{ \left(\frac{1}{1+\rho}\right)^t u(c_t) + \lambda_t \left[k_t + g\left(c_t, k_t\right) - k_{t+1}\right] \right\} .$$

The first order conditions are

$$\begin{aligned} \left(\frac{1}{1+\rho}\right)^t u'(c_t) + \lambda_t g_{c_t}\left(c_t, k_t\right) &= 0 \\ \left(\frac{1}{1+\rho}\right)^{t+1} u'(c_{t+1}) + \lambda_{t+1} g_{c_{t+1}}\left(c_{t+1}, k_{t+1}\right) &= 0 \\ -\lambda_t + \lambda_{t+1}\left(1 + g_{k_{t+1}}\left(c_{t+1}, k_{t+1}\right)\right) &= 0 \end{aligned}$$

Re-arranging gives

$$
\begin{aligned}
\lambda_{t+1}\left(1+g_{k_{t+1}}\left(c_{t+1},k_{t+1}\right)\right) &= \lambda_t \\
\frac{\lambda_{t+1}}{\lambda_t}\left(1+g_{k_{t+1}}\left(c_{t+1},k_{t+1}\right)\right) &= 1 \\
\frac{\lambda_{t+1}-\lambda_t}{\lambda_t}\left(1+g_{k_{t+1}}\left(c_{t+1},k_{t+1}\right)\right) &= -g_{k_{t+1}}\left(c_{t+1},k_{t+1}\right) \\
\frac{\lambda_{t+1}-\lambda_t}{\lambda_t} &= -\frac{g_{k_{t+1}}\left(c_{t+1},k_{t+1}\right)}{\left(1+g_{k_{t+1}}\left(c_{t+1},k_{t+1}\right)\right)} \\
\frac{\lambda_{t+1}-\lambda_t}{\lambda_t} &= -\frac{\lambda_{t+1}}{\lambda_t}g_{k_{t+1}}\left(c_{t+1},k_{t+1}\right)
\end{aligned}
$$

Continuing with c_t and c_{t+1} gives the Euler equation in discrete time

$$
\begin{aligned}
\left(\frac{1}{1+\rho}\right)^t u'(c_t) &= -\, \lambda_t g_{c_t}\left(c_t, k_t\right) \\
\left(\frac{1}{1+\rho}\right)^t u'(c_t) &= -\, \lambda_{t+1}\left(1 + g_{k_{t+1}}\left(c_{t+1}, k_{t+1}\right)\right) g_{c_t}\left(c_t, k_t\right) \\
\left(\frac{1}{1+\rho}\right)^t u'(c_t) &= \left(\frac{1}{1+\rho}\right)^{t+1} u'(c_{t+1}) \frac{g_{c_t}\left(c_t, k_t\right)}{g_{c_{t+1}}\left(c_{t+1}, k_{t+1}\right)}\left(1 + g_{k_{t+1}}\left(c_{t+1}, k_{t+1}\right)\right)
\end{aligned}
$$

$$
\frac{u'(c_t)}{u'(c_{t+1})} = \frac{1 + g_{k_{t+1}}\left(c_{t+1}, k_{t+1}\right)}{1+\rho} \frac{g_{c_t}\left(c_t, k_t\right)}{g_{c_{t+1}}\left(c_{t+1}, k_{t+1}\right)}
$$

Dynamic Optimization in Continuous Time

The optimization problem in continuous time is similar to the one in discrete time (where we can use Lagrange to solve for a sequence of c_t), but we need a different tool to solve it.

Pontryagin's Maximum Principle:

- Set-up a constrained optimization problem with co-state variables (Hamiltonian function).
- Determine the necessary conditions for a maximum (Maximum Principle).

Dynamic Optimization in Continuous Time

Generic set-up:

- Households maximize their lifetime utility (in intensive form)

$$U = \int_0^\infty e^{-\rho t} u\left((c(t)\right) dt \ ,$$

where

  - $u\left((c(t)\right)$: instantaneous utility.
  - ρ: the rate of time preference.

- The higher ρ, the less weight is given to future consumption. It can be shown that

$$\left(\frac{1}{1+\rho\Delta t}\right)^{\frac{t}{\Delta t}} \to e^{-\rho t} \quad \text{for } \Delta t \to 0 \ .$$

- Households choose their consumption path $c(t)$ to maximize their lifetime utility under the law of motion

$$\dot{k}(t) = g\left(c(t), k(t)\right) \ .$$

  - $c(t)$: control variable.
  - $k(t)$: state variable.

Pontryagin's Maximum Principle:

Consider the stationary dynamic problem

$$\max_{c(t),k(t)} \int_0^\infty e^{-\rho t} u\left(c(t)\right)\, dt \qquad \text{s.t.} \qquad \dot{k}(t) = g\left(c(t), k(t)\right)$$

plus an initial condition $k(0) = k_0$ and terminal value constraint (more on that later). For the optimal path $(c(t)^*, k(t)^*)$ it holds that there exists a function $\lambda(t)$ such that for the *present-value Hamiltonian*

$$\mathcal{H}\left(c(t), k(t), \mu(t)\right) = e^{-\rho t} u\left(c(t)\right) + \lambda(t) g\left(c(t), k(t)\right)$$

the following necessary conditions are satisfied along the optimal path

$$\begin{aligned}
\frac{\partial \mathcal{H}\left(c(t), k(t), \lambda(t)\right)}{\partial c(t)} =& 0 \\
\frac{\partial \mathcal{H}\left(c(t), k(t), \lambda(t)\right)}{\partial k(t)} =& -\dot{\lambda}(t) \\
\frac{\partial \mathcal{H}\left(c(t), k(t), \lambda(t)\right)}{\partial \lambda(t)} =& \dot{k}(t) \\
\lim_{t\to\infty} \lambda(t) k(t) =& 0
\end{aligned}$$

Maximum principle for the present-value Hamiltonian:

$$\mathcal{H}\left(c(t), k(t), \lambda(t)\right) = e^{-\rho t} u\left(c(t)\right) + \lambda(t) g\left(c(t), k(t)\right)$$

The necessary conditions for a maximum are

$$e^{-\rho t} u'\left(c(t)\right) + \lambda(t) \underbrace{\frac{\partial g\left(c(t), k(t)\right)}{\partial c(t)}}_{g_c(c(t), k(t))} = 0$$

$$\lambda(t) \underbrace{\frac{\partial g\left(c(t), k(t)\right)}{\partial k(t)}}_{g_k(c(t), k(t))} = -\dot{\lambda}(t)$$

$$g\left(c(t), k(t)\right) = \dot{k}(t)$$

$$\lim_{t \to \infty} \lambda(t) k(t) = 0$$

Re-arranging gives

$$
\begin{aligned}
e^{-\rho t}u'\left(c(t)\right) =& -\lambda(t)g_c\left(c(t),k(t)\right)\\
-\rho e^{-\rho t}u'\left(c(t)\right)+e^{-\rho t}u''\left(c(t)\right)\dot{c}(t) =& -\dot{\lambda}(t)g_c\left(c(t),k(t)\right)-\lambda(t)\dot{g}_c\left(c(t),k(t)\right)\\
-\rho+\underbrace{\frac{u''\left(c(t)\right)c(t)}{u'\left(c(t)\right)}}_{\equiv-\varepsilon_u(c(t))}\frac{\dot{c}(t)}{c(t)} =& \frac{\dot{\lambda}(t)}{\lambda(t)}+\frac{\dot{g}_c\left(c(t),k(t)\right)}{g_c\left(c(t),k(t)\right)}\\
\frac{\dot{c}(t)}{c(t)} =& -\frac{1}{\varepsilon_u\left(c(t)\right)}\left(\rho+\frac{\dot{\lambda}(t)}{\lambda(t)}+\frac{\dot{g}_c\left(c(t),k(t)\right)}{g_c\left(c(t),k(t)\right)}\right)
\end{aligned}
$$

$$
\begin{aligned}
\lambda(t)g_k\left(c(t),k(t)\right) =& -\dot{\lambda}(t)\\
\frac{\dot{\lambda}(t)}{\lambda(t)} =& -g_k\left(c(t),k(t)\right)
\end{aligned}
$$

Putting both conditions together gives the Euler equation in continuous time

$$
\frac{\dot{c}(t)}{c(t)}=-\frac{1}{\varepsilon_u\left(c(t)\right)}\left(\rho-g_k\left(c(t),k(t)\right)+\frac{\dot{g}_c\left(c(t),k(t)\right)}{g_c\left(c(t),k(t)\right)}\right)
$$

- The conditions with respect to $c(t)$ and $\lambda(t)$ are (relatively) straightforward. However, the conditions

$$\lambda(t) g_k\left(c(t), k(t)\right) = -\dot{\lambda}(t)$$

$$\lim_{t \to \infty} \lambda(t) k(t) = 0$$

are somewhat unusual.

