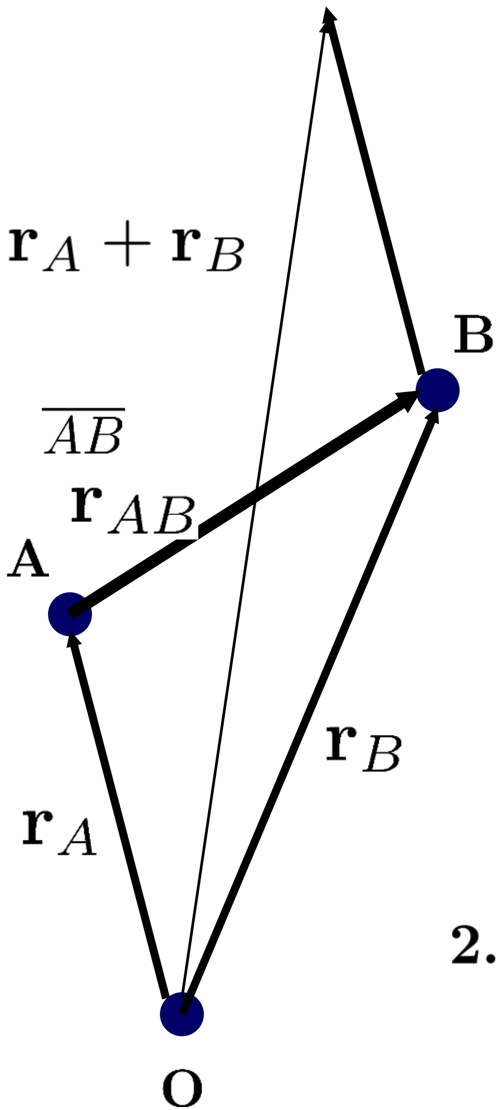
Last Time:



Note $\mathbf{r}_A = \mathbf{r}_{OA}$

Recall Triangle Law (or Parallelogram Law)

1. Position vector and relative position vector $\mathbf{r}_{AB} = \mathbf{r}_B - \mathbf{r}_A =: \mathbf{r}_{B/A}$

$$\mathbf{r}_{AB} = (X_B - X_A)\mathbf{i} + (Y_B - Y_A)\mathbf{j} + (Z_B - Z_A)\mathbf{k}$$

Length:

$$|\mathbf{r}_{AB}| = \sqrt{(X_B - X_A)^2 + (Y_B - Y_A)^2 + (Z_B - Z_A)^2}$$

Unit vector along direction $\overline{AB}$:

2. Unit direction vector $\boldsymbol{\lambda} = \dfrac{\mathbf{r}_{AB}}{|\mathbf{r}_{AB}|};$

HW2

2.78 The boom OA carries a load $\mathbf{P}$ and is supported by two cables, as shown. Knowing that the tension in cable AB is 183 lb and that the resultant of the load $\mathbf{P}$ and of the forces exerted at A by the two cables must be directed along OA, determine the tension in cable AC.

