Macroeconomics - Growth

Endogenous Growth

Gerrit Meyerheim

Economics Department
LMU München

Winter Semester 20/21

Steady-State vs. Balanced Growth

Candidates for endogenous growth:

- Constant marginal product of capital. (Why?)
- Examples:
  1. production is linear in capital (AK).
  2. externalities in the accumulation of physical capital (Romer).
  3. productive government expenditure (Barro).
  4. human capital (second source of accumulation (Lucas).
- Knowledge/Technology is accumulated similar to physical capital.

The AK-Model

- Consider a neo-classical growth model where total output is linear in capital, that is

$$Y(t) = AK(t) ,$$

  with $A > 0$.
- Under some parametric restrictions this implies that the economy grows at a constant rate.
- Problems:
  - The assumption directly generates the result.
  - No role for labor at all.
  - No diminishing returns to capital.
  - No transitional dynamics.

Capital Externalities

Romer model:

- P. Romer (1986): "Increasing Returns and Long-Run Growth", *Journal of Political Economy* 94(5), pp. 1002–1037.
- Neo-classical growth model with production externalities.
- Total factor productivity is proportional to the accumulated capital stock per worker, $k(t) = \frac{K(t)}{L(t)}$.
- As a result any capital use by a firm produces an externality in the form of an increased productivity for all other firms in the economy.
- The size of a firm is assumed to be very small. As such the firm does not take this positive spillover into account.
- Technological progress is thus explained within the model as the result of a production externality.
- Important: the marginal product of capital is not necessarily decreasing.

Production:

The production of a firm is constant returns to scale if the externality is ignored, that is

$$F\left(K(t), L(t), k(t)\right) = \underbrace{A(t)}_{=k(t)^{\psi}} K(t)^{\alpha} L(t)^{1-\alpha} ,$$

where $k(t)^{\psi}$ is the externality of physical capital. Re-writing in intensive form yields

$$f\left(k(t)\right) = A(t)k(t)^{\alpha} = k(t)^{\alpha+\psi} .$$

Important:

Firms ignore their influence on the aggregate capital stock. Therefore the private return to capital is computed as if $k(t)^{\psi}$ were an exogenous constant, that is

$$R(t)_{\text{private}} = \frac{\partial A(t)k(t)^{\alpha}}{\partial k(t)} = \alpha A(t)k(t)^{\alpha-1} = \alpha k(t)^{\alpha+\psi-1}$$

$$R(t)_{\text{social}} = \frac{\partial k(t)^{\alpha+\psi}}{\partial k(t)} = (\alpha+\psi)k(t)^{\alpha+\psi-1} > R(t)_{\text{private}}$$

Detour: Externalities

Profit maximization of firms not taking the externality into account is given by

$$\max_{K(t),L(t)} \pi\left(K(t),L(t)\right) = A(t)K(t)^{\alpha}L(t)^{1-\alpha} - R(t)K(t) - w(t)L(t)\ .$$

In intensive form

$$\max_{k(t)} \pi\left(k(t)\right) = A(t)k(t)^{\alpha} - R(t)k(t) - w(t)\ .$$

As a result the return to capital is given by

$$\frac{\partial \pi\left(k(t)\right)}{\partial k(t)} = \alpha A(t)k(t)^{\alpha-1} - R(t) = 0 \quad \Rightarrow \; R(t) = \alpha A(t)k(t)^{\alpha-1}\ ,$$

with the wage $w(t)$ given by

$$w(t) = A(t)k(t)^{\alpha} - R(t)k(t) = (1-\alpha)A(t)k(t)^{\alpha}\ .$$

Detour: Externalities

Inserting $A(t) = k(t)^{\psi}$ gives

$$
\begin{aligned}
R(t) =& \alpha k(t)^{\alpha+\psi-1} \\
w(t) =& (1-\alpha) k(t)^{\alpha+\psi}
\end{aligned}
$$

$$
\Rightarrow \; f\left(k(t)\right) = k(t)^{\alpha+\psi} = A(t)k(t)^{\alpha} = R(t)k(t) + w(t)
$$

The profit maximization if the externality is taken into account is given by

$$
\max_{k(t)} \; \pi\left(k(t)\right) = k(t)^{\alpha+\psi} - R(t)k(t) - w(t) \; .
$$

As a result the return to capital and the wage are given by

$$
\begin{aligned}
R(t) =& (\alpha+\psi) k(t)^{\alpha+\psi-1} \\
w(t) =& (1-\alpha-\psi) k(t)^{\alpha+\psi}
\end{aligned}
$$

$$
\Rightarrow \; f\left(k(t)\right) = k(t)^{\alpha+\psi} = R(t)k(t) + w(t)
$$

Detour: Externalities

As a result the total amount of factor renumeration is identical regardless of taking the externality into account.

However, what has changed is the allocation of resources. When taking the externality into account capital is re-numerated at a higher rate than without the externality (as the indirect effect is taken into account)

$$(\alpha + \psi)k(t)^{\alpha+\psi-1} > \alpha k(t)^{\alpha+\psi-1}$$

As the total amount of factor renumeration is identical the wage is lower when taking the externality into account

$$(1 - \alpha - \psi)k(t)^{\alpha+\psi} < (1 - \alpha)k(t)^{\alpha+\psi}$$

Discussion:

- The production function has constant returns to scale on the firm level but increasing returns to scale on the aggregate level.
- The requirement that firms take $k(t)$ (i.e. the aggregate capital stock per worker) as given is needed since otherwise the production function would exhibit increasing returns to scale at the firm level.
- The assumption is justifiable for small firms.

Capital Externalities

Equilibrium:

- Optimization on the household side is one-to-one identical to the baseline neo-classical growth model without technological progress.
- As a result the economy is again fully described by the evolution of capital and consumption

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\theta}$$
$$\dot{k}(t) = f\left(k(t)\right) - (n+\delta)k(t) - c(t)$$

- The equilibrium conditions for the economy are also identical apart from the fact that the return to capital must be given by the private return (in a competitive equilibrium).
- Balanced growth implies

$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{k}(t)}{k(t)} = \gamma\ ,$$

  where γ is constant.
- What condition(s) need to be fulfilled to admit a balanced growth path?

Inserting the private return to capital in to the Euler equation gives

$$\frac{\dot{c}(t)}{c(t)} = \frac{\alpha k(t)^{\alpha+\psi-1} - \delta - \rho}{\theta} \,.$$

Note that if $\psi = 0$ this corresponds to the standard neo-classical growth model. Additionally, there are three possible cases

- $\alpha + \psi < 1$: diminishing returns. That is, as $k(t)$ increases, consumption growth goes to zero. Identical in behaviour to the neo-classical growth model.
- $\alpha + \psi > 1$: increasing returns. This implies exploding growth, i.e. the economy grows at an ever increasing rate.
- $\alpha + \psi = 1$: linear returns. This the only case that admits a balanced growth path.

If $\alpha + \psi = 1$ the Euler equation becomes

$$\frac{\dot{c}(t)}{c(t)} = \frac{\alpha - \delta - \rho}{\theta} \ .$$

In order to ensure positive consumption growth $\alpha > \rho + \delta$ has to hold.

