Macroeconomics - Growth

The Solow Model

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Winter Semester 20/21

What is Macroeconomics?

- "Macroscopic": length scale on which objects are large enough to be visible with the naked eye (μακρός: makrós "wide, large").
- Macroeconomics is the branch of economics that deals with "large" behavior, i.e. aggregates.
- Macroeconomists seek to understand the determinants of aggregate trends in the economy with particular focus on national income, (un)employment, inflation, investment, and international trade.
- While macroeconomics is a broad field, it can be divided into two main areas:
  1. Determinants of long-run economic growth (this semester).
  2. Causes and consequences of short-run fluctuations in aggregate outcomes (next semester).

Some Empirical Facts

What we want to explain first: Economic Growth

Economic growth is a central topic in macroeconomics.

Ultimately, we would like to answer questions like:

Why is Germany so much richer than D.R. Congo, and why has growth been so much lower in Europe (Germany) than in the USA (and what can we do about it?)?

Some Empirical Facts

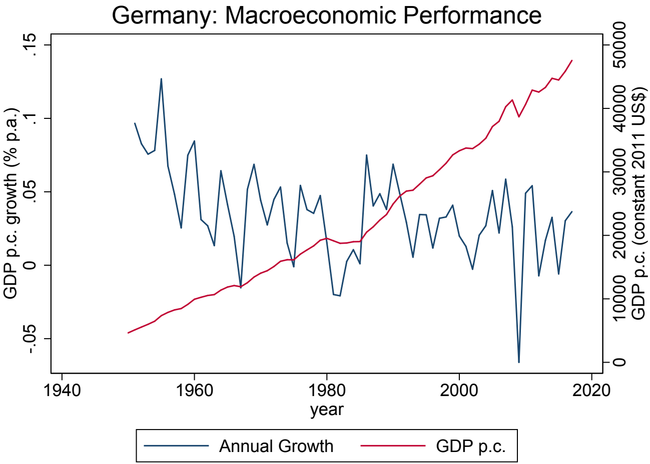


Figure: Real GDP p.c. Germany (constant 2011 US$, output-side), Source: Penn World Tables (v9.1)

Some Empirical Facts

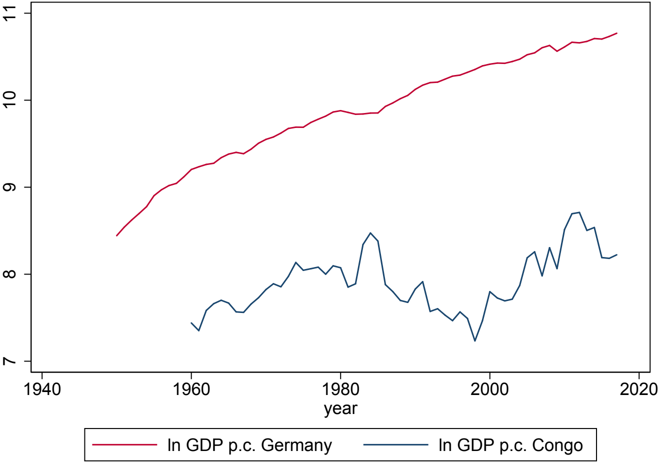


Figure: Real GDP p.c. Germany and D.R. Congo (constant 2011 US$, output-side), Source: Penn World Tables (v9.1)

Detour: Some Thoughts on Growth Rates

"1.5% growth is not that much..."

Yes and no. Depends on the time horizon.

Consider two countries with identical GDP per capita, one growing at 0.5% per year, the other at 1.5%:

$$
\begin{aligned}
(1+0.005) &= 1.005 & (1+0.015) &= 1.015 \\
(1+0.005)^{10} &\approx 1.0051 & (1+0.015)^{10} &\approx 1.16 \\
(1+0.005)^{20} &\approx 1.1 & (1+0.015)^{20} &\approx 1.35 \\
(1+0.005)^{50} &\approx 1.28 & (1+0.015)^{50} &\approx 2.1 \\
(1+0.005)^{100} &\approx 1.64 & (1+0.015)^{100} &\approx 4.43 \\
(1+0.005)^{200} &\approx 2.71 & (1+0.015)^{200} &\approx 19.64
\end{aligned}
$$

Some Empirical Facts

- Persistent difference in growth rates leads to large divergence of income per capita levels.
- There are enormous differences in growth rates across countries.
- What (types of) countries growth more rapidly?
- What country-specific characteristics have a causal effect on growth?
- Investment rates? Human capital accumulation? Technology?

Some Empirical Facts

Income and Welfare:

- Why should we care about income differences?
- GDP is not a welfare measure (does not capture: pollution, non-market activities, etc.).
- But: GDP is highly correlated with other important measures of quality of life, health, and standard of living.
- High income levels reflect high standards of living and health (note: correlation not causation!).
- High income differences $\Rightarrow$ high differences in welfare.

Income and Consumption:

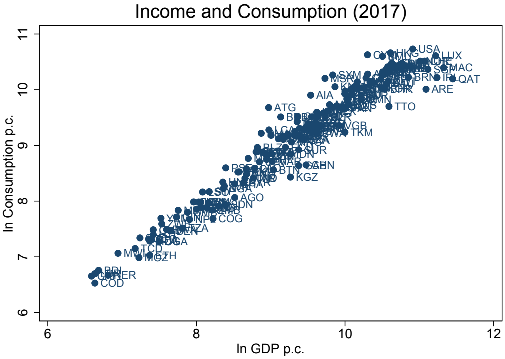
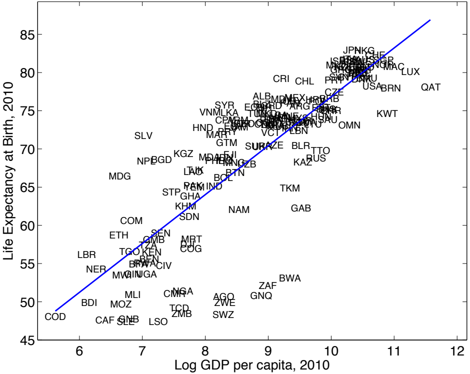


FIGURE: Real GDP p.c. and real consumption of households and government (at current PPPs, constant 2011 US$, income is output-side), Source: Penn World Tables (v9.1)

Some Empirical Facts

Income and Health:



- Large correlation between life expectancy at birth and GDP.
- Major outliers: HIV/AIDS in sub-Saharan Africa.

