PS 2. OLG model

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1.1 Budget constraints in two periods of life are

$$c_t^1 + s_t \leq w_t$$
$$c_{t+1}^2 \leq (1+r_{t+1}) s_t$$

Alternatively if written in saving rates:

$$c_t^1 + s_t w_t \leq w_t$$
$$c_{t+1}^2 \leq (1+r_{t+1}) s_t w_t$$

Substitution of the constraint into the utility function: # in lecture derived with Lagrange

$$\ln(c_t^1) + \beta \ln[(1+r_{t+1})(w_t - c_t^1)]$$

First order condition

$$\frac{1}{c_t^1} - \beta \frac{(1+r_{t+1})}{(1+r_{t+1})(w_t - c_t^1)} = 0$$

Rearranging and solving for the Euler

$$\frac{1}{c_t^1} = \frac{\beta}{w_t - c_t^1}$$
$$\beta c_t^1 = w_t - c_t^1$$
$$c_t^1 = \frac{1}{1+\beta} \cdot w_t$$
$$\Rightarrow c_{t+1}^2 = (1+r_{t+1})(w_t - c_t^1)$$
$$c_{t+1}^2 = (1+r_{t+1})(w_t - \frac{1}{1+\beta} w_t)$$
$$c_{t+1}^2 = (1+r_{t+1}) \frac{\beta}{1+\beta} w_t$$
$$\Rightarrow \frac{c_{t+1}^2}{c_t^1} = (1+r_{t+1})\beta$$

1.2 Optimal savings are given by

$$s_t = w_t - c_t^1 = w_t - \frac{1}{1+\beta} w_t = \frac{\beta}{1+\beta} w_t$$

Another option is to argue that consumption in t+1 is determined ~~by~~ solely by savings in t because of log-utility. Then of course the amount invested into consumption in t+1 must equal the relative weight of the log times the income generated in t i.e.:

$$s_t = \frac{\text{utility weight of future consumption}}{\text{sum of utility weights}} \cdot \text{income in } t$$

$$s_t = \frac{\beta}{1+\beta} w_t$$ important: only applies to log preferences!

1.3 The evolution of capital stock is given by

$$K_{t+1} = s_t L_t$$

with s_t being savings per capita. Therefore the evolution of capital per capita is given by

$$\frac{K_{t+1}}{L_{t+1}} = \frac{s_t L_t}{L_{t+1}}$$

Insert savings per capita gives

$$k_{t+1} = \frac{\beta}{1+\beta} \cdot (1-\alpha) \cdot \left[\frac{K_t}{L_t}\right]^{\alpha} \frac{L_t}{L_{t+1}}$$

wage is the marginal product of labor

$$k_{t+1} = \frac{\beta}{1+\beta} (1-\alpha) \cdot k_t^{\alpha} \frac{L_t}{L_{t+1}}$$

The value L_{t+1} can be obtained via the definition of its growth rate

$$\frac{L_{t+1} - L_t}{L_t} = n$$

$$\Rightarrow L_{t+1} = (1+n) L_t$$

Inserting these results gives

$$k_{t+1} = \frac{\beta}{1+\beta} \cdot \frac{1-\alpha}{1+n} \cdot k_t^{\alpha}$$

In a steady state $k_{t+1} = k_t = k$, so

$$k = \frac{\beta}{1+\beta} \frac{1-\alpha}{1+n} \cdot k^{\alpha}$$

$$k^{1-\alpha} = \frac{\beta}{1+\beta} \cdot \frac{1-\alpha}{1+n}$$

k^* depends positively on β and negatively on n

$$k^* = \left[\frac{\beta}{1+\beta} \frac{1-\alpha}{1+n}\right]^{\frac{1}{1-\alpha}}$$

1.4 Draw the k_{t+1} curve and its intersect with the $k_{t+1} = k_t$ (45° line) locus. The x-axis is k_t, the y-axis is k_{t+1}

a) Effect on the steady state level of capital per capita k^*

$$\frac{\partial k^*}{\partial n} = -\frac{1}{(1-\alpha)(1+n)} \cdot \left[\frac{\beta}{1-\beta} \frac{1-\alpha}{1+n}\right]^{\frac{1}{1-\alpha}} < 0$$