Integrating this differential equation and eliminating the constant of integration gives the particular solution

$$c(t) = c(0) \cdot e^{\frac{\alpha - \delta - \rho}{\theta} t} \ .$$

Inserting the result into the law of motion for capital yields

$$\begin{aligned}
\dot{k}(t) =& f\left(k(t)\right) - (n+\delta)k(t) - c(t) \\
\dot{k}(t) =& k(t)^{\alpha+\psi} - (n+\delta)k(t) - c(t) \\
\dot{k}(t) =& k(t) - (n+\delta)k(t) - c(t) \\
\dot{k}(t) =& \underbrace{[1 - (n+\delta)]}_{\equiv b} k(t) - c(0) \cdot e^{\frac{\alpha - \delta - \rho}{\theta} t}
\end{aligned}$$

The differential equation

$$\dot{k}(t) = bk(t) - c(0) \cdot e^{\frac{\alpha-\delta-\rho}{\theta}t}$$

can be solved by multiplying both sides with the **integrating factor** e^{-bt}

$$\begin{aligned}
\dot{k}(t)e^{-bt} =& bk(t)e^{-bt} - c(0) \cdot e^{\frac{\alpha-\delta-\rho}{\theta}t}e^{-bt} \\
\dot{k}(t)e^{-bt} - bk(t)e^{-bt} =& - c(0) \cdot e^{\frac{\alpha-\delta-\rho-\theta b}{\theta}t} \\
\int \dot{k}(t)e^{-bt} - bk(t)e^{-bt}\, dt =& - c(0) \int e^{\frac{\alpha-\delta-\rho-\theta b}{\theta}t}\, dt \\
k(t)e^{-bt} + a_1 =& - \frac{\theta c(0)}{\alpha-\delta-\rho-\theta b} e^{\frac{\alpha-\delta-\rho-\theta b}{\theta}t} + a_2 \\
k(t) =& \frac{\theta c(0)}{\theta b - (\alpha-\delta-\rho)} e^{\frac{\alpha-\delta-\rho}{\theta}t} + a_3 e^{bt} \\
k(t) =& \frac{\theta c(0)}{\rho+\delta-\{\alpha-[1-(n+\delta)]\,\theta\}} e^{\frac{\alpha-\delta-\rho}{\theta}t} + a_3 e^{[1-(n+\delta)]t}
\end{aligned}$$

Parametric restrictions:

In order for capital not to shrink the following conditions have to hold

$$\begin{aligned} \rho + \delta &< \alpha \\ \rho + \delta &> \alpha - [1 - (n + \delta)]\, \theta \end{aligned}$$

As a result

$$\rho + \delta < \alpha < \rho + \delta - [1 - (n + \delta)]\, \theta$$

Note that $n + \delta < 1$ as $\rho + \delta < \alpha$ with $\alpha \in (0, 1)$ and $\rho > n$.

Determining initial values:

Inserting consumption into the transversality condition yields

$$
\begin{aligned}
\lim_{t\to\infty} e^{-(\rho-n)t}\mu(t)k(t) =& 0\\
\lim_{t\to\infty} e^{-(\rho-n)t}c(t)^{-\theta}k(t) =& 0\\
\lim_{t\to\infty} e^{-(\rho-n)t}\left(c(0)\cdot e^{\frac{\alpha-\delta-\rho}{\theta}t}\right)^{-\theta}k(t) =& 0\\
\lim_{t\to\infty} e^{-(\rho-n)t}c(0)^{-\theta}e^{-(\alpha-\delta-\rho)t}k(t) =& 0\\
\lim_{t\to\infty} e^{-(\alpha-n-\delta)t}c(0)^{-\theta}k(t) =& 0\\
\lim_{t\to\infty} e^{-(\alpha-n-\delta)t}k(t) =& 0
\end{aligned}
$$

where the last line follows from the fact that $c(0)$ is always greater than zero for any $k(0) > 0$.

Inserting the result for capital gives

$$\lim_{t\to\infty} e^{-(\alpha-n-\delta)t}\left[\frac{\theta c(0)}{\rho+\delta-\{\alpha-[1-(n+\delta)]\,\theta\}}e^{\frac{\alpha-\delta-\rho}{\theta}t}+a_3e^{-[1-(n+\delta)]t}\right]=0$$

$$\lim_{t\to\infty}\left[\underbrace{\frac{\theta c(0)}{\rho+\delta-\{\alpha-[1-(n+\delta)]\,\theta\}}e^{\frac{(1-\theta)(\alpha-\delta)-\rho+\theta n}{\theta}t}}_{\equiv x(t)}+\underbrace{a_3e^{(1-\alpha)t}}_{\equiv z(t)}\right]=0$$

In order for $x(t)$ to converge to zero $(1-\theta)(\alpha-\delta)-\rho+\theta n<0$ has to hold.

This implies a stricter restriction than previously applied, namely

$$\alpha-[\alpha-(n+\delta)]\,\theta<\rho+\delta\ .$$

Additionally, $z(t)$ becomes zero if and only if $a_3=0$.

Capital Externalities

As a result the path of capital is given by

$$k(t) = \frac{\theta c(0)}{\rho + \delta - \{\alpha - [1 - (n + \delta)]\, \theta\}} e^{\frac{\alpha - \delta - \rho}{\theta} t}$$

Solving for $c(0)$ at $t = 0$ gives

$$c(0) = \frac{\rho + \delta - \{\alpha - [1 - (n + \delta)]\, \theta\}}{\theta} k(0) \; .$$

Inserting the result back into the paths of consumption and capital yields

$$c(t) = \frac{\rho + \delta - \{\alpha - [1 - (n + \delta)]\, \theta\}}{\theta} k(0) \cdot e^{\frac{\alpha - \delta - \rho}{\theta} t}$$

$$k(t) = k(0) \cdot e^{\frac{\alpha - \delta - \rho}{\theta} t}$$

Capital Externalities

Balanced growth path:

The growth rates of consumption and capital on the balanced growth path are given by

$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{k}(t)}{k(t)} = \gamma_{\text{equilibrium}} = \frac{\alpha - \delta - \rho}{\theta}$$

This is the result of overcoming diminishing returns by linearizing the returns to accumulation.

As the externality is not internalised the growth rate in a decentralized equilibrium differs from the growth rate in a social planner allocation, which is given by

$$\gamma_{\text{planner}} = \frac{\alpha + \psi - \delta - \rho}{\theta} > \frac{\alpha - \delta - \rho}{\theta} = \gamma_{\text{equilibrium}}$$

Capital Externalities

Implications:

- The decentralized market outcome is not Pareto optimal. This is more pronounced the greater the externality. In fact a planner economy would be $e^{\frac{\psi}{\theta}t}$ times richer at time t, as (remember that $\alpha + \psi = 1$ on a balanced growth path)

$$\frac{y(t)_{\text{planner}}}{y(t)_{\text{equilibrium}}} = \frac{[k(t)_{\text{planner}}]^{\alpha+\psi}}{[k(t)_{\text{equilibrium}}]^{\alpha+\psi}} = \left(\frac{k(0)\cdot e^{\frac{\alpha+\psi-\delta-\rho}{\theta}t}}{k(0)\cdot e^{\frac{\alpha-\delta-\rho}{\theta}t}}\right)^{\alpha+\psi} = e^{\frac{\psi}{\theta}t}$$

- Similarly, the model implies divergent incomes across countries for countries with larger capital shares (or higher patience). A country with a higher capital share α is $e^{\frac{\Delta\alpha}{\theta}t}$ richer at time t. A country with higher patience (lower ρ) is $e^{\frac{\Delta\rho}{\theta}t}$ richer at time t.