Some Empirical Facts

Income Levels vs. Income Growth:

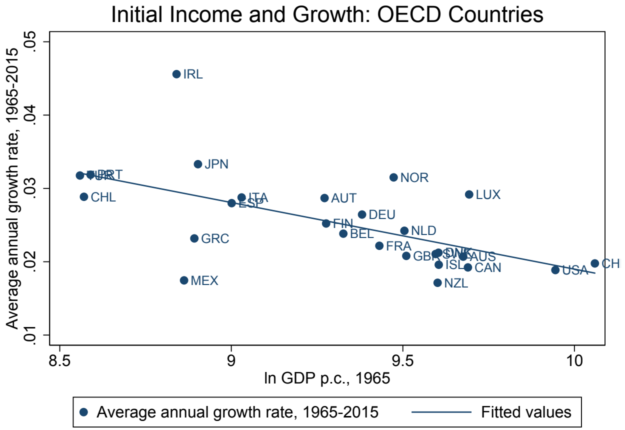


Figure: Real GDP p.c. (at current PPPs, constant 2011 US$, income is output-side), Source: Penn World Tables (v9.1)

Some Empirical Facts

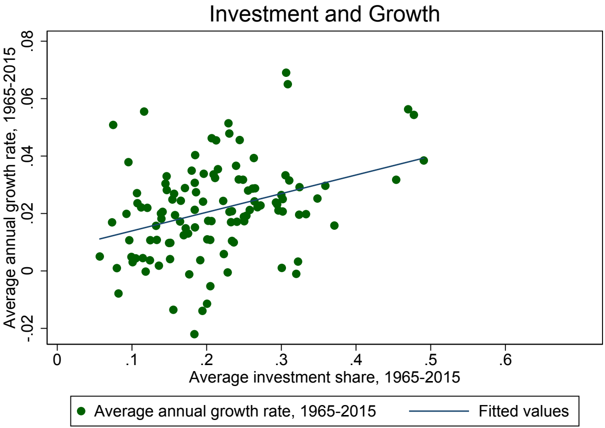
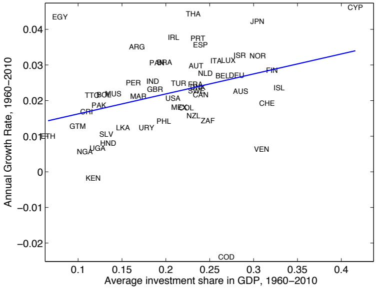
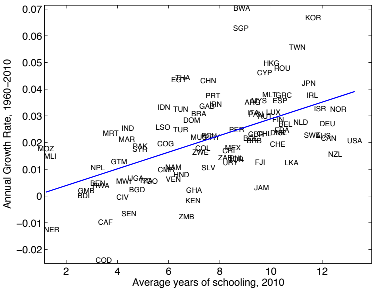


Figure: Real GDP p.c. and share of gross capital formation (at current PPPs, constant 2011 US$, income is output-side), Source: Penn World Tables (v9.1)

Investment, Schooling and GDP Growth:





Some Empirical Facts

- These factors are only "proximate" causes of economic growth.
- Investment or human capital choices are driven by other factors.

"Fundamental" causes of economic growth:

1. Geography: affecting natural resources and productivity, imposing constraints on individual behavior.
2. Institutions: laws, regulation, property rights shape individual incentives.
3. Culture: determines individual preferences, values and beliefs.
4. Luck (multiple equilibria): two economies with identical characteristics (preferences, market structures, etc.) end up on divergent paths (e.g. due to shocks).

Some Empirical Facts

- Three major questions:
  1. Why are there such large differences in income per capita and productivity across countries?
  2. Why do some countries grow more rapidly than others?
  3. What sustains economic growth over long periods of time?
- Analysis of each of these questions requires well-formulated economic models (why?).
- First step: the Solow growth model.
- The emphasis lies on physical capital accumulation. Technology is a black box – endogenous growth models are explicit about technology.

Some Empirical Facts

Starting point: Kaldor's (1961) stylized growth facts

1. The shares of national income accruing to labor and capital are roughly constant over long periods of time.
2. Capital stock per worker grows at a roughly constant rate over long periods of time.
3. Output per worker grows at a roughly constant rate over long periods of time.
4. The capital-output ratio is roughly constant over long periods of time.
5. The rate of return on capital is roughly constant over long periods of time.
6. There is considerable variation in the rate of growth of labor productivity and total output between countries.

