PS 3: Neoclassical Growth

1 Discrete vs Continuous Time:

Consider an interval of time t. Divide the interval into $\frac{t}{\Delta t}$ equally sized sub-intervals. Assume that one unit of savings ~~at~~ pays an interest rate $r\Delta t$ at each subinterval (this is important as otherwise the renumeration would not be constant over time). As a result the value of one unit of savings at t is given by

$$V(t|\Delta t) = (1 + r\Delta t)^{\frac{t}{\Delta t}}.$$

The key difference between discrete and continuous time is the size of Δt. In discrete time $\Delta t = 1$, in continuous time $\Delta t \to 0$. Therefore the value of one unit of savings at t in discrete time is

$$v(t) = (1 + r)^t$$

while in continuous time it is given by

$$\begin{aligned} v(t) = \lim_{\Delta t \to 0} v(t|\Delta t) &= \lim_{\Delta t \to 0} (1 + r\Delta t)^{\frac{t}{\Delta t}} \\ &= \lim_{\Delta t \to 0} e^{\ln (1+r\Delta t)^{\frac{t}{\Delta t}}} \\ &= \lim_{\Delta t \to 0} e^{\frac{t}{\Delta t} \cdot \ln(1 + r\Delta t)} \end{aligned}$$

The limit of $\frac{t}{\Delta t} \ln(1 + r\Delta t)$ is given by

$$\lim_{\Delta t \to 0} \frac{\ln(1 + r\Delta t)}{\frac{\Delta t}{t}} \overset{\text{L'Hôpital's Rule}}{=} \lim_{\Delta t \to 0} \frac{\frac{r}{1 + r\Delta t}}{\frac{1}{t}} = rt$$

As a result the value of one unit of saving at t in continuous time is given by

$$v(t) = e^{rt}.$$

Further intuition: Euler's number e is defined as

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

with

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n,$$

This can be inverted such that

$$e^x = \lim_{n \to 0} (1 + xn)^{\frac{1}{n}}.$$

Using this result, setting $x = r$, $n = \Delta t$ and taking the exponent t on both sides yields

$$e^{rt} = \lim_{\Delta t \to 0} (1 + r\Delta t)^{\frac{t}{\Delta t}}$$

2 Constant Elasticity of Substitution Production (constant σ)

general case that nests Cobb-Douglas, complements or substitutes
↳ depends on value of σ

Taking the log of $Y(T)$ yields

$$\ln Y(t) = \frac{\sigma}{\sigma - 1} \ln \left[\alpha (K(t))^{\frac{\sigma-1}{\sigma}} + (1 - \alpha)(A(t)L(t))^{\frac{\sigma-1}{\sigma}}\right].$$

In the limit $\sigma \to 1$ this becomes

$$\lim_{\sigma \to 1} \ln Y(t) = \lim_{\sigma \to 1} \frac{\sigma \ln\left[\alpha (K(t))^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(A(t)L(t))^{\frac{\sigma-1}{\sigma}}\right]}{\sigma - 1}$$

where $\lim_{\sigma \to 1} \frac{\sigma \ln\left[\alpha (K(t))^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(A(t)L(t))^{\frac{\sigma-1}{\sigma}}\right]}{\sigma - 1}$ is given by

$$\lim_{\sigma \to 1} \frac{\ln\left[\alpha (K(t))^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(A(t)L(t))^{\frac{\sigma-1}{\sigma}}\right] + \frac{\sigma\left[\alpha \cdot \frac{\sigma-(\sigma-1)}{\sigma^2}(K(t))^{\frac{\sigma-1}{\sigma}} \ln K(t) + (1-\alpha)\frac{\sigma-(\sigma-1)}{\sigma^2}(A(t)L(t))^{\frac{\sigma-1}{\sigma}} \ln(A(t)L(t))\right]}{\alpha (K(t))^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(A(t)L(t))^{\frac{\sigma-1}{\sigma}}}}{1}$$

which becomes

$$\frac{\ln 1 + \frac{\alpha \ln K(t) + (1-\alpha)\ln(A(t)L(t))}{1}}{1} = \alpha \ln K(t) + (1-\alpha)\ln(A(t)L(t))$$

