Acemoglu

Transition to Clean Technology

... Technical Change

## (6) Closed-form solution to Neo-Classical

6.1 identical to former problems

6.2 A steady state in consumption implies

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta} = 0$$

Hence in a steady state it must hold

$$\frac{f'(k) - \delta - \rho}{\theta} = 0$$

$$\alpha k^{\alpha-1} = \rho + \delta$$

$$k^{1-\alpha} = \frac{\alpha}{\rho+\delta}$$

$$k^* = \left[\frac{\alpha}{\rho+\delta}\right]^{\frac{1}{1-\alpha}}$$

Combining the law of motion ~~yields~~ and perfect competition yields

$$\dot{k}(t) = f'(k(t)) \cdot k(t) - \delta k(t) + w(t) - c(t)$$

$$\dot{k}(t) = f'(k(t)) \cdot k(t) - \delta k(t) + \left[f(k(t)) - f'(k(t)) \cdot k(t)\right] - c(t)$$

$$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t)$$

A steady state implies

$$0 = f(k) - \delta k - c$$

$$c = f(k) - \delta k$$

$$c = (k^*)^\alpha - \delta k^*$$

$$c^* = \left[\frac{\alpha}{\rho+\delta}\right]^{\frac{\alpha}{1-\alpha}} - \delta\left[\frac{\alpha}{\rho+\delta}\right]^{\frac{1}{1-\alpha}}$$

6.3 The law of motion for capital per capita is given by

$$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t)$$

$$\dot{k}(t) = k(t)^\alpha - \delta k(t) - c(t)$$

The law of motion for output per capita is given by

$$\dot{y}(t) = \alpha k(t)^{\alpha-1} \dot{k}(t) = \alpha \dot{k}(t) \frac{y(t)}{k(t)}$$ # chain rule applied here

The capital-output ratio is defined as

$$z(t) = \frac{k(t)}{y(t)},$$

with its law of motion given by

$$\dot{z}(t) = \frac{\dot{k}(t) y(t) - k(t)\dot{y}(t)}{[y(t)]^2}$$ # quotient rule

$$\dot{z}(t) = \frac{\dot{k}(t)}{y(t)} - \frac{\dot{y}(t)}{y(t)} \cdot \frac{k(t)}{y(t)} = \frac{\dot{k}(t)}{y(t)} - \frac{\alpha \dot{k}(t)}{y(t)} = \frac{(1-\alpha)\dot{k}(t)}{y(t)}$$

$$\dot{z}(t) = \frac{(1-\alpha)\left[k(t)^\alpha - \delta k(t) - c(t)\right]}{k(t)^\alpha} = (1-\alpha)\left\{1 - \delta k(t)^{1-\alpha} - \frac{c(t)}{k(t)}\frac{k(t)}{k(t)^\alpha}\right\}$$

$$\dot z(t) = (1-\alpha)\{1-[\delta + x(t)]z(t)\}$$

The consumption-capital ratio is defined as

$$x(t) = \frac{c(t)}{k(t)}$$

with its growth rate given by

$$\ln x(t) = \ln c(t) - \ln k(t)$$

$$\frac{\dot x(t)}{x(t)} = \frac{\dot c(t)}{c(t)} - \frac{\dot k(t)}{k(t)}$$

$$\frac{\dot x(t)}{x(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta} - \frac{f(k(t)) - \delta k(t) - c(t)}{k(t)} = \frac{\alpha k(t)^{\alpha-1} - \delta - \rho}{\theta} - k(t)^{\alpha-1} + \delta + x(t)$$

$$\frac{\dot x(t)}{x(t)} = \frac{1}{z(t)}\left(\frac{\alpha}{\theta} - 1\right) + \delta\left(1 - \frac{1}{\theta}\right) - \frac{\rho}{\theta} + x(t)$$

↳ if $\alpha = \theta$, we avoid non-linearities (in $z(t)$)