Labor Productivity: Y/L for Germany

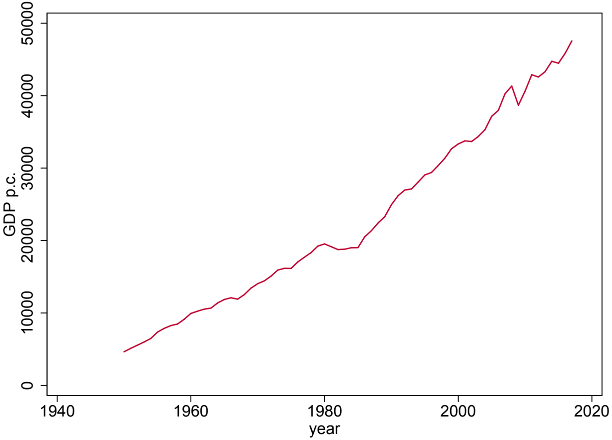


Figure: Source: Penn World Tables 9.1

Labor Productivity: Y/L for Germany

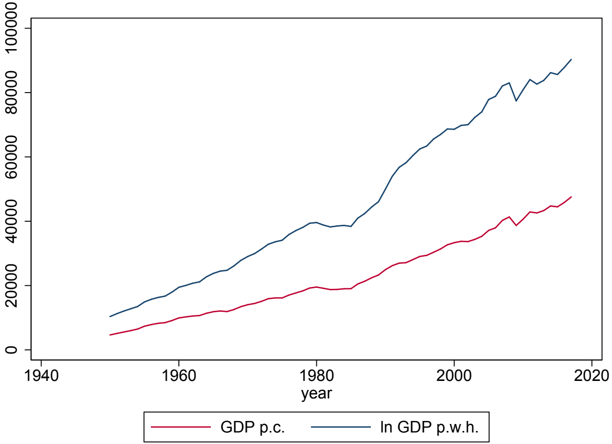


Figure: Source: Penn World Tables 9.1

Capital per Worker: K/L for Germany

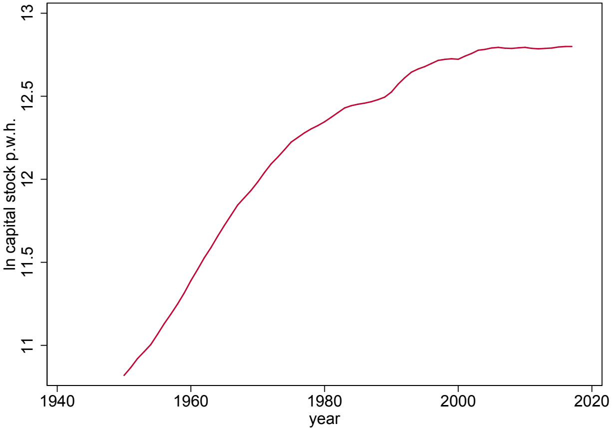


Figure: Source: Penn World Tables 9.1

Capital-Output Ratio: K/Y for Germany

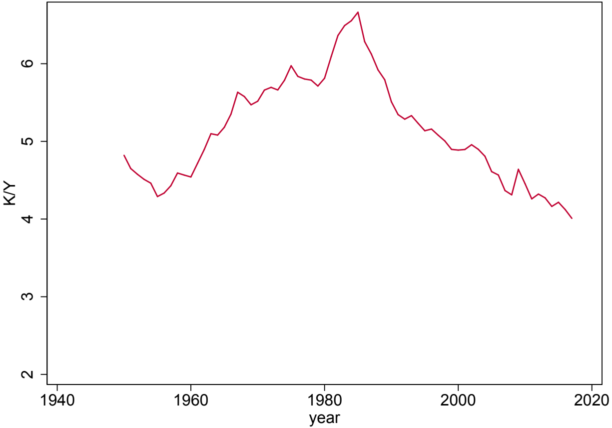


Figure: Source: Penn World Tables 9.1

Labor Income Share: Germany

Figure: Source: Penn World Tables 9.1

Growth in Output per capita: Germany

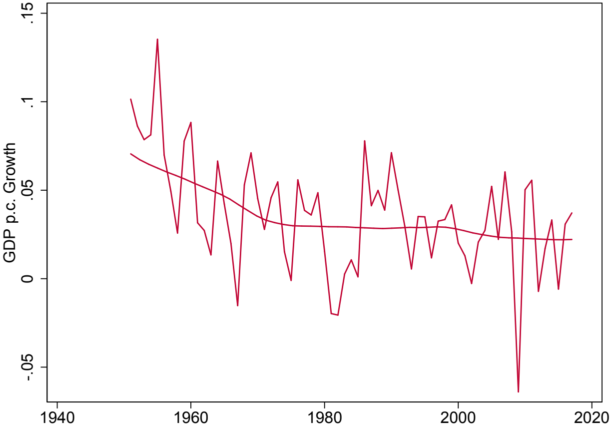


Figure: Source: Penn World Tables 9.1

<u>What is an economic model?</u>

- A model is an approximation of the real world constructed to explain a specific (set of) facts or questions.
- A model will always be "wrong" in certain dimensions because it abstracts from some features of the real world (descriptive realism is not the central objective).
- How can we measure the success of an economic model?
- A good model:
  1. Is logically valid (internally consistent).
  2. Is consistent with the facts it aims to explain.
  3. Does not openly contradict other (relevant) facts.
  4. Can be modified and adapted to other/broader questions.

The Solow Model: Preliminaries

All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive. A "crucial" assumption is one on which the conclusions do depend sensitively, and it is important that crucial assumptions be reasonably realistic. When the results of a theory seem to flow specifically from a special crucial assumption, then if the assumption is dubious, the results are suspect.

Robert M. Solow - A Contribution to the Theory of Economic Growth (1956)

The Solow Model: Preliminaries

What are the elements of an economic model?

- A model is a set of assumptions ("ifs") about primitives (e.g., preferences, technologies, endowments, information, market structure) that lead to a set of results/implications ("thens").
- Exogenous and endogenous variables:
    - Exogenous variables: elements that do not change or mechanically change from outside the model.
    - Endogenous Variables: things that have to be solved for within the model.
    - Solving a model means expressing endogenous variables in terms of exogenous variables.
    - You should always be aware of what is exogenous and endogenous!

Discrete vs. Continuous Time:

- All models we will cover are dynamic, that is variables change over time.
- Most models can be cast either in discrete or continuous time (each has their advantages and disadvantages).
    - Discrete time: variables are **indexed** by discrete time intervals, i.e. consumption at time t is c_t and consumption at time $t+1$ is c_{t+1}. Example: spreadsheets of (yearly) data.
    - Continuous time: variables are a **function** of time, i.e. consumption at point t is given by $c(t)$. Example: stock market prices calculated at (virtually) every second.

The Solow Model: Preliminaries

Growth rates:

Since we are ultimately interested in growth rates it is useful to define them in advance. The growth g rate of a variable x is:

- Discrete time:

$$\begin{aligned} \frac{x_{t+1} - x_t}{x_t} =& g \\ x_{t+1} =& (1+g)x_t \ . \end{aligned}$$

- Continuous time:

$$\begin{aligned} \frac{\dot{x}(t)}{x(t)} =& g \quad \text{where} \quad \dot{x}(t) = \frac{d\,x(t)}{d\,t} \\ \dot{x}(t) =& g x(t) \ . \end{aligned}$$

The Solow Model: Preliminaries

Difference and Differential equations:

Autonomous (does not depend on time itself), one-dimensional (one variable), first-order (only depends on the previous period or current time), linear difference equation

$$x_{t+1} = (1+g)x_t \ .$$

Solution (given constant g and x_0): iterate forward.

$$\begin{aligned} x_1 =& (1+g)x_0 \\ x_2 =& (1+g)x_1 = (1+g)(1+g)x_0 = (1+g)^2 x_0 \\ &\vdots \\ x_t =& (1+g)^t x_0 \end{aligned}$$

Autonomous, one-dimensional, first-order, linear differential equation

$$\dot{x}(t) = gx(t) \ .$$

Solution: a bit more complicated.

The Solow Model: Preliminaries

Solution for simple differential equations:

General solution:

$$\begin{aligned}
\frac{\dot{x}(t)}{x(t)} =& g \\
\int \frac{\dot{x}(t)}{x(t)}\, dt =& \int g\, dt \\
\ln\left(x(t)\right) + a_0 =& gt + a_1 \\
\ln\left(x(t)\right) =& gt + a_1 - a_0 \\
x(t) =& e^{gt + a_1 - a_0} \\
x(t) =& e^{gt} \cdot e^{a_1 - a_0} = A \cdot e^{gt}
\end{aligned}$$

Particular solution:

$$\begin{aligned}
x(0) =& A \cdot e^{g0} = A \cdot e^{0} = A \\
\Rightarrow x(t) =& x(0) \cdot e^{gt}\ .
\end{aligned}$$

Note that both discrete and continuous time imply *exponential* growth!

The Solow Model

Simple dynamic continuous time general equilibrium model going back to

- Robert M. Solow (1956), "A Contribution to the Theory of Economic Growth", *Quarterly Journal of Economics* 70(1), 65 – 94
- Trevor W. Swan (1956), "Economic Growth and Capital Accumulation", *Economic Record* 32(2), 334 – 361
- Nobel Prize to Solow in 1987.
- Highly abstract representation of an aggregate economy, serves as a starting point for future analysis.
- Little reference to individual decisions, tastes, abilities.
- Despite its simplicity, it is consistent with many features of the data (Kaldor facts).

Households:

- Closed economy producing a single final good that can be used for consumption or investment.
- Large number of identical individuals/households (representative agents) $N(t)$, each supplying one unit of labor (time) inelastically $N(t) = L(t)$.
- No preferences, households save an exogenous constant fraction $s \in (0, 1)$ of disposable income.
- Large number of (identical) firms producing output $Y(t)$.

Firms:

The final good $Y(t)$ is produced by a large number of identical firms (representative firm) with the *neoclassical production function*:

$$Y(t) = F\left(K(t), L(t)\right) \ .$$

- $Y(t)$: total production of the final good at time t.
- $K(t)$: physical capital stock at time t.
- $L(t)$: labor input at time t.

