# PS 4: Endogenous Growth Models

1

1.1 see earlier problems

1.2

Differences:

- linear returns to accumulation; no diminishing returns to capital
- no role for labor.

1.3

The aggregate production function is

$$Y(t) = A\,K(t)$$

The per capita production function is

$$y(t) = A\,k(t)$$

Therefore, the return to capital is given by

$$R(t) = \frac{\partial y(t)}{\partial k(t)} = A$$

As a result the return to labor is

$$w(t) = y(t) - R(t)\,k(t) = y(t) - Ak(t) = 0$$ # no labor in the model

Subsequently, the law of motion is given by

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t)$$

$$\dot{a}(t) = (A - \delta - n)a(t) - c(t)$$

Inserting the return to capital into the Euler equation yields

$$\frac{\dot{c}(t)}{c(t)} = \frac{A - \delta - \rho}{\theta}$$

Integrating both sides gives the general solution

$$\int \frac{\dot{c}(t)}{c(t)}\,dt = \int \frac{A-\delta-\rho}{\theta}\,dt$$

$$\ln c(t) + a_1 = \frac{A-\delta-\rho}{\theta}\,t + a_2$$

$$c(t) = A \cdot e^{\frac{A-\delta-\rho}{\theta}t}$$

Setting t=0 eliminates the constant of integration yielding the particular solution

$$c(0) = A \cdot e^{\frac{A-\delta-\rho}{\theta} \cdot 0} = A$$

$$c(t) = c(0) \cdot e^{\frac{A-\delta-\rho}{\theta} \cdot t}$$

how does the growth rate change with a percentage change in $c(t)$

↳ 59 in Solow

↳ speed of convergence

effective labor units have no real meaning

$\varepsilon_u(c(t))$ → how do we exchange consumption over time?

↳ $r(t)$ is the only factor that changes over time

↳ how does change in interest rate change consumption growth?

↳ related to curvature and risk aversion

↳ risk averse → afraid of the future

No-Ponzi Game condition

↳ whenever discounting is assumed

1.4 The simplified law of motion is given by

$$\dot{a}(t) = (A-\delta-n)a(t) - c(t)$$

In equilibrium, $a(t) = k(t)$, hence

$$\dot{k}(t) = (A-\delta-n)k(t) - c(t)$$

Inserting the result for consumption gives

$$\dot{k}(t) = (A-\delta-n)k(t) - c(0)\cdot e^{\frac{A-\delta-\rho}{\theta}t}$$

Rearranging and multiplying with the integration factor $e^{-(A-\delta-n)t}$ gives

$$\dot{k}(t)e^{-(A-\delta-n)t} - (A-\delta-n)k(t)e^{-(A-\delta-n)t} = -c(0)\cdot e^{\frac{A-\delta-\rho}{\theta}t}\cdot e^{-(A-\delta-n)t}$$

$$\dot{k}(t)e^{-(A-\delta-n)t} - (A-\delta-n)k(t)e^{-(A-\delta-n)t} = -c(0)\cdot e^{\frac{(1-\theta)(A-\delta)-\rho+\theta n}{\theta}t}$$

integrating both sides gives the general solution

$$\int \dot{k}(t)e^{-(A-\delta-n)t} - (A-\delta-n)k(t)e^{-(A-\delta-n)t}\,dt = -c(0)\int e^{\frac{(1-\theta)(A-\delta)-\rho+\theta n}{\theta}t}\,dt$$

$$k(t)e^{-(A-\delta-n)t} + a_1 = \frac{\theta c(0)}{(1-\theta)(A-\delta)-\rho+\theta n}\cdot e^{\frac{(1-\theta)(A-\delta)-\rho+\theta n}{\theta}t} + a_2$$

$$k(t)e^{-(A-\delta-n)t} = \frac{\theta c(0)}{\rho-(1-\theta)(A-\delta)-\theta n}\cdot e^{\frac{A-\delta-\rho}{\theta}t}e^{-(A-\delta-n)t} + a_3$$

$$k(t) = \frac{\theta c(0)}{\rho-(1-\theta)(A-\delta)-\theta n}\cdot e^{\frac{A-\delta-\rho}{\theta}t} + a_3 e^{(A-\delta-n)t}$$

