PS 1: Solow model

(1)

1.1 Show k^*!

Given are $\dot{K}(t) = sY(t) - \delta K(t)$ and $k(t) = \frac{K(t)}{A(t) \cdot L(t)}$

Therefore, the law of motion of capital per effective unit of labor is

$$\dot{k}(t) = \frac{\dot{K}(t)}{A(t)L(t)} - \frac{\dot{L}(t)}{L(t)} \cdot \frac{K(t)}{A(t)L(t)} - \frac{\dot{A}(t)}{A(t)} \cdot \frac{K(t)}{A(t)L(t)}$$

$$\dot{k}(t) = sf(k(t)) - (n+g+\delta)k(t).$$

Derivation of the steady state value is

$$\dot{k}(t) = 0$$

$$0 = sk^\alpha - (n+g+\delta)k$$

$$k^* = \left[\frac{s}{n+g+\delta}\right]^{\frac{1}{1-\alpha}}.$$

1.2 Steady-state consumption is defined as

$$c = (1-s)k^\alpha = k^\alpha - sk^\alpha$$

Given the result from the steady state condition

$$sk^\alpha = (n+g+\delta)k$$

consumption is maximized at

$$\frac{\partial c}{\partial k} = \alpha k^{\alpha-1} - (n+g+\delta) = 0$$

$$k^{GR} = \left[\frac{\alpha}{n+g+\delta}\right]^{\frac{1}{1-\alpha}}$$

The golden-rule savings rate is given by

$$s^{GR}[k^{GR}]^\alpha = (n+g+\delta)k^{GR}$$

$$s^{GR} = (n+g+\delta)[k^{GR}]^{1-\alpha}$$

$$s^{GR} = (n+g+\delta)\frac{\alpha}{n+g+\delta} = \alpha$$

Consumption is maximized when the economy saves exactly its capital share in total output.

1.3 (also in oral exam!)

There is no effect on the break-even line $(n+g+\delta)k(t)$ The effect of a rise in α on the $s \cdot f(k(t))$ locus is given by

$$\frac{\partial s\, k(t)^{\alpha}}{\partial \alpha} = s\, k(t)^{\alpha} \ln(k(t)) = \begin{cases} <0 & \text{if } k(t)<1 \\ =0 & \text{if } k(t)=1 \\ >0 & \text{if } k(t)>1 \end{cases}$$

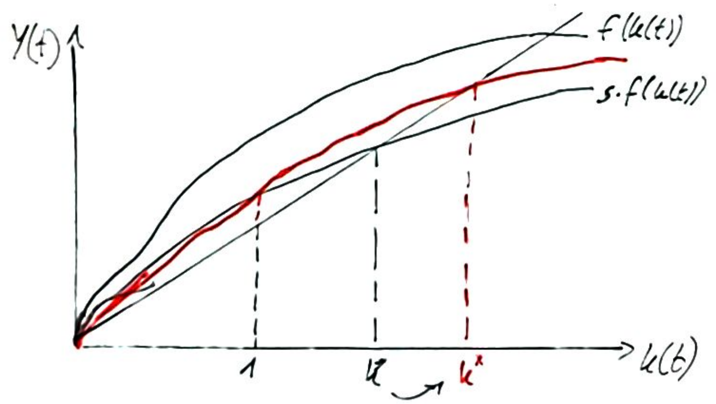
This means that the new locus lies below the old for $k(t)<1$, intersects the old at $k(t)=1$ and lies above the old for $k(t)>1$

The effect on the steady state is given by

$$\frac{\partial k^*}{\partial \alpha} = \frac{\partial \left[\frac{s}{n+g+\delta}\right]^{\frac{1}{1-\alpha}}}{\partial \alpha} = \frac{1}{(1-\alpha)^2}\left[\frac{s}{n+g+\delta}\right]^{\frac{1}{1-\alpha}} \ln\left[\frac{s}{n+g+\delta}\right] = \begin{cases} <0 & \text{if } s<n+g+\delta \\ =0 & \text{if } s=n+g+\delta \\ >0 & \text{if } s>n+g+\delta \end{cases}$$

As $s>n+g+\delta$ the steady state increases (shifts to the right)

Explanation: because of higher capital intensity more production is possible at higher amounts of inputs; the function becomes less concave.