Assumptions regarding the neoclassical production function:

1. $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is twice differentiable in $K(t)$ and $L(t)$.
2. Positive and diminishing marginal products of $K(t)$ and $L(t)$:

$$\begin{aligned} F_K\left(K(t), L(t)\right) > 0 \quad & F_{KK}\left(K(t), L(t)\right) < 0 \\ F_L\left(K(t), L(t)\right) > 0 \quad & F_{LL}\left(K(t), L(t)\right) < 0 \end{aligned}$$

3. Constant returns to scale (CRS) in $K(t)$ and $L(t)$, i.e.

$$\lambda F\left(K(t), L(t)\right) = F\left(\lambda K(t), \lambda L(t)\right) \ .$$

4. *Inada* conditions:
   - $F\left(0, L(t)\right) = 0$ for all $L(t)$ and $F\left(K(t), 0\right) = 0$ for all $K(t)$.
   - For all $L(t)$: $\lim_{K(t)\to 0} F_K\left(K(t), L(t)\right) = \infty$ and $\lim_{K(t)\to\infty} F_K\left(K(t), L(t)\right) = 0$.
   - For all $K(t)$: $\lim_{L(t)\to 0} F_L\left(K(t), L(t)\right) = \infty$ and $\lim_{L(t)\to\infty} F_L\left(K(t), L(t)\right) = 0$.

Labor market:

- Competitive markets: households and firms are *price-takers* and prices clear the markets.
- Factors of production (capital and labor) are owned by households.
- Labor is supplied at price $w(t)$.

Capital market:

- Households own the capital stock and rent it out to firms at the rental rate $R(t)$.
- Market clearing implies that capital demand equals supply.
- The initial capital stock $K(0) > 0$ is given
- Physical capital depreciates at constant rate $\delta \in (0, 1)$.

Interest Rate vs. Rental Rate:

- Presence of depreciation introduces a wedge between the rental rate of capital and the return to savings.
- Net interest rate $r(t)$ faced by households is $r(t) = R(t) - \delta$.

Firm optimization:

- Firm problem: taking $R(t)$ and $w(t)$ as given, maximize profits as

$$\max_{K(t)\geq 0, L(t)\geq 0} F\left(K(t), L(t)\right) - R(t)K(t) - w(t)L(t)\ .$$

- First order conditions

$$w(t) = F_L\left(K(t), L(t)\right) \quad \text{and} \quad R(t) = F_K\left(K(t), L(t)\right)\ .$$

- No profits as consequence of Euler's theorem.

Dynamic evolution of the economy:

- Law of motion of the (total) capital stock

$$\dot{K}(t) = I(t) - \delta K(t) \ .$$

- National accounting identity (I):

$$Y(t) = C(t) + I(t) \ .$$

- National accounting identity (II):

$$I(t) = S(t) = s \cdot Y(t) \quad \text{and} \quad C(t) = (1 - s) \cdot Y(t) \ .$$

- Fundamental law of motion of the Solow model:

$$\dot{K}(t) = s \cdot F\left(K(t), L(t)\right) - \delta K(t) \ .$$

- This is a nonlinear differential equation. Together with the law of motion for $L(t)$, it describes the equilibrium of the Solow growth model.

Additional assumptions:

- Population growth:

$$\dot{L}(t) = nL(t) \ .$$

which implies $L(t) = L(0) \cdot e^{nt}$.

- Capital intensity: $k(t) = \frac{K(t)}{L(t)}$ which (because of CRS) implies $y(t) = \frac{Y(t)}{L(t)} = F\left(\frac{K_t}{L_t}, 1\right) = f\left(k(t)\right)$ and

$$R(t) = f'\left(k(t)\right) \ .$$

In general equilibrium this implies

$$\begin{aligned} Y(t) =& R(t)K(t) + w(t)L(t) \\ y(t) =& R(t)k(t) + w(t) \\ w(t) =& y(t) - R(t)k(t) \\ w(t) =& f\left(k(t)\right) - f'\left(k(t)\right)k(t) \ . \end{aligned}$$

Evolution of capital per capita:

$$
\begin{aligned}
k(t) &= \frac{K(t)}{L(t)} \\
\dot{k}(t) &= \frac{\dot{K}(t)L(t) - K(t)\dot{L}(t)}{L(t)^2} \\
\dot{k}(t) &= \frac{\dot{K}(t)}{L(t)} - \underbrace{\frac{\dot{L}(t)}{L(t)}}_{=n} \frac{K(t)}{L(t)} \\
\dot{k}(t) &= \frac{s \cdot F\left(K(t), L(t)\right) - \delta K(t)}{L(t)} - nk(t) \\
\dot{k}(t) &= s \cdot f\left(k(t)\right) - (n+\delta)k(t)
\end{aligned}
$$

Competitive equilibrium in the Solow model:

Definition (Equilibrium Path)

For an economy with population growth n and an initial capital stock $K(0)$, an equilibrium path is a sequence $\{K(t), L(t), Y(t), C(t), w(t), R(t)\}_{t=0}^{\infty}$ such that

- $k(t)$ satisfies $\dot{k}(t) = s \cdot f(k(t)) - (n+\delta)k(t)$.
- $y(t) = f(k(t))$.
- $c(t) = (1-s) \cdot y(t)$
- $R(t) = f'(k(t))$ and $w(t) = f(k(t)) - f'(k(t))\,k(t)$.

Balanced growth:

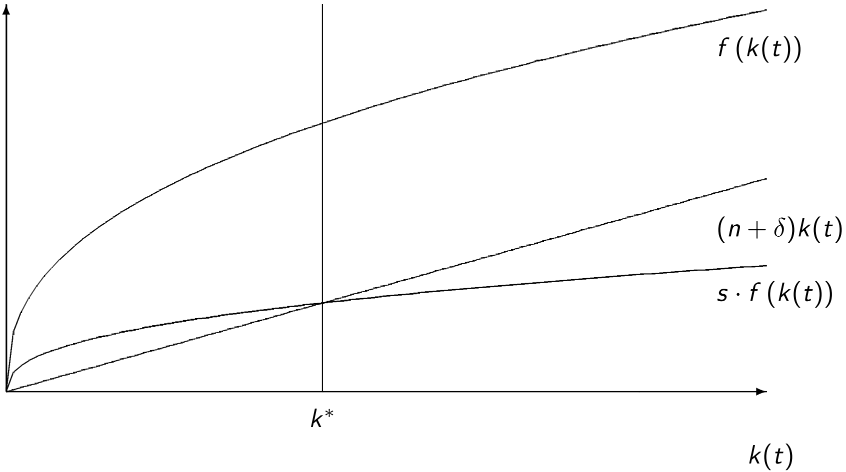
Definition (Balanced Growth Path)

A balanced growth equilibrium is a characterized by constant growth rates of all state variables (this can be zero). Without technological progress, the balanced growth path is an equilibrium path in which $k(t) = k^*$ for all t (steady-state).

