Macroeconomics - Growth

Growth and Overlapping Generations

Gerrit Meyerheim

Economics Department
LMU München

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Infinite-Horizon vs. Overlapping Generations

- The Solow model is useful for describing the long-run growth paths of an economy (likewise the Ramsey/neo-classical growth model).
- Infinitely-lived household could be understood as dynasties; savings are not just individual savings, but also include bequests that are handed over to subsequent generations.
- Overlapping Generations (OLG) models are different
  1. Saving over the life-cycle is very different from living forever (or passing on bequests).
  2. Explicit treatment of labor supply (and basis for other decisions like education, fertility, social security...).
  3. Different **welfare implications**, comparison of welfare across generations important for evaluation of policy that has differential effects on different generations (e.g. pension reforms).

Infinite-Horizon vs. Overlapping Generations

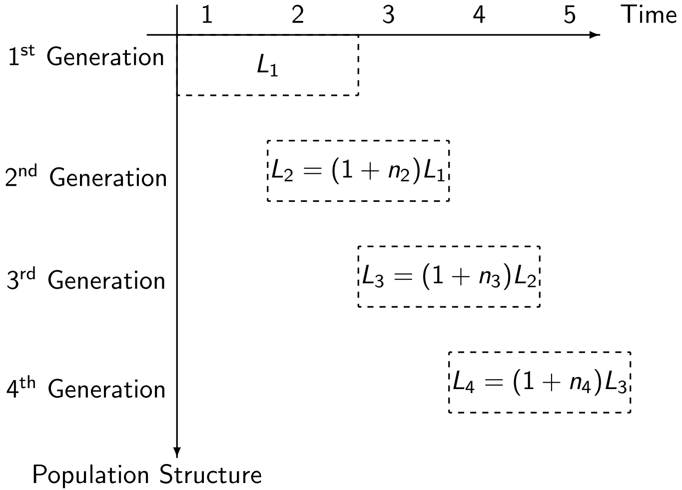
- The OLG model is useful for gaining insights to dynamic optimization in a simple way.
- The OLG model is not just a variation of the Solow (or later the neo-classical) model. Both its steady-state and efficiency implications are qualitatively different.
- Decisions made by the old generation lead to (pecuniary) externalities that affect the young but are not internalized by the old.

OLG Model: main concepts.

- Aggregation of a simple life-cycle model.
- Individuals maximize their utility, taking prices (wage, interest rate) as given.
- As in Solow, prices are determined in equilibrium.
- **New:** population structure with heterogeneity.

A Basic OLG Framework

Basic structure (two periods):



A Basic OLG Framework

The idea goes back to Allais (1947; "Économie et intérêt"), seminal work was provided by Samuelson (1958) and Diamond (1965).

- Time is discrete and runs from $t = 0, 1, ...\infty$.
- A generation lives for two periods.
- The generation born at time t is alive in periods t and $t + 1$.
- At time $t + 1$ the next generation is born. Therefore at any point in time two generations co-exist.
- We assume that death is deterministic and occurs at the end of the second period of life (alternatives: stochastic death, conditional survival curves, etc.).
- No heterogeneity within a cohort. Therefore each generation is represented by a single individual.

A Basic OLG Framework

Households and population structure:

- N_t individuals are born in period t. They supply one unit of labor inelastically to the labor market.
- Population grows at rate n, that is

$$N_{t+1} = (1+n)N_t = (1+n)^{t+1} \cdot N_0 \ .$$

  Note: in general, n does not need to be constant (see chapter 5).
- Individuals only work in the first period of life. As a result

$$L_t = N_t \ \forall t \ .$$

- Individuals receive the market wage w_t when young, which is allocated between consumption and savings.
- Key difference to Solow: saving is decided optimally.
- In the second period individuals do not work and only consume their savings (including any interest).

A Basic OLG Framework

Preferences:

- Total life-time utility of an individual born at date t

$$U(c_t^1, c_{t+1}^2) = u(c_t^1) + \beta u(c_{t+1}^2) \ .$$

- Instantaneous utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is
  1. Strictly increasing and concave.
  2. Twice differentiable with $u'(\cdot) > 0$ and $u''(\cdot) < 0$ for $c \in \mathbb{R}_+$
- c_t^1 is consumption of an individual when young (at date t).
- c_{t+1}^2 is consumption of an individual when old (at date $t + 1$).
- $\beta \in (0, 1)$ is the discount factor (measure of impatience).