- In order to get an intuition for the results remember the solutions from discrete time.

Dynamic Optimization in Continuous Time

Discrete vs. continuous time:

- Differentiating with respect to the state variable has yielded (for continuous and discrete time)

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = -\, g_k\left(c(t), k(t)\right)$$

$$\frac{\lambda_{t+1} - \lambda_t}{\lambda_t} = -\, \frac{\lambda_{t+1}}{\lambda_t} g_{k_{t+1}}\left(c_{t+1}, k_{t+1}\right)$$

- Now reformulate the discrete time expression

$$\lambda_{t+1} - \lambda_t = -\, \lambda_{t+1} g_{k_{t+1}}\left(c_{t+1}, k_{t+1}\right)$$

$$\lambda(t+\Delta t) - \lambda(t) = -\, \lambda(t+\Delta t) g_{k(t+\Delta t)}\left(c(t+\Delta t), k(t+\Delta t)\right)\Delta t$$

$$\frac{\lambda(t+\Delta t) - \lambda(t)}{\Delta t} = -\, \lambda(t+\Delta t) g_{k(t+\Delta t)}\left(c(t+\Delta t), k(t+\Delta t)\right)$$

Detour: L'Hôpital's Rule

Theorem (L'Hôpital's Rule)

Let $f : [a, b] \to \mathbb{R}$ *and* $g : [a, b] \to \mathbb{R}$ *be differentiable functions on* $[a, b]$ *and* $g'(x) \neq 0$ *for* $x \in (a, b)$ *and let* $c \in [a, b]$*. If*

$$\lim_{x \to c} \frac{f'(x)}{g'(x)}$$

exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \quad \text{or} \quad \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \infty \ ,$$

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \ .$$

Applying L'Hôpital's rule and taking the limit yields

$$\lim_{\Delta t \to 0} \frac{\lambda(t+\Delta t) - \lambda(t)}{\Delta t} = - \lim_{\Delta t \to 0} \lambda(t+\Delta t) g_{k(t+\Delta t)} \left(c(t+\Delta t), k(t+\Delta t)\right)$$

$$\lim_{\Delta t \to 0} \frac{\frac{d\lambda(t+\Delta t)}{d(t+\Delta t)} \cdot 1}{1} = - \lim_{\Delta t \to 0} \lambda(t+\Delta t) g_{k(t+\Delta t)} \left(c(t+\Delta t), k(t+\Delta t)\right)$$

$$\frac{d\lambda(t)}{dt} = - \lambda(t) g_k \left(c(t), k(t)\right)$$

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = - g_k \left(c(t), k(t)\right)$$

As a result discrete and continuous time solution are (conceptually) equivalent.

Interpretation:

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = -g_k\left(c(t), k(t)\right)$$

- The interpretation of the co-state variable $\lambda(t)$ is similar to the Lagrange multiplier in discrete time.
- Key difference: continuous time. Therefore $\lambda(t)$ represents the value of an infinitesimal increase of $k(t)$ at time t.
- First intuition behind the terminal value constraint:

$$\lim_{t\to\infty} \lambda(t)k(t) = 0$$

There should be no value of having more (or less) $k(t)$ at the end of the planning horizon.

Present- vs. current-value Hamiltonian:

Most of the time it is easier to formulate the problem in *current-value* terms. In order to do so, define

$$\mu(t) \equiv \lambda(t)e^{\rho t} \;\Rightarrow\; e^{-\rho t}\mu(t) = \lambda(t)\ ,$$

and re-write the current-value Hamiltonian as

$$\hat{\mathcal{H}}\left(c(t), k(t), \mu(t)\right) = u\left(c(t)\right) + \mu(t)g\left(c(t), k(t)\right)$$

Note that this implies the following

$$\dot{\lambda}(t) = \frac{d\, e^{-\rho t}\mu(t)}{d\, t} = -\rho e^{-\rho t}\mu(t) + e^{-\rho t}\dot{\mu}(t)$$

$$\Rightarrow\; -\dot{\lambda}(t) = \rho e^{-\rho t}\mu(t) - e^{-\rho t}\dot{\mu}(t)$$

Subsequently, the derivative with respect to the state variable must be transformed

$$
\begin{aligned}
\lambda(t) g_k\left(c(t), k(t)\right) &= -\dot{\lambda(t)} \\
e^{-\rho t}\mu(t) g_k\left(c(t), k(t)\right) &= \rho e^{-\rho t}\mu(t) - e^{-\rho t}\dot{\mu}(t) \\
\mu(t) g_k\left(c(t), k(t)\right) &= \rho\mu(t) - \dot{\mu}(t) \\
\frac{\partial\hat{\mathcal{H}}\left(c(t), k(t), \mu(t)\right)}{\partial k(t)} &= \rho\mu(t) - \dot{\mu}(t)
\end{aligned}
$$

Maximum principle for the current-value Hamiltonian:

$$\hat{\mathcal{H}}\left(c(t), k(t), \mu(t)\right) = u\left(c(t)\right) + \mu(t) g\left(c(t), k(t)\right)$$

The necessary conditions for a maximum are

$$\frac{\partial \hat{\mathcal{H}}\left(c(t), k(t), \mu(t)\right)}{\partial c(t)} = 0$$

$$\frac{\partial \hat{\mathcal{H}}\left(c(t), k(t), \mu(t)\right)}{\partial k(t)} = \rho\mu(t) - \dot{\mu}(t)$$

$$\frac{\partial \hat{\mathcal{H}}\left(c(t), k(t), \mu(t)\right)}{\partial \lambda(t)} = \dot{k}(t)$$

$$\lim_{t\to\infty} e^{-\rho t}\mu(t)k(t) = 0$$

Terminal value constraints:

- Needed in infinite horizon problems.
- Finite horizon problems (e.g. OLG) have a finite number of first-order conditions determining optimality.
- The transversality condition rules out beneficial simultaneous changes in an infinite number of choice variables.
- Intuition: as $t \to \infty$, the value of the Hamiltonian must approach zero

$$\begin{aligned} \lim_{t\to\infty}\left[e^{-\rho t}u\left(c(t)\right)+\lambda(t)g\left(c(t),k(t)\right)\right] =& 0 \\ \lim_{t\to\infty}\lambda(t)g\left(c(t),k(t)\right) =& 0 \\ \lim_{t\to\infty}\lambda(t)\dot{k}(t) =& 0\ , \end{aligned}$$

where the second line uses the fact that the value of consumption is finite.

- An interior continuous solution demands that either

$$\lim_{t\to\infty} k(t) = k^* \iff \dot{k}(t) = 0$$

or

$$\lim_{t\to\infty} \frac{\dot{k}(t)}{k(t)} = \kappa \Rightarrow \lim_{t\to\infty} \dot{k}(t) = \lim_{t\to\infty} \kappa k(t) = \kappa \lim_{t\to\infty} k(t)$$

This corresponds to a steady state or balanced growth path in the context of growth models.

- If $\dot{k}(t) = 0$, then

$$\begin{aligned} \frac{\dot{\lambda}(t)}{\lambda(t)} =& - g_k\left(c^*, k^*\right) = -r \\ \lambda(t) =& \lambda(0)e^{-rt} \end{aligned}$$

- Consequently, for an interior continuous solution the transversality condition is either

$$\lim_{t\to\infty} \lambda(t)\dot{k}(t) = 0 = \lim_{t\to\infty} \lambda(0)e^{-rt}k(t) = \lim_{t\to\infty} \lambda(t)k(t)$$

or

$$\lim_{t\to\infty} \lambda(t)\dot{k}(t) = \lim_{t\to\infty} \lambda(t)\kappa k(t) = \kappa \lim_{t\to\infty} \lambda(t)k(t) = 0 = \lim_{t\to\infty} \lambda(t)k(t)$$

- As a result the strict version of the transversality condition can be written as

$$\lim_{t\to\infty} \lambda(t)k(t) = 0$$

or

$$\lim_{t\to\infty} e^{-\rho t}\mu(t)k(t) = 0$$

in current-value form.