The steady state is a stationary point with $\dot{k}(t) = 0$, that is

$$\begin{aligned} \dot{k}(t) =& s \cdot f\left(k(t)\right) - (n+\delta)k(t) = 0 \\ 0 =& s \cdot f(k^*) - (n+\delta)k^* \\ s \cdot f(k^*) =& (n+\delta)k^* \end{aligned}$$

Steady state:



Dynamic efficiency: savings and consumption

- Steady-state per capita consumption is given by $c^* = (1-s)f(k^*)$
- c^* is not monotonic in the savings rate s:

$$c^*(s) = (1-s)f\left(k^*(s)\right) = f\left(k^*(s)\right) - (n+\delta)k^*(s)\ .$$

- Then c^* changes with s according to:

$$\frac{\partial c^*(s)}{\partial s} = \left[f'\left(k^*(s)\right) - (n+\delta)\right]\frac{\partial k^*}{\partial s}\ .$$

- As $\frac{\partial k^*}{\partial s} > 0$, we get that

$$\frac{\partial c^*(s)}{\partial s} = \begin{cases} f'\left(k^*(s)\right) - (n+\delta) > 0 & \text{for small values of } s \\ f'\left(k^*(s)\right) - (n+\delta) < 0 & \text{for large values of } s \end{cases}$$

Intuition: diminishing returns of saving as $f''\left(k^*(s)\right) < 0$.

Dynamic Efficiency: "optimal" k^*

- Optimality as dynamic concept: maximize long-run (steady-state) consumption!
- **Golden Rule**:

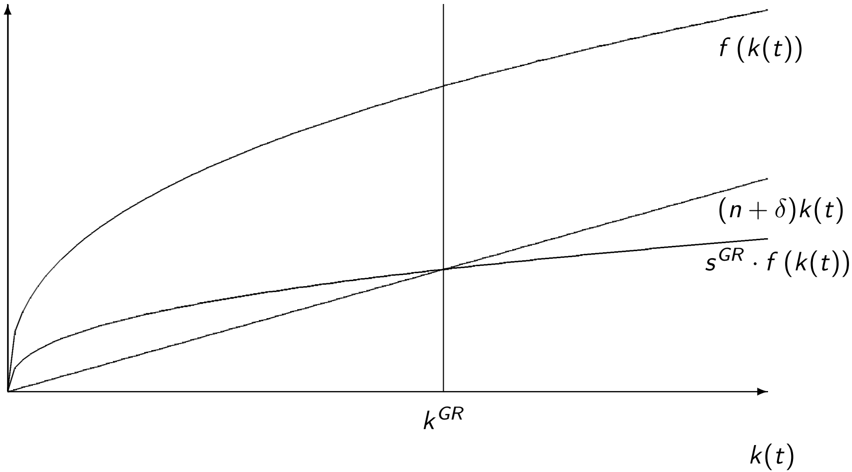
$$k^{GR} : \; \frac{\partial c^*(s)}{\partial s} = 0 \iff f'(k^{GR}) = (n+\delta) \, .$$

- Possibility of dynamic inefficiency (over-saving) if $s > s^{GR}$ where

$$s^{GR} = \frac{(n+\delta)k^{GR}}{f(k^{GR})} \, .$$

- But would people optimally choose s^{GR}?

Golden rule:



Growth rates:

- Growth of capital per capita:

$$\gamma_k := \frac{\dot{k}(t)}{k(t)} = \frac{s \cdot f\left(k(t)\right)}{k(t)} - (n+\delta) \,.$$

- Growth of total capital:

$$\begin{aligned}\gamma_K := \frac{\dot{K}(t)}{K(t)} &= \frac{s \cdot F\left(K(t), L(t)\right) - \delta K(t)}{K(t)} = \frac{s \cdot F\left(K(t), L(t)\right)}{K(t)} - \delta \\ &= \frac{s \cdot F\left(K(t), L(t)\right)}{K(t)} \cdot \frac{L(t)}{L(t)} - \delta \frac{L(t)}{L(t)} = \frac{s \cdot f\left(k(t)\right)}{k(t)} - \delta \\ &= \frac{s \cdot f\left(k(t)\right)}{k(t)} - (n+\delta) + n = \gamma_k + n \,.\end{aligned}$$

- Growth of output per capita (and hence consumption per capita):

$$\begin{aligned}\gamma_y := \frac{\dot{y}(t)}{y(t)} &= \frac{f'\left(k(t)\right) \cdot \dot{k}(t)}{f\left(k(t)\right)} = \frac{f'\left(k(t)\right)}{f\left(k(t)\right)} \cdot \left[s \cdot f\left(k(t)\right) - (n+\delta)k(t)\right] \\ &= \underbrace{\frac{f'\left(k(t)\right) \cdot k(t)}{f\left(k(t)\right)}}_{\equiv \varepsilon_k(k(t)) \, \in (0,1)} \cdot \frac{s \cdot f\left(k(t)\right) - (n+\delta)k(t)}{k(t)} = \varepsilon_k\left(k(t)\right) \cdot \gamma_k \,.\end{aligned}$$

Technological progress:

- If $\gamma_k = 0$ (which holds true in steady-state) there is no output growth. This is counterfactual.
- Solution: technological progress.
- There are various ways to introduce technological progress:
    1. $Y = A(t)F(K(t), L(t))$: "Hicks-neutral"
    2. $Y = F(A(t)K(t), L(t))$: "Solow-neutral"
    3. $Y = F(K(t), A(t)L(t))$: "Harrod-neutral"
- Balanced growth requires "Harrod-neutral" technological progress. However, in Cobb-Douglas production functions "Hicks-neutral" and "Solow-neutral" technological progress can be expressed in "Harrod-neutral" form (for details see Acemoglu, 2008, chapter 2.7).

- Consider Harrod-neutral technological progress with

$$\dot{A}(t) = gA(t) \iff A(t) = A(0) \cdot e^{gt} \ .$$

- Now use efficiency units of labor $A(t)L(t)$ as normalization (before: $L(t)$)

$$\tilde{y}(t) = \frac{Y(t)}{A(t)L(t)} = \frac{F\left(K(t), A(t)L(t)\right)}{A(t)L(t)} = F\left(\frac{K(t)}{A(t)L(t)}, 1\right) \equiv f\left(\tilde{k}(t)\right) \ .$$

Evolution of capital per effective unit of labor:

$$
\begin{aligned}
\tilde{k}(t) =& \frac{K(t)}{A(t)L(t)} \\
\dot{\tilde{k}}(t) =& \frac{\dot{K}(t)A(t)L(t) - K(t)\dot{A}(t)L(t) - K(t)A(t)\dot{L}(t)}{[A(t)L(t)]^2} \\
\dot{\tilde{k}}(t) =& \frac{\dot{K}(t)}{A(t)L(t)} - \underbrace{\frac{\dot{A}(t)}{A(t)}}_{=g} \cdot \frac{K(t)}{A(t)L(t)} - \underbrace{\frac{\dot{L}(t)}{L(t)}}_{=n} \cdot \frac{K(t)}{A(t)L(t)} \\
\dot{\tilde{k}}(t) =& \frac{s \cdot F\left(K(t), L(t)\right) - \delta K(t)}{A(t)L(t)} - (n+g)\tilde{k}(t) \\
\dot{\tilde{k}}(t) =& s \cdot f\left(\tilde{k}(t)\right) - (n+g+\delta)\tilde{k}(t)
\end{aligned}
$$