② fixed factor important device to model poverty traps

2.1

Decreasing returns to scale

$$\lambda Y(t) = [\lambda K(t)]^{\alpha} \cdot [\lambda L(t)]^{\beta} \cdot [Z]^{1-\alpha-\beta}$$
$$= \lambda^{\alpha}\lambda^{\beta} K(t)^{\alpha} L(t)^{\beta} Z^{1-\alpha-\beta}$$
$$< \lambda Y(t)$$

Inada conditions

$$f(k(t)\ z(t)) = \frac{F(K(t), L(t), Z)}{L(t)} = k(t)^{\alpha} z(t)^{1-\alpha-\beta}$$
$$f(0, z(t)) = 0^{\alpha} z(t)^{1-\alpha-\beta} = 0$$
$$f(k(t), 0) = k(t)^{\alpha} 0^{1-\alpha-\beta} = 0$$

$$\lim_{k(t)\to 0} \frac{\partial f(k(t),z(t))}{\partial k(t)} = \lim_{k(t)\to 0} \alpha \left[\frac{1}{k(t)}\right]^{1-\alpha} z(t)^{1-\alpha-\beta} = \infty$$

$$\lim_{k(t)\to \infty} \frac{\partial f(k(t),z(t))}{\partial k(t)} = \lim_{k(t)\to \infty} \alpha \left[\frac{1}{k(t)}\right]^{1-\alpha} z(t)^{1-\alpha-\beta} = 0$$

$$\lim_{z(t)\to 0} \frac{\partial f(k(t),z(t))}{\partial z(t)} = \lim_{z(t)\to 0} (1-\alpha-\beta) k(t)^{\alpha} \left[\frac{1}{z(t)}\right]^{\alpha+\beta} = \infty$$

$$\lim_{z(t)\to \infty} \frac{\partial f(k(t),z(t))}{\partial z(t)} = \lim_{z(t)\to \infty} (1-\alpha-\beta) k(t)^{\alpha} \left[\frac{1}{z(t)}\right]^{\alpha+\beta} = 0$$

2.2 Total accumulation is deducible from the PS

$$\dot{K}(t) = sY(t) - \delta K(t)$$

and the capital-labor ratio is given by

$$k(t) = \frac{K(t)}{L(t)}$$

Therefore the law of motion of capital per capita is

$$k(t) = \frac{\dot{K}(t)L(t) - K(t)\cdot \dot{L}(t)}{[L(t)]^2}$$

$$\dot{k}(t) = \frac{\dot{K}(t)}{L(t)} - \frac{\dot{L}(t)\; K(t)}{L(t)\cdot L(t)}$$

$$\dot{k}(t) = s\frac{Y(t)}{L(t)} - (n+\delta)k(t)$$

$$\dot{k}(t) = s\,k(t)^{\alpha} z(t)^{1-\alpha-\beta} - \delta k(t)$$

Derivation of the steady state capital-labor ratio

$$\dot{k}(t) = 0$$

$$0 = s k(t)^{\alpha} z(t)^{1-\alpha-\beta} - \delta k(t)$$

$$k^{1-\alpha} = \frac{s}{\delta}\cdot z^{1-\alpha-\beta}$$

$$k^{*} = \left[\frac{s}{\delta}\cdot z^{1-\alpha-\beta}\right]^{\frac{1}{1-\alpha}}$$

2.3 Land per worker is given by $z(t) = \frac{Z}{L(t)}$

Subsequently the growth rate of land per worker is given by $\frac{\dot{z}(t)}{z(t)}$,

where $\dot{z}(t)$ is given by $\dot{z}(t) = \frac{\dot{Z}L(t) - \dot{L}(t)Z}{(L(t))^2}$

Since Z is fixed, $\dot{Z}=0$ and $z(t)$ becomes

$$z(t) = -\frac{\dot{L}(t)}{L(t)} \quad \frac{Z(t)}{L(t)}$$

$$z(t) = -n\cdot z(t)$$

As a result the growth rate of land per worker is given by

$$\frac{\dot{z}(t)}{z(t)} = -n$$

The returns to land and labor are given by

$$p(t) = \frac{\partial F(K(t), L(t), Z)}{\partial Z} = (1-\alpha-\beta)\frac{K(t)^\alpha L(t)^\beta}{Z^{\alpha+\beta}}$$

$$w(t) = \frac{\partial F(K(t), L(t), Z)}{\partial L(t)} = \beta\frac{K(t)^\alpha Z^{1-\alpha-\beta}}{L(t)^{1-\beta}}$$