Interpretation:

- Two-point boundary value problem: given an initial $k(0)$ and the transversality condition we can find a trajectory the control variable given by the Euler equation, where
  - $k(0)$ gives a starting point $c(0)$
  - the transversality condition imposes a bound from above
  - the Euler equation describes the optimal control path (optimal control growth rate)
- Application: the neo-classical growth model.

Basic structure of the neo-classical growth model:

- Continuous time, infinite-horizon, one-good economy.
- Representative firm and representative households.
- Population grows at rate n, that is

$$\frac{\dot{L}(t)}{L(t)} = n \;\Rightarrow\; L(t) = L(0) \cdot e^{nt} \ .$$

Firms:

- Firms are owned by households.
- Firms produce output using physical capital $K(t)$ and labor $L(t)$ as inputs in a neo-classical production function that satisfies the Inada conditions

$$Y(t) = F\left(K(t), L(t)\right) \ ,$$

or in intensive form

$$y(t) = f\left(k(t)\right) \ .$$

- Firms maximize profits $\pi = F\left(K(t), L(t)\right) - R(t)K(t) - w(t)L(t)$.
- Firms operate in a competitive market with free entry. As a result factors are paid their marginal products

$$R(t) = f'\left(k(t)\right) \quad \text{and} \quad w(t) = f\left(k(t)\right) - f'\left(k(t)\right) k(t)$$

- Physical capital depreciates at rate $\delta \in (0,1)$. Total physical capital evolves according to the law of motion

$$\dot{K}(t) = F\left(K(t), L(t)\right) - \delta K(t) - c(t)L(t) \ .$$

Households:

- Household grow at the rate population grows.
- Households earn labor and capital income by renting their labor and savings (as capital) to firms on competitive markets receiving the wage $w(t)$ and capital return $R(t)$, respectively.
- The income is split between consumption and savings in each (infinitesimal) period.
- Objective: maximize lifetime utility by choosing an adequate consumption/savings pattern.

Aggregation:

Important: the representative agent assumption (i.e. a representative firm and a representative household) requires at least one of the following assumptions to be justified

- All individuals are identical in preferences, endowments etc.
- Preferences are homothetic and of "Gorman"-type (linear Engel-curves).

Preferences:

Households maximize their lifetime utility

$$U = \int_0^\infty e^{-\rho t} L(t) u\left(c(t)\right)\, dt \ ,$$

where $L(t) = L(0) \cdot e^{nt}$ and therefore ($L(0)$ can be normalised here)

$$U = \int_0^\infty e^{-(\rho - n)t} u\left(c(t)\right)\, dt \ .$$

- $u\left(c(t)\right)$: instantaneous per capita utility.
- ρ: rate of time preference.
- **Important:** $\rho > n$. (Why?)

Instantaneous utility:

- CRRA/CIES utility

$$u\left(c(t)\right) = \frac{c(t)^{1-\theta} - 1}{1-\theta}$$

- Constant relative risk aversion (CRRA)

$$\begin{aligned} u'\left(c(t)\right) =& c(t)^{-\theta} \\ u''\left(c(t)\right) =& -\theta c(t)^{-\theta-1} \\ \Rightarrow \varepsilon_u\left(c(t)\right) \equiv -\frac{u''\left(c(t)\right) c(t)}{u'\left(c(t)\right)} =& \theta \end{aligned}$$

- The elasticity of the marginal utility is constant and equal to θ. The intertemporal elasticity of substitution is given by $\frac{1}{\theta}$.
- Special case: log utility ($\theta \to 1$).

Detour: CIES/CRRA

- Imagine two possible pay-offs: 50$ guaranteed, gamble between 100$ and 0$ both with chance $\frac{1}{2}$.
- The expected value of both pay-offs is 50$
  - Risk averse: rather take less than 50$ than gamble (concave utility).
  - Risk neutral: indifferent between 50$ gambling (linear utility).
  - Risk loving: gamble even if offered more than 50$ (convex utility).
- The more concave the utility function, the more risk averse the individual.
- Problem: utility is not unique. The utility functions

$$a \ln c \quad \text{and} \quad b \ln c \quad \{a, b\} > 0,\ a \neq b$$

have the same preference ordering but different degrees of concavity as their second derivatives

$$-\frac{a}{c^2} \quad \text{and} \quad -\frac{b}{c^2}$$

are not identical.

Detour: CIES/CRRA

- A measure that does not depend on the affine transformation of utility is the coefficient of absolute risk aversion, given by

$$-\frac{u''(\cdot)}{u'(\cdot)} \quad \Rightarrow \quad -\left(-\frac{\frac{a}{c^2}}{\frac{a}{c}}\right) = -\left(-\frac{\frac{b}{c^2}}{\frac{b}{c}}\right) = \frac{1}{c}$$

- This describes at what rate marginal utility decreases with a marginal increase in consumption.
- Problem: not unitless, i.e. depends on how consumption is measured.
- How does marginal utility change if consumption changes by one percent, i.e. with a relative change? A measure of this is the elasticity of marginal utility with respect to consumption

$$-\frac{u''(x)x}{u'(x)} \quad \Rightarrow \quad \frac{c}{c} = 1$$

which is also called the coefficient of relative risk aversion.

DETOUR: CIES/CRRA

If utility is given by

$$u\left(c(t)\right) = \frac{c(t)^{1-\theta} - 1}{1-\theta}$$

the coefficient of relative risk aversion is

$$\frac{u''\left(c(t)\right)c(t)}{u'\left(c(t)\right)} = \theta$$

The intertemporal elasticity of substitution describes the percentage change of consumption growth over time with respect to a percentage change in the interest rate over time (which is equivalent to the percentage change in marginal utility over time)

$$\frac{\partial \frac{\dot{c}(t)}{c(t)}}{\partial \frac{\dot{u}'(c(t))}{u'(c(t))}} = \frac{\partial \frac{\dot{c}(t)}{c(t)}}{\partial \frac{u''(c(t))\dot{c}(t)}{u'(c(t))}} = \frac{\partial \frac{\dot{c}(t)}{c(t)}}{\partial \frac{u''(c(t))c(t)}{u'(c(t))} \frac{\dot{c}(t)}{c(t)}} = \frac{1}{\frac{u''(c(t))c(t)}{u'(c(t))}} = \frac{1}{\theta}$$

Optimization:

- Profit maximization of firms gives $R(t)$ and $w(t)$.
- Households choose their consumption path $c(t)$ to maximize their intertemporal utility given the law of motion of assets $A(t)$

$$\begin{aligned} \dot{A}(t) =& R(t)A(t) + w(t)L(t) - c(t)L(t) - \delta A(t) \\ \Rightarrow \quad \dot{a}(t) =& \underbrace{(R(t) - \delta)}_{=r(t)} a(t) - na(t) + w(t) - c(t) \\ \dot{a}(t) =& (r(t) - n)\, a(t) + w(t) - c(t) \end{aligned}$$

  - $c(t)$: control variable.
  - $a(t)$: state variable.