Steady-state and growth rates:

- The steady-state condition now is: $s \cdot f\left(\tilde{k}(t)\right) = (n+g+\delta)\tilde{k}(t)$
- Growth rates of capital and output per efficiency unit of labor:

$$\gamma_{\tilde{k}} := \frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{s \cdot f\left(\tilde{k}(t)\right)}{\tilde{k}(t)} - (n+g+\delta)$$

$$\gamma_{\tilde{y}} := \frac{\dot{\tilde{y}}(t)}{\tilde{y}(t)} = \underbrace{\frac{f'\left(\tilde{k}(t)\right) \cdot \tilde{k}(t)}{f\left(\tilde{k}(t)\right)}}_{\equiv \varepsilon_{\tilde{k}}\left(\tilde{k}(t)\right) \in (0,1)} \cdot \gamma_{\tilde{k}} \, .$$

- Growth rates of capital and output per capita (note: $k(t) = A(t) \cdot \tilde{k}(t)$):

$$\begin{aligned}
\gamma_k := \frac{\dot{k}(t)}{k(t)} =& \frac{\dot{A}(t) \cdot \tilde{k}(t) + A(t) \cdot \dot{\tilde{k}}(t)}{k(t)} \\
=& \frac{\dot{A}(t)}{A(t)} \frac{k(t)}{k(t)} + \frac{A(t)}{k(t)} \cdot \dot{\tilde{k}}(t) = g + \frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} \\
=& \, g + \gamma_{\tilde{k}} \\
\gamma_y := \frac{\dot{y}(t)}{y(t)} =& \frac{\dot{A}(t) \cdot f\left(\tilde{k}(t)\right) + A(t) \cdot f'\left(\tilde{k}(t)\right) \cdot \dot{\tilde{k}}(t)}{f(k(t))} \\
=& \frac{\dot{A}(t)}{A(t)} \frac{f\left(\tilde{k}(t)\right)}{f\left(\tilde{k}(t)\right)} + \frac{A(t)}{f(k(t))} \cdot f'\left(\tilde{k}(t)\right) \cdot \tilde{k}(t) \cdot \frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} \\
=& g + \varepsilon_{\tilde{k}}\left(\tilde{k}(t)\right) \cdot \frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = g + \varepsilon_{\tilde{k}}\left(\tilde{k}(t)\right) \cdot \gamma_{\tilde{k}} \, .
\end{aligned}$$

- Per capita units grow with the rate of technological progress (even in steady-state). Sustained growth!

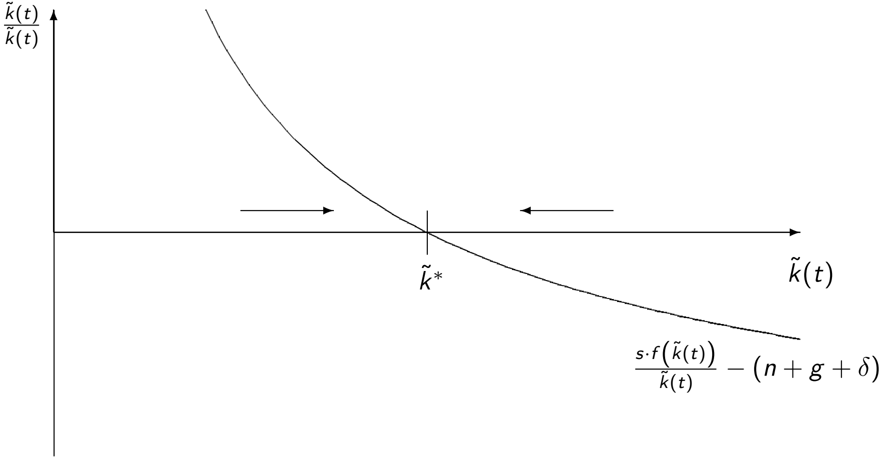
Transitional dynamics: general results to remember

- The steady state of an autonomous, linear differential equation $\dot{x}(t) = ax(t)$ is globally asymptotically stable if $a < 0$.
- The steady state of an autonomous, non-linear differential equation $\dot{x}(t) = g\left(x(t)\right)$, where $g(\cdot)$ is differentiable and $g'(\cdot) < 0$ in the neighborhood of the steady state x^*, is locally asymptotically stable.
- The steady state of a non-linear differential equation $\dot{x}(t) = g\left(x(t)\right)$, where $g(x^*) = 0$ and $g(x) < 0$ for all $x > x^*$ and $g(x) > 0$ for all $x < x^*$, is globally asymptotically stable: starting with any $x(0)$, $x(t) \to x^*$.

Proof: follows from stability results for linear and non-linear differential equations (see, e.g., Acemoglu, 2008, Theorems 2.4, 2.5).

Convergence Dynamics

Convergence: $\frac{\partial \gamma_{\tilde{k}}}{\partial \tilde{k}(t)} = \frac{\partial \left(\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} \right)}{\partial \tilde{k}(t)} = s \cdot \frac{f'\big(\tilde{k}(t)\big)\tilde{k}(t) - f\big(\tilde{k}(t)\big)}{\tilde{k}(t)^2} < 0$



Transitional dynamics in the Solow model:

- Characterising convergence is hard as the system is highly non-linear.
- Linearization around the steady state offers a potential solution.
- A useful rule to keep in mind

$$\frac{\partial y}{\partial \ln x} \cdot \frac{\partial \ln x}{\partial x} = \frac{\partial y}{\partial x} \Rightarrow \frac{\partial y}{\partial \ln x} \cdot \frac{1}{x} = \frac{\partial y}{\partial x} \Rightarrow \frac{\partial y}{\partial \ln x} = \frac{\partial y}{\partial x} \cdot x$$

$$\frac{\partial y}{\partial \ln x} \cdot \frac{\partial \ln y}{\partial y} = \frac{\partial \ln y}{\partial \ln x} \Rightarrow \frac{\partial y}{\partial x} \cdot x \cdot \frac{1}{y} = \frac{\partial \ln y}{\partial \ln x} .$$

Detour: Taylor Series

The Taylor series of a real-valued, differentiable function $f(x)$ with respect to "x" around a number "a" is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \ .$$

As a result a first-order Taylor expansion is given by

$$f(a) + \frac{f'(a)}{1!}(x-a)^1 \ .$$

Intuition: if a real-valued function $f(x)$ is differentiable at point a, then it has a linear approximation near this point. More precisely, there exists a function $h(x)$ such that

$$f(x) = f(a) + f'(a)(x-a) + h(x)(x-a) \ ,$$

with $\lim_{x \to a} h(x) = 0$.

This allows using linear approximations of non-linear functions when in the vicinity of a certain point. The linear approximation becomes less accurate the further one moves away from the point.