As a result the production function becomes

$$\lim_{\sigma \to 1} \ln Y(t) = \alpha \ln K(t) + (1-\alpha)\ln(A(t)L(t))$$
$$= \ln\left[K(t)^{\alpha}(A(t)L(t))^{1-\alpha}\right]$$

$$\Rightarrow \lim_{\sigma \to 1} Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha}$$

3 Constant Relative Risk Aversion Utility

The condition for constant relative risk aversion

$$-\frac{u''(c) \cdot c}{u'(c)} = \theta,$$

is a second-order differential equation. However, defining $x(c) = u'(c)$ allows transforming it into a first order differential equation as

$$x'(c) = u''(c).$$

Using this result yields a general solution

$$-\frac{x'(c) \cdot c}{c} = \theta$$

$$\frac{x'(c)}{x(c)} = -\frac{\theta}{c}$$

$$\int \frac{x'(c)}{x(c)} dc = -\int \frac{\theta}{c} dc$$

$$\ln x(c) + a_1 = -\theta \ln c + a_2$$

$$\ln x(c) = -\theta \ln c + a_3$$

$$e^{\ln x(c)} = e^{-\theta \ln c + a_3} = e^{a_3} e^{-\theta \ln c} \qquad \# \; e^{a_3} = A \quad > 0$$

$$x(c) = A \cdot c^{-\theta}$$

$$u'(c) = A \cdot c^{-\theta}$$

$$\int u'(c) dc = A \cdot \int c^{-\theta} dc$$

$$u(c) + a_5 = A \cdot \frac{c^{1-\theta}}{1-\theta} + a_6$$

$$u(c) = A \cdot \frac{c^{1-\theta}}{1-\theta} + a_7$$

As utility is unique up to a monotone transformation (same preference ranking, same marginal utility) a possible particular solution is given by $A = 1$ and $a_7 = -\frac{1}{1-\theta}$. Using this particular solution has the appealing property that for $\theta \to 1$ it converges to

$$\lim_{\theta \to 1} \frac{c^{1-\theta} - 1}{1-\theta} = \lim_{\theta \to 1} \frac{-c^{1-\theta} \ln c}{-1} = \ln c \qquad \text{\# again L'Hôpital}$$

↳ exam: know about concepts but not derivations itself

④ Canonical Neo-Classical Growth Model

4.1 The law of motion is

$$\dot{a}(t) = r(t) a(t) + w(t) - c(t), \quad \text{with } r(t) = f'(k(t)) - \delta.$$

The current-value Hamiltonian is

$$\hat{\mathcal{H}}(c(t), a(t), \mu(t)) = \frac{c(t)^{1-\theta} - 1}{1-\theta} + \mu(t) [r(t) a(t) + w(t) - c(t)]$$

The FOCs are

$$\frac{\partial \hat{\mathcal{H}}(c(t), a(t), \mu(t))}{\partial c(t)} = c(t)^{-\theta} - \mu(t) = 0$$

$$\frac{\partial \hat{\mathcal{H}}(c(t), a(t), \mu(t))}{\partial a(t)} = \mu(t) r(t) = \rho \mu(t) - \dot{\mu}(t)$$

$$\frac{\partial \hat{\mathcal{H}}(c(t), a(t), \mu(t))}{\partial \mu(t)} = r(t) a(t) + w(t) - c(t) = \dot{a}(t)$$