Therefore an increase in population growth n pushes the k_{t+1} curve downwards, the intersection with the 45° line gives a lower steady state

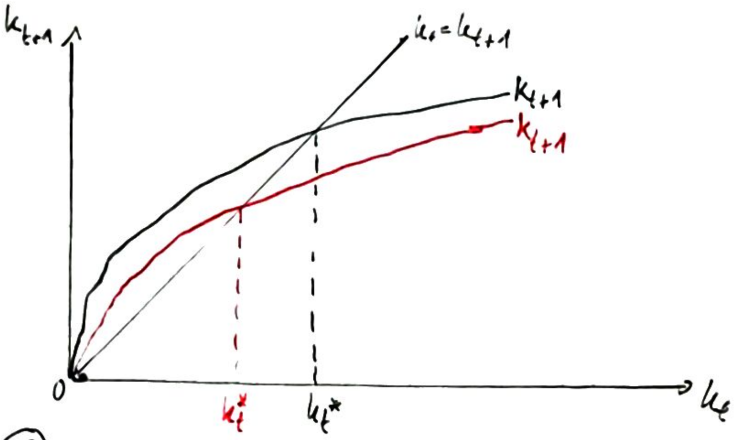
Explanation: higher population growth leads to higher effective depreciation and hence lower return on capital.

b) Effect on the steady state level of capital per capita k^*:

$$\frac{\partial k^*}{\partial \beta} = \frac{1}{(1-\alpha)(1+\beta)} \left[\frac{\beta}{1+\beta} \cdot \frac{1-\alpha}{1+n}\right]^{\frac{1}{1-\alpha}} > 0$$

Therefore a drop in the discount factor β pushes the k_{t+1} curve downwards; the intersection with the 45° line gives a lower steady state.

Explanation: a lower discount factor implies less preference weight on future consumption leading to a reduction in savings, which lowers the capital stock.



② 2.1 Analytic explanation. working full-time implies $l_t = 1$. This results in

$$\ln(1-1) = \ln(0),$$

which would generate a utility of $-\infty$.

Verbal explanation: there is utility from not working. Therefore given positive marginal utility at least some time should be spent not working.

2.2 Budget constraints in the two periods of life are

$$c_t^1 + s_t \leq w_t \cdot l_t$$

$$c_{t+1}^2 \leq (1 + r_{t+1}) s_t$$

$$\Rightarrow \frac{c_{t+1}^2}{1+r_{t+1}} \leq w_t l_t - c_t^1$$

Setting up the Lagrangian (or correct substitution into the objective function)

$$\mathcal{L}(c_t^1, c_{t+1}^2, l_t, \lambda) = \ln(c_t^1) + \beta \ln(c_{t+1}^2) + \phi \ln(1 - l_t) + \lambda \left[w_t l_t - c_t^1 - \frac{c_{t+1}^2}{1+r_{t+1}} \right]$$

FOCs are

$$\frac{\partial \mathcal{L}}{\partial c_t^1} = \frac{1}{c_t^1} - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}^2} = \frac{\beta}{c_{t+1}^2} - \frac{\lambda}{1+r_{t+1}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial l_t} = -\frac{\phi}{1-l_t} + \lambda w_t = 0$$