- Since there is only one investment good market clearing implies

$$a(t) = k(t) \iff \dot{a}(t) = \dot{k}(t) \; .$$

Current-value Hamiltonian:

$$\hat{\mathcal{H}}\left(c(t), a(t), \mu(t)\right) = \frac{c(t)^{1-\theta} - 1}{1-\theta} + \mu(t)\left[\left(r(t) - n\right)a(t) + w(t) - c(t)\right]$$

The first-order conditions are

- Optimal consumption

$$\frac{\partial \hat{\mathcal{H}}\left(c(t), a(t), \mu(t)\right)}{\partial c(t)} = c(t)^{-\theta} - \mu(t) = 0$$

- Asset pricing equation

$$\begin{aligned}\dot{\mu}(t) =& (\rho - n)\mu(t) - \frac{\partial \hat{\mathcal{H}}\left(c(t), a(t), \mu(t)\right)}{\partial a(t)} \\ =& (\rho - n)\mu(t) - \left(r(t) - n\right)\mu(t) = \left(\rho - r(t)\right)\mu(t)\end{aligned}$$

- Intertemporal budget constraint

$$\dot{a}(t) = \frac{\partial \hat{\mathcal{H}}\left(c(t), a(t), \mu(t)\right)}{\partial \mu(t)} = \left(r(t) - n\right)a(t) + w(t) - c(t)$$

Re-writing optimal consumption yields

$$-\theta\frac{\dot{c}(t)}{c(t)} = \frac{\dot{\mu}(t)}{\mu(t)} \; .$$

Re-writing the asset pricing equation yields

$$\frac{\dot{\mu}(t)}{\mu(t)} = \rho - r(t) \; .$$

Combining the two gives the Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\theta} \; .$$

Transversality condition

$$\lim_{t\to\infty} e^{-(\rho-n)t}\mu(t)a(t) = 0 \; .$$

The Neo-Classical Growth Model

Discussion:

Substituting $r(t) = f'(k(t)) - \delta$ into the Euler equation and the law of motion (using $a(t) = k(t)$) gives the full description of the dynamics of the neo-classical growth model

$$\begin{aligned}\frac{\dot{c}(t)}{c(t)} &= \frac{f'(k(t)) - \delta - \rho}{\theta} \\ \dot{k}(t) &= f(k(t)) - (n+\delta)k(t) - c(t)\end{aligned}$$

- The Euler equation shows the optimal consumption growth over time.
- This system generally has no analytic solution (only for CIES utility, Cobb-Douglas production **and** $\alpha = \theta$). However, it is relatively straightforward to characterise and simulate.
- Optimal consumption growth does not depend on the wage profile. (Where is the wage?)
- The optimality conditions have been derived in a decentralized economy. Thus they are not a priori Pareto optimal.

Natural debt limit:

- The law of motion is not really a budget constraint as $a(t)$ could be negative (i.e. borrowing). In fact there is an incentive for the individual to make $a(t)$ arbitrarily negative. (Why?)
- However, as $a(t) = k(t)$ this would imply arbitrarily negative capital, which is not feasible as $k(t) \geq 0 \; \forall t$.
- Intuition: ensure that at the end-of-life there is no debt.
- Problem: infinite horizon, there is no end-of-life.

Lifetime budget constraint:

The lifetime budget constraint is given by

$$\int_0^T c(t)L(t)e^{\int_t^T r(s)\,ds}\,dt + A(T) = \int_0^T w(t)L(t)e^{\int_t^T r(s)\,ds}\,dt + A(0)e^{\int_0^T r(s)\,ds}\,,$$

where $e^{\int_0^T r(s)\,ds}$ is the stream of all interest rates accrued over time. Dividing both sides by $L(0)e^{\int_0^T r(s)\,ds}$ yields

$$\int_0^T c(t)e^{-\int_0^t r(s)-n\,ds}\,dt + e^{-\int_0^T r(s)-n\,ds}a(t) = \int_0^T w(t)e^{-\int_0^t r(s)-n\,ds}\,dt + a(0)\,.$$

Discounted lifetime expenditure has to be smaller than or equal to discounted lifetime income stream as $T \to \infty$, that is

$$\int_0^\infty c(t)e^{-\int_0^t r(s)-n\,ds}\,dt \leq \int_0^\infty w(t)e^{-\int_0^t r(s)-n\,ds}\,dt + a(0)\,.$$

The only way for this to hold is if

$$\lim_{T\to\infty} e^{-\int_0^T r(s)-n\,ds}a(t) \geq 0\,.$$

- The condition

$$\lim_{T \to \infty} e^{-\int_0^T r(s)-n\,ds} a(t) \geq 0 \ .$$

  is also called the "No-Ponzi Game" condition (after Charles Ponzi, one of the first people convicted of chain-letter/pyramid-scheme fraud).
- The condition ensures that the representative household does not (asymptotically) tend to negative wealth.
- In the optimum the condition holds with strict equality. (Why?)
- **Important:** do not confuse the No-Ponzi Game condition with the transversality condition. The transversality condition is an optimality condition that the individual wants to satisfy, the No-Ponzi Game condition is an additional budget constraint that the individual would love to break.

The Neo-Classical Growth Model

Labor efficiency units:

- Introducing Harrod-neutral technological progress is uncomplicated. If technology grows at constant rate g, the law of motion becomes

$$\dot{\tilde{k}}(t) = f\left(\tilde{k}(t)\right) - (n+g+\delta)\tilde{k}(t) - \tilde{c}(t) ,$$

with $\tilde{k}(t) = \frac{k(t)}{A(t)}$, $\tilde{c}(t) = \frac{c(t)}{A(t)}$.

- The Euler equation can be derived via using $\tilde{c}(t) = \frac{c(t)}{A(t)}$

$$\begin{aligned}
\tilde{c}(t) =& \frac{c(t)}{A(t)} \\
\dot{\tilde{c}}(t) =& \frac{\dot{c}(t)A(t) - c(t)\dot{A}(t)}{[A(t)]^2} = \frac{\dot{c}(t)}{A(t)} - \underbrace{\frac{\dot{A}(t)}{A(t)}}_{=g}\underbrace{\frac{c(t)}{A(t)}}_{=\tilde{c}(t)} \\
\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} =& \frac{\dot{c}(t)}{c(t)} - g = \frac{f'\left(k(t)\right) - \delta - \rho}{\theta} - g \\
\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} =& \frac{f'\left(k(t)\right) - \delta - \rho - \theta g}{\theta}
\end{aligned}$$

Competitive equilibrium in the neo-classical growth model:

DEFINITION

For an economy with population growth n and an initial capital stock $K(0)$, an equilibrium path is a sequence $\{K(t), L(t), Y(t), c(t), w(t), R(t)\}_{t=0}^{\infty}$ such that

- $k(t)$ satisfies $\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t)$.
- $y(t) = f(k(t))$.
- $R(t) = f'(k(t))$ and $w(t) = f(k(t)) - f'(k(t))\,k(t)$.
- The representative household maximizes utility subject to the law of motion of assets and the No-Ponzi Game condition taking factor prices and initial $K(0)$ as given.

Detour: Effects of Assumptions

Remember the result from the generic set-up (important: the state variable is now $a(t)$ and discounting in the utility function is $\rho - n$ instead of just ρ)

$$\frac{\dot{c}(t)}{c(t)} = -\frac{1}{\varepsilon_u(c(t))}\left(\rho - n - g_a(c(t), a(t)) + \frac{\dot{g}_c(c(t), a(t))}{g_c(c(t), a(t))}\right)$$

Given the utility function of the neo-classical growth model

$$u(c(t)) = \frac{c(t)^{1-\theta} - 1}{1-\theta}$$
$$\varepsilon_u(c(t)) \equiv -\frac{u''(c(t))\, c(t)}{u'(c(t))} = \theta$$

this simplifies to

$$\frac{\dot{c}(t)}{c(t)} = \frac{g_a(c(t), a(t)) - \rho + n - \frac{\dot{g}_c(c(t),a(t))}{g_c(c(t),a(t))}}{\theta}\ .$$

Detour: Effects of Assumptions

The law of motion is given by

$$\dot{a}(t) = g\left(c(t), a(t)\right) = \left(r(t) - n\right) a(t) + w(t) - c(t)$$

Therefore the marginal effects on the law of motion are given by

$$\begin{aligned} g_a\left(c(t), a(t)\right) =& r(t) - n \\ g_c\left(c(t), a(t)\right) =& -1 \ , \end{aligned}$$

where the linearity of the law of motion in consumption implies

$$\dot{g}_c\left(c(t), a(t)\right) = 0 \ ,$$

and therefore

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\theta} = \frac{f'\left(k(t)\right) - \delta - \rho}{\theta} \ .$$

Detour: Effects of Assumptions

Summary:

- CIES utility ensures that there is no income (or time) dependent effect of the curvature of the utility function.
- Linear cost of the control variable in terms of the state variable ensures that there are no higher order effects of the control variable.
- Benefits vs. costs of accumulating the state variable

$$f'(k(t)) - \delta - \rho$$

  Concave benefit as $f''(k(t)) < 0$ and linear cost $-(\delta + \rho)$. This ensures the uniqueness of a steady state. (Why? What could happen if benefits are linear?)

- Inada conditions

$$\lim_{k(t)\to\infty} f'(k(t)) = 0 \;\text{ and }\; \lim_{k(t)\to 0} f'(k(t)) = \infty$$

  This ensures the existence of the steady state. (Why?)