Barro model:

- R. Barro (1990): "Government Spending in a Simple Model of Endogenous Growth", *Journal of Political Economy* 98(5), pp. 103–125.
- Neo-classical growth model with externalities from government expenditure.
- Total output is given by

$$F\left(K(t), L(t), g(t)\right) = \underbrace{A(t)}_{=g(t)^{\beta}} L(t)^{\alpha} K(t)^{1-\alpha} ,$$

  where $g(t)$ is government expenditure per capita.
- Re-writing in intensive form yields

$$f\left(k(t), g(t)\right) = A(t)k(t)^{1-\alpha} = g(t)^{\beta} k(t)^{1-\alpha} .$$

- Government expenditure is financed by a tax τ on output, that is

$$g(t) = \tau f\left(k(t), g(t)\right) .$$

- Households take the level of taxes and the level of government expenditure as given.

Government Spending

Production with internalised government spending:

Government expenditure is a function of $k(t)$ since

$$
\begin{aligned}
g(t) =& \tau f\left(k(t), g(t)\right) = \tau g(t)^{\beta} k(t)^{1-\alpha} \\
g(t) =& \tau^{\frac{1}{1-\beta}} k(t)^{\frac{1-\alpha}{1-\beta}}
\end{aligned}
$$

Inserting the result back into output gives

$$
y(t) = \left(\tau^{\frac{1}{1-\beta}} k(t)^{\frac{1-\alpha}{1-\beta}}\right)^{\beta} k(t)^{1-\alpha} = \tau^{\frac{\beta}{1-\beta}} k(t)^{\frac{1-\alpha}{1-\beta}}
$$

Important:

As households take the level of government expenditure as given the private return to capital is computed as if $g(t)^{\beta}$ were an exogenous constant

$$
\begin{aligned}
R(t)_{\text{private}} =& \frac{\partial g(t)^{\beta} k(t)^{1-\alpha}}{\partial k(t)} = (1-\alpha)\tau^{\frac{\beta}{1-\beta}} k(t)^{\frac{1-\alpha}{1-\beta}-1} \\
R(t)_{\text{social}} =& \frac{\partial \tau^{\frac{\beta}{1-\beta}} k(t)^{\frac{1-\alpha}{1-\beta}}}{\partial k(t)} = \frac{1-\alpha}{1-\beta}\tau^{\frac{\beta}{1-\beta}} k(t)^{\frac{1-\alpha}{1-\beta}-1}
\end{aligned}
$$

Optimization:

- The law of motion for assets is given by

$$\dot{a}(t) = [(1-\tau)R(t) - \delta - n]\, a(t) + (1-\tau)w(t) - c(t)\ ,$$

  where $(1-\tau)R(t)$ and $(1-\tau)w(t)$ are the after-tax returns to capital and labor, respectively.

- Solving the maximization problem gives the Euler equation as

$$\frac{\dot{c}(t)}{c(t)} = \frac{(1-\tau)R(t) - \delta - \rho}{\theta}\ .$$

- The equilibrium conditions for the economy are again identical to the neo-classical growth model apart from the fact that the return to capital must be given by the private return (in a competitive equilibrium).

Government Spending

Equilibrium:

Inserting the private return to capital in to the Euler equation gives

$$\frac{\dot{c}(t)}{c(t)} = \frac{(1-\tau)(1-\alpha)\tau^{\frac{\beta}{1-\beta}} k(t)^{\frac{1-\alpha}{1-\beta}-1} - \delta - \rho}{\theta} \, .$$

Note that if $\beta = 0$ this corresponds to the standard neo-classical growth model. Again, there are three possible cases

- $\beta < \alpha$: diminishing returns. That is, as $k(t)$ increases, consumption growth goes to zero. Identical in behaviour to the neo-classical growth model.
- $\beta > \alpha$: increasing returns. This implies exploding growth, i.e. the economy grows at an ever increasing rate.
- $\beta = \alpha$: linear returns. This the only case that admits a balanced growth path.

Balanced growth path:

Applying the same solution techniques as in the Romer model gives the paths of consumption and capital as

$$c(t) = \frac{\rho + \delta - \left[(1-\tau)(1-\alpha-\theta)\tau^{\frac{\beta}{1-\beta}} + (n+\delta)\theta\right]}{\theta} k(0) \cdot e^{\frac{(1-\tau)(1-\alpha)\tau^{\frac{\beta}{1-\beta}} - \delta - \rho}{\theta} t}$$

$$k(t) = k(0) \cdot e^{\frac{(1-\tau)(1-\alpha)\tau^{\frac{\beta}{1-\beta}} - \delta - \rho}{\theta} t}$$

The growth rates of consumption and capital on the balanced growth path are given by

$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{k}(t)}{k(t)} = \gamma_{\text{equilibrium}} = \frac{(1-\tau)(1-\alpha)\tau^{\frac{\beta}{1-\beta}} - \delta - \rho}{\theta} .$$

Important:

The growth rates now depend on policy (size of τ).

Optimal equilibrium tax rate:

Differentiating $\gamma_{\text{equilibrium}}$ with respect to the tax rate yields

$$\frac{\partial \gamma_{\text{equilibrium}}}{\partial \tau} = \frac{1}{\theta}\left(\frac{\beta}{\tau} - 1\right)\frac{1-\alpha}{1-\beta}\tau^{\frac{\beta}{1-\beta}} ,$$

with second derivative

$$\frac{\partial^2 \gamma_{\text{equilibrium}}}{\partial \tau^2} = \frac{1}{\theta}\left(\frac{2\beta - 1}{\tau} - 1\right)\frac{\beta(1-\alpha)}{(1-\beta)^2}\tau^{\frac{\beta}{1-\beta}-1} .$$

Solving $\frac{\partial \gamma_{\text{equilibrium}}}{\partial \tau} = 0$ for τ gives

$$\tau^*_{\text{equilibrium}} = \beta ,$$

which is a maximum for all $\beta \in (0,1)$ as

$$\left.\frac{\partial^2 \gamma_{\text{equilibrium}}}{\partial \tau^2}\right|_{\tau=\tau^*_{\text{equilibrium}}} = \frac{1}{\theta}\left(1 - \frac{1}{\beta}\right)\frac{\beta(1-\alpha)}{(1-\beta)^2}\beta^{\frac{\beta}{1-\beta}-1} < 0 \quad \forall \beta \in (0,1)$$

Government Spending

Optimal growth:

The decentralized equilibrium is not Pareto optimal as the growth rate of the social planner is given by (remember that $\beta = \alpha$ on a balanced growth path)

$$\gamma_{\text{planner}} = \frac{(1-\tau)\frac{1-\alpha}{1-\beta}\tau^{\frac{\beta}{1-\beta}} - \delta - \rho}{\theta} = \frac{(1-\tau)\tau^{\frac{\beta}{1-\beta}} - \delta - \rho}{\theta} > \gamma_{\text{equilibrium}}$$

The decentralized growth rate can be larger than the planner rate given optimal taxation in a decentralized equilibrium and sub-optimal taxation under the planner, namely

$$\begin{aligned} \gamma_{\tau=\tau^*_{\text{equilibrium}}} = \frac{(1-\beta)(1-\alpha)\beta^{\frac{\beta}{1-\beta}} - \delta - \rho}{\theta} &> \frac{(1-\tau)\tau^{\frac{\beta}{1-\beta}} - \delta - \rho}{\theta} = \gamma_{\text{planner}} \\ (1-\alpha)(1-\beta)\beta^{\frac{\beta}{1-\beta}} &> (1-\tau)\tau^{\frac{\beta}{1-\beta}} \\ (1-\beta)^2\beta^{\frac{\beta}{1-\beta}} &> (1-\tau)\tau^{\frac{\beta}{1-\beta}} \end{aligned}$$

This holds for some $\tau \in (0,1)$. However, the planner selecting a sub-optimal τ is not convincing.