Optimization:

- The life-cycle is structured as follows

$$\begin{aligned} t&: \text{ earn wage } w_t \text{, decide on consumption } c_t^1 \text{ and savings } s_t. \\ t+1&: \text{ earn and consume returns on savings } (1+r_{t+1})s_t \end{aligned}$$

- The objective of the household is maximizing lifetime utility

$$\max_{c_t^1, c_{t+1}^2} \; u(c_t^1) + \beta u(c_{t+1}^2)$$

subject to first- and second-period budget constraints

$$\begin{aligned} c_t^1 + s_t &\leq w_t \\ c_{t+1}^2 &\leq (1+r_{t+1})s_t \end{aligned}$$

- Since $u(\cdot)$ is strictly increasing, both constraints hold as equalities.

A Basic OLG Framework

- Substituting $s_t = w_t - c_t^1$ into the second-period budget constraint gives the lifetime budget constraint as

$$w_t = c_t^1 + \frac{c_{t+1}^2}{1 + r_{t+1}} .$$

- The Lagrangian for this problem now is

$$\mathcal{L} = u(c_t^1) + \beta u(c_{t+1}^2) + \lambda \left[w_t - c_t^1 + \frac{c_{t+1}^2}{1 + r_{t+1}} \right] .$$

- The first order conditions are

$$\begin{aligned} u'(c_t^1) =& \lambda \\ \beta u'(c_{t+1}^2) =& \frac{\lambda}{1 + r_{t+1}} \end{aligned}$$

$$\Rightarrow \; u'(c_t^1) = \beta(1 + r_{t+1}) u'(c_{t+1}^2)$$

Optimal consumption:

- The optimality condition for consumption is

$$u'(c_t^1) = \beta(1 + r_{t+1})u'(c_{t+1}^2) \ .$$

- This is the **Euler equation**, that is at the heart of a lot of macroeconomic theory. It describes the basic trade-off of:

  consumption today vs. consumption tomorrow

- Illustration: increase s_t by one marginal unit, then
  - Marginal cost: less consumption today, utility loss in terms of current period utility $u'(c_t^1)$.
  - Marginal benefit: more resources $(1 + r_{t+1}) > 1$ and therefore consumption tomorrow, but at discounted utility value $\beta u'(c_{t+1}^2)$.
- Euler equation: in the optimum the costs and benefits are equalized.
- As the optimization problem is strictly concave the Euler equation is sufficient to characterize an optimal consumption path (c_t^1, c_{t+1}^2)

Savings:

- Combine the Euler equation with the budget constraints to obtain

$$u'(w_t - s_t) = \beta(1 + r_{t+1})u'\left((1 + r_{t+1})s_t\right) \ .$$

- This is an implicit function that determines individual savings as

$$s_t = s(w_t, 1 + r_{t+1}) \ .$$

- Savings are increasing in the wage (why?), the effect of the return to capital is a priori not clear (why?).

An Example with CIES Utility

- Same set-up as before. The lifetime utility of a young individual is given by

$$\frac{\left(c_t^1\right)^{1-\theta}-1}{1-\theta}+\beta\cdot\frac{\left(c_{t+1}^2\right)^{1-\theta}-1}{1-\theta}\quad \text{with } \theta>0 \ .$$

- The budget constraints in both periods are

$$\begin{aligned} c_t^1 &= w_t - s_t \\ c_{t+1}^2 &= (1+r_{t+1})s_t \end{aligned}$$

An Example with CIES Utility

- The lifetime budget constraint as

$$w_t = c_t^1 + \frac{c_{t+1}^2}{1 + r_{t+1}} .$$

- The Lagrangian is

$$\mathcal{L} = \frac{\left(c_t^1\right)^{1-\theta} - 1}{1-\theta} + \beta \cdot \frac{\left(c_{t+1}^2\right)^{1-\theta} - 1}{1-\theta} + \lambda \left[w_t - c_t^1 + \frac{c_{t+1}^2}{1 + r_{t+1}} \right] .$$