Optimal Growth

Is the decentralised market equilibrium equivalent to that of a benevolent social planner?

The social planner maximizes the sum of all utilities (which is equivalent to maximising the utility of the representative household) subject to the resource constraint

$$\dot{k}(t) = f\left(k(t)\right) - (n + \delta)k(t) - c(t)\ .$$

The current-value Hamiltonian of the social planner problem is

$$\hat{\mathcal{H}}\left(c(t), k(t), \mu(t)\right) = \frac{c(t)^{1-\theta} - 1}{1-\theta} + \mu(t)\left[f\left(k(t)\right) - (n+\delta)k(t) - c(t)\right]$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial \hat{\mathcal{H}}\left(c(t), k(t), \mu(t)\right)}{\partial c(t)} =& c(t)^{-\theta} - \mu(t) = 0 \\
\frac{\partial \hat{\mathcal{H}}\left(c(t), k(t), \mu(t)\right)}{\partial k(t)} =& \mu(t)\left[f'\left(k(t)\right) - (n+\delta)\right] = (\rho - n)\mu(t) - \dot{\mu}(t) \\
\frac{\partial \hat{\mathcal{H}}\left(c(t), k(t), \mu(t)\right)}{\partial \mu(t)} =& f\left(k(t)\right) - (n+\delta)k(t) - c(t) = \dot{k}(t)
\end{aligned}$$

Re-writing optimal consumption yields

$$-\theta \frac{\dot{c}(t)}{c(t)} = \frac{\dot{\mu}(t)}{\mu(t)} \; .$$

Re-writing the asset pricing equation yields

$$\frac{\dot{\mu}(t)}{\mu(t)} = \rho - f'\left(k(t)\right) + \delta \; .$$

Combining the two gives the Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'\left(k(t)\right) - \delta - \rho}{\theta} \; .$$

Transversality condition

$$\lim_{t \to \infty} e^{-(\rho - n)t} \mu(t) k(t) = 0 \; .$$

Proposition

In a neo-classical growth model with

1. *perfect competition*
2. *neo-classical production*
3. *strictly increasing, strictly concave, twice continuously differentiable utility*
4. *discounting*

the decentralized equilibrium is Pareto optimal and coincides with the optimal growth path chosen by the social planner.

Detour: Leibniz's Rule

Theorem (Leibniz's Rule)

Let $f(s,t)$ be continuous in s on $[a,b]$ and suppose the functions $a(t)$ and $b(t)$ are differentiable with $-\infty < a(t), b(t) < \infty$, then

$$\frac{d}{d\,t}\int_{a(t)}^{b(t)} f(s,t)\,ds = f\,(b(t),t)\cdot\frac{d}{d\,t}b(t) - f\,(a(t),t)\cdot\frac{d}{d\,t}a(t) + \int_{a(t)}^{b(t)}\frac{\partial}{\partial t}f(s,t)\,ds$$

As a result

$$\int r(t)\,dt = \int_0^t r(s)\,ds + a_1$$

since

$$\begin{aligned}\frac{d}{d\,t}\int r(t)\,dt =& \frac{d}{d\,t}\int_0^t r(s)\,ds + \frac{d}{d\,t}a_1\\ r(t) =& r(t)\cdot\frac{d}{d\,t}t - r(0)\cdot\frac{d}{d\,t}0 + \int_0^t \frac{\partial}{\partial t}r(s)\,ds = r(t)\end{aligned}$$

Transversality vs. No Ponzi Game:

The first-order condition with respect to consumption and the asset-pricing equation are given by

$$
\begin{aligned}
\mu(t) =& c(t)^{-\theta} \\
\frac{\dot{\mu}(t)}{\mu(t)} =& -\left(r(t) - \rho\right)
\end{aligned}
$$

Integrating the asset-pricing equation and inserting the first-order condition for consumption at $t = 0$ yields

$$
\begin{aligned}
\mu(t) =& \mu(0) \cdot e^{-\int_0^t r(s) - \rho \, ds} \\
\mu(t) =& c(0)^{-\theta} \cdot e^{-\int_0^t r(s) - \rho \, ds}
\end{aligned}
$$

A Closer Look at Equilibrium Growth

Inserting the result for $\mu(t)$ into the transversality condition gives

$$
\begin{aligned}
\lim_{t\to\infty} e^{-(\rho-n)t}\mu(t)a(t) &= 0\\
\lim_{t\to\infty} e^{-(\rho-n)t}c(0)^{-\theta}e^{-\int_0^t r(s)-\rho\, ds}a(t) &= 0\\
\lim_{t\to\infty} c(0)^{-\theta}e^{-\int_0^t r(s)-n\, ds}a(t) &= 0\\
\lim_{t\to\infty} e^{-\int_0^t r(s)-n\, ds}a(t) &= 0
\end{aligned}
$$

where the last line follows from the fact that $c(0)$ is always greater than zero for any $k(0) > 0$.

As a result the first-order conditions in combination with the transversality condition imply that the No-Ponzi Game condition holds with equality.

A Closer Look at Equilibrium Growth

Consumption path:

The Euler equation is given by

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\theta} \ .$$

Integrating gives

$$c(t) = c(0) \cdot e^{\frac{1}{\theta} \int_0^t r(s) - \rho \, ds} \ .$$

Define $\bar{r}(t)$ as the average interest rate between dates 0 and t

$$\bar{r}(t) = \frac{1}{t} \int_0^t r(s) \, ds \ ,$$

which allows re-writing $c(t)$ as

$$c(t) = c(0) \cdot e^{\frac{\bar{r}(t) - \rho}{\theta} t} \ .$$

Initial consumption:

The lifetime budget constraint (that adheres to the No-Ponzi Game condition) is given by

$$\int_0^\infty c(t)e^{-\int_0^t r(s)-n\,ds}\,dt = \int_0^\infty w(t)e^{-\int_0^t r(s)-n\,ds}\,dt + a(0)\,.$$

Again using $\bar{r}(t) = \frac{1}{t}\int_0^t r(s)\,ds$ and inserting the result for $c(t)$ gives

$$\begin{aligned}
\int_0^\infty c(t)e^{-(\bar{r}(t)-n)t}\,dt &= \int_0^\infty w(t)e^{-(\bar{r}(t)-n)t}\,dt + a(0)\\
\int_0^\infty c(0)e^{\frac{\bar{r}(t)-\rho}{\theta}t}e^{-(\bar{r}(t)-n)t}\,dt &= \int_0^\infty w(t)e^{-(\bar{r}(t)-n)t}\,dt + a(0)\\
\int_0^\infty c(0)e^{\left(\frac{(1-\theta)\bar{r}(t)-\rho}{\theta}+n\right)t}\,dt &= \int_0^\infty w(t)e^{-(\bar{r}(t)-n)t}\,dt + a(0)
\end{aligned}$$

Solving for initial consumption yields

$$c(0) = \left[\int_0^\infty e^{-\left(\frac{(1-\theta)\bar{r}(t)-\rho}{\theta} + n \right) t} \, dt \right] \cdot \left[\int_0^\infty w(t) e^{-(\bar{r}(t)-n)t} \, dt + a(0) \right] .$$

As a result the consumption path is given by

$$c(t) = \left[\int_0^\infty e^{-\left(\frac{(1-\theta)\bar{r}(t)-\rho}{\theta} + n \right) t} \, dt \right] \cdot \left[\int_0^\infty w(t) e^{-(\bar{r}(t)-n)t} \, dt + a(0) \right] \cdot e^{\frac{\bar{r}(t)-\rho}{\theta} t} .$$

Steady state: Euler equation

The Euler equation is given by

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\theta} \,.$$

A steady state implies $\dot{c}(t) = 0$, that is

$$r(t) - \rho = 0 \iff r(t) = \rho$$

Note that $r(t) = \rho > n$, which is needed for the No-Ponzi Game condition to hold (i.e. wealth not to explode). In equilibrium $r(t) = f'(k(t)) - \delta$, hence

$$f'(k(t)) = \rho + \delta \ \Rightarrow f'(k^{*}) = \rho + \delta$$

In a $k(t) - c(t)$ diagram this represents a numeric value.