↳ $\frac{\dot x(t)}{x(t)}$ does, thus, not depend on $z(t)$

↳ only on constants and $x(t)$ itself

↳ this is then a closed-form solution

6.4

With $\theta = \alpha$ the growth rate of the capital-consumption ratio becomes

$$\frac{\dot x(t)}{x(t)} = \frac{1}{z(t)}\left(\frac{\alpha}{\theta} - 1\right) + \delta\left(1 - \frac{1}{\theta}\right) - \frac{\rho}{\theta} + x(t)$$

$$\frac{\dot x(t)}{x(t)} = \delta\left(1 - \frac{1}{\alpha}\right) - \frac{\rho}{\alpha} + x(t)$$

$$\frac{\dot x(t)}{x(t)} = \frac{\rho + \delta(1-\alpha)}{\alpha} + x(t)$$

≠ logistic differential equation → not exam material

Rewriting this as pure differential equation yields

$$\dot x(t) = -x(t)\left(\frac{\rho + \delta(1-\alpha)}{\alpha} - x(t)\right)$$

Separating the time derivative using the fact that $\dot x(t) = \frac{dx(t)}{dt}$ gives

$$dt = \frac{1}{-x(t)\left(\frac{\rho+\delta(1-\alpha)}{\alpha} - x(t)\right)}\, dx(t)$$

The denominator of the RHS needs to be separated such that

$$\frac{1}{-x(t)\left(\frac{\rho+\delta(1-\alpha)}{\alpha} - x(t)\right)} = \frac{A}{x(t)} + \frac{B}{\frac{\rho+\delta(1-\alpha)}{\alpha} - x(t)},$$

where $A$ and $B$ are constants. Re-arranging gives

$$1 = -A\cdot\left(\frac{\rho+\delta(1-\alpha)}{\alpha} - x(t)\right) - Bx(t)$$

$$1 = -\frac{\rho+\delta(1-\alpha)}{\alpha}A + (A-B)x(t)$$

A solution is given by $A = B$ and $A = -\frac{\alpha}{\rho+\delta(1-\alpha)}$

As a result the denominator can be transformed to

$$\frac{1}{-x(t)\left(\frac{\rho+\delta(1-\alpha)}{\alpha} - x(t)\right)} = -\frac{\alpha}{\rho+\delta(1-\alpha)}\frac{1}{x(t)} - \frac{\alpha}{\rho+\delta(1-\alpha)}\cdot\frac{1}{\frac{\rho+\delta(1-\alpha)}{\alpha} - x(t)}$$

The differential equation is now given by

$$dt = \left(-\frac{\alpha}{\rho+\delta(1-\alpha)}\frac{1}{x(t)} - \frac{\alpha}{\rho+\delta(1-\alpha)}\frac{1}{\frac{\rho+\delta(1-\alpha)}{\alpha} - x(t)}\right)dx(t)$$

$$\frac{\rho+\delta(1-\alpha)}{\alpha}dt = \left(-\frac{1}{x(t)} - \frac{1}{\frac{\rho+\delta(1-\alpha)}{\alpha} - x(t)}\right)dx(t)$$

Integrating both sides gives

$$\int \frac{\rho+\delta(1-\alpha)}{\alpha}dt = \int\left(-\frac{1}{x(t)} - \frac{1}{\frac{\rho+\delta(1-\alpha)}{\alpha} - x(t)}\right)dx(t)$$

$$\frac{\rho+\delta(1-\alpha)}{\alpha}t + a_1 = -\ln x(t) + a_2 + \ln\left[\frac{\rho+\delta(1-\alpha)}{\alpha} - x(t)\right] + a_3$$

$$\ln x(t) - \ln\left[\frac{\rho+\delta(1-\alpha)}{\alpha} - x(t)\right] = -\frac{\rho+\delta(1-\alpha)}{\alpha}t + a_4$$

$$\ln\left[\frac{x(t)}{\frac{\rho+\delta(1-\alpha)}{\alpha} - x(t)}\right] = \frac{\rho+\delta(1-\alpha)}{\alpha}t + a_4$$

$$\frac{x(t)}{\frac{\rho+\delta(1-\alpha)}{\alpha} - x(t)} = Ae^{-\frac{\rho+\delta(1-\alpha)}{\alpha}t}$$