$\frac{\partial \mathcal{L}}{\partial \lambda}$ omitted because not important

Rearranging and solving for the Euler equation

$$\frac{c_{t+1}^2}{c_t^1} = \beta(1 + r_{t+1})$$

Rearranging the Euler equation

$$c_{t+1}^2 = c_t^1 \cdot \beta (1 + r_{t+1})$$

Inserting back into the second-period BC

$$c_t^1 \beta (1 + r_{t+1}) = (1 + r_{t+1}) s_t$$

$$s_t = \beta c_t^1$$

Inserting back into the first period BC

$$c_t^1 (1+\beta) = w_t l_t$$

$$c_t^1 = \frac{w_t l_t}{1+\beta}$$

Using $\frac{1}{c_t^1} = \lambda$ and inserting the result for c_t^1 back into the FOC w.r.t. l_t and solving for labor supply

$$-\frac{\phi}{1-l_t} + \lambda w_t = 0$$

$$\frac{\phi}{1-l_t} = \frac{w_t}{c_t^1}$$

$$\frac{\phi}{1-l_t} = \frac{w_t(1+\beta)}{w_t l_t}$$

$$\phi l_t = 1+\beta-(1+\beta) l_t$$

$$l_t = \frac{1+\beta}{1+\beta+\phi}$$

interesting: in this setup, there is an optimal labor supply and it is constant

Inserting back into consumption

$$c_t^1 = \frac{w_t l_t}{1+\beta}$$

$$c_t^1 = \frac{w_t}{1+\beta} \cdot \frac{1+\beta}{1+\beta+\phi}$$

$$c_t^1 = \frac{w_t}{1+\beta+\phi}$$

Solving for savings

$$s_t = \beta c_t^1 = \beta \frac{w_t}{1+\beta+\phi} = \frac{\beta}{1+\beta+\phi} w_t$$

2.3 The evolution of the capital stock is given by

$$K_{t+1} = s_t N_t$$

with s_t being savings per capita. Therefore the evolution of capital per capita is given by

$$\frac{K_{t+1}}{N_{t+1}} = \frac{s_t N_t}{N_{t+1}}$$

$$k_{t+1} = \frac{s_t N_t}{(1+n) N_t}$$

$$k_{t+1} = \frac{\beta}{(1+\beta+\phi)(1+n)} w_t$$

The wage is given by

$$w_t = \frac{\partial Y_t}{\partial L_t} = (1-\alpha)\left(\frac{K_t}{L_t}\right)^\alpha$$

$$w_t = (1-\alpha)\left(\frac{K_t}{l_t N_t}\right)^\alpha = (1-\alpha)\left(\frac{K_t}{N_t}\right)^\alpha \cdot \left(\frac{1}{l_t}\right)^\alpha$$

$$w_t = (1-\alpha) k_t^\alpha \cdot \left(\frac{1+\beta+\phi}{1+\beta}\right)^\alpha$$

Therefore

$$k_{t+1} = \frac{\beta}{(1+\beta+\phi)(1+n)} \cdot (1-\alpha) k_t^\alpha \cdot \left(\frac{1+\beta+\phi}{1+\beta}\right)^\alpha$$

$$k_{t+1} = \frac{\beta}{(1+\beta+\phi)^{1-\alpha}(1+\beta)^\alpha} \cdot \frac{1-\alpha}{1+n} \cdot k_t^\alpha$$

In a steady state $k_{t+1} = k_t = k$, so

$$k^* = \frac{1}{1+\beta+\phi} \cdot \left[\frac{\beta}{(1+\beta)^\alpha} \cdot \frac{1-\alpha}{1+n}\right]^{\frac{1}{1-\alpha}}.$$

2.4 Setup as before.

Effect on the steady state level of capital per capita k^*

$$\frac{\partial k^*}{\partial \phi} = -\frac{1}{(1+\beta+\phi)^2} \cdot \left[\frac{\beta}{(1+\beta)^{\alpha}} \quad \frac{1-\alpha}{1+n}\right]^{\frac{1}{1-\alpha}} < 0.$$

Therefore an increase in the preference for leisure ϕ pushes the k_{t+1} curve downwards; the intersection with the $45°$ line gives a lower steady state.