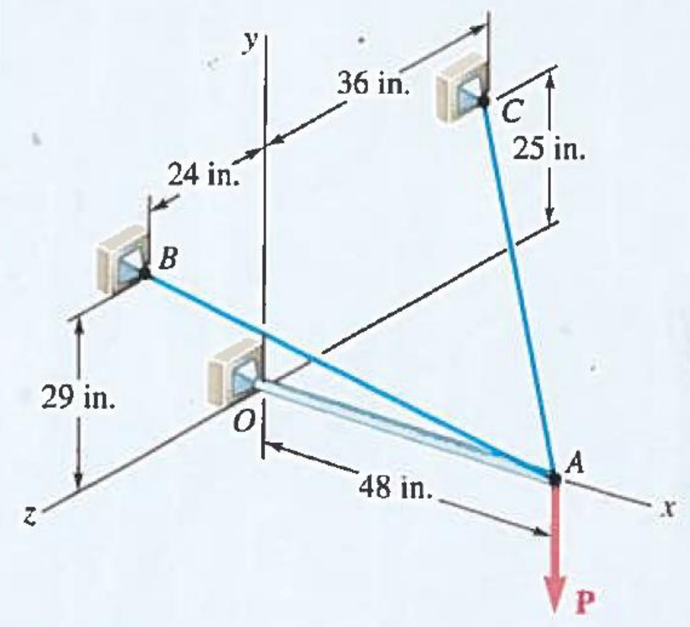
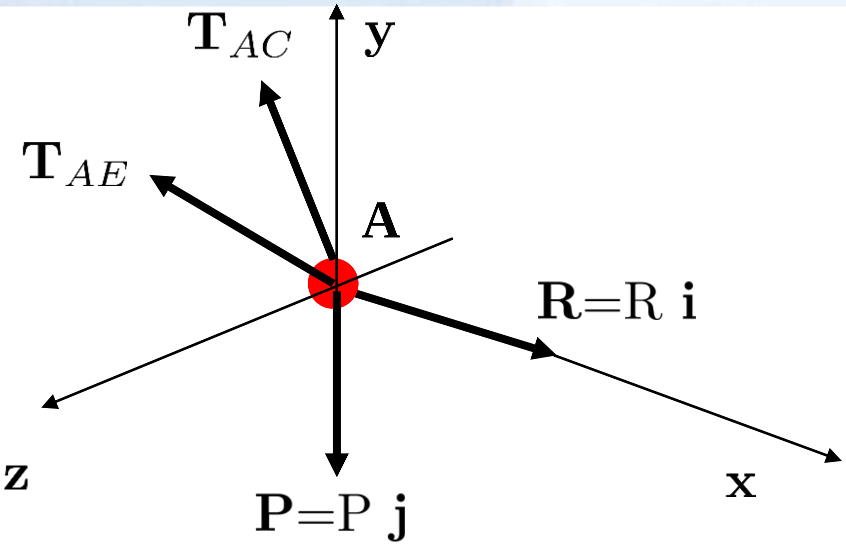


Fig. P2.78

$$\mathbf{R} = \mathbf{T}_{AB} + \mathbf{T}_{AC} + \mathbf{P}$$

$$\textbf{Find } |\mathbf{T}_{AC}| \textbf{ ?}$$



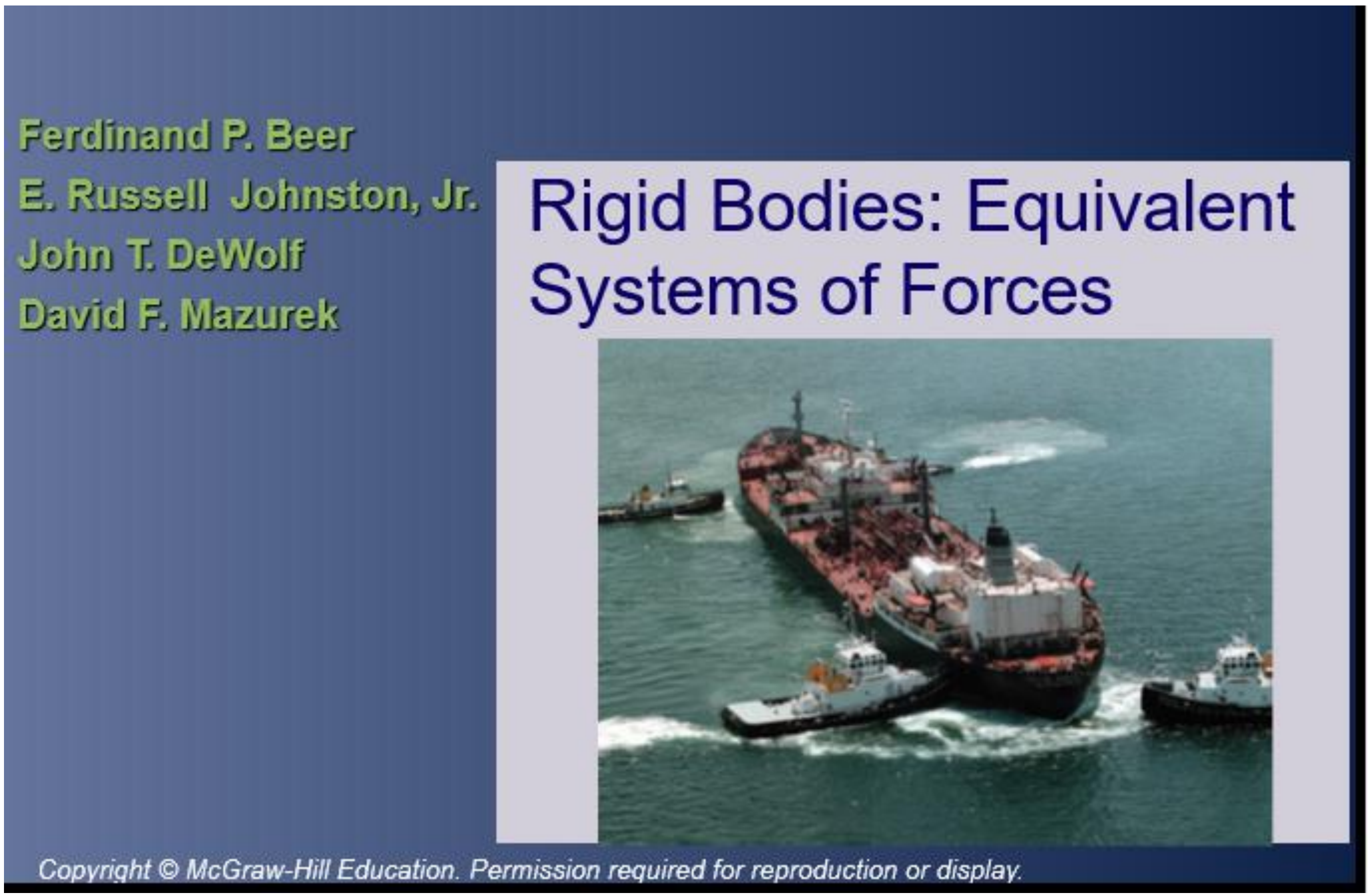
A: (48, 0, 0)