$$\left[1 + Ae^{-\frac{\rho+\delta(1-\alpha)}{\alpha}t}\right]x(t) = \frac{\rho+\delta(1-\alpha)}{\alpha}Ae^{-\frac{\rho+\delta(1-\alpha)}{\alpha}t}$$

$$x(t) = \frac{\frac{\rho+\delta(1-\alpha)}{\alpha}Ae^{-\frac{\rho+\delta(1-\alpha)}{\alpha}t}}{1 + Ae^{-\frac{\rho+\delta(1-\alpha)}{\alpha}t}}$$

$$x(t) = \frac{\frac{\rho+\delta(1-\alpha)}{\alpha}}{1 + \tilde{A}e^{\frac{\rho+\delta(1-\alpha)}{\alpha}t}}$$

Re-write the solution for $x(t) = \frac{c(t)}{k(t)}$ in terms of capital per capita

$$k(t) = \frac{1 + \tilde{A}e^{\frac{\rho+\delta(1-\alpha)}{\alpha}t}}{\frac{\rho+\delta(1-\alpha)}{\alpha}}c(t)$$

Remember the FOC w.r.t. consumption

$$c(t)^{-\theta} = \mu(t)$$

Combining the results in the transversality condition yields

$$\lim_{t\to\infty} e^{-\rho t}\mu(t)k(t) = 0$$

$$\lim_{t\to\infty} e^{-\rho t}c(t)^{1-\theta}\left(\frac{1 + \tilde{A}e^{\frac{\rho+\delta(1-\alpha)}{\alpha}t}}{\frac{\rho+\delta(1-\alpha)}{\alpha}}\right) = 0$$

$$\lim_{t\to\infty} e^{-\rho t}c(t)^{1-\theta}\left(\frac{1 + \tilde{A}e^{\frac{\rho+\delta(1-\alpha)}{\alpha}t}}{\frac{\rho+\delta(1-\alpha)}{\alpha}}\right) = 0$$

$$\lim_{t\to\infty} c(t)^{1-\theta}\left(\frac{e^{-\rho t} + \tilde{A}e^{\frac{(\rho+\delta)(1-\alpha)}{\alpha}t}}{\frac{\rho+\sigma(1-\alpha)}{\alpha}}\right) = 0$$

$$\lim_{t\to\infty}\left(\frac{e^{-\rho t}+\tilde{A}e^{\frac{(\rho+\delta)(1-\alpha)}{\alpha}t}}{\frac{\rho+\delta(1-\alpha)}{\alpha}}\right)=0$$

Note that this holds iff $\tilde{A}=0$

Substituting the result into $x(t)$ gives

$$x(t)=\frac{\frac{\rho+\delta(1-\alpha)}{\alpha}}{1+0\cdot e^{\frac{\rho+\delta(1-\alpha)}{\alpha}t}}$$

$$x(t)=\frac{\rho+\delta(1-\alpha)}{\alpha}$$ # constant over time

$$\Rightarrow c(t)=\frac{\rho+\delta(1-\alpha)}{\alpha}\cdot k(t)$$

The law of motion of the capital-output ratio is given by

$$\dot{z}(t)=(1-\alpha)\{1-[\delta+x(t)]z(t)\}$$

$$z(t)=(1-\alpha)\left(1-\frac{\rho+\delta}{\alpha}z(t)\right)$$

Again separating the time derivatives using the fact that $\dot{z}(t)=\frac{dz(t)}{dt}$ gives

$$(1-\alpha)\,dt=\frac{1}{1-\frac{\rho+\delta}{\alpha}z(t)}\,dz(t)$$

Integrating both sides and simplifying yields the general solution

$$(1-\alpha)\,dt=\frac{1}{1-\frac{\rho+\delta}{\alpha}z(t)}\,dz(t)$$

$$(1-\alpha)t+a_1=-\frac{\alpha}{\rho+\delta}\ln\left[1-\frac{\rho+\delta}{\alpha}z(t)\right]+a_2$$