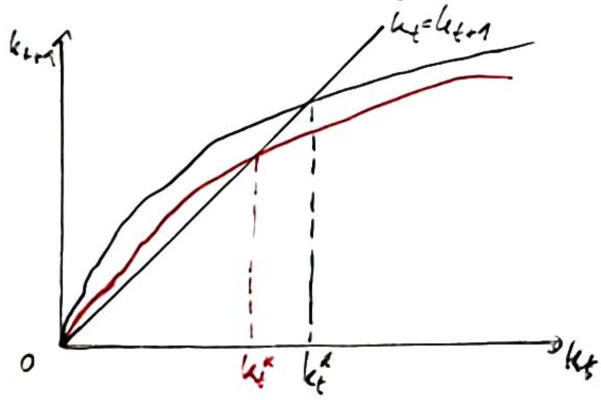
Verbal explanation: higher preference for leisure leads to less labor supply, which leads to higher wages per unit of labor but smaller earnings; labor supply effect dominates wage effect as

$$w_t = (1-\alpha)\, k_t^{\alpha} \cdot \left(\frac{1}{l_t}\right)^{\alpha}$$

increases with smaller l_t but

$$w_t l_t = (1-\alpha)\, k_t^{\alpha} \quad l_t^{1-\alpha}$$

decreases with decreasing l_t.



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3.1 The maximization problem is

$$\max_{c_t^i,\, e_t^i} \; (1-\delta) \ln c_t^i + \delta \ln e_t^i \quad s.t. \; c_t^i + e_t^i = w_t$$

Re-arranging the constraint for e_t^i and substituting into the objective function gives

$$\max_{e_t^i} \; (1-\delta) \cdot \ln(w_t^i - e_t^i) + \delta \ln e_t^i$$

The FOC is given by

$$-(1-\delta)\frac{1}{w_t^i - e_t^i} + \delta \cdot \frac{1}{e_t^i} = 0$$

Solving e_t^i yields

$$\frac{\delta}{e_t^i} = \frac{1-\delta}{w_t^i - e_t^i}$$

$$\delta(w_t^i - e_t^i) = (1-\delta) e_t^i$$

$$e_t^i = \delta w_t^i$$

$$\Rightarrow c_t^i = (1-\delta) w_t^i$$

As a result equilibrium education investment is given by

$$e_t^i = \delta A h_t^i ,$$

which, together with $\delta A > 1 > \delta A \bar{h}$ implies

$$e_t^i \begin{cases} > 1 & \text{if } h_t^i > \frac{1}{\delta A} \\ < 1 & \text{if } h_t^i < \frac{1}{\delta A} \end{cases}$$

This result immediately gives human capital accumulation as $e_t^i < 1$ implies $h_{t+1}^i = \bar{h}$, which implies $e_{t+1}^i = \delta A h_{t+1}^i = \delta A \bar{h} < 1$

$$h_{t+1} = \begin{cases} (\delta A h_t^i)^\gamma & \text{if } h_i^t > \frac{1}{\delta A} \\ \bar{h} & \text{if } h_i^t < \frac{1}{\delta A} \end{cases} \Rightarrow h_{t+1}^i = \begin{cases} (\delta A h_t^i)^\gamma & \text{if } h_0^i > \frac{1}{\delta A} \\ \bar{h} & \text{if } h_0^i < \frac{1}{\delta A} \end{cases}$$

with steady states

$$h^* = \begin{cases} (\delta A)^{\frac{\gamma}{1-\gamma}} & \text{if } h_0^i > \frac{1}{\delta A} \\ \bar{h} & \text{if } h_0^i < \frac{1}{\delta A} \end{cases}$$

There exist two steady states because the system has a non-convexity in the accumulation of human capital that allows for a poverty trap.

\# possibility of a vicious circle ($\bar{h}$) and a virtuous circle (poverty trap)

3.2 - 3.5 For details on simulation, see .py file. The deciding factor in which steady state is achieved is the initial condition for human capital h_0. Possible breakouts of the poverty trap could be achieved via

- larger preference for education (not appealing policy-wise and also ~~that~~ has a hard limit $\delta \to 1$).
- Better technology, technological progress (again, not appealing, but no hard limit per se.)
- Allow individuals to borrow against the future, i.e. the existence of financial markets (allows ~~for~~ "over-investing" in e_t^i, more appealing policy-wise, but not without problems if there are multiple investment opportunities.