The transversality condition is

$$\lim_{t \to \infty} e^{-\rho t} \mu(t) a(t) = 0$$

Use $\frac{\partial \hat{\mathcal{H}}}{\partial c(t)} = 0$

$$c(t)^{-\theta} - \mu(t) = 0$$

$$-\theta \ln(c(t)) = \ln(\mu(t))$$

$$-\theta \frac{\dot{c}(t)}{c(t)} = \frac{\dot{\mu}(t)}{\mu(t)}$$

$$\frac{\dot{c}(t)}{c(t)} = -\frac{1}{\theta} \frac{\dot{\mu}(t)}{\mu(t)}$$

From $\frac{\partial \hat{\mathcal{H}}}{\partial a(t)} = \rho \mu(t) - \dot{\mu}(t)$

$$r(t) \mu(t) = \rho \mu(t) - \dot{\mu}(t)$$

$$-\dot{\mu}(t) = \mu(t)(r(t) - \rho)$$

$$-\frac{\dot{\mu}(t)}{\mu(t)} = r(t) - \rho$$

Combining the two yields

$$\frac{\dot{c}(t)}{c(t)} = -\frac{1}{\theta}\frac{\dot{\mu}(t)}{\mu(t)}$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t)-\rho}{\theta}$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t))-\delta-\rho}{\theta}$$

4.2

The general solution to the differential equation is

$$\dot{L}(t) = n L(t)$$

$$\frac{\dot{L}(t)}{L(t)} = n$$

$$\int \frac{\dot{L}(t)}{L(t)}\,dt = \int n\,dt$$

$$\ln(L(t)) + a_1 = nt + a_2$$

$$\ln(L(t)) = nt + a_3$$

$$L(t) = e^{nt+a_3}$$

$$L(t) = e^{a_3} e^{nt}$$

$$L(t) = A e^{nt}$$

Given that $L(0) = 1$ (from the problem sheet) the particular solution is

$$L(0) = A e^{n0}$$

$$1 = A e^{0}$$

$$A = 1$$

$$\Rightarrow L(t) = e^{nt}$$

The utility part of the maximization is now given by

$$\max_{[c(t),\,a(t)]_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} L(t) \frac{c(t)^{1-\theta}-1}{1-\theta}\,dt$$

inserting $L(t) = e^{nt}$ gives

$$\int_0^{\infty} e^{-\rho t} L(t) \frac{c(t)^{1-\theta}-1}{1-\theta}\,dt$$

$$\int_0^{\infty} e^{-\rho t} e^{nt} \frac{c(t)^{1-\theta}-1}{1-\theta}\,dt$$

$$\int_0^{\infty} e^{-(\rho-n)t} \frac{c(t)^{1-\theta}-1}{1-\theta}\,dt$$

The law of motion of total assets is now given by

$$\dot{A}(t) = r(t)a(t)L(t) + w(t)L(t) - c(t)L(t).$$

As a result the law of motion for capita assets is

$$a(t) = \frac{A(t)}{L(t)}$$

$$\dot{a}(t) = \frac{\dot{A}(t)L(t) - A(t)\dot{L}(t)}{(L(t))^2}$$

$$\dot{a}(t) = \frac{\dot{A}(t)}{L(t)} - \frac{\dot{L}(t)}{L(t)} \cdot \frac{A(t)}{L(t)}$$

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t) - na(t)$$

$$\dot{a}(t) = [f'(k(t)) - n - \delta]a(t) + w(t) - c(t)$$

In order to ensure discounting the following has to hold

$$\rho - n > 0 \iff \rho > n$$

The new transversality condition is

$$\lim_{t \to \infty} e^{-(\rho-n)t} \mu(t) a(t) = 0.$$

The Euler Equation is unchanged as

$$\frac{\partial \hat{H}}{\partial a(t)} = \mu(t)[r(t) - n] = (\rho - n)\mu(t) - \dot{\mu}(t)$$

$$\Rightarrow -\frac{\dot{\mu}(t)}{\mu(t)} = r(t) - \rho$$

Verbal: population growth enters both (effective) discounting and assets per capita meaning that the maximization problem is still cast in per capita terms.