Steady State and Transitional Dynamics

Dynamics:

The resting point is given by

$$f'(k^*) = \rho + \delta \,.$$

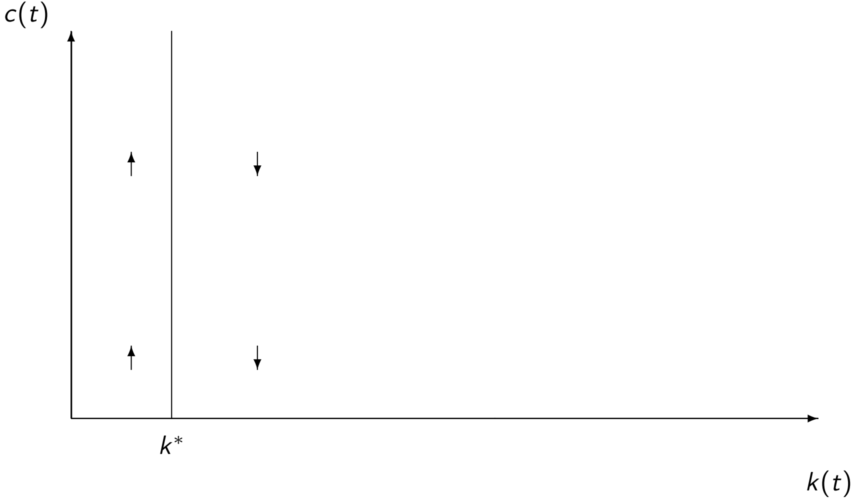
Subsequently the full system is given by

$$f'(k(t)) \begin{cases} < \rho + \delta & \text{if } k(t) > k^* \\ = \rho + \delta & \text{if } k(t) = k^* \\ > \rho + \delta & \text{if } k(t) < k^* \end{cases}$$

As this determines the direction of consumption growth it immediately follows that

$$\frac{\dot{c}(t)}{c(t)} \begin{cases} < 0 & \text{if } k(t) > k^* \\ = 0 & \text{if } k(t) = k^* \\ > 0 & \text{if } k(t) < k^* \end{cases}$$

Steady State and Transitional Dynamics



Steady state: capital accumulation

Equilibrium capital accumulation is given by

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t) \ .$$

A steady state implies $\dot{k}(t) = 0$, that is

$$f(k(t)) - (n+\delta)k(t) - c(t) = 0 \iff c(t) = f(k(t)) - (n+\delta)k(t)$$

As an immediate result steady-state consumption is given by

$$c^* = f(k^*) - (n+\delta)k^* \ .$$

However, the condition

$$c(t) = f(k(t)) - (n+\delta)k(t) \ ,$$

is a function (and not just a value) in the $k(t) - c(t)$ space.

Characterisation of steady-state accumulation:

Technically all points on the

$$c(t) = f\,(k(t)) - (n+\delta)k(t)\ ,$$

locus are steady states in $\dot{k}(t) = 0$. For comparative dynamics exercises it is useful to know the shape and characteristics of the locus. If $k(t) = 0$, then

$$c(t) = f(0) - (n+\delta)\cdot 0 = 0\ .$$

Likewise if $c(t) = 0$, then

$$\frac{f\,(k(t))}{k(t)} = n+\delta\ .$$

The derivative with respect to $k(t)$ is given by

$$\frac{d\,c(t)}{d\,k(t)} = f'(k(t)) - (n+\delta)\ ,$$

with

$$\frac{d^2\,c(t)}{d\,k(t)^2} = f''(k(t)) < 0\ .$$

As a result the slope of the $\dot{k} = 0$ locus is

$$\frac{d\,c(t)}{d\,k(t)} \begin{cases} < 0 & \text{if } f'(k(t)) < n+\delta \\ = 0 & \text{if } f'(k(t)) = n+\delta \\ > 0 & \text{if } f'(k(t)) > n+\delta \end{cases}$$

The condition

$$f'(k(t)) = n + \delta \, ,$$

implicitly defines the Golden-Rule capital stock as

$$f'\left(k^{GR}\right) = n + \delta \, .$$

Note that $f'(k^*) = \rho + \delta$ and thus

$$\rho > n \;\Rightarrow\; k^* < k^{GR}$$

Intuition: it is the sum of **discounted** utility from consumption that is maximized not consumption itself.

Lastly, note that $f(k(t)) - f'(k(t))\,k(t) > 0$ and $\frac{f(k(t))}{k(t)} = n + \delta$ at $c(t) = 0$, thus

$$\frac{f(k(t))}{k(t)} > f'(k(t)) \;\Rightarrow k^{MAX} > k^{GR}$$

As a result the $\dot{k}(t) = 0$ locus starts at $k(t) = c(t) = 0$ with slope

$$\frac{d\,c(t)}{d\,k(t)} \begin{cases} < 0 & \text{if } k(t) > k^{GR} \\ = 0 & \text{if } k(t) = k^{GR} \\ > 0 & \text{if } k(t) < k^{GR} \end{cases}$$

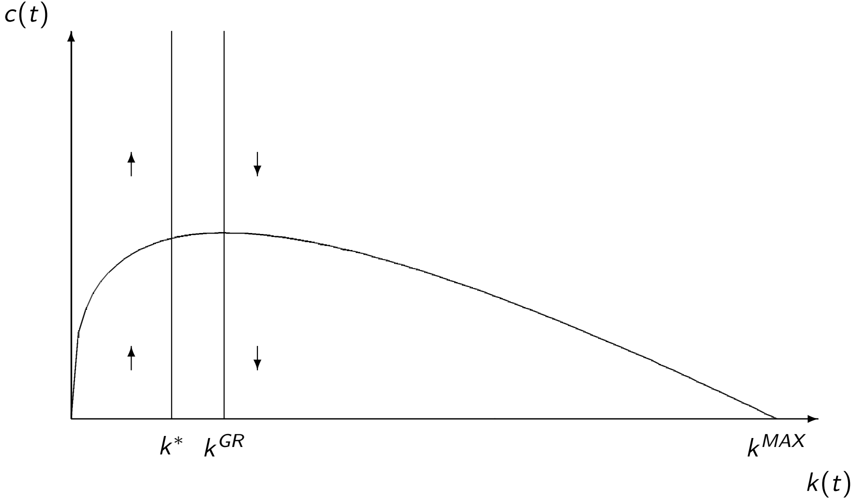
It reaches a maximum at

$$k(t) = k^{GR} \ ,$$

implicitly defined by $f'(k(t)) = n + \delta$ and intersects the $k(t)$-axis again at

$$k(t) = k^{MAX} > k^{GR} \ ,$$

implicitly defined by $\frac{f(k(t))}{k(t)} = n + \delta$.



$c(t)$
k^*
k^{GR}
k^{MAX}
$k(t)$

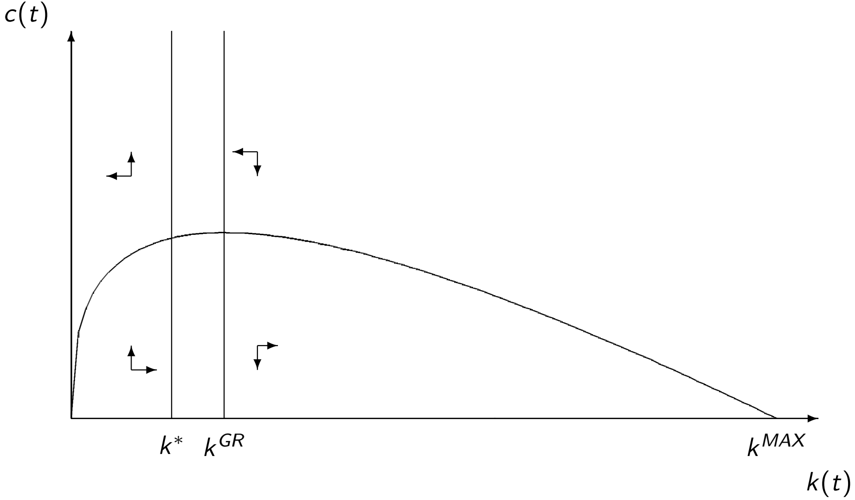
Dynamics:

The steady-state locus is given by

$$c(t) = f\,(k(t)) - (n+\delta)k(t)\;,$$

Subsequently the full system is

$$\dot{k}(t)\begin{cases} <0 & \text{if } c(t) > f\,(k(t)) - (n+\delta)k(t) \\ =0 & \text{if } c(t) = f\,(k(t)) - (n+\delta)k(t) \\ >0 & \text{if } c(t) < f\,(k(t)) - (n+\delta)k(t) \end{cases}$$


$c(t)$
k^*
k^{GR}
k^{MAX}
$k(t)$

Saddle path:

- The only path compatible with the transversality condition is called the saddle path of the maximization problem.
- As already derived, the transversality condition determines the initial value of consumption (as a function of the initial capital stock).
- The saddle path always passes through the steady state $\dot{k}(t) = \dot{c}(t) = 0$.
- The optimal consumption path is time consistent.