Inserting the result into the transversality condition yields

$$\lim_{t\to\infty} e^{-(\rho-n)t}\mu(t)k(t) = 0$$

$$\lim_{t\to\infty} e^{-(\rho-n)t}c(t)^{-\theta}k(t) = 0$$

$$\lim_{t\to\infty} e^{-(\rho-n)t}c(0)^{-\theta}e^{-(A-\delta-\rho)t}k(t) = 0$$

$$\lim_{t\to\infty} c(0)^{-\theta}e^{-(A-\delta-n)t}k(t) = 0$$

As $c(0) > 0$ for any $k(0) > 0$ this simplifies to

$$\lim_{t\to\infty} e^{-(A-\delta-n)t}k(t) = 0$$

Inserting the result for $k(t)$ gives

$$\lim_{t\to\infty} e^{-(A-\delta-n)t}k(t) = 0$$

$$\lim_{t\to\infty} e^{-(A-\delta-n)t}\left[\frac{\theta c(0)}{\rho-(1-\theta)(A-\delta)-\theta n}\cdot e^{\frac{A-\delta-\rho}{\theta}t} + a_3 e^{(A-\delta-n)t}\right] = 0$$

$$\lim_{t\to\infty}\left[\frac{\theta c(0)}{\rho-(1-\theta)(A-\delta)-\theta n}\cdot e^{\frac{(1-\theta)(A-\delta)-\rho+\theta n}{\theta}t} + a_3\right] = 0$$

$$\lim_{t\to\infty}\left[\frac{\theta c(0)}{\rho-(1-\theta)(A-\delta)-\theta n}\cdot e^{-\frac{\rho-(1-\theta)(A-\delta)-\theta n}{\theta}t} + a_3\right] = 0$$

In order for the transversality condition to hold, two conditions have to be satisfied

$$\rho > (1-\theta)(A-\delta) + \theta n$$

$$a_3 = 0$$

where the first condition ensures the first term converges to zero while the second condition is the only constant of integration that can satisfy the transversality condition.

As a result the path of capital is given by

$$k(t) = \frac{\theta c(0)}{\rho-(1-\theta)(A-\delta)-\theta n} \cdot e^{\frac{A-\delta-\rho}{\theta}t}$$

# no transitional dynamics in AK model
↳ only one rate of growth (no slower or faster growth)

Evaluating at $t=0$ gives the result for $c(0)$

$$c(0) = \frac{\rho-(1-\theta)(A-\delta)-\theta n}{\theta} k(0)$$

Inserting back into the solution for the path of capital gives

$$k(t) = k(0) \cdot e^{\frac{A-\delta-\rho}{\theta}t}$$

For $c(0)$ to be positive the following has to hold

$$\rho > (1-\theta)(A-\delta) + \theta n$$

which is already required for the transversality condition to hold. In order to ensure positive growth the growth rate has to be positive, that is

$$A > \rho + \delta$$

As a result the following parametric restriction has to hold

$$A > \rho + \delta > (1-\theta)(A-\delta) + \theta n + \delta$$

② 

2.1 The return to capital in the consumption sector is given by

$$\frac{\partial C(t)}{\partial K_c(t)} = \alpha \cdot A_c \cdot \left(\frac{L(t)}{K_c(t)}\right)^{1-\alpha} = \alpha \cdot A_c \cdot \left(\frac{L}{(1-\kappa(t))K(t)}\right)^{1-\alpha}$$

The return to capital in the investment sector is given by

$$\frac{\partial I(t)}{\partial K_I(t)} = A_I .$$

As capital is free to move between sectors it will allocate such that prices are equalized

$$p_I(t) \cdot \frac{\partial I(t)}{\partial K_I(t)} = p_c(t) \cdot \frac{\partial C(t)}{\partial K_c(t)}$$

Inserting the marginal products, using the fact that $p_c(t) \equiv 1$ and rearranging gives

$$p_I(t) = \alpha \cdot \frac{A_c}{A_I} \cdot \left(\frac{L}{(1-\kappa(t))K(t)}\right)^{1-\alpha}$$

otherwise people would only invest in investment sector in the long run

# only way to ensure marginal products are the same is by introducing relative wages/prices

2.2 Inserting $r_c(t)$ in the Euler equation gives

$$\frac{\dot{c}(t)}{c(t)} = \frac{A_I - \delta + \frac{\dot{p}_I(t)}{p_I(t)} - \frac{\dot{p}_c(t)}{p_c(t)} - \rho}{\theta},$$

where $\frac{\dot{p}_c(t)}{p_c(t)} = 0$ as $p_c(t) \equiv 1$

Use the equilibrium condition $p_I(t) = \alpha \cdot \frac{A_c}{A_I} \cdot \left(\frac{L}{(1-x(t))K(t)}\right)^{1-\alpha}$ to derive

$$\ln p_I(t) = \left[\ln\left(\alpha \cdot \frac{A_c}{A_I}\right) + (1-\alpha)\ln L\right] - (1-\alpha)\left(\ln K(t) + \ln(1-x(t))\right)$$

$$\frac{\dot{p}_I(t)}{p_I(t)} = -(1-\alpha) \cdot \left(\frac{\dot{K}(t)}{K(t)} - \frac{\dot{x}(t)}{1-x(t)}\right)$$

Using the equilibrium condition $\frac{\dot{c}(t)}{c(t)} = \alpha \cdot \left[\frac{\dot{K}(t)}{K(t)} - \frac{\dot{x}(t)}{1-x(t)}\right]$ gives

$$\frac{\dot{p}_I(t)}{p_I(t)} = -\frac{1-\alpha}{\alpha} \cdot \frac{\dot{c}(t)}{c(t)}.$$