4.3

A steady state in consumption implies

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta} = 0.$$

Hence in a steady state it must hold

$$\frac{f'(k) - \delta - \rho}{\theta} = 0$$

$$\alpha k^{\alpha - 1} = \rho + \delta$$

$$k^{1-\alpha} = \frac{\alpha}{\rho + \delta}$$

$$k^* = \left[\frac{\alpha}{\rho + \delta}\right]^{\frac{1}{1-\alpha}}$$

Combining the law of motion and perfect competition yields

$$\dot{k}(t) = f'(k(t)) \cdot k(t) - nk(t) - \delta k(t) + w(t) - c(t)$$

$$\dot{k}(t) = f'(k(t))k(t) - nk(t) - \delta k(t) + [f(k(t)) - f'(k(t))k(t)] - c(t)$$

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t)$$

A steady state implies $0 = f(k) - (n+\delta)k - c$

$$c = f(k) - (n+\delta)k$$

$$c = [k^*]^\alpha - (n+\delta)k^*$$

$$c^* = \left[\frac{\alpha}{\rho \cdot \delta}\right]^{\frac{\alpha}{1-\alpha}} - (n+\delta)\left[\frac{\alpha}{\rho+\delta}\right]^{\frac{1}{1-\alpha}}$$

Consumption is defined by

$$c(t) = (1 - s(t)) \cdot f(k(t)) = f(k(t)) - s(t) f(k(t))$$

As a result the steady-state savings rate is given by

$$s = 1 - \frac{c}{f(k)}$$

$$s^* = 1 - \frac{c^*}{[k^*]^\alpha}$$

$$s^* = 1 - \frac{[k^*]^\alpha}{[k^*]^\alpha} + (n+\delta)\frac{k^*}{[k^*]^\alpha}$$

$$s^* = \frac{n+\delta}{\rho+\delta} \cdot \alpha$$

4.4

Steady state consumption is given by

$$c = f(k) - (n+\delta)k$$

Therefore consumption is maximized at

$$\frac{\partial c}{\partial k} = f'(k) - (n+\delta) = 0$$

$$f'(k) = n+\delta$$

$$\alpha k^{\alpha-1} = n+\delta$$

$$k^{GR} = \left[\frac{\alpha}{n+\delta}\right]^{\frac{1}{1-\alpha}}$$

This is larger than k^* given that the assumption to ensure discounting $(\rho > n)$ holds. Alternative explanation via comparing optimal and Golden Rule savings rates

$$s^{GR} = \alpha \neq \frac{n+\delta}{\rho+\delta} \cdot \alpha = s^*$$

Therefore

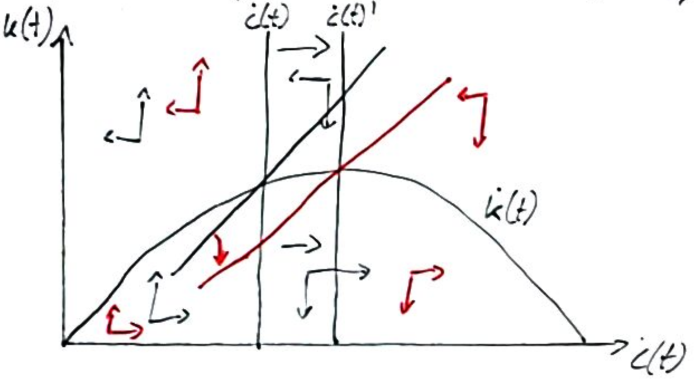
$$s^* \begin{cases} < s^{GR} & \text{if } \rho > n \\ = s^{GR} & \text{if } \rho = n \\ > s^{GR} & \text{if } \rho < n \end{cases}$$

Verbal explanation: the optimal steady state capital stock in an economy with discounting will always be lower than the Golden Rule since impattence induces lower savings than what would be needed to maximice steady-state consumption.

4.5

Draw the $\dot{k}(t)$ (reverse U-shaped with origin at zero) and $\dot{c}(t)$ (straight parallel to the y-axis) loci correctly. The x-axis is $k(t)$, the y-axis is $c(t)$. Draw correct phase arrows.

a) The $\dot{c}(t)$ locus shifts to the right. More precisely it shifts to the maximum of the $\dot{k}(t)$ locus. The consumption of the economy drops immediately because of the decrease in ρ and then converges to the new steady state at the new intersection of the two loci.



b) Yes, as the savings rate is now given by

$$s^* = \frac{n+\delta}{\rho+\delta}\cdot\alpha$$

$$s^* = \frac{n+\delta}{n+\delta}\cdot\alpha$$

$$s^* = \alpha = s^{GR}$$

Alternatively, discounting is now $e^{-(\rho-n)t} = 1$, which means that utility maximization implies behavior like the Golden Rule in the Solow Model.