- The first order conditions are

$$\begin{aligned} \left(c_t^1\right)^{-\theta} =& \lambda \\ \beta \cdot \left(c_{t+1}^2\right)^{-\theta} =& \frac{1}{1 + r_{t+1}} \cdot \lambda \\ \Rightarrow \frac{c_{t+1}^2}{c_t^1} =& \left[\beta(1 + r_{t+1})\right]^{\frac{1}{\theta}} \end{aligned}$$

An Example with CIES Utility

- Inserting the result back into the second-period budget constraint gives

$$\underbrace{\left[\beta(1+r_{t+1})\right]^{\frac{1}{\theta}} \cdot c_t^1}_{=c_{t+1}^2} = (1+r_{t+1})\underbrace{\left(w_t - c_t^1\right)}_{=s_t}$$

$$c_t^1 = \frac{w_t}{1+\beta^{\frac{1}{\theta}}(1+r_{t+1})^{\frac{1-\theta}{\theta}}}$$

- This allows solving for the remaining c_{t+1}^2 and s_t

$$c_{t+1}^2 = \left[\beta(1+r_{t+1})\right]^{\frac{1}{\theta}} \cdot c_t^1 = \frac{(1+r_{t+1})w_t}{1+\beta^{-\frac{1}{\theta}}(1+r_{t+1})^{-\frac{1-\theta}{\theta}}}$$

$$s_t = \frac{c_{t+1}^2}{1+r_{t+1}} = \frac{w_t}{1+\beta^{-\frac{1}{\theta}}(1+r_{t+1})^{-\frac{1-\theta}{\theta}}} < w_t$$

An Example with CIES Utility

Comparative statics:

- Effect of wages on savings

$$\frac{\partial s_t}{\partial w_t} = \frac{1}{1+\beta^{-\frac{1}{\theta}}(1+r_{t+1})^{-\frac{1-\theta}{\theta}}} \in (0,1) \, .$$

- Effect of the return on capital on savings

$$\begin{aligned}\frac{\partial s_t}{\partial(1+r_{t+1})} =& \frac{-\left[\beta^{-\frac{1}{\theta}}\left(-\frac{1-\theta}{\theta}\right)(1+r_{t+1})^{-\frac{1-\theta}{\theta}-1}\right] w_t}{\left[1+\beta^{-\frac{1}{\theta}}(1+r_{t+1})^{-\frac{1-\theta}{\theta}}\right]^2} \\ =& \frac{1-\theta}{\theta}\frac{s_t\left[\beta(1+r_{t+1})\right]^{-\frac{1}{\theta}}}{1+\beta^{-\frac{1}{\theta}}(1+r_{t+1})^{-\frac{1-\theta}{\theta}}} \gtreqless 0\end{aligned}$$

An Example with CIES Utility

- The sign of the effect depends on θ.

$$\frac{\partial s_t}{\partial(1+r_{t+1})} = \begin{cases} >0 & \text{if } \theta < 1 \\ =0 & \text{if } \theta = 1 \\ <0 & \text{if } \theta > 1 \end{cases} \qquad \frac{c^2_{t+1}}{c^1_t} = [\beta(1+r_{t+1})]^{\frac{1}{\theta}}$$

- There are two effects of an increase in $(1+r_{t+1})$: income effect & substitution effect.
  - Substitution effect: young-age consumption becomes more expensive relative to old-age consumption therefore reducing c^1_t; effect dominates when $\theta < 1$.
  - Income effect: same amount of saving delivers more old-age income therefore increasing c^1_t; effect dominates when $\theta > 1$.
  - Income and substitution cancel each other for $\theta = 1$ (log preferences).

Production and Aggregate Dynamics

Production:

- Production is the same as in the Solow model

$$
\begin{aligned}
Y_t =& F(K_t, L_t) \\
y_t \equiv \frac{Y_t}{L_t} =& F\left(\frac{K_t}{L_t},\ 1\right) = f(k_t)
\end{aligned}
$$

  where $k_t \equiv \frac{K_t}{L_t}$ is capital stock per capita.

- $F(\cdot)$ satisfies the assumptions of a neoclassical production function.
- Factor prices are determined on competitive markets

$$
\begin{aligned}
R_t =& f'(k_t) \\
r_t =& R_t - \delta \\
w_t =& f(k_t) - f'(k_t)k_t
\end{aligned}
$$

Aggregate variables:

- Total savings in the economy are equal to

$$S_t = s_t N_t \ .$$

- Aggregate capital stock evolves according to

$$K_{t+1} = S_t + (1-\delta)K_t = s_t N_t + (1-\delta)K_t \ ,$$

where $\delta \in (0,1)$ is the rate of depreciation.

