LMU München, Winter Term 2020/2021
Macroeconomics - Growth
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Macroeconomics - Growth, Problem Set 3: Neo-Classical Growth Models

Problem 1: Discrete vs. Continuous Time

Show that the formulation for discounting in discrete time $(1+r)^t$ is equivalent to e^{rt} in continuous time.

Problem 2: Constant Elasticity of Substitution Production

Show that for $\sigma \to 1$ the production function

$$Y(t) = \left[\alpha \left(K(t)\right)^{\frac{\sigma-1}{\sigma}} + (1-\alpha)\left(A(t)L(t)\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} ,$$

converges (up to a constant) to

$$Y(t) = K(t)^{\alpha}\left(A(t)L(t)\right)^{1-\alpha} .$$

Problem 3: Constant Relative Risk Aversion Utility

Show that the generic class of utility functions with constant relative risk aversion, i.e. utility functions with

$$-\frac{u''(c)\cdot c}{u'(c)} = \theta ,$$

is given by

$$\frac{c^{1-\theta}-1}{1-\theta} .$$

What happens if $\theta \to 1$?

Problem 4: The Canonical Neo-Classical Growth Model

Consider the standard neo-classical growth model in continuous time. Output is produced according to a Cobb-Douglas production function

$$Y(t) = K(t)^{\alpha} L(t)^{1-\alpha} \quad \alpha \in (0,1) \, .$$

Markets are perfectly competitive and capital depreciates at rate $\delta \in (0,1)$. However, there is no population growth, i.e. $n = 0$.

Assume the economy admits a representative household that is infinitely lived and maximises lifetime utility by allocating disposable resources to either consumption $c(t)$ or investment in assets $a(t)$

$$\max_{[c(t),a(t)]_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \, \frac{c(t)^{1-\theta} - 1}{1-\theta} \, dt \quad \{\rho, \theta\} > 0 \, .$$

4.1 Write down the law of motion for asset holdings of the household. Use the law of motion to set up the Hamiltonian of your choice and derive all first order conditions. Write down the corresponding transversality condition and show that the Euler equation is given by:

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta} \, .$$

4.2 For the rest of the problem assume that population growth is governed by the differential equation:

$$\dot{L}(t) = nL(t) \quad \text{with} \quad n > 0, \; L(0) = 1$$

Solve this differential equation for $L(t)$ and show that the maximisation problem of the household is now given by:

$$\max_{[c(t),a(t)]_{t=0}^{\infty}} \int_0^{\infty} e^{-(\rho-n)t} \, \frac{c(t)^{1-\theta} - 1}{1-\theta} \, dt$$

$$\text{s.t.} \quad \dot{a}(t) = \left[f'(k(t)) - \delta - n \right] a(t) + w(t) - c(t)$$

What additional assumption do you have to make to ensure discounting? Write down the new transversality condition and give a theoretical or verbal intuition why the Euler equation is unchanged.

4.3 Derive that in a competitive equilibrium the values of steady state capital stock per capita, consumption per capita as well as the savings rate are given by:

$$k^* = \left[\frac{\alpha}{\rho + \delta} \right]^{\frac{1}{1-\alpha}}$$

$$c^* = \left[\frac{\alpha}{\rho + \delta} \right]^{\frac{\alpha}{1-\alpha}} - (n + \delta) \left[\frac{\alpha}{\rho + \delta} \right]^{\frac{1}{1-\alpha}}$$

$$s^* = \frac{n + \delta}{\rho + \delta} \cdot \alpha$$

4.4 Show that the steady state level of capital stock per capita is smaller than what the Golden Rule level in this economy would demand and give a theoretical or verbal intuition for the result.

Hint: think about the optimal savings rate vs. the one implied by the Golden Rule.

4.5 Draw a phase diagram of the economy including the steady state conditions for capital stock per capita and consumption per capita (be careful to label all curves, axes and values correctly!).

Assume there is drop in the discount rate, such that:

$$\rho = n$$

a) Show graphically how this would affect the saddle path. Explain verbally how the economy converges to the new equilibrium.

b) Does the new steady state capital stock per capita satisfy the Golden Rule? If so why, if not why not? Give a theoretical or verbal intuition for the result.

4.6 Using the previous result without technological progress (you do not need to derive this result), show that for positive technology growth, given by

$$\frac{\dot{A}(t)}{A(t)} = g \quad \text{with} \quad \{g, A(0)\} > 0 \ ,$$

the Euler equation in units of consumption per effective labour $\tilde{c}(t)$ is given by:

$$\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{f'\left(k(t)\right) - \delta - \rho - \theta g}{\theta}$$

Problem 5: The Neo-Classical Growth and Balanced Growth

Consider a modified neo-classical growth model with the following production function

$$Y(t) = K(t)^{\alpha} L(t)^{1-\alpha} + AK(t) \quad \alpha \in (0,1) ,$$

where $K(t)$ is the total capital stock at time t, $L(t)$ the number of workers and A the exogenous fixed amount of technology, with $\{K(0), L(0), A\} > 0$. Markets are perfectly competitive and capital depreciates at rate $\delta \in (0,1)$; there is no population growth, i.e. $n = 0$. Preferences of the representative household are given by

$$\max_{[c(t),a(t)]_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{c(t)^{1-\theta} - 1}{1-\theta} dt \quad \{\rho, \theta\} > 0 .$$