Saddle path stability:

A first-order Taylor approximation of

$$\dot{c}(t) = \frac{f'(k(t)) - \delta - \rho}{\theta} c(t)$$
$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t)$$

around the steady state $\{k^*, c^*\}$ is given by (note that $f'(k^*) = \rho + \delta$)

$$\dot{c}(t) \approx \underbrace{\frac{f'(k^*) - \delta - \rho}{\theta}}_{=0} c^* + \frac{f''(k^*) c^*}{\theta} (k(t) - k^*) + \underbrace{\frac{f'(k^*) - \delta - \rho}{\theta}}_{=0} (c(t) - c^*)$$
$$\dot{c}(t) \approx \frac{f''(k^*) c^*}{\theta} (k(t) - k^*)$$

$$\dot{k}(t) \approx \underbrace{f(k^*) - (n+\delta)k^* - c^*}_{=0} + \left(f'(k^*) - \delta - n\right)(k(t) - k^*) - 1 \cdot (c(t) - c^*)$$
$$\dot{k}(t) \approx (\rho - n)(k(t) - k^*) - (c(t) - c^*)$$

Combining both equations and writing them in matrix notation yields

$$\begin{bmatrix} \dot{c}(t) \\ \dot{k}(t) \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} k(t) - k^* \\ c(t) - c^* \end{bmatrix}$$

with

$$\boldsymbol{A} = \begin{bmatrix} \frac{f''(k^*)c^*}{\theta} & 0 \\ \rho - n & -1 \end{bmatrix}$$

The solution to this system is (locally) stable and unique if and only if both eigenvalues of $\boldsymbol{A}$ are negative.

Detour: Eigenvalues

Consider a $n \times n$ matrix $\boldsymbol{A}$. It is non-singular or invertible if its determinant is not zero. This implies that the only solution to

$$\boldsymbol{A}\boldsymbol{v} = 0 \,,$$

is $\boldsymbol{v}$ being the zero vector. Furthermore it implies that $\boldsymbol{A}^{-1}$ exists and that

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I} \,,$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix. On the other hand, if the determinant of $\boldsymbol{A}$ is zero there exists a non-zero solution to

$$\boldsymbol{A}\boldsymbol{v} = 0$$

and $\boldsymbol{A}$ is not invertible.

An eigenvalue is a transformation of $\boldsymbol{A}$ such that

$$\boldsymbol{A}\boldsymbol{v} = \xi \boldsymbol{v} \ ,$$

where $\boldsymbol{A}$ is a $n \times n$ matrix, $\boldsymbol{v}$ a (non-zero) $n \times 1$ vector and ξ a scalar.

This can be re-arranged such that

$$(\boldsymbol{A} - \xi \boldsymbol{I})\boldsymbol{v} = 0 \ ,$$

This has a non-zero solution for $\boldsymbol{v}$ if and only if the determinant of $\boldsymbol{A} - \xi \boldsymbol{I}$ is zero, that is

$$|\boldsymbol{A} - \xi \boldsymbol{I}| = 0 \ ,$$

where the left-hand side is a polynomial of degree n (of the order of matrix $\boldsymbol{A}$). The solution to this polynomial gives the eigenvalues of the problem (which may have complex parts). Given the solution to the eigenvalues, $\boldsymbol{v}$ is the eigenvector that satisfies

$$(\boldsymbol{A} - \xi \boldsymbol{I})\boldsymbol{v} = 0 \ .$$

Eigenvalues and eigenvectors can be used for diagonalizing a non-diagonal square matrix. If $\boldsymbol{A}$ has n distinct real eigenvalues then

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D} \ ,$$

where $\boldsymbol{D}$ is a diagonal matrix with the distinct eigenvalues on the diagonal and $\boldsymbol{P}$ a matrix of the corresponding eigenvectors.

Application to systems of differential equations:

Given a system of differential equations

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{t}) \ ,$$

define $\boldsymbol{z}(\boldsymbol{t}) = \boldsymbol{P}^{-1}\boldsymbol{x}(\boldsymbol{t})$, then

$$\begin{aligned}
\dot{\boldsymbol{z}}(\boldsymbol{t}) =& \boldsymbol{P}^{-1}\dot{\boldsymbol{x}}(\boldsymbol{t}) \\
\dot{\boldsymbol{z}}(\boldsymbol{t}) =& \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{x}(\boldsymbol{t}) \\
\dot{\boldsymbol{z}}(\boldsymbol{t}) =& \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}\boldsymbol{z}(\boldsymbol{t}) \\
\dot{\boldsymbol{z}}(\boldsymbol{t}) =& \boldsymbol{D}\boldsymbol{z}(\boldsymbol{t})
\end{aligned}$$

This allows solving for all elements of $\dot{\boldsymbol{z}}(\boldsymbol{t})$ and hence $\dot{\boldsymbol{x}}(\boldsymbol{t})$ if the eigenvalues and eigenvectors are known. Additionally, this implies that the steady state of $\dot{\boldsymbol{z}}(\boldsymbol{t})$ and hence $\dot{\boldsymbol{x}}(\boldsymbol{t})$ is stable if and only if (the real parts of) all eigenvalues are negative. (Why?)

Applied to the growth problem

$$
\begin{aligned}
\boldsymbol{A}\boldsymbol{v} =& \xi \boldsymbol{v} \\
(\boldsymbol{A} - \xi \boldsymbol{I})\boldsymbol{v} =& 0 \\
\Rightarrow \quad |\boldsymbol{A} - \xi \boldsymbol{I}| =& 0
\end{aligned}
$$

where $|\boldsymbol{A} - \xi \boldsymbol{I}|$ is given by

$$
\left| \begin{bmatrix} \frac{f''\left(k^*\right)c^*}{\theta} & 0 \\ \rho - n & -1 \end{bmatrix} - \xi \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|
$$

$$
= \left| \begin{bmatrix} \frac{f''\left(k^*\right)c^*}{\theta} - \xi & 0 \\ \rho - n & -1 - \xi \end{bmatrix} \right|
$$

$$
\begin{aligned}
=& -\left(\frac{f''\left(k^*\right)c^*}{\theta} - \xi \right)(1 + \xi) - (\rho - n) \cdot 0 \\
=& \xi^2 + \left(1 - \frac{f''\left(k^*\right)c^*}{\theta} \right)\xi - \frac{f''\left(k^*\right)c^*}{\theta}
\end{aligned}
$$

The eigenvalues are given by the solution to

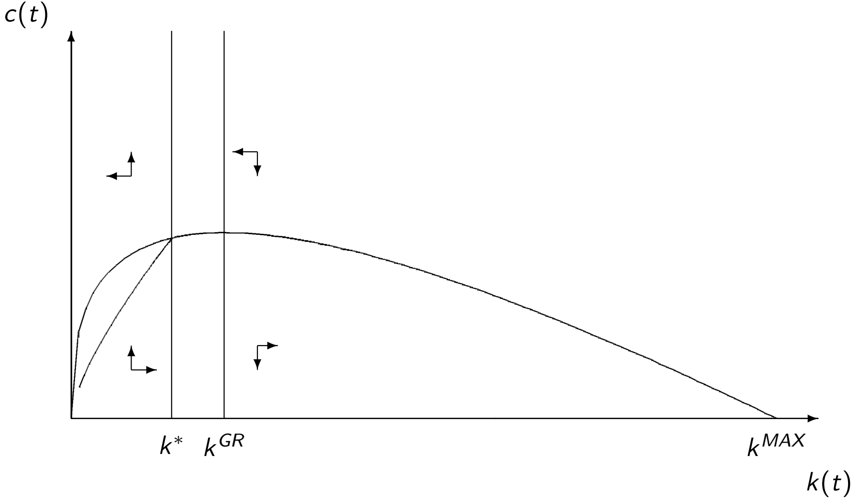
$$\xi_{1/2} = \frac{1}{2}\left\{-\left(1-\frac{f''(k^*)\,c^*}{\theta}\right) \pm \left[\left(1-\frac{f''(k^*)\,c^*}{\theta}\right)^2 - 4\left(-\frac{f''(k^*)\,c^*}{\theta}\right)\right]^{\frac{1}{2}}\right\}$$

The solutions are

$$\xi_1 = \frac{f''(k^*)\,c^*}{\theta}$$
$$\xi_2 = -1$$

As $f''(k^*) < 0$ and $\{c^*, \theta\} > 0$ both eigenvalues are smaller than zero and the solution to the system is (locally) stable and unique.