- Remember: population grows at rate n

$$N_{t+1} = (1+n)N_t \ .$$

Definition (Equilibrium Path)

A competitive equilibrium in the OLG model is a sequence of aggregate capital stocks, household consumption, and factor prices

$$\left\{K_t, c_t^1, c_t^2, R_t, w_t\right\}_{t=0}^{\infty}$$

such that

- the factor price sequence $\{R_t, w_t\}_{t=0}^{\infty}$ is given by

$$R_t = f'(k_t) \qquad w_t = f(k_t) - f'(k_t)k_t$$

- individual consumption decisions $\left\{c_t^1, c_{t+1}^2\right\}_{t=0}^{\infty}$ are given by

$$u'(c_t^1) = \beta(1 + r_{t+1})u'(c_{t+1}^2) \qquad s_t = s(w_t, 1 + r_{t+1})$$

- and the aggregate capital stock $\{K_t\}_{t=0}^{\infty}$ evolves according to

$$K_{t+1} = s_t N_t + (1 - \delta)K_t$$

Dynamics in the OLG Framework

General case:

- The evolution of the total capital stock is given by

$$K_{t+1} = s_t N_t + (1-\delta)K_t \ .$$

- Divide both sides by $N_{t+1} = (1+n)N_t$ to get the evolution of capital per capita

$$\begin{aligned} k_{t+1} =& \frac{1}{1+n}\left[s_t + (1-\delta)k_t\right] \\ =& \frac{1}{1+n}\left[s(w_t, 1+r_{t+1}) + (1-\delta)k_t\right] \end{aligned}$$

- A steady state would be characterized by $k_{t+1} = k_t$. However, no explicit characterization is possible, depending on the production function and preferences all the following can be equilibrium outcomes
  - A unique, stable steady state.
  - Poverty trap: k_{t+1} intersects three times with the $k_{t+1} = k_t$ line with two stable and one instable steady state.
  - Multiple equilibria: k_{t+1} intersects three times with the $k_{t+1} = k_t$ line, but there is a region for which k_{t+1} is not uniquely defined.

Dynamics in the OLG Framework

- Cobb-Douglas production, CIES utility, full depreciation (i.e. $\delta = 1$).
- Cobb-Douglas production

$$\begin{aligned} R_t =& \frac{\alpha}{k_t^{1-\alpha}} \\ r_t =& R_t - \delta = R_t - 1 \\ \Rightarrow 1 + r_t =& R_t \\ w_t =& (1-\alpha) k_t^{\alpha} \end{aligned}$$

- CIES utility

$$s_t = \frac{w_t}{1 + \beta^{-\frac{1}{\theta}} (1 + r_{t+1})^{-\frac{1-\theta}{\theta}}} = \frac{(1-\alpha) k_t^{\alpha}}{1 + \beta^{-\frac{1}{\theta}} \left(\frac{\alpha}{k_{t+1}^{1-\alpha}} \right)^{-\frac{1-\theta}{\theta}}} .$$

- Full depreciation

$$k_{t+1} = \frac{1}{1+n} \left[s_t + (1-\delta) k_t \right] = \frac{s_t}{1+n} = \frac{(1-\alpha) k_t^{\alpha}}{(1+n) \left[1 + \beta^{-\frac{1}{\theta}} \left(\frac{\alpha}{k_{t+1}^{1-\alpha}} \right)^{-\frac{1-\theta}{\theta}} \right]} .$$

Dynamics in the OLG Framework

- A steady state is a point such that $k_{t+1} = k_t = k^*$

$$k^* = \frac{(1-\alpha)(k^*)^\alpha}{(1+n)\left[1+\beta^{-\frac{1}{\theta}}\left(\frac{\alpha}{(k^*)^{1-\alpha}}\right)^{-\frac{1-\theta}{\theta}}\right]}$$

$$\frac{1-\alpha}{(k^*)^{1-\alpha}} = (1+n)\left[1+\beta^{-\frac{1}{\theta}}\left(\frac{\alpha}{(k^*)^{1-\alpha}}\right)^{-\frac{1-\theta}{\theta}}\right]$$

- An obvious steady state is given by $k_{t+1} = k_t = 0$, but that is not economically interesting. Does another exist?