Government Spending

Optimal planner tax rate:

Differentiating γ_{planner} with respect to the tax rate yields

$$\frac{\partial \gamma_{\text{planner}}}{\partial \tau} = \frac{1}{\theta}\left(\frac{\beta}{1-\beta}\frac{1-\tau}{\tau} - 1\right)\tau^{\frac{\beta}{1-\beta}},$$

with second derivative

$$\frac{\partial^2 \gamma_{\text{planner}}}{\partial \tau^2} = -\frac{1}{\theta}\frac{1+\tau-2\beta}{1-\beta}\frac{\beta}{\tau(1-\beta)}\tau^{\frac{\beta}{1-\beta}-1}.$$

Solving $\frac{\partial \gamma_{\text{planner}}}{\partial \tau} = 0$ for τ gives

$$\tau^*_{\text{planner}} = \beta = \tau^*_{\text{equilibrium}}.$$

which is a maximum for all $\beta \in (0,1)$ as

$$\left.\frac{\partial^2 \gamma_{\text{planner}}}{\partial \tau^2}\right|_{\tau=\tau^*_{\text{planner}}} = -\frac{1}{\theta}\frac{1}{1-\beta}\beta^{\frac{\beta}{1-\beta}-1} < 0 \quad \forall \beta \in (0,1)$$

Discussion:

- Positive externalities of government expenditure lead to growth.
- The efficiency of government expenditure determines the growth path. As expenditure is given by

$$g(t) = \tau^{\frac{1}{1-\beta}} k(t)^{\frac{1-\alpha}{1-\beta}} \ ,$$

  this implies

  1. $\beta < \alpha$: government expenditure exhibits diminishing returns.
  2. $\beta > \alpha$: government expenditure exhibits increasing returns.
  3. $\beta = \alpha$: government expenditure exhibits linear returns.

- Deviation from optimal policies leads to divergent growth paths as in the Romer model.

Human Capital

Lucas model:

- R.E. Lucas (1988): "On the Mechanics of Economic Development", *Journal of Monetary Economics* 22(1), pp. 3–42.
- Neo-classical growth model with a second source of accumulation: human capital.
- Population: N workers with skill level $h \in (0, \infty)$

$$N(t) = \int_0^\infty N(h)\, dh \,.$$

- Average skill level

$$h_a(t) = \frac{\int_0^\infty hN(h)\, dh}{\int_0^\infty N(h)\, dh} \,.$$

- Labor supply: fraction $u(t)$ of time devoted to production, $1 - u(t)$ devoted to human capital accumulation.
- Effective labor force

$$N_e(t) = u(t) \int_0^\infty hN(h)\, dh = u(t) h_a(t) N(t) \,.$$

Production:

- Neo-classical production function with human capital externalities

$$F\left(K(t), N_e(t), h_a(t)\right) = \underbrace{A(t)}_{=h_a(t)^\gamma} K(t)^\alpha N_e(t)^{1-\alpha} .$$

- Accumulation of physical capital

$$\dot{K}(t) = F\left(K(t), N_e(t), h_a(t)\right) - \delta K(t) - c(t)N(t) .$$

- Accumulation of human capital

$$\dot{h}_a(t) = [1 - u(t)]\, \psi h_a(t) .$$

- The household side is identical to the standard neo-classical growth model without population growth.

Optimization:

The current-value Hamiltonian is given by

$$
\begin{aligned}
\hat{\mathcal{H}}\left(c(t), u(t), K(t), h_a(t), \mu_1(t), \mu_2(t)\right) = & N(t)\frac{c(t)^{1-\theta}-1}{1-\theta} \\
& + \mu_1(t)\left[F\left(K(t), N_e(t), h_a(t)\right) - \delta K(t) - c(t)N(t)\right] \\
& + \mu_2(t)\left[\left[1-u(t)\right]\psi h(t)\right]
\end{aligned}
$$

Note that this problem contains two control variables $(c(t), u(t))$ and two state variables $(K(t), h_a(t))$.

Important:

The human capital externality has an effect on the wage but is not internalised in the accumulation process.

The optimality conditions are given by the first-order conditions

$$
\begin{aligned}
c(t)^{-\theta} =& \mu_1(t) \\
\mu_2(t)\psi h_a(t) =& \mu_1(t)(1-\alpha)\left(\frac{K(t)}{N_e(t)}\right)^{\alpha} N(t)h_a(t)^{1+\gamma} \\
\dot{\mu}_1(t) =& \rho\mu_1(t) - \mu_1(t)\left[\alpha\left(\frac{N_e(t)}{K(t)}\right)^{1-\alpha} h_a^{\gamma} - \delta\right] \\
\dot{\mu}_2(t) =& \rho\mu_2(t) - \mu_1(t)(1-\alpha+\gamma)\left(\frac{K(t)}{N_e(t)}\right)^{\alpha} u(t)N(t)h_a(t)^{\gamma} - \mu_2(t)\left[1-u(t)\right]\psi
\end{aligned}
$$

and transversality conditions

$$
\begin{aligned}
\lim_{t\to\infty} e^{-\rho t}\mu_1(t)K(t) =& 0 \\
\lim_{t\to\infty} e^{-\rho t}\mu_2(t)h_a(t) =& 0
\end{aligned}
$$

Combining first-order conditions gives the Euler equation for consumption

$$\frac{\dot{c}(t)}{c(t)} = \frac{\alpha\left(\frac{N_e(t)}{K(t)}\right)^{1-\alpha} h_a^{\gamma} - \delta - \rho}{\theta} .$$