5.1 Write down the law of motion for asset holdings of the household. Use the law of motion to set up the Hamiltonian of your choice and derive all first order conditions. Write down the corresponding transversality condition and show that the Euler Equation is given by

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta} .$$

5.2 Define the consumption-capital ratio as

$$x(t) = \frac{c(t)}{k(t)} .$$

Show that the growth rate of the consumption-capital ratio is given by

$$\frac{\dot{x}(t)}{x(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta} - \left[\frac{f(k(t))}{k(t)} - x(t) - \delta\right] .$$

5.3 For the remaining questions of this part assume that $\theta = \alpha$, $A > \delta$ and that $x(t)$ is **constant at every point in time**.

a) Use this information to show that the solution for $c(t)$ is given by

$$c(t) = \frac{\rho - (A-\delta)(1-\alpha)}{\alpha} k(t) .$$

b) What is the shape of the saddle path implied by this solution?

c) What condition has to hold for the economy to admit a balanced growth path?

5.4 For the last question assume that $\rho > A - \delta$. Draw a phase diagram of the economy including the steady state conditions for capital stock per capita and consumption per capita (be careful to label all curves, axes and values correctly!).

Show graphically how an increase in the exogenous technology A would affect the saddle path and describe verbally how the economy converges to the new equilibrium.

Problem 6: A Closed-Form Solution to the Neo-Classical Growth Model

(This problem is significantly more difficult than a standard exam question.)

Consider the standard neo-classical growth model in continuous time. Output is produced according to a Cobb-Douglas production function

$$Y(t) = K(t)^{\alpha} L(t)^{1-\alpha} \qquad \alpha \in (0,1) \ .$$

Markets are perfectly competitive and capital depreciates at rate $\delta \in (0,1)$; there is no population growth, i.e. $n = 0$. Preferences of the representative household are given by

$$\max_{[c(t),a(t)]_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \, \frac{c(t)^{1-\theta} - 1}{1-\theta} \, dt \qquad \{\rho, \theta\} > 0 \ .$$

6.1 Write down the law of motion for asset holdings of the household. Use the law of motion to set up the Hamiltonian of your choice and derive all first order conditions. Write down the corresponding transversality condition and compute the Euler equation.

6.2 Derive the steady state values of capital and consumption per capita $\{k^*, c^*\}$.

6.3 Define the capital-output and consumption-capital ratios as

$$\begin{aligned} z(t) =& \frac{k(t)}{y(t)} \\ x(t) =& \frac{c(t)}{k(t)} \ . \end{aligned}$$

Write down the law of motion of capital and output per capita, respectively. Use these expressions to derive the law of motion of the capital-output ratio and the growth rate of the consumption-capital ratio.

6.4 For the remainder of this problem assume that $\theta = \alpha$ and that the initial capital-labor ratio is given by $k(0) = k_0$. Use this information in combination with the transversality condition to show that the particular solutions for $c(t)$ and $k(t)$ are given by

$$\begin{aligned} c(t) =& \frac{\rho + \delta(1-\alpha)}{\alpha} \cdot k(t) \\ k(t) =& \left\{ (k^*)^{1-\alpha} - \left[(k^*)^{1-\alpha} - k_0^{1-\alpha} \right] e^{-(1-\alpha)\frac{\rho+\delta}{\alpha} \cdot t} \right\}^{\frac{1}{1-\alpha}} \end{aligned}$$

6.5 What do the solutions in (6.4) imply about the shape of the saddle path? Use the solutions to show that the growth rate of capital per capita and the savings rate are given by

$$\begin{aligned} \frac{\dot{k}(t)}{k(t)} =& -\frac{\rho + \delta}{\alpha} \cdot \left[1 - \left(\frac{k^*}{k(t)} \right)^{1-\alpha} \right] \\ s(t) =& s^* - (s^* - s_0) \, e^{-(1-\alpha)\frac{\rho+\delta}{\alpha} \cdot t} \ , \end{aligned}$$

where s^* and s_0 are the steady-state and initial savings rate, respectively.