Dynamics in the OLG Framework

- If $k^* \to 0$, then

$$\frac{1-\alpha}{(k^*)^{1-\alpha}} \to \infty$$

$$(1+n)\left[1+\beta^{-\frac{1}{\theta}}\left(\frac{\alpha}{(k^*)^{1-\alpha}}\right)^{-\frac{1-\theta}{\theta}}\right] \to \begin{cases} 1+n<\infty & \text{if } \theta<1 \\ \frac{(1+n)(1+\beta)}{\beta}<\infty & \text{if } \theta=1 \\ \lim_{k^*\to 0} \mathcal{G}(k^*) < \lim_{k^*\to 0} \frac{1-\alpha}{(k^*)^{1-\alpha}} & \text{if } \theta>1 \end{cases}$$

where $\mathcal{G}(k^*)$

$$\mathcal{G}(k^*) = \frac{(1+n)\beta^{-\frac{1}{\theta}}\alpha^{-\frac{1-\theta}{\theta}}}{(k^*)^{-\frac{(1-\alpha)(1-\theta)}{\theta}}}\ .$$

- If $k^* \to \infty$, then

$$\frac{1-\alpha}{(k^*)^{1-\alpha}} \to 0$$

$$(1+n)\left[1+\beta^{-\frac{1}{\theta}}\left(\frac{\alpha}{(k^*)^{1-\alpha}}\right)^{-\frac{1-\theta}{\theta}}\right] \to \begin{cases} \infty>0 & \text{if } \theta<1 \\ \frac{(1+n)(1+\beta)}{\beta}>0 & \text{if } \theta=1 \\ 1+n>0 & \text{if } \theta>1 \end{cases}$$

- The derivatives of both sides are given by

$$-\frac{1-\alpha}{(k^*)^{2-\alpha}} < 0$$

$$\frac{(1+n)(1-\alpha)(1-\theta)}{\alpha^{\frac{1-\theta}{\theta}}\beta^{\frac{1}{\theta}}\theta}\frac{1}{(k^*)^{\frac{1-\alpha}{\theta}-(2-\alpha)}} = \begin{cases} >0 & \text{if } \theta < 1 \\ =0 & \text{if } \theta = 1 \\ <0 & \text{if } \theta > 1 \end{cases}$$

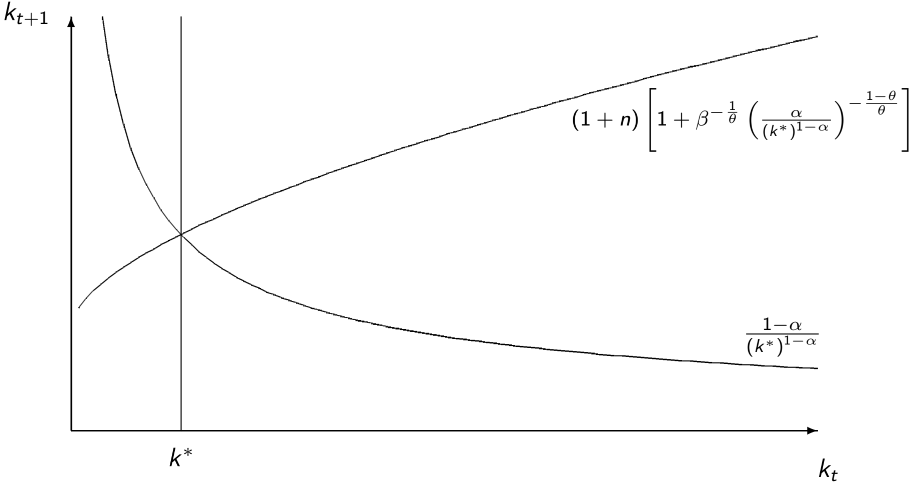
- Note that for $k^* > 0$ the right-hand side is monotonically increasing (decreasing) for $\theta < 1$ ($\theta > 1$).
- As a result there exists a unique, stable steady state for all $\{k_0, \theta\} > 0$ where the result follows from the Intermediate Value Theorem.