Taking the log of the first order condition with respect to $u(t)$ and differentiating with respect to time yields

$$\begin{aligned}
\mu_2(t)\psi h_a(t) =& \mu_1(t)(1-\alpha)K(t)^{\alpha}N(t)^{1-\alpha}u(t)^{-\alpha}h_a(t)^{1-\alpha+\gamma} \\
\frac{\dot{\mu}_2(t)}{\mu_2(t)} + \frac{\dot{h}_a(t)}{h_a(t)} =& \frac{\dot{\mu}_1(t)}{\mu(t)} + \alpha\frac{\dot{K}(t)}{K(t)} + (1-\alpha)\underbrace{\frac{\dot{N}(t)}{N(t)}}_{=0} - \alpha\frac{\dot{u}(t)}{u(t)} + (1-\alpha+\gamma)\frac{\dot{h}_a(t)}{h_a(t)} \\
\frac{\dot{\mu}_2(t)}{\mu_2(t)} - \frac{\dot{\mu}_1(t)}{\mu(t)} =& \alpha\left(\frac{\dot{K}(t)}{K(t)} - \frac{\dot{u}(t)}{u(t)}\right) + (\gamma-\alpha)\frac{\dot{h}_a(t)}{h_a(t)} \\
\frac{\dot{\mu}_2(t)}{\mu_2(t)} + \theta\frac{\dot{c}(t)}{c(t)} =& \alpha\left(\frac{Y(t)}{K(t)} - \delta - \frac{c(t)N(t)}{K(t)} - \frac{\dot{u}(t)}{u(t)}\right) + (\gamma-\alpha)\left[1-u(t)\right]\psi \\
\frac{\dot{\mu}_2(t)}{\mu_2(t)} + \alpha\frac{Y(t)}{K(t)} - \delta - \rho =& \alpha\left(\frac{Y(t)}{K(t)} - \delta - \frac{c(t)N(t)}{K(t)} - \frac{\dot{u}(t)}{u(t)}\right) + (\gamma-\alpha)\left[1-u(t)\right]\psi
\end{aligned}$$

Continued:

$$
\begin{aligned}
\frac{\dot{\mu}_2(t)}{\mu_2(t)} - \rho =& \delta(1-\alpha) - \alpha\left(\frac{c(t)}{k(t)} + \frac{\dot{u}(t)}{u(t)}\right) + (\gamma - \alpha)\left[1 - u(t)\right]\psi \\
\rho - \frac{\mu_1(t)}{\mu_2(t)}(1-\alpha+\gamma)\frac{Y(t)}{h_a(t)} - \left[1-u(t)\right]\psi - \rho =& \delta(1-\alpha) - \alpha\left(\frac{c(t)}{k(t)} + \frac{\dot{u}(t)}{u(t)}\right) + (\gamma - \alpha)\left[1 - u(t)\right]\psi \\
\alpha\frac{\dot{u}(t)}{u(t)} - \frac{\mu_1(t)}{\mu_2(t)}(1-\alpha+\gamma)\frac{Y(t)}{h_a(t)} =& \delta(1-\alpha) - \alpha\frac{c(t)}{k(t)} + (1-\alpha+\gamma)\left[1-u(t)\right]\psi \\
\alpha\frac{\dot{u}(t)}{u(t)} - \frac{\psi h_a(t)}{(1-\alpha)\frac{Y(t)}{u(t)}}(1-\alpha+\gamma)\frac{Y(t)}{h_a(t)} =& \delta(1-\alpha) - \alpha\frac{c(t)}{k(t)} + (1-\alpha+\gamma)\left[1-u(t)\right]\psi \\
\alpha\frac{\dot{u}(t)}{u(t)} - \frac{(1-\alpha+\gamma)\psi u(t)}{1-\alpha} =& \delta(1-\alpha) - \alpha\frac{c(t)}{k(t)} + (1-\alpha+\gamma)\left[1-u(t)\right]\psi
\end{aligned}
$$

Euler equation for education

$$
\frac{\dot{u}(t)}{u(t)} = \frac{(1-\alpha+\gamma)\psi + (1-\alpha)\delta}{\alpha} - \frac{c(t)}{k(t)} + \frac{(1-\alpha+\gamma)\psi u(t)}{1-\alpha} .
$$

Human Capital

As a result the system is characterised by four differential equations

$$\frac{\dot{c}(t)}{c(t)} = \frac{\alpha \frac{Y(t)}{K(t)} - \delta - \rho}{\theta}$$

$$\frac{\dot{u}(t)}{u(t)} = \frac{(1-\alpha+\gamma)\psi + (1-\alpha)\delta}{\alpha} - \frac{c(t)}{k(t)} + \frac{(1-\alpha+\gamma)\psi u(t)}{1-\alpha}$$

$$\frac{\dot{K}(t)}{K(t)} = \frac{Y(t)}{K(t)} - \delta - \frac{c(t)N(t)}{K(t)}$$

$$\frac{\dot{h}_a(t)}{h_a(t)} = [1-u(t)]\,\psi$$

Human Capital

Balanced growth path:

Balanced growth requires that human capital grows at a constant rate

$$\frac{\dot{h}_a(t)}{h_a(t)} = [1 - u(t)]\,\psi = \gamma_h \; .$$

This is only possible if education (i.e. the fraction of time spent acquiring human capital) is constant, that is $u(t) = \bar{u}$

$$(1 - \bar{u})\psi = \gamma_h \;\Rightarrow\; \bar{u} = 1 - \frac{\gamma_h}{\psi} \; .$$

Constant education implies that $\dot{u}(t) = 0$, that is

$$0 = \frac{(1 - \alpha + \gamma)\psi + (1 - \alpha)\delta}{\alpha} - \frac{c(t)}{k(t)} + \frac{(1 - \alpha + \gamma)\psi\bar{u}}{1 - \alpha}$$

Re-arranging gives

$$\frac{c(t)}{k(t)} = \frac{c(t)N(t)}{K(t)} = \frac{(1-\alpha+\gamma)\left[1-\alpha\left(1-\bar{u}\right)\right]\psi}{\alpha(1-\alpha)} + \frac{(1-\alpha)\delta}{\alpha} .$$

As the right-hand side contains only constants differentiating with respect to time implies

$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{K}(t)}{K(t)}$$

$$\frac{\alpha\frac{Y(t)}{K(t)} - \delta - \rho}{\theta} = \frac{Y(t)}{K(t)} - \delta - \frac{c(t)N(t)}{K(t)}$$

$$(\alpha-\theta)\frac{Y(t)}{K(t)} = \rho + (1-\theta)\delta - \theta\frac{c(t)N(t)}{K(t)}$$

Continued:

$$\frac{(\alpha-\theta)N_e(t)^{1-\alpha}h_a(t)^{\gamma}}{K(t)^{1-\alpha}} = \rho + (1-\theta)\delta - \frac{\theta(1-\alpha+\gamma)\left[1-\alpha\left(1-\bar{u}\right)\right]\psi}{\alpha(1-\alpha)} - \frac{\theta(1-\alpha)\delta}{\alpha}$$

$$\frac{(\alpha-\theta)N_e(t)^{1-\alpha}h_a(t)^{\gamma}}{K(t)^{1-\alpha}} = \underbrace{\frac{\alpha(1-\alpha)\rho - \theta(1-\alpha+\gamma)\left[1-\alpha\left(1-\bar{u}\right)\right]\psi - (1-\alpha)(\alpha-\theta)\delta}{\alpha(1-\alpha)}}_{\equiv\tilde{\chi}(\bar{u})}$$

$$K(t) = \left(\frac{\alpha-\theta}{\tilde{\chi}(\bar{u})}\right)^{\frac{1}{1-\alpha}} h_a(t)^{\frac{\gamma}{1-\alpha}} N_e(t)$$

$$K(t) = \left(\frac{\alpha-\theta}{\tilde{\chi}(\bar{u})}\right)^{\frac{1}{1-\alpha}} h_a(t)^{\frac{\gamma}{1-\alpha}} \bar{u} h_a(t) N(t)$$

$$K(t) = \left(\frac{\alpha-\theta}{\tilde{\chi}(\bar{u})}\right)^{\frac{1}{1-\alpha}} \bar{u} N(t) h_a(t)^{\frac{1-\alpha+\gamma}{1-\alpha}}$$

Taking the log and differentiating with respect to time yields the growth rate

$$\frac{\dot{K}(t)}{K(t)} = \frac{1-\alpha+\gamma}{1-\alpha}\frac{\dot{h}_a(t)}{h(t)} .$$

As a result the growth rates of consumption and capital on a balanced growth path are given by

$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{K}(t)}{K(t)} = \frac{1-\alpha+\gamma}{1-\alpha}\frac{\dot{h}_a(t)}{h(t)}$$

Subsequently, consumption and capital grow with the growth rate of human capital.