6.6 Show that the speed of convergence in this economy is given by

$$\beta_k = (1-\alpha) \, \frac{\rho + \delta}{\alpha} \cdot \left(\frac{k^*}{k(t)} \right)^{1-\alpha}$$

Preface to Problem 7: The Euler Forward Method for Numerical Simulation

In order to simulate a continuous time model for which we do not have the particular solution we have to approximate the model in discrete time. Thankfully, there are a number of ways to do this, the simplest one being the Euler forward method. The intuition is the following: take a local derivative of a differential equation and accumulate small increases/decreases of its argument $x(t)$. This method will produce a sequence of discretized expressions that will approach the true solution of the original differential equation as the increases/decreases become very small. More formally, suppose you have a discrete time process

$$x_t = F(x_{t-1}) \ .$$

Set up a related continuous time process

$$\dot{x}(t) = \frac{dx(t)}{dt} = G(x(t)) \ ,$$

and use the following approximation

$$\frac{dx(t)}{dt} \approx \frac{\Delta x}{\Delta t} \ ,$$

where the left-hand side denotes a ratio of infinitesimally small quantities, while the right-hand is a ratio of small but discrete quantities. Rewriting the expression gives

$$\frac{\Delta x}{\Delta t} = \frac{x\,(t+\Delta t) - x(t)}{\Delta t} \approx G(x(t)) \ ,$$

which can be re-arranged to a difference equation that we can simulate with normal discrete-time methods

$$x\,(t+\Delta t) \approx x(t) + G(x(t))\Delta t = x(t) + \dot{x}(t)\Delta t \ .$$

Note that if $\Delta t \to 0$ the original expression becomes

$$\lim_{\Delta t \to 0} \frac{x\,(t+\Delta t) - x(t)}{\Delta t} = \frac{0}{0} \quad \Rightarrow \quad \lim_{\Delta t \to 0} \frac{\frac{dx(t+\Delta t)}{d(t+\Delta t)} \cdot 1}{1} = \dot{x}(t) \ ,$$

and we are back to continuous time. As a result, the smaller the Δt we choose, the more exact we can approximate the true continuous-time value discretely.

Problem 7: A Numerical Implementation of the Neo-Classical Growth Model

Consider the standard neo-classical growth model in continuous time. Output is produced according to a Cobb-Douglas production function

$$Y(t) = K(t)^{\alpha} \left(A(t)L(t)\right)^{1-\alpha} \qquad \alpha \in (0,1) \ .$$

Markets are perfectly competitive and capital depreciates at rate $\delta \in (0,1)$; population and labor-augmenting technological progress grow at rates $\{n, g\} > 0$, respectively. The initial values of capital, labor and technology are $K_0 = L_0 = A_0 = 1$. Preferences of the representative household are given by

$$\max_{[c(t),a(t)]_{t=0}^{\infty}} \quad \int_0^{\infty} e^{-(\rho-n)t} \, \frac{c(t)^{1-\theta} - 1}{1-\theta} \, dt \qquad \{\rho, \theta\} > 0 \ .$$

7.1 Compute the law of motion, the Euler equation as well as the steady state values of capital and consumption in efficiency units of labor.

7.2 Assume parameter values of $\alpha = 0.4$, $\delta = 0.05$, $n = 0.02$, $g = 0.01$, $\rho = 0.04$, $\theta = 0.8$ and $\Delta t = 1$. Simulate the model using the Euler forward method for $t = 100$ (i.e. $\sum_{\Delta t} = 100$). Does the steady state fit the analytical prediction? Repeat the exercise for $\Delta t = \{0.1, 0.01\}$ and comment on the results.

Hint: in oder to set the saddle-path compatible value of $c(0)$, use the following algorithm:

(a) *Set values $\underline{c}(0)$ and $\bar{c}(0)$, with*

$$\underline{c}(0) = \begin{cases} 0 & \text{if } k(0) < k^* \\ y(0) - (\delta + n + g)k(0) & \text{if } k(0) \geq k^* \end{cases}$$

for $\underline{c}(0)$ and

$$\bar{c}(0) = \begin{cases} y(0) - (\delta + n + g)k(0) & \text{if } k(0) < k^* \\ y(0) - (\delta + n + g)k(0) + k(0) & \text{if } k(0) \geq k^* \end{cases}$$

for $\bar{c}(0)$.

(b) *Set initial consumption $c(0)$ to $c(0) = \frac{1}{2}\left[\underline{c}(0) + \bar{c}(0)\right]$.*

(c) *Simulate the model and restart the simulation with the two rules (note: the order of execution must be sequential)*

$$\text{if } k(t) > k^* \Rightarrow \begin{cases} \underline{c}(0) & = c(0) \\ c(0) & = \frac{1}{2}\left[\underline{c}(0) + \bar{c}(0)\right] \end{cases}$$

and

$$\text{if } c(t) > c^* \Rightarrow \begin{cases} \bar{c}(0) & = c(0) \\ c(0) & = \frac{1}{2}\left[\underline{c}(0) + \bar{c}(0)\right] \end{cases}$$

(d) *Stop the simulation if* $\sqrt{[c^* - c(t)]^2 + [k^* - k(t)]^2} < \tau \equiv 0.0001$.

7.3 Using the set-up from (7.2), simulate the effects of

- a sudden drop in the elasticity of substitution, given by $\theta^{\text{new}} = 1.5$,
- a sudden increase in the rate of depreciation, given by $\delta^{\text{new}} = 0.1$,
- a sudden drop in the discount rate, given by $\rho^{\text{new}} = 0.03$,

for $t \geq 50$ and comment on the result.

7.4 Let $n = g = 0$ and set $\alpha = \theta = 0.4$. Use the solutions from (6.4) to re-simulate the model without resorting to the Euler method.