Detour: Intermediate Value Theorem

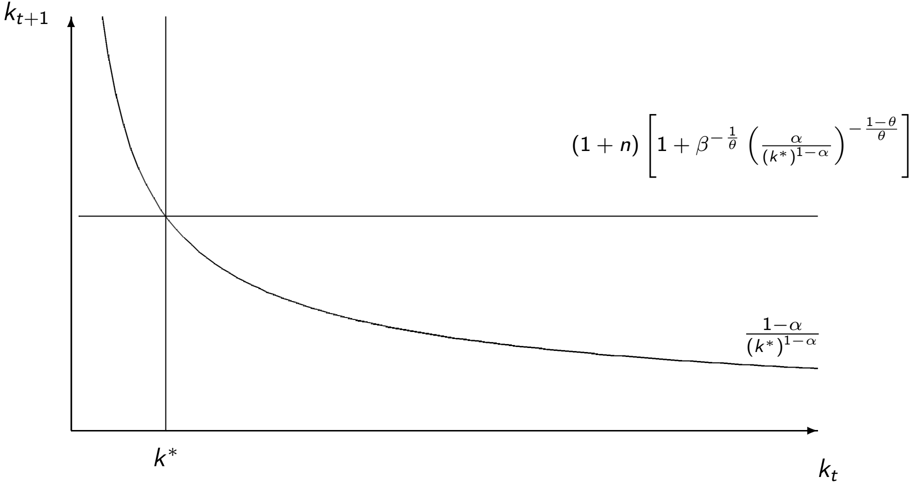
Theorem (Intermediate Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose that $f(a) \neq f(b)$. Then for c intermediate between $f(a)$ and $f(b)$ (e.g., $c \in (f(a), f(b))$ if $f(a) < f(b)$), there exists a $x^* \in (a, b)$ such that $f(x^*) = c$.

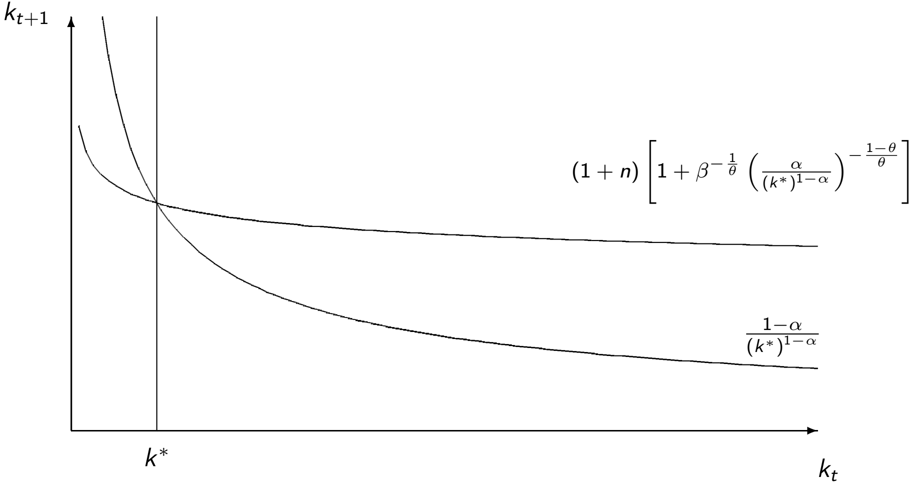
Intermediate Value Theorem: $(\theta < 1)$



Intermediate Value Theorem: ($\theta = 1$)



Intermediate Value Theorem: $(\theta > 1)$



Graphical representation:

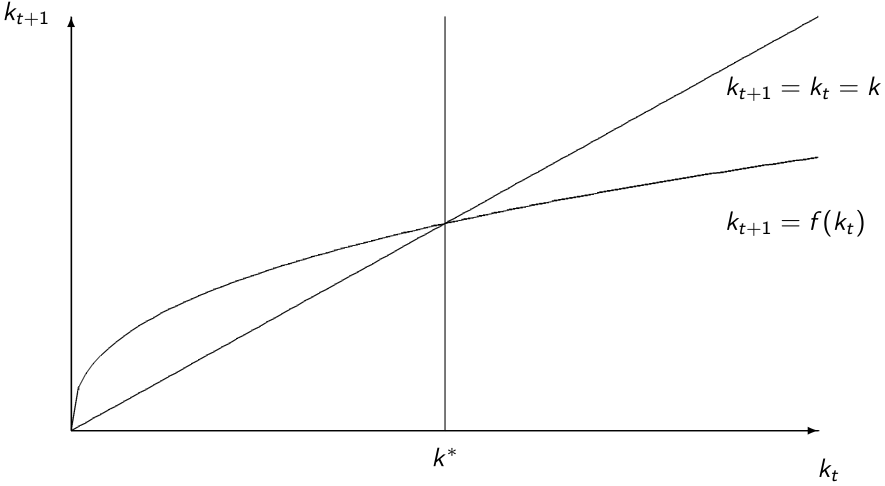
- Representation of capital accumulation in a k_{t+1} – k_t diagram: k_t is on the horizontal axis k_{t+1} on the vertical axis.
- The steady state lies on the $k_{t+1} = k_t$ locus (45° line from the origin).
- Capital accumulation is given by

$$k_{t+1} = \frac{(1-\alpha)k_t^{\alpha}}{(1+n)\left[1+\beta^{-\frac{1}{\theta}}\left(\frac{\alpha}{k_{t+1}^{1-\alpha}}\right)^{-\frac{1-\theta}{\theta}}\right]}\ .$$

with $k_{t+1} = 0$ for $k_t = 0$ and

$$\frac{\partial k_{t+1}}{\partial k_t} > 0 \quad \frac{\partial^2 k_{t+1}}{\partial k_t^2} < 0$$

- The intersection of k_{t+1} with the $k_{t+1} = k_t$ locus defines the steady state.


k_{t+1}
$k_{t+1} = k_t = k$
$k_{t+1} = f(k_t)$
k^*
k_t

Dynamic (In-)Efficiency

- How to evaluate the welfare properties of an OLG economy?
- An economy is *dynamically efficient* if no Pareto improvements are possible.
- The steady state in an OLG economy is not necessarily the best possible outcome (why?).

Social planner:

- In the OLG model the social planner allocation $\neq$ competitive equilibrium allocation.
- This is due to a missing market: the young generation cannot contract with the old as lifetime is finite.
- This leads to a pecuniary externality:
  1. The old generation receives a return on capital determined by their savings decisions.
  2. However, the young generation faces a wage **determined by the savings decision made by the old**.
- Planner objective: maximize the social welfare function, which is the weighted average of all generations' utilities:

$$\sum_{t=0}^{\infty} \zeta_t \left(\underbrace{u(c_t^1) + \beta u(c_{t+1}^2)}_{\text{utility of generation born in } t} \right)$$

- ζ_t: utility weight of generation t with $\sum_{t=0}^{\infty} \zeta_t < \infty$.

Resource constraint:

- The social planner maximizes subject to the total resource constraint

$$F(K_t, L_t) = c_t^1 N_t + c_t^2 N_{t-1} + K_{t+1} \ .$$

- Dividing by N_t yields

$$f(k_t) = c_t^1 + \frac{c_t^2}{1+n} + (1+n)k_{t+1} \ .$$

Dynamic (In-)Efficiency

Optimization:

- The Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty} \zeta_t \left(u(c_t^1) + \beta u(c_{t+1}^2) + \lambda_t \left(f(k_t) - c_t^1 - \frac{c_t^2}{1+n} - (1+n)k_{t+1} \right) \right)$$

- The social planner chooses: $c_t^1, c_{t+1}^2, k_{t+1}$.
- The first order conditions are

$$\begin{aligned} u'(c_t^1) =& \lambda_t \\ \zeta_t \beta u'(c_{t+1}^2) =& \frac{\zeta_{t+1}\lambda_{t+1}}{1+n} \\ \zeta_{t+1}\lambda_{t+1} f'(k_{t+1}) =& \zeta_t \lambda_t (1+n) \end{aligned}$$

- Re-arranging the first order condition for k_{t+1} yields

$$\frac{\zeta_{t+1}\lambda_{t+1}}{(1+n)\zeta_t} = \frac{\lambda_t}{f'(k_{t+1})}$$

- Plugging the result in the first order condition for c^2_{t+1} and combining with the first order condition for c^1_t gives

$$u'(c^1_t) = \beta f'(k_{t+1}) u'(c^2_{t+1}) \ .$$

- This is identical to the competitive equilibrium (if $\delta = 1$).
- So what is the problem?