Combining the Euler equation for consumption

$$\frac{\dot{c}(t)}{c(t)} = \frac{\alpha\frac{Y(t)}{K(t)} - \delta - \rho}{\theta} = \frac{1-\alpha+\gamma}{1-\alpha}\frac{\dot{h}_a(t)}{h(t)}$$

with the fact that $\alpha\frac{Y(t)}{K(t)}$ and $\frac{\dot{h}_a(t)}{h(t)}$ are given by

$$\begin{aligned}\alpha\frac{Y(t)}{K(t)} =& \frac{\alpha}{\alpha-\theta}\cdot\tilde{\chi}(\bar{u}) \\ \frac{\dot{h}_a(t)}{h_a(t)} =& (1-\bar{u})\psi\end{aligned}$$

allows solving for $\bar{u}$

$$\bar{u} = 1 - \frac{1-\frac{(1-\alpha)\rho}{(1-\alpha+\gamma)\psi}}{\theta}\ .$$

Human Capital

Parametric restrictions:

In order for $\bar{u} > 0$ the following has to hold

$$1 - \frac{1 - \frac{(1-\alpha)\rho}{(1-\alpha+\gamma)\psi}}{\theta} > 0 \iff \rho > \frac{(1-\theta)(1-\alpha+\gamma)\psi}{1-\alpha}$$

Conversely, in order for $\bar{u} < 1$ the following has to hold

$$1 - \frac{1 - \frac{(1-\alpha)\rho}{(1-\alpha+\gamma)\psi}}{\theta} < 1 \iff \rho < \frac{(1-\alpha+\gamma)\psi}{1-\alpha}$$

As result the following parametric restriction ensures $\bar{u} \in (0, 1)$

$$\frac{(1-\theta)(1-\alpha+\gamma)\psi}{1-\alpha} < \rho < \frac{(1-\alpha+\gamma)\psi}{1-\alpha}$$

Note that this is not sensitive to $\theta \gtreqless 1$.

As a result the growth rate of human capital on the balanced growth path is given by

$$\frac{\dot{h}_a(t)}{h_a(t)} = \frac{\psi - \frac{(1-\alpha)\rho}{1-\alpha+\gamma}}{\theta},$$

with the growth rates of consumption and capital given by

$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{K}(t)}{K(t)} = \frac{(1-\alpha+\gamma)\psi - (1-\alpha)\rho}{(1-\alpha)\theta}$$

Finally, the growth rate of output is

$$\begin{aligned}
\ln Y(t) =& \alpha \ln K(t) + (1-\alpha)\ln \bar{u} + (1-\alpha)\ln N(t) + (1-\alpha+\gamma)\ln h_a(t) \\
\frac{\dot{Y}(t)}{Y(t)} =& \alpha \frac{\dot{K}(t)}{K(t)} + (1-\alpha)\underbrace{\frac{\dot{N}(t)}{N(t)}}_{=0} + (1-\alpha+\gamma)\frac{\dot{h}_a(t)}{h_a(t)} \\
\frac{\dot{Y}(t)}{Y(t)} =& \frac{(1-\alpha+\gamma)\psi - (1-\alpha)\rho}{(1-\alpha)\theta}
\end{aligned}$$

Discussion:

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{(1-\alpha+\gamma)\psi - (1-\alpha)\rho}{(1-\alpha)\theta}$$

- The growth rate of output on the balanced growth path is positive even in the absence of externalities as, given $\gamma = 0$, $\psi > \rho$ is needed for an interior solution of $\bar{u}$.
- This result stems from the linear accumulation of human capital

$$\dot{h}_a(t) = (1-\bar{u})\psi h_a(t) \ .$$

- Ultimately, how the accumulation of human capital is modeled is a (micro-)empirical question.
- But are physical and human capital acutally important?

Evidence: Human Capital and Externalities

Growth empirics:

- N.G. Mankiw, D. Romer and D. Weil (1992): "A Contribution to the Empirics of Economic Growth", *Quarterly Journal of Economics* 107, pp. 407–437.
- Estimation of the Solow model extended for human capital

$$
\begin{aligned}
Y(t) =& K(t)^{\alpha} H(t)^{\beta} \left(A(t)L(t)\right)^{1-\alpha-\beta} \\
\tilde{y}(t) =& \tilde{k}(t)^{\alpha} \tilde{h}(t)^{\beta} \\
L(t) =& L(0) \cdot e^{nt} \\
A(t) =& A(0) \cdot e^{gt} \\
\dot{\tilde{k}}(t) =& s_k \tilde{y}(t) - (n+g+\delta)\tilde{k}(t) \\
\dot{\tilde{h}}(t) =& s_h \tilde{y}(t) - (n+g+\delta)\tilde{h}(t)
\end{aligned}
$$

where $\tilde{y}(t) = \frac{Y(t)}{A(t)L(t)}$, $\tilde{k}(t) = \frac{K(t)}{A(t)L(t)}$, $\tilde{h}(t) = \frac{H(t)}{A(t)L(t)}$ and $\{\alpha, \beta\} \in (0,1)$ with $\alpha + \beta < 1$.

- Assume a steady-state in the data and the model as the data generating process.

Steady state:

- Solving the laws of motion for the steady-state values of $\tilde{k}(t)$ and $\tilde{h}(t)$ gives

$$\tilde{k}^* = \left(\frac{s_k^{1-\beta} s_h^{\beta}}{n+g+\delta} \right)^{\frac{1}{1-\alpha-\beta}} \qquad \tilde{h}^* = \left(\frac{s_k^{\alpha} s_h^{1-\alpha}}{n+g+\delta} \right)^{\frac{1}{1-\alpha-\beta}}$$

- Income per capita is given by

$$\frac{Y(t)}{L(t)} = A(t)\tilde{k}(t)^{\alpha}\tilde{h}(t)^{\beta} \ .$$

- As a result log income per capita is

$$\ln \frac{Y(t)}{L(t)} = \ln A(0) + gt + \alpha \ln \tilde{k}(t) + \beta \ln \tilde{h}(t) \ .$$

- In a steady state this implies

$$\ln y^* = \ln A(0) + gt + \frac{\alpha}{1-\alpha-\beta} \ln s_k + \frac{\beta}{1-\alpha-\beta} \ln s_h - \frac{\alpha+\beta}{1-\alpha-\beta} \ln(n+g+\delta) \ .$$

Evidence: Human Capital and Externalities

Decomposing $A(0)$ into a constant and a country-specific term ε yields the specification

$$\ln y^* = a + gt + \frac{\alpha}{1-\alpha-\beta}\ln s_k + \frac{\beta}{1-\alpha-\beta}\ln s_h - \frac{\alpha+\beta}{1-\alpha-\beta}\ln(n+g+\delta) + \varepsilon \ ,$$

where income per capita depends on

1. The rate of investment in physical capital s_k given by the ratio of investment to GDP.
2. The rate of investment in human capital s_h given by the enrollment rate of the working age population in secondary schools.
3. The rate of population growth n given by the growth rate of the working age population.