4.6

Consumption per effective unit of labor is given by

$$\tilde{c}(t) = \frac{c(t)}{A(t)}.$$

As a result the Euler equation in consumption per efficiency units of labor is given by

$$\dot{\tilde{c}}(t) = \frac{\dot{c}(t)A(t) - c(t)\dot{A}(t)}{(A(t))^2}$$

$$\dot{\tilde{c}}(t) = \frac{\dot{c}(t)}{A(t)} - \frac{\dot{A}(t)}{A(t)}\cdot\frac{c(t)}{A(t)}$$

$$\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{\dot{c}(t)}{A(t)}\,\frac{A(t)}{c(t)} - \frac{\dot{A}(t)}{A(t)}\,\frac{c(t)}{A(t)}\,\frac{A(t)}{c(t)}$$

$$\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{\dot{c}(t)}{c(t)} - g$$

$$\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{f'(k(t)) - \delta - \rho - \theta g}{\theta}$$

⑤ Neo-Classical Growth and Balanced Growth

5.1

see problem 4.1

5.2 possible: taking the log of both sides and plugging in

OR: The consumption-capital ratio $x(t)$ is given by

$$x(t) = \frac{c(t)}{k(t)}$$

Therefore the law of motion of $x(t)$ is given by

$$\dot{x}(t) = \frac{\dot{c}(t)k(t) - c(t)\cdot\dot{k}(t)}{[k(t)]^2}$$

$$\dot{x}(t) = \frac{\dot{c}(t)}{k(t)} - \frac{c(t)}{k(t)}\frac{\dot{k}(t)}{k(t)}$$

Remember that

$$x(t) = \frac{c(t)}{k(t)}$$

As a result

$$\dot{x}(t) = \frac{\dot{c}(t)}{c(t)}\cdot x(t) - \frac{\dot{k}(t)}{k(t)}\cdot x(t)$$

$$\frac{\dot{x}(t)}{x(t)} = \frac{\dot{c}(t)}{c(t)} - \frac{\dot{k}(t)}{k(t)}$$

The growth rate of the capital stock is given by

$$\dot{k}(t) = [f'(k(t)) - \delta]k(t) + w(t) - c(t)$$

$$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t)$$

$$\Rightarrow \frac{\dot{k}(t)}{k(t)} = \frac{f(k(t))}{k(t)} - x(t) - \delta$$

The Euler equation is given by

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta}$$

Combining the two yields

$$\frac{\dot{x}(t)}{x(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta} - \left[\frac{f(k(t))}{k(t)} - x(t) - \delta\right]$$

5.3 (no matter the t, $\dot{x}(t)$ will be 0)

The growth rate of the consumption-capital ratio is given by

$$\frac{\dot{x}(t)}{x(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta} - \left[\frac{f(k(t))}{k(t)} - x(t) - \delta\right].$$

Given that $x(t)$ is constant for every point in time, this implies

$$\dot{x}(t) = 0 \quad \forall t$$

Subsequently, the growth rate of the consumption-capital ratio can be solved for $x(t)$

$$x(t) = \left[\frac{f(k(t))}{k(t)} - \delta\right] - \frac{f'(k(t)) - \delta - \rho}{\theta},$$

which gives a solution for $c(t)$ given that $x(t) = \frac{c(t)}{k(t)} \Rightarrow c(t) = x(t)k(t)$