Steady State and Transitional Dynamics



The Role of Policy and Comparative Dynamics

- As savings choice is now endogenous the neo-classical growth model can be used for both policy evaluation and studying cross-country comparative development.
- Possible variations across countries could include
  - different elasticity of substitution: $\frac{1}{\theta}$
  - different rate of time preference: ρ
  - different rate of depreciation: δ
  - different rate of population growth: n
  - different rate of technology growth: g
  - different production function: $f(\cdot)$
- However, most of these are not appealing as:
  - data can be hard to find ($\frac{1}{\theta}$, ρ, δ).
  - exogeneity is disputable (n and g).
  - rationalizing differences between countries can be hard ($f(\cdot)$).
- Instead, try to evaluate the result of different policy measures.

Suppose that the capital gains are taxed at rate $\tau \in (0,1)$ such that

$$\begin{aligned}\tilde{R}(t) =& (1-\tau)R(t) \\ \tilde{r}(t) =& (1-\tau)R(t) - \delta\end{aligned}$$

and that the tax revenue is distributed back lump-sum to the household (note: this assumes no cost of taxation). As a result the household law of motion becomes

$$\dot{a}(t) = \left(\tilde{r}(t) - n\right) a(t) + w(t) + b(t) - c(t) \; ,$$

where $b(t)$ is the per capita transfer received by the household from taxation revenue (taken as given).

Solving the optimization problem yields the Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{\tilde{r}(t) - \rho}{\theta} .$$

In equilibrium this becomes

$$\frac{\dot{c}(t)}{c(t)} = \frac{(1-\tau)f'(k(t)) - \delta - \rho}{\theta} .$$

Solving for the steady state gives

$$f'(k^*_\tau) = \frac{\rho + \delta}{1 - \tau} .$$

The steady states without and with taxation are implicitly defined by

$$f'(k^*) = \rho + \delta \quad \text{and} \quad f'(k^*_\tau) = \frac{\rho + \delta}{1 - \tau}$$

As $\tau \in (0, 1)$ it follows that

$$\rho + \delta < \frac{\rho + \delta}{1 - \tau}$$

and hence

$$k^* > k^*_\tau \ .$$

As a result capital taxation leads to lower steady state capital (and thus output) per capita.

The Role of Policy and Comparative Dynamics

Discussion:

- Influencing the incentives for accumulation affects the growth path.
- Even without costs of taxation capital taxation decreases steady-state levels.
- However, this is driven by the assumptions of
  1. Fully competitive, frictionless (capital) markets.
  2. Representative household (i.e. no inequality).

Quantitative evaluation of policy:

Assume that the production function is Cobb-Douglas, that is

$$Y(t) = F\left(K(t), L(t)\right) = K(t)^{\alpha} L(t)^{1-\alpha} .$$

As a result the intensive form is given by

$$y(t) = f\left(k(t)\right) = k(t)^{\alpha} ,$$

with factor rents given by

$$R(t) = f'\left(k(t)\right) = \frac{\alpha}{k(t)^{1-\alpha}}$$

$$\Rightarrow \; r(t) = \frac{\alpha}{k(t)^{1-\alpha}} - \delta$$

$$w(t) = f\left(k(t)\right) - f'\left(k(t)\right) k(t) = (1-\alpha) k(t)^{\alpha}$$

Assume that accumulation of total assets is still given by

$$\dot{A}(t) = I(t) - \delta A(t) \ .$$

However, now there is taxation in investment implying that the price of assets is now $1 + \tau$ with $\tau \in (0, 1)$

$$(1 + \tau) I(t) + C(t) \le Y(t) \ \Rightarrow \ I(t) \le \frac{1}{1 + \tau} \left(Y(t) - C(t) \right) \ .$$

As a result the law of motion of assets per capita is given by

$$\dot{a}(t) = \left(\frac{1}{1 + \tau} R(t) - \delta - n \right) a(t) + \frac{1}{1 + \tau} \left(w(t) - c(t) \right) \ .$$

The Role of Policy and Comparative Dynamics

Maximization yields the Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{\frac{1}{1+\tau} R(t) - \delta - \rho}{\theta}$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{\frac{1}{1+\tau} \frac{\alpha}{k(t)^{1-\alpha}} - \delta - \rho}{\theta}$$

Solving for the steady state gives

$$\frac{\alpha}{(k_\tau^*)^{1-\alpha}} = (1+\tau)(\rho+\delta)$$

$$k_\tau^* = \left(\frac{\alpha}{(1+\tau)(\rho+\delta)}\right)^{\frac{1}{1-\alpha}}$$

Therefore countries with a higher relative price of investment have lower steady-state capital per capita.

As a result steady-state output per capita is given by

$$y_\tau^* = (k_\tau^*)^\alpha = \left(\frac{\alpha}{(1+\tau)(\rho+\delta)}\right)^{\frac{\alpha}{1-\alpha}} \quad .$$

Assume two countries that differ in their tax rate on investment (but are otherwise identical). Then the ratio of their steady-state outputs per capita is given by

$$\frac{y_{\tau_1}^*}{y_{\tau_2}^*} = \left(\frac{1+\tau_2}{1+\tau_1}\right)^{\frac{\alpha}{1-\alpha}} \quad .$$

Can the neoclassical growth model account for quantitatively large cross-country income differences?

Parameter selection:

- The parameter α represents the capital share in total production. Too see this note

$$\frac{f'(k(t))\,k(t)}{f(k(t))} = \frac{\frac{\alpha}{k(t)^{1-\alpha}}k(t)}{k(t)^{\alpha}} = \frac{\alpha k(t)^{\alpha}}{k(t)^{\alpha}} = \alpha\ .$$

- The plausible range of this parameter in the data is $\alpha \in (0.25, 0.4)$.
- Estimates of τ can be obtained from the data by noting that in a closed-economy neoclassical growth model the price of investment goods relative to consumption goods is $1+\tau$ (in order to get the same effect per unit of input the taxation costs have to be offset).
- The price of investment goods relative to consumption goods has considerable variation in the data depending on the countries observed.
- Assume a sevenfold difference (roughly equal to the maximum distance observed in the data) for exposition.

- Using these parameter values implies

$$\frac{y^*_{\tau_1}}{y^*_{\tau_2}} = 7^{\frac{0.33}{1-0.33}} \approx 2.61$$

- As a result, even large differences in the price of investment goods (taxation) cannot account for the large comparative development differences observed in the data.
- However, it is important to keep in mind that α not only represents the capital share in total output but indirectly also the share of production factors that are accumulated by individual decisions (population growth and/or technology growth are exogenous).
- If, for example, α represents both physical and human capital (education) than a more plausible size might be $\alpha = 0.66$ (0.33 as capital and 0.33 as human capital share) and the ratio becomes

$$\frac{y^*_{\tau_1}}{y^*_{\tau_2}} = 7^{\frac{0.66}{1-0.66}} \approx 43.70$$

Wrap-Up

- The one-sector, closed economy neo-classical growth model is (arguably) the most important model in macroeconomics.
- The neo-classical growth model endogenizes the savings decision as function of preferences, production technology and prices. In contrast to the Solow model this allows room for policy evaluation and comparison between equilibrium and optimal growth.
- One key reason the neo-classical growth model has remained relevant is its simplicity at the core and endless extendability.
- Extensions:
  1. Endogenous labor supply (utility from leisure $\rightarrow$ RBC models).
  2. Effects of government expenditure.
  3. Open economy (free capital mobility).
  4. Investment costs.
  5. Multiple sectors.

  $\vdots$
- Two black boxes of growth are still left: technology (endogenous growth) and population growth (unified growth).