Note: g and δ are assumed to be equal across countries.

If human capital is accumulated and an important factor of production, estimating the specification without accounting for human capital (i.e. $\beta = 0$) will lead to biased results.

Evidence: Human Capital and Externalities

TABLE I
ESTIMATION OF THE TEXTBOOK SOLOW MODEL

Dependent variable: log GDP per working-age person in 1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	5.48	5.36	7.97
	(1.59)	(1.55)	(2.48)
ln(I/GDP)	1.42	1.31	0.50
	(0.14)	(0.17)	(0.43)
$\ln(n + g + \delta)$	−1.97	−2.01	−0.76
	(0.56)	(0.53)	(0.84)
$\overline{R}^2$	0.59	0.59	0.01
s.e.e.	0.69	0.61	0.38
Restricted regression:			
CONSTANT	6.87	7.10	8.62
	(0.12)	(0.15)	(0.53)
$\ln(I/GDP) - \ln(n + g + \delta)$	1.48	1.43	0.56
	(0.12)	(0.14)	(0.36)
$\overline{R}^2$	0.59	0.59	0.06
s.e.e.	0.69	0.61	0.37
Test of restriction:			
p-value	0.38	0.26	0.79
Implied α	0.60	0.59	0.36
	(0.02)	(0.02)	(0.15)

Note. Standard errors are in parentheses. The investment and population growth rates are averages for the period 1960–1985. $(g + \delta)$ is assumed to be 0.05.

TABLE II
ESTIMATION OF THE AUGMENTED SOLOW MODEL

Dependent variable: log GDP per working-age person in 1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	6.89	7.81	8.63
	(1.17)	(1.19)	(2.19)
ln(I/GDP)	0.69	0.70	0.28
	(0.13)	(0.15)	(0.39)
$\ln(n + g + \delta)$	−1.73	−1.50	−1.07
	(0.41)	(0.40)	(0.75)
ln(SCHOOL)	0.66	0.73	0.76
	(0.07)	(0.10)	(0.29)
$\overline{R}^2$	0.78	0.77	0.24
s.e.e.	0.51	0.45	0.33
Restricted regression:			
CONSTANT	7.86	7.97	8.71
	(0.14)	(0.15)	(0.47)
$\ln(I/GDP) - \ln(n + g + \delta)$	0.73	0.71	0.29
	(0.12)	(0.14)	(0.33)
$\ln(SCHOOL) - \ln(n + g + \delta)$	0.67	0.74	0.76
	(0.07)	(0.09)	(0.28)
$\overline{R}^2$	0.78	0.77	0.28
s.e.e.	0.51	0.45	0.32
Test of restriction:			
p-value	0.41	0.89	0.97
Implied α	0.31	0.29	0.14
	(0.04)	(0.05)	(0.15)
Implied β	0.28	0.30	0.37
	(0.03)	(0.04)	(0.12)

Note. Standard errors are in parentheses. The investment and population growth rates are averages for the period 1960–1985. $(g + \delta)$ is assumed to be 0.05. SCHOOL is the average percentage of the working-age population in secondary school for the period 1960–1985.

Evidence: Human Capital and Externalities

Convergence:

- The Solow model predicts convergence to the steady state, not across countries (with different s, n, g etc.).
  - Absolute convergence: the steady state y^* is the same for all countries.
  - Conditional convergence: the steady states differ across countries, but the determinants of the steady states can be accounted for.
- The Solow model makes quantitative predictions about the speed of convergence.
- Endogenous growth models do not predict convergence. In fact they predict divergence.

Evidence: Human Capital and Externalities

TABLE III
Tests for Unconditional Convergence

Dependent variable: log difference GDP per working-age person 1960–1985

Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	−0.266	0.587	3.69
	(0.380)	(0.433)	(0.68)
ln(Y60)	0.0943	−0.00423	−0.341
	(0.0496)	(0.05484)	(0.079)
$\overline{R}^2$	0.03	−0.01	0.46
s.e.e.	0.44	0.41	0.18
Implied λ	−0.00360	0.00017	0.0167
	(0.00219)	(0.00218)	(0.0023)

Note. Standard errors are in parentheses. Y60 is GDP per working-age person in 1960.

TABLE IV
TESTS FOR CONDITIONAL CONVERGENCE

Dependent variable: log difference GDP per working-age person 1960–1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	1.93	2.23	2.19
	(0.83)	(0.86)	(1.17)
ln(Y60)	−0.141	−0.228	−0.351
	(0.052)	(0.057)	(0.066)
ln(I/GDP)	0.647	0.644	0.392
	(0.087)	(0.104)	(0.176)
$\ln(n + g + \delta)$	−0.299	−0.464	−0.753
	(0.304)	(0.307)	(0.341)
$\overline{R}^2$	0.38	0.35	0.62
s.e.e.	0.35	0.33	0.15
Implied λ	0.00606	0.0104	0.0173
	(0.00182)	(0.0019)	(0.0019)

Note. Standard errors are in parentheses. Y60 is GDP per working-age person in 1960. The investment and population growth rates are averages for the period 1960–1985. $(g + \delta)$ is assumed to be 0.05.

TABLE V
TESTS FOR CONDITIONAL CONVERGENCE

Dependent variable: log difference GDP per working-age person 1960–1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	3.04	3.69	2.81
	(0.83)	(0.91)	(1.19)
ln(Y60)	−0.289	−0.366	−0.398
	(0.062)	(0.067)	(0.070)
ln(I/GDP)	0.524	0.538	0.335
	(0.087)	(0.102)	(0.174)
$\ln(n + g + \delta)$	−0.505	−0.551	−0.844
	(0.288)	(0.288)	(0.334)
ln(SCHOOL)	0.233	0.271	0.223
	(0.060)	(0.081)	(0.144)
$\overline{R}^2$	0.46	0.43	0.65
s.e.e.	0.33	0.30	0.15
Implied λ	0.0137	0.0182	0.0203
	(0.0019)	(0.0020)	(0.0020)

Note. Standard errors are in parentheses. Y60 is GDP per working-age person in 1960. The investment and population growth rates are averages for the period 1960–1985. $(g + \delta)$ is assumed to be 0.05. SCHOOL is the average percentage of the working-age population in secondary school for the period 1960–1985.

Evidence: Human Capital and Externalities

Summary:

- The Solow model appears consistent with the data.
- Augmenting for human capital is important.
- There is evidence for convergence.
- There is no (or weak) evidence for capital externalities ($\alpha \approx 1/3$).