Substituting the result back into the Euler equation, rearranging gives

$$\frac{\dot{c}(t)}{c(t)} = \frac{A_I - \delta - \frac{1-\alpha}{\alpha} \cdot \frac{\dot{c}(t)}{c(t)} - \rho}{\theta}$$

$$\left(\frac{1-\alpha}{\alpha} + \theta\right) \cdot \frac{\dot{c}(t)}{c(t)} = A_I - \delta - \rho$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{\alpha(A_I - \delta - \rho)}{1 - \alpha(1-\theta)}$$

Solving the differential equation gives

$$\int \frac{\dot{c}(t)}{c(t)}\, dt = \int \frac{\alpha(A_I - \delta - \rho)}{1-\alpha(1-\theta)}\, dt$$

$$c(t) = B \cdot e^{\frac{\alpha(A_I - \delta - \rho)}{1-\alpha(1-\theta)} t}$$

Evaluation at $t=0$ eliminates the constant of integration

$$c(0) = B$$

leading to the particular solution

$$c(t) = c(0) \cdot e^{\frac{\alpha(A_I - \delta - \rho)}{1-\alpha(1-\theta)} t}.$$

$\#$ it only depends on $A_I$: $r_c(t)$ looks like diminishing returns but investment sector enables linear returns
↳ as long as there is one factor that can be accumulated without diminishing returns you can grow constantly

2.3 Having $x(t) \equiv \bar{x}$ implies $\dot{x}(t) = 0$, therefore

$$\frac{\dot{c}(t)}{c(t)} = \alpha\left[\frac{\dot{K}(t)}{K(t)} - 0\right] = \alpha \cdot \frac{\dot{K}(t)}{K(t)}$$

Using the fact that $\frac{c(t)}{c(t)} = \frac{\alpha(A_I - \delta - \rho)}{1-\alpha(1-\theta)}$

$$\alpha \frac{\dot{K}(t)}{K(t)} = \frac{\alpha(A_I - \delta - \rho)}{1-\alpha(1-\theta)}$$

$$\frac{\dot{K}(t)}{K(t)} = \frac{A_I - \delta - \rho}{1-\alpha(1-\theta)}$$

and solving the resulting differential equation yields

$$\int \frac{\dot{K}(t)}{K(t)}\, dt = \int \frac{A_I - \delta - \rho}{1-\alpha(1-\theta)}\, dt$$

$$K(t) = D \cdot e^{\frac{A_I - \delta - \rho}{1-\alpha(1-\theta)} t}$$

Evaluation at $t=0$ eliminates the constant of integration

$$K(0) = D \cdot e^{\dots \cdot 0} = D$$

leading to the particular solution

$$K(t) = K(0) \cdot e^{\frac{A_I - \delta - \rho}{1-\alpha(1-\theta)} t}.$$

# growth rate of capital and consumption are not the same

The law of motion for capital is given by

$$\dot{K}(t) = I(t) - \delta K(t) = A_I K_I(t) - \delta K(t) = A_I \bar{x} K(t) - \delta K(t)$$

Rearranging gives

$$\frac{\dot{K}(t)}{K(t)} = A_I \bar{x} - \delta.$$

Substituting & solving for $\bar{x}$ yields

$$A_I \bar{x} - \delta = \frac{A_I - \delta - \rho}{1-\alpha(1-\theta)}$$

$$\bar{x} = \frac{1}{A_I} \cdot \left( \frac{A_I - \delta - \rho}{1-\alpha(1-\theta)} + \delta \right)$$

4

-> see code

2.4 In order to have a balanced growth path consumption and capital must grow at a constant rate, that is

$$\frac{\dot{c}(t)}{c(t)} = \frac{\alpha(A_I - \delta - \rho)}{1-\alpha(1-\theta)} > 0$$

$$\frac{\dot{K}(t)}{K(t)} = \frac{A_I - \delta - \rho}{1-\alpha(1-\theta)} > 0$$

Note that $1-\alpha(1-\theta) > 0 \quad \forall \alpha \in (0,1),\ \theta > 0$. Therefore a necessary and sufficient condition is

$$A_I - \delta - \rho > 0 \iff A_I > \delta + \rho.$$

The economy admits a 'degenerate' steady state iff $A_I = \delta + \rho$

$$\frac{\dot{c}(t)}{c(t)} = 0$$

$$c(t) = c(0) = c^*$$

$$\frac{\dot{K}(t)}{K(t)} = 0$$

$$K(t) = K(0) = K^*$$

Exams: how can we get y by choosing x?

↳ interpretation questions

↳ no derivation required, no computation

↳ graphical interpretation!

final key conditions, assumptions!

↳ interpretation of conditions