$$c(t) = f(k(t)) - \delta k(t) - \frac{f'(k(t)) - \delta - \rho}{\theta} k(t).$$

Output per capita $f(k(t))$ and return to capital $f'(k(t))$ are given by

$$f(k(t)) = k(t)^\alpha + Ak(t)$$
$$f'(k(t)) = \alpha k(t)^{\alpha-1} + A$$

Combining the results and using $\theta = \alpha$ yields

$$c(t) = k(t)^\alpha + Ak(t) - \delta k(t) - \frac{\alpha k(t)^{\alpha-1} + A - \delta - \rho}{\alpha} k(t)$$
$$c(t) = k(t)^\alpha + Ak(t) - \delta k(t) - k(t)^\alpha - \frac{A-\delta-\rho}{\alpha} k(t)$$
$$c(t) = \frac{\alpha(A-\delta) - A + \delta + \rho}{\alpha} k(t)$$
$$c(t) = \frac{\rho - (A-\delta)(1-\alpha)}{\alpha} k(t)$$

linear function of $k(t)$

A assumed to be larger than δ

The saddle path is given by the solution to $c(t)$. Since the phase diagram is drawn in the $k(t) - c(t)$ space this implies a linear shape as $c(t)$ is a linear function of $k(t)$.

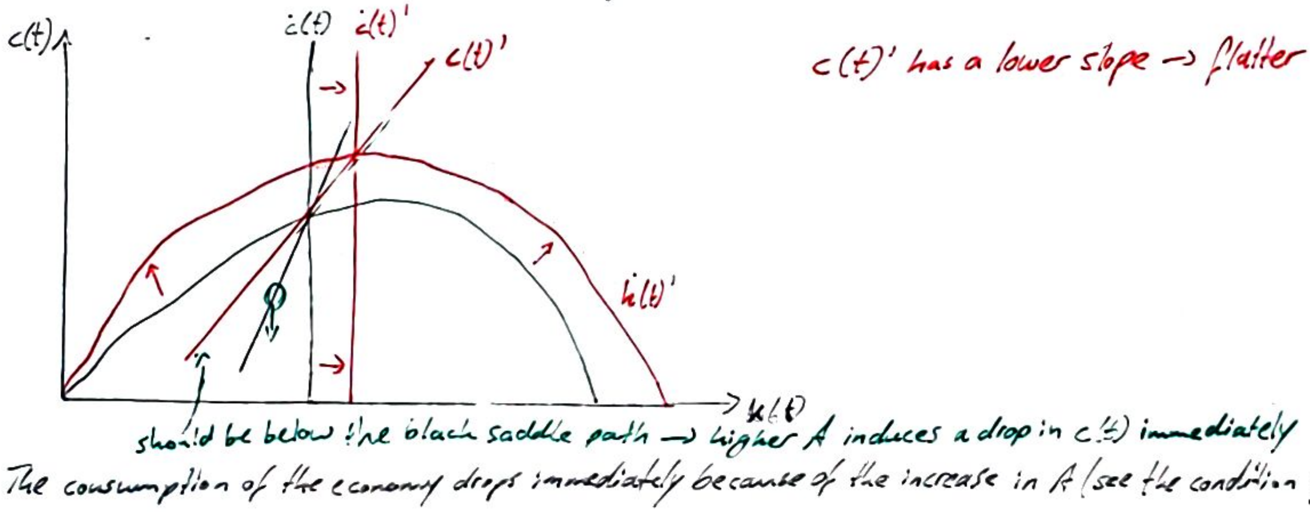
For a balanced growth path, $c(t)$ has to be greater or equal to zero for all t. Since $A > \delta$ this implies

$$\rho - (A-\delta)(1-\alpha) \geq 0 \iff \rho \geq (A-\delta)(1-\alpha)$$

5.4

Draw the $\dot{k}(t)$ and $\dot{c}(t)$ loci. The x-axis is $k(t)$, the y-axis is $c(t)$.

The $\dot{k}(t)$ locus shifts outwards and to the right, the $\dot{c}(t)$ locus shifts to the right



The consumption of the economy drops immediately because of the increase in A (see the condition for $c(t)$) and then converges to the new steady state at the new intersection of the two loci.