Summary: Endogenous Neo-Classical Growth

Endogenous growth can emerge from neo-classical growth models via means of (among others)

- externalities from physical capital investment (Romer, 1986).
- externalities from government expenditures (Barro, 1990).
- introducing a second source of accumulation (Lucas, 1988).

All include some mechanism that makes aggregate returns to accumulated factors linear, i.e. there is always some

$$\dot{x}(t) = gx(t) ,$$

that is either assumed (AK model) or arises from the model set-up. Models with strong concavity (especially with Inada conditions) cannot generate sustained growth.

Summary: Endogenous Neo-Classical Growth

Problems:

- Evidence for externalities is weak.
- While the standard neo-classical model has problems generating large income differences neo-classical endogenous growth models cannot generate convergence.
- Treating countries as separated islands without interaction is unlikely to be a good assumption.
- Technological improvements are usually the results of ideas (and investment) not purely learning-by-doing externalities.

Summary: Endogenous Neo-Classical Growth

Alternative modeling strategies:

How can the mapping from ideas into technology be modeled?

- The production function is the result of an ideas frontier (Jones, 2005).
- Innovation arises as creating ideas generates monopoly rents (e.g. Romer, 1990 and Aghion & Howitt, 1992).

$\vdots$

From Ideas to Production

Ideas and aggregate production:

- Houthakker (1955): the aggregate production function is the upper envelope of different ideas.
- In idea is a particular way of combining capital and labor in an individual production technique.
- The individual technique is Leontief, that is

$$Y(t) = \min\{b_i X(t), a_i Z(t)\} \ .$$

- When the set of ideas becomes very large, the resulting envelope is not Leontief.
- If ideas are Pareto distributed the envelope becomes the Cobb-Douglas production function.
- Jones (2005): the long-run production function is the upper envelope of a large number of ideas generated over time.
- This result implies a major difference between short-run and long-run production functions.

Set-up:

- An idea i is a vector (a_i, b_i) that allows producing the single final good with the Leontief production function

$$Y(t) = \min\{a_i L(t), b_i K(t)\} \ ,$$

where $K(t)$ and $L(t)$ are the amounts of capital and labor, respectively.

- Assume the components a_i and b_i are independently drawn from two Pareto distributions, that is

$$Pr(a_i \leq a) = 1 - \left(\frac{a}{\gamma_a}\right)^{-\alpha} \quad \text{and} \quad Pr(b_i \leq b) = 1 - \left(\frac{b}{\gamma_b}\right)^{-\beta}$$

with $a \geq \gamma_a$, $b \geq \gamma_b$, $\{\alpha, \beta\} > 0$ and $\alpha + \beta > 1$.

- As a result the joint probability of $a_i \geq a$ and $b_i \geq b$ is given by

$$\mathbb{G}(b, a) \equiv Pr[a_i \geq a \ \wedge \ b_i \geq b] = \left(\frac{a}{\gamma_a}\right)^{-\alpha} \left(\frac{b}{\gamma_b}\right)^{-\beta} \ .$$

Aggregate production:

Aggregate production of idea i is a random variable. As the production function is Leontief the probability of an idea i producing output below a threshold y is given by

$$\begin{aligned} H(y) \equiv Pr(\tilde{Y}_i(t) \leq y) =& 1 - Pr[a_i L(t) \geq y \ \wedge \ b_i K(t) \geq y] \\ =& 1 - \mathbb{G}\left(\frac{y}{L(t)}, \frac{y}{K(t)}\right) \\ =& 1 - \left(\frac{y}{\gamma_a L(t)}\right)^{-\alpha} \left(\frac{y}{\gamma_b K(t)}\right)^{-\beta} \\ =& 1 - \underbrace{\gamma_a \gamma_b}_{\equiv \gamma} L(t)^{\alpha} K(t)^{\beta} y^{-(\alpha+\beta)} \\ =& 1 - \gamma L(t)^{\alpha} K(t)^{\beta} y^{-(\alpha+\beta)} \end{aligned}$$

Expanding ideas:

The probability of $N(t)$ ideas producing output below a threshold y is given by

$$Pr\left[\tilde{Y}_i\left(t; N(t)\right) \leq y\right] = H(y)^{N(t)} = \left(1 - \gamma L(t)^{\alpha} K(t)^{\beta} y^{-(\alpha+\beta)}\right)^{N(t)} ,$$

as the ideas are independently drawn.

As $N(t) \to \infty$, the probability of producing below y converges to zero.

Limit distribution:

In order to determine the (stationary) limit distribution y has to be normalized (as the Pareto distribution has unbounded support. Define $\tilde{y}$ as

$$\tilde{y} = y\left(\gamma N(t) L(t)^{\alpha} K(t)^{\beta}\right)^{\frac{1}{\alpha+\beta}} ,$$

then $Pr\left[\tilde{Y}_i\left(t; N(t)\right) \leq \tilde{y}\right]$ is given by

$$\begin{aligned} Pr\left[\tilde{Y}_i\left(t; N(t)\right) \leq \tilde{y}\right] &= \left(1 - \gamma L(t)^{\alpha} K(t)^{\beta} \tilde{y}^{-(\alpha+\beta)}\right)^{N(t)} \\ &= \left(1 - \frac{y^{-(\alpha+\beta)}}{N(t)}\right)^{N(t)} \end{aligned}$$

Expanding $N(t) \to \infty$ gives

$$\lim_{N(t)\to\infty} Pr\left[\tilde{Y}_i\left(t; N(t)\right) \leq \tilde{y}\right] = e^{-y^{-(\alpha+\beta)}} ,$$

for $y > 0$.

From Ideas to Production

As a result

$$\frac{\tilde{Y}_i\left(t; N(t)\right)}{\left(\gamma N(t) L(t)^{\alpha} K(t)^{\beta}\right)^{\frac{1}{\alpha+\beta}}} \sim \text{Fréchet}(\alpha+\beta)\ .$$

This implies the normalised long-run distribution of output follows a Fréchet distribution with shape parameter $\alpha + \beta$. Consequently, the long-run total production is approximately given by

$$\tilde{Y}_i\left(t; N(t)\right) \approx \varepsilon(t) \left(\gamma N(t) L(t)^{\alpha} K(t)^{\beta}\right)^{\frac{1}{\alpha+\beta}}\ ,$$

where $\varepsilon(t)$ is a random variable drawn from the Fréchet distribution with shape parameter $\alpha + \beta$. Define $\tilde{\alpha} = \frac{\beta}{\alpha+\beta}$ and $\left(\gamma N(t)\right)^{\frac{1}{\alpha+\beta}} = \tilde{A}(t)$, then

$$\tilde{Y}_i\left(t; N(t)\right) \approx \varepsilon(t) \tilde{A}(t) K(t)^{\tilde{\alpha}} L(t)^{1-\tilde{\alpha}}\ .$$

From Ideas to Production

Discussion:

- The Cobb-Douglas production function arises purely from aggregation: this is both a strength and a weakness.
- Strength: relies only on the assumptions of Leontief production and independent draws of ideas from a Pareto distribution.
- Weakness: no equilibrium interactions, price or market size effects.
- Weakness: unit of observation. Is it single firm, industry, region, country?
- The implication that every long-run production function is Cobb-Douglas implies constant factor shares in the long run.
- However, there is considerable variation in factor shares across industries (and over time).
- How do ideas emerge and why?