B: (0, 29, 24)

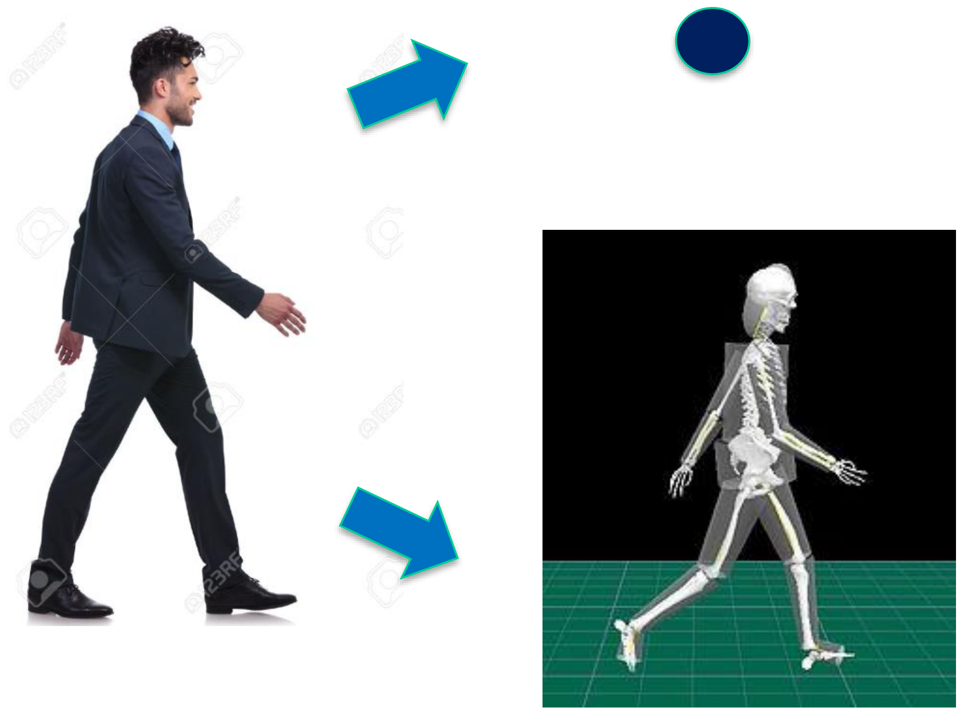
C: (0, 25, -36)

O: (0, 0, 0)