with their time derivatives being

$$\dot{p}(t) = \frac{1-\alpha-\beta}{Z^{\alpha+\beta}}\left[\alpha K(t)^\alpha L(t)^\beta \frac{\dot{K}(t)}{K(t)} + \beta K(t)^\alpha L(t)^\beta \frac{\dot{L}(t)}{L(t)}\right]$$

$$= \alpha p(t)\frac{\dot{K}(t)}{K(t)} + \beta p(t)\frac{\dot{L}(t)}{L(t)}$$

$$= \alpha p(t)\left(\frac{\dot{k}(t)}{k(t)} + n\right) + \beta p(t) n$$

$$\dot{w}(t) = \frac{\beta}{[L(t)^{1-\beta}]^2}\left[\alpha K(t)^\alpha Z^{1-\alpha-\beta} L(t)^{1-\beta}\frac{\dot{K}(t)}{K(t)} - (1-\beta)K(t)^\alpha Z^{1-\alpha-\beta} L(t)^{1-\beta}\frac{\dot{L}(t)}{L(t)}\right]$$

$$= \alpha w(t)\frac{\dot{K}(t)}{K(t)} - (1-\beta)w(t)\frac{\dot{L}(t)}{L(t)}$$

$$= \alpha w(t)\left(\frac{\dot{k}(t)}{k(t)} + n\right) - (1-\beta)w(t) n$$

As a result the growth rates are given by

$$\frac{\dot{p}(t)}{p(t)} = \alpha\frac{\dot{k}(t)}{k(t)} - (\alpha+\beta)n$$

$$\frac{\dot{w}(t)}{w(t)} - \alpha\frac{\dot{k}(t)}{k(t)} - (1-\alpha-\beta)n$$

As $\lim_{t\to\infty}\frac{\dot{k}(t)}{k(t)} = 0$

$$\lim_{t\to\infty}\frac{\dot{p}(t)}{p(t)} = (\alpha+\beta)n > 0$$

$$\lim_{t\to\infty}\frac{\dot{w}(t)}{w(t)} = -(1-\alpha-\beta)n < 0$$

Therefore the returns to land and labor in the limit become

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} p(0)e^{(\alpha+\beta)nt} = \infty$$

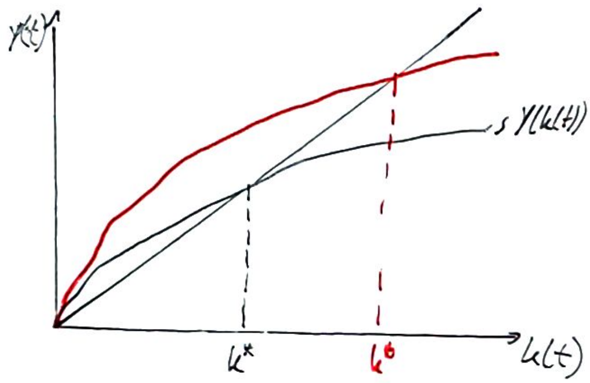
$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty} w(0)e^{-(1-\alpha-\beta)nt} = 0$$

Intuition: as time goes to infinity, $L(t)$ increases without bound. As a result the return to labor will converge to zero since the abundance of extra work leads to a smaller marginal product of labor. Since factors are paid their marginal product in perfect competition the return to labor (and therefore the wage rate) will be zero. The opposite is true for the return to land, where the marginal product will approach infinity.

2.4 The x-axis is $k(t)$, the y axis is $y(t)$

The break-even line is unaffected. The investment curve shifts upwards leading to a higher steady-state capital-labor ratio. Sufficient explanations are:

- more output at each and every level of capital is possible thus leading to higher investment
- an increase in land acts as a one time exogenous technology increase that is permanent thus leading to higher investment



3. 3.1 The law of motion of capital per capita is given by

$$\frac{K_{t+1}}{L_{t+1}} = k_{t+1} = \frac{s \cdot Y_t + (1-\delta) K_t}{(1+n) L_t} = \frac{s}{1+n} \cdot y_t + \frac{1-\delta}{1+n} k_t$$

which simplifies to $k_{t-1} = \frac{s}{1+n} k_t^{\alpha} + \frac{1-d}{1+n} k_t$

Therefore the steady-state capital-labor ratio is given by

$$k^* = \left[\frac{s}{n+\delta}\right]^{\frac{1}{1-\alpha}}$$

3.2 - 3.4 The model needs initial values $K(0)$ and $L(0)$. For details see Python file.