$$\frac{\alpha}{\rho+\delta}\ln\left[1-\frac{\rho+\delta}{\alpha}z(t)\right]=-(1-\alpha)t+a_3$$

$$\ln\left[1-\frac{\rho+\delta}{\alpha}z(t)\right]=-(1-\alpha)\frac{\rho+\delta}{\alpha}t+a_3$$

$$1-\frac{\rho+\delta}{\alpha}z(t)=Ae^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}$$

$$z(t)=\frac{\alpha}{\rho+\delta}\left(1-Ae^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}\right)$$

$$k(t)^{1-\alpha}=\left(\frac{\alpha}{\rho+\delta}-\frac{\alpha}{\rho+\delta}Ae^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}\right)$$

$$k(t)=\left(\frac{\alpha}{\rho+\delta}-\frac{\alpha}{\rho+\delta}Ae^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}\right)^{\frac{1}{1-\alpha}}$$

Setting $t=0$ and using the fact that $k^* = \left[\frac{\alpha}{\rho+\delta}\right]^{\frac{1}{1-\alpha}}$ gives the constant of integration as

$$k(0)^{1-\alpha} = k_0^{1-\alpha} = \frac{\alpha}{\rho+\delta} - \frac{\alpha}{\rho+\delta} A e^{-(1-\alpha)\frac{\rho+\delta}{\alpha}0}$$

$$A = 1 - \frac{k_0^{1-\alpha}}{\frac{\alpha}{\rho+\delta}}$$

$$A = \frac{1}{(k^*)^{1-\alpha}}\left[(k^*)^{1-\alpha} - k_0^{1-\alpha}\right]$$

Combining the general solution with the solution for the constant of integration and again using the fact that $k^* = \left[\frac{\alpha}{\rho+\delta}\right]^{\frac{1}{1-\alpha}}$ gives the particular solution

$$k(t) = \left(\frac{\alpha}{\rho+\delta} - \frac{\alpha}{\rho+\delta} A e^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}\right)^{\frac{1}{1-\alpha}}$$

$$k(t) = \left\{(k^*)^{1-\alpha} - \left[(k^*)^{1-\alpha} - k_0^{1-\alpha}\right] e^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}\right\}^{\frac{1}{1-\alpha}}$$

# exact illustration how Neo behaves as $t \to 0, t \to \infty$

# assume $k_0$ smaller $k^*$ or greater $k^*$ to understand dynamics

# rate of convergence contained in exponent

# no "escape" from the steady state as long as $k_0$ is not negative

6.5

As $c(t)$ is a linear function in $k(t)$, the saddle path must be linear. Taking the log of the solution for $k(t)$ yields

$$\ln k(t) = \frac{1}{1-\alpha} \ln\left\{(k^*)^{1-\alpha} - \left[(k^*)^{1-\alpha} - k_0^{1-\alpha}\right] e^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}\right\}$$

Differentiating w.r.t. time gives the growth rate of capital

$$\frac{\dot k(t)}{k(t)} = \frac{1}{1-\alpha} \cdot \frac{(1-\alpha)\frac{\rho+\delta}{\alpha}\left[(k^*)^{1-\alpha} - k_0^{1-\alpha}\right] e^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}}{(k^*)^{1-\alpha} - \left[(k^*)^{1-\alpha} - k_0^{1-\alpha}\right] e^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}}$$

$$\frac{\dot k(t)}{k(t)} = \frac{\frac{\rho+\delta}{\alpha}\left[(k^*)^{1-\alpha} - k_0^{1-\alpha}\right] e^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}}{(k^*)^{1-\alpha} - \left[(k^*)^{1-\alpha} - k_0^{1-\alpha}\right] e^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}}$$

Remember the solution for $k(t)$

$$k(t) = \left\{(k^*)^{1-\alpha} - \left[(k^*)^{1-\alpha} - k_0^{1-\alpha}\right] e^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}\right\}^{\frac{1}{1-\alpha}}$$

Re-arranging gives the following expressions

$$k(t)^{1-\alpha} = (k^*)^{1-\alpha} - \left[(k^*)^{1-\alpha} - k_0^{1-\alpha}\right] e^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}$$

$$(k^*)^{1-\alpha} - k(t)^{1-\alpha} = \left[(k^*)^{1-\alpha} - k_0^{1-\alpha}\right] e^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}$$