Lecture 3
Rigid Bodies: Equivalent Systems of Forces



The Major Difference between Rigid-body Model and Particle Model

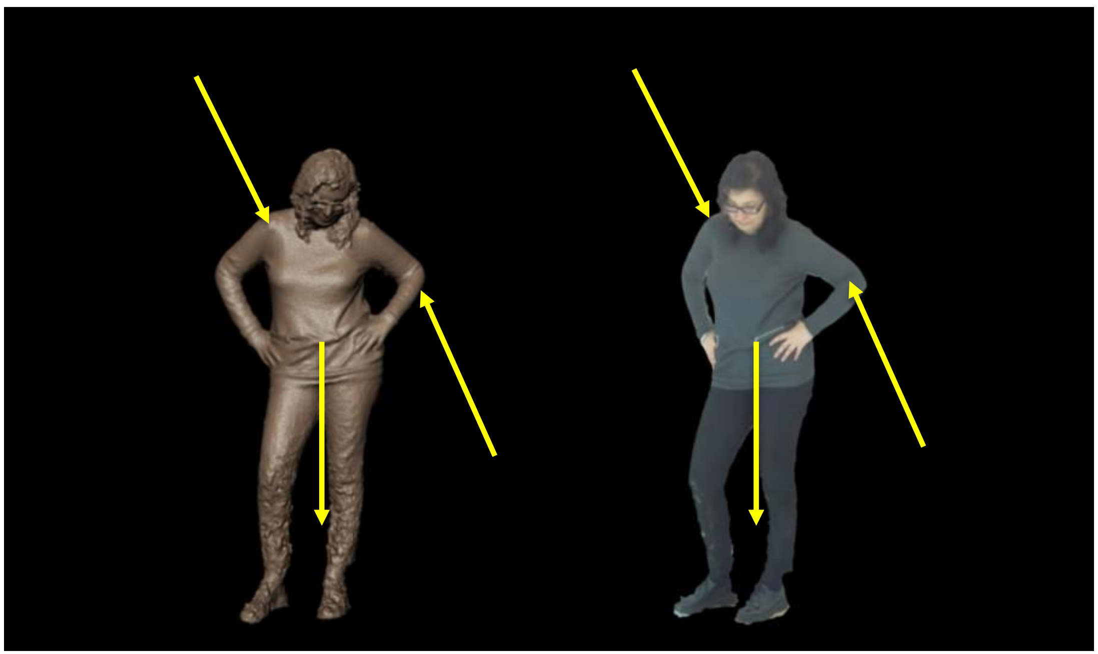


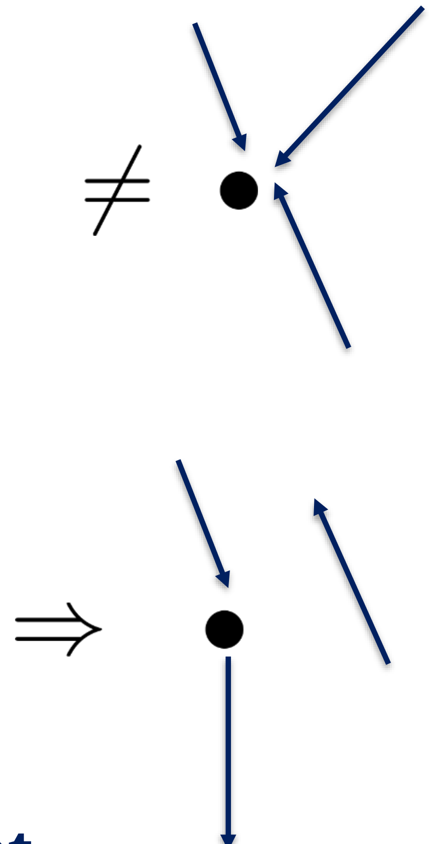
Rigid-body has many points (infinitely many points), so that external forces acting on it may not be concurrent.

How can we solve the rigid-body equilibrium?

Rigid Body Model

- Treatment of a body as a single particle is not accurate. In general, the size of the body and the specific points of application of the forces must be considered.





The forces may NOT be concurrent.

How to study equilibrium of rigid bodies ?