Convergence Dynamics

- The growth rate of capital per effective unit of labor is

$$\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} = \frac{s \cdot f\left(\tilde{k}(t)\right)}{\tilde{k}(t)} - (n + g + \delta) .$$

- A first-order Taylor expansion with respect to $\tilde{k}(t)$ around $\tilde{k}^*$ yields

$$\begin{aligned}
\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} &\approx \underbrace{\frac{s \cdot f\left(\tilde{k}^*\right)}{\tilde{k}^*} - (n + g + \delta)}_{=0} - \left(1 - \varepsilon_{\tilde{k}}\left(\tilde{k}^*\right)\right) \underbrace{\frac{s \cdot f\left(\tilde{k}^*\right)}{\tilde{k}^*}}_{=n+g+\delta} \frac{1}{\tilde{k}^*}\left(\tilde{k}(t) - \tilde{k}^*\right) \\
&\approx -\left(1 - \varepsilon_{\tilde{k}}\left(\tilde{k}^*\right)\right)(n + g + \delta)\left(\frac{\tilde{k}(t) - \tilde{k}^*}{\tilde{k}^*}\right)
\end{aligned}$$

- A first-order Taylor expansion of $\ln \tilde{k}(t)$ with respect to $\tilde{k}(t)$ around $\tilde{k}^*$ gives

$$\begin{aligned}
\ln \tilde{k}(t) &\approx \ln \tilde{k}^* + \frac{1}{\tilde{k}^*}\left(\tilde{k}(t) - \tilde{k}^*\right) \\
\Rightarrow \ \ln \tilde{k}(t) - \ln \tilde{k}^* &\approx \frac{\tilde{k}(t) - \tilde{k}^*}{\tilde{k}^*}
\end{aligned}$$

Convergence Dynamics

- A first-order Taylor expansion of $\ln y(t)$ with respect to $\ln \tilde{k}(t)$ around $\ln \tilde{k}^*$ noting that $y^*(t) \equiv A(t) f\left(\tilde{k}^*\right)$ yields

$$\begin{aligned} \ln y(t) &\approx \ln y^*(t) + \varepsilon_{\tilde{k}}\left(\tilde{k}^*\right)\left(\ln \tilde{k}(t) - \ln \tilde{k}^*\right) \\ \ln y(t) - \ln y^*(t) &\approx \varepsilon_{\tilde{k}}\left(\tilde{k}^*\right)\left(\ln \tilde{k}(t) - \ln \tilde{k}^*\right) \end{aligned}$$

- Substituting into the growth rate of output per capita gives

$$\begin{aligned} \frac{\dot{y}(t)}{y(t)} &\approx g + \varepsilon_{\tilde{k}}\left(\tilde{k}^*\right) \cdot \frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)} \\ &\approx g - \varepsilon_{\tilde{k}}\left(\tilde{k}^*\right)\left(1 - \varepsilon_{\tilde{k}}\left(\tilde{k}^*\right)\right)(n + g + \delta)\left(\frac{\tilde{k}(t) - \tilde{k}^*}{\tilde{k}^*}\right) \\ &\approx g - \left(1 - \varepsilon_{\tilde{k}}\left(\tilde{k}^*\right)\right)(n + g + \delta)\varepsilon_{\tilde{k}}\left(\tilde{k}^*\right)\left(\ln \tilde{k}(t) - \ln \tilde{k}^*\right) \\ &\approx g - \left(1 - \varepsilon_{\tilde{k}}\left(\tilde{k}^*\right)\right)(n + g + \delta)\left(\ln y(t) - \ln y^*(t)\right) \end{aligned}$$

- The growth rate of output per capita depends on the rate of technological progress and the "gap" between current and (implied) steady-state output per capita.

Detour: Convergence Speed

The speed of convergence is given by

$$\beta = -\frac{\partial \frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)}}{\partial \ln \tilde{k}(t)} = \left(1 - \varepsilon_{\tilde{k}}\left(\tilde{k}(t)\right)\right) \frac{s \cdot f\left(\tilde{k}(t)\right)}{\tilde{k}(t)}$$

Around the steady state this simplifies to

$$\beta = \left(1 - \varepsilon_{\tilde{k}}\left(\tilde{k}^*\right)\right) \frac{s \cdot f\left(\tilde{k}^*\right)}{\tilde{k}^*} = \left(1 - \varepsilon_{\tilde{k}}\left(\tilde{k}^*\right)\right)(n + g + \delta)$$

The speed of convergence measures how quickly $\tilde{k}(t)$ (and hence $\tilde{y}(t)$) increases (decreases) if $\tilde{k}(t) < \tilde{k}^*$ ($\tilde{k}(t) > \tilde{k}^*$).

Note: the speed of convergence only has an interpretation if $\tilde{k}(t) \neq \tilde{k}^*$ as otherwise the system is resting.

Convergence Dynamics

Growth in transition?

The steady state is implicitly characterized by

$$\frac{f(\tilde{k}^*)}{\tilde{k}^*} = \frac{n+g+\delta}{s} .$$

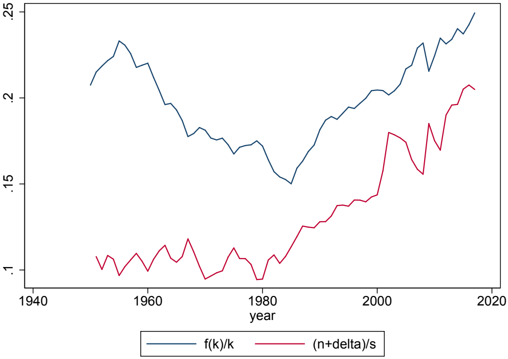


FIGURE: Germany, Source: Penn World Tables 9.1

Convergence? Germany vs. USA

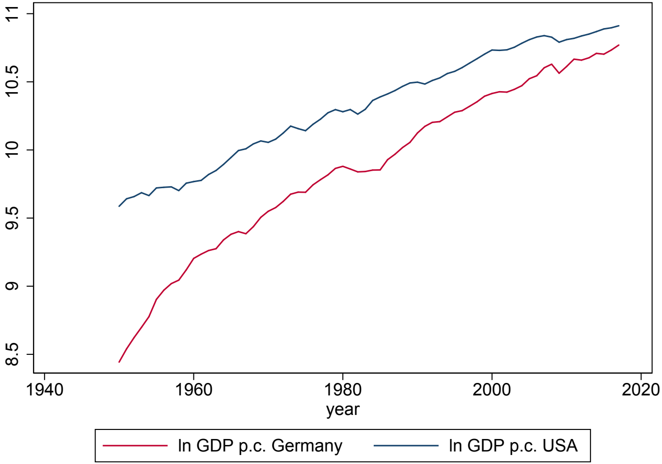


Figure: Germany, Source: Penn World Tables 9.1

Convergence? Germany vs. Congo

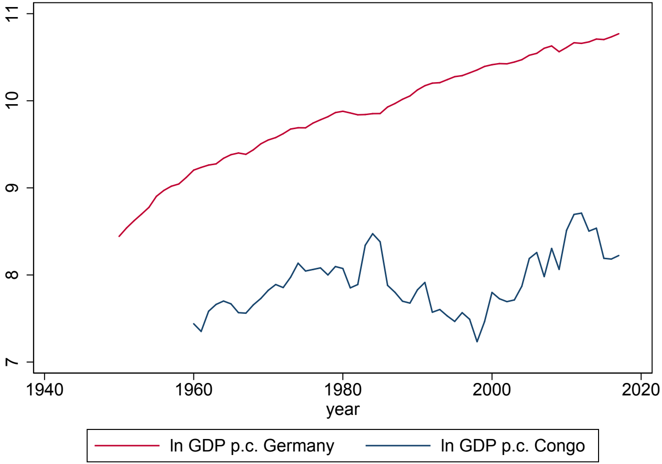


Figure: Germany, Source: Penn World Tables 9.1

Convergence Dynamics

Concepts of convergence:

- Absolute convergence:
  - Consider two countries with $k_1(0) < k_2(0)$, then $\gamma_{k_1} > \gamma_{k_2}$.
- Conditional convergence ("β" – convergence):
  - Consider two countries with $k_1(0) < k_2(0)$ and $k_1^* \neq k_2^*$, then $\gamma_{k_1} \gtrless \gamma_{k_2}$.
  - Depends on $k_1^* - k_1(0) \gtrless k_2^* - k_2(0)$, i.e. the distance to the steady-state.

A Closed-Form Solution

An analytical solution:

- Consider the Solow model with a Cobb-Douglas production function

$$Y(t) = K(t)^{\alpha} \left[A(t)L(t)\right]^{1-\alpha} \quad \text{with } \alpha \in (0,1) .$$

- Population and technology grow at constant rates n and g

$$\begin{aligned} L(t) =& L(0) \cdot e^{nt} \\ A(t) =& A(0) \cdot e^{gt} . \end{aligned}$$

- Writing the model in labor efficiency units ($\tilde{y}(t) = \frac{Y(t)}{A(t)L(t)}$) gives:

$$\begin{aligned} \tilde{y}(t) =& \tilde{k}(t)^{\alpha} \\ \dot{\tilde{k}}_t =& s\tilde{k}(t)^{\alpha} - (n+g+\delta)\tilde{k}(t) . \end{aligned}$$

Solution:

Use

$$
\begin{aligned}
z(t) \equiv& \tilde{k}(t)^{1-\alpha} \\
\dot{z}(t) =& (1-\alpha)\frac{\dot{\tilde{k}}(t)}{\tilde{k}(t)^{\alpha}} \\
\dot{z}(t) =& (1-\alpha)s - (1-\alpha)(n+g+\delta)z(t) \, .
\end{aligned}
$$

Next define

$$
x(t) = z(t) - \frac{s}{n+g+\delta} \, .
$$

Note that

$$
\begin{aligned}
\dot{x}(t) =& \dot{z}(t) \\
z(t) =& x(t) + \frac{s}{n+g+\delta} \, .
\end{aligned}
$$

A Closed-Form Solution

Substituting back gives

$$\dot{x}(t) = (1-\alpha)s - (1-\alpha)(n+g+\delta)\left(x(t) + \frac{s}{n+g+\delta}\right)$$
$$\dot{x}(t) = -(1-\alpha)(n+g+\delta)x(t)\ .$$

Applying the known solution procedure gives

$$x(t) = x(0) \cdot e^{-(1-\alpha)(n+g+\delta)t}\ .$$

Substituting back for $z(t)$ gives

$$z(t) = \frac{s}{n+g+\delta} + \left(z(0) - \frac{s}{n+g+\delta}\right) \cdot e^{-(1-\alpha)(n+g+\delta)t}\ .$$

Interpretation: the capital-output ratio $z(t) = \tilde{k}(t)^{1-\alpha}$ is a weighted average between its steady state and its initial level with weights being exponential functions of time.

Substituting back for $\tilde{k}(t)$ gives

$$\tilde{k}(t) = \left[\frac{s}{n+g+\delta} + \left(\tilde{k}(0)^{1-\alpha} - \frac{s}{n+g+\delta}\right) \cdot e^{-(1-\alpha)(n+g+\delta)t}\right]^{\frac{1}{1-\alpha}}\ .$$

The Solow Model: Summary

- Simple and tractable framework to study capital accumulation and the implications of technological progress.
- Without technological progress, there will be no sustained growth.
- Strong economic growth is only a temporary phenomenon that occurs along the transition path.
- Technological progress is a black box. (What generates $g > 0$? What makes some firms invent better technologies?)
- Organizing framework to think about the mechanics of economic growth.
- Important message: to understand growth, we have to understand physical (and human) capital accumulation and the origins of technological progress.

Growth accounting:

- What are the sources of economic growth?
- Assume a neo-classical production function and competitive markets

$$Y(t) = F\left(K(t), L(t), A(t)\right) .$$

- The growth rate is given by

$$\begin{aligned}\underbrace{\frac{\dot{Y}(t)}{Y(t)}}_{\equiv g_Y(t)} = & \underbrace{\frac{F_K\left(K(t), L(t), A(t)\right) \cdot K(t)}{Y(t)}}_{\equiv \varepsilon_K(t)} \underbrace{\frac{\dot{K}(t)}{K(t)}}_{\equiv g_K(t)} \\ & + \underbrace{\frac{F_L\left(K(t), L(t), A(t)\right) \cdot L(t)}{Y(t)}}_{\equiv \varepsilon_L(t)} \underbrace{\frac{\dot{L}(t)}{L(t)}}_{\equiv g_L(t)} \\ & + \underbrace{\frac{F_A\left(K(t), L(t), A(t)\right) \cdot A(t)}{Y(t)} \frac{\dot{A}(t)}{A(t)}}_{\equiv x(t)} .\end{aligned}$$

$$x(t) = g_Y(t) - \varepsilon_K(t) g_K(t) - \varepsilon_L(t) g_L(t) \ .$$

- How much of economic growth can be attributed to increased factor inputs and how much is due to technological progress?
- Solow's conclusion: a large part of growth is due to technological progress ("$x(t)$"):
    - Contribution of technological progress to economic growth.
    - "Solow residual"
    - "Total Factor Productivity (TFP) growth"
- Problem of measuring factor inputs (capital stocks data, relative prices, quality changes, etc.).

From Solow to Neoclassical Growth

- Solow model assumes that households have fixed labor supply and save at a constant rate.
- Not satisfactory as it has important effects on the model predictions.
- Next: introduce utility-maximizing households that decide on how much to consume/save (labor supply still being exogenous).
- Related to “Lucas critique”: suppose we introduce taxes in the Solow model, predictions of the reaction of the economy based on the historically observed savings rate might be wrong, because optimizing households will adjust their savings rate

Outlook: Neoclassical Growth Model

Ramsey (1928) – Cass (1965) – Koopmans (1965) model

- The growth model is "the" benchmark model of modern macroeconomics.
- Also a great laboratory for teaching tools of macro...
- Many other models in macroeconomics build on the growth model. Examples:
  - Real business cycle (RBC) model = growth model with aggregate productivity shocks.
  - New Keynesian model = RBC model + sticky prices (and/or wages).
  - Incomplete markets model (Aiyagari-Bewley-Huggett) = growth model + heterogeneity in form of uninsurable idiosyncratic shocks.