Substituting the expressions into the growth rate yields

$$\frac{\dot k(t)}{k(t)} = \frac{\frac{\rho+\delta}{\alpha}\left[(k^*)^{1-\alpha} - k(t)^{1-\alpha}\right]}{k(t)^{1-\alpha}}$$

$$\frac{\dot{k}(t)}{k(t)} = -\frac{\rho+\delta}{\alpha}\cdot\left[1-\left(\frac{k^*}{k(t)}\right)^{1-\alpha}\right]$$

The savings rate is defined as

$$c(t) = (1-s(t))f(k(t)) \Rightarrow s(t) = 1-\frac{c(t)}{f(k(t))}$$

Substituting for the solution for $c(t)$ yields

$$s(t) = 1-\frac{\rho+\delta(1-\alpha)}{\alpha}\cdot k(t)^{1-\alpha}$$

Substituting the result for $k(t)$ gives

$$s(t) = 1-\frac{\rho+\delta(1-\alpha)}{\alpha}\cdot k(t)^{1-\alpha}$$

$$s(t) = 1-\frac{\rho+\delta(1-\alpha)}{\alpha}\cdot\left\{(k^*)^{1-\alpha}-\left[(k^*)^{1-\alpha}-k_0^{1-\alpha}\right]e^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}\right\}$$

The steady-state savings rate is given by

$$c^* = (1-s^*)f(k^*)$$

$$s^* = 1-\frac{c^*}{f(k^*)}$$

$$s^* = 1-\frac{(k^*)^\alpha-\delta k^*}{(k^*)^\alpha}$$

$$s^* = \delta(k^*)^{1-\alpha}$$

Applying this result and using the fact that $k^* = \left[\frac{\alpha}{\rho+\delta}\right]^{\frac{1}{1-\alpha}}$ yields

$$s(t) = 1-\left(\frac{\rho+\delta}{\alpha}-\delta\right)\left\{(k^*)^{1-\alpha}-\left[(k^*)^{1-\alpha}-k_0^{1-\alpha}\right]e^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}\right\}$$

$$s(t) = 1-1+s^*+\left(\frac{\rho+\delta}{\alpha}-\delta\right)\left[(k^*)^{1-\alpha}-k_0^{1-\alpha}\right]e^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}$$

$$s(t) = s^*+\left(\frac{\rho+\delta}{\alpha}-\delta\right)\left[(k^*)^{1-\alpha}-k_0^{1-\alpha}\right]e^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}$$

$$s(t) = s^*+\left[1-s^*-\frac{\rho+\delta(1-\alpha)}{\alpha}k_0^{1-\alpha}\right]e^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}$$

Initial savings is given by

$$s(0) = s_0 = 1-\frac{\rho+\delta(1-\alpha)}{\alpha}\cdot k_0^{1-\alpha}$$

Combining the results gives the savings rate

$$s(t) = s^*-(s^*-s_0)e^{-(1-\alpha)\frac{\rho+\delta}{\alpha}t}$$ # gives savings rate dynamics

6.6 The speed of convergence is defined as

$$\beta_k = \frac{\partial\frac{\dot{k}(t)}{k(t)}}{\partial\ln k(t)}$$ # percentage change

Remember that

$$\frac{\partial y}{\partial\ln x} = \frac{\partial y}{\partial x}\cdot x$$

As a result the speed of convergence is given by

$$\beta_k = \frac{\partial \frac{\dot{k}(t)}{k(t)}}{\partial k(t)} \cdot k(t)$$

$$\beta_k = \frac{-\partial \frac{\rho+\delta}{\alpha} \cdot \left[1-\left(\frac{k^*}{k(t)}\right)^{1-\alpha}\right]}{\partial k(t)} \cdot k(t)$$

$$\beta_k = (1-\alpha)\frac{\rho+\delta}{\alpha} \cdot \left(\frac{k^*}{k(t)}\right)^{1-\alpha}$$