To gain new knowledge by reviewing the old first

------Confucius

Equilibrium of a particle:

$$\sum_i \mathbf{F}_i = \mathbf{R} = 0 \ ,$$

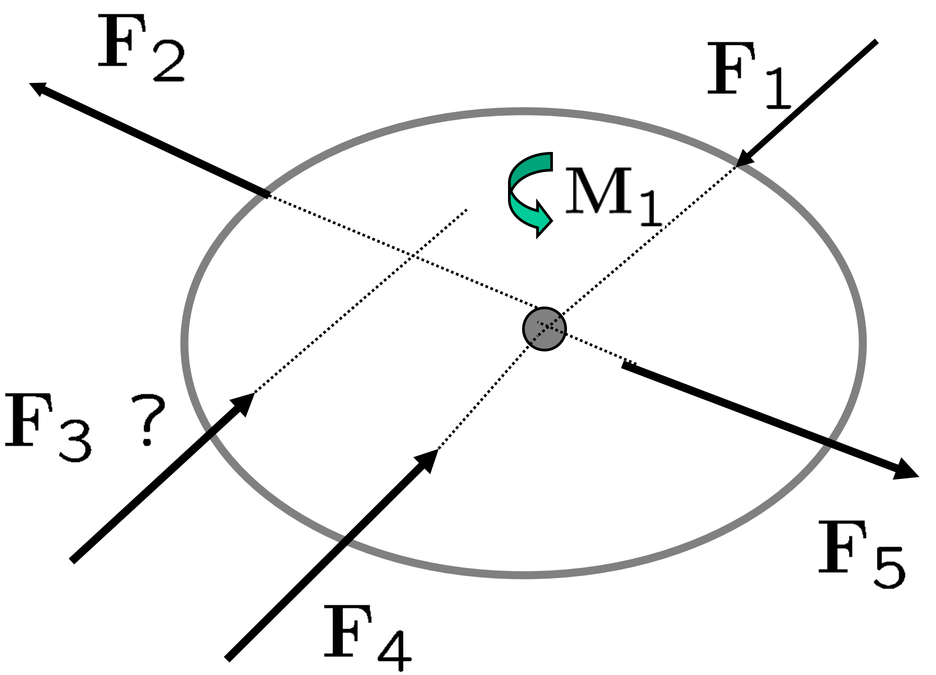
or

$$\sum_i \mathbf{F}_{ix} = 0, \quad \sum_i \mathbf{F}_{iy} = 0, \ \text{and} \ \sum_i \mathbf{F}_{iz} = 0 \ .$$

All these forces are con-current, i.e. They are acting at a same point !

How to study equilibrium of rigid bodies ?

We want to ***move*** all the forces into **ONE** point.



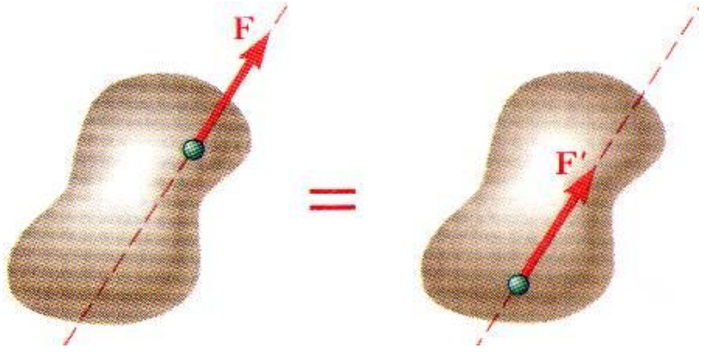
What is the issue ?

In mechanics, all the force vectors are fixed vectors, i.e. they have magnitude, line of action, point of application, and sense of direction.
Since a rigid body cannot deform, one can move a force along its line of action without change the equilibrium condition of the rigid body. Therefore, we can view all force vectors in **rigid body model** as sliding vectors. This will help us simplify the problem.

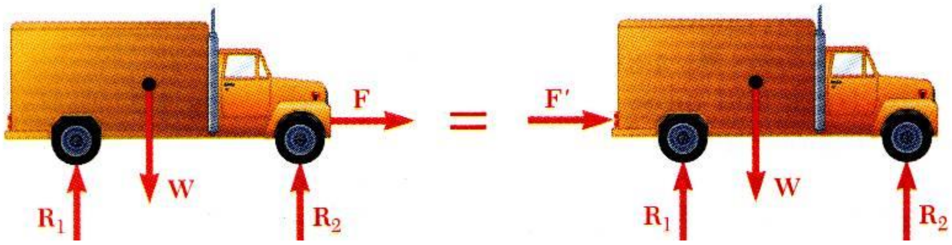
How to move all the forces into One point?

How to move a force vector in a rigid body?

- *Principle of Transmissibility* - Conditions of equilibrium or motion are not affected by *transmitting* a force along its line of action (within the body).
  NOTE: **F** and **F