Dynamic (In-)Efficiency

Dynamic inefficiency:

- The steady state of the OLG economy implies

$$
\begin{aligned}
f(k^*) =& c_1^* + \frac{c_2^*}{1+n} + (1+n)k^* \\
f(k^*) - (1+n)k^* =& \underbrace{c_1^* + \frac{c_2^*}{1+n}}_{\equiv c^*}
\end{aligned}
$$

  where c^* is total (young & old) steady-state consumption.

- A necessary condition for maximum steady-state consumption is

$$
\frac{\partial c^*}{\partial k^*} = f'(k^*) - (1+n) = 0 \ .
$$

- As a result, the golden rule capital-labor ratio k^{GR} is given by

$$
\frac{\partial c^*}{\partial k^*} = 0 \qquad \Rightarrow \ f'(k^{GR}) = 1 + n \ .
$$

Dynamic (In-)Efficiency

- If $k^* > k^{GR}$, then $\frac{\partial c^*}{\partial k^*} < 0$ and the economy is dynamically inefficient (i.e. over-saving).
- For $\delta = 1$ this implies that

$$f'(k^*) = 1 + r^* < 1 + n \;\Rightarrow\; r^* < n \; .$$

- Intuition:
  - Suppose at date T we have that $k^* > k^{GR}$.
  - Now reduce the capital-labor ratio to $k^* - \Delta k \;\in (k_{GR}, k^*)$.
  - This leads to a direct increase in consumption (due to the decrease in saving)

$$\Delta c(T) = (1+n)\Delta k \; .$$

  - But also increases consumption for $t > T$

$$\Delta c(t) = -\left[f'(k^* - \Delta k) - (1+n)\right]\Delta k \; .$$

- Increase in (c^1, c^2) and utility of all generations, implying a Pareto improvement.

Dynamic (In-)Efficiency

- Despite competitive markets and absence of externalities, the competitive equilibrium in the OLG model is potentially inefficient.
- Reason: pecuniary externality between generations t and $t + 1$.
- When the capital returns $f'(k)$ fall below the "human" returns n, a social arbitrage opportunity emerges that the planner can exploit:
  - The planner prevents savings, and dictates that the young support the consumption of the old.
  - The unsaved capital is used to increase consumption of both generations.
  - Population growth creates more wealth than the savings of the old would if $r < n$.

Implications for social security:

- Social security can be used as a way of dealing with oversaving.
- Fully-funded vs. pay-as-you-go:
  - Fully-funded: pay contributions when young and receive them back (with interest) when old.
  - Pay-as-you-go: direct transfer from the young to the old each period.

Fully-funded:

$$\max_{c_t^1, c_{t+1}^2} \; u(c_t^1) + \beta u(c_{t+1}^2)$$

subject to

$$\begin{aligned} c_t^1 + s_t + d_t &= w_t \\ c_{t+1}^2 &= (1 + r_{t+1})(s_t + d_t) \end{aligned}$$

- d_t: individual contribution to social security.
- No change in fundamentals as long as d_t is not exogenously set to be larger than the equilibrium s_t without social security.
- Therefore the equilibrium with social security is either identical to the competitive equilibrium or features even higher accumulation.

Dynamic (In-)Efficiency

Pay-as-you-go:

$$\max_{c_t^1, c_{t+1}^2} \; u(c_t^1) + \beta u(c_{t+1}^2)$$

subject to

$$\begin{aligned} c_t^1 + s_t + d_t =& w_t \\ c_{t+1}^2 =& (1 + r_{t+1}) s_t + (1 + n) d_{t+1} \end{aligned}$$

- d_t: contribution to social security at time t.
- Now: pure transfer system – discourages savings and therefore reduces overaccumulation. As a result, it can lead to a Pareto improvement of a dynamically inefficient equilibrium.
- Sustainability here depends on $n \geq 0$ (similar to a pyramid scheme).

Wrap-Up

- Overlapping generations models offer a tractable alternative to infinite horizon, representative agent models.
- Equilibria in OLG models can have scope for Pareto improvements due to dynamic inefficiencies arising from pecuniary externalities.
- From a growth perspective: important not to over-emphasize dynamic inefficiency. A central question is why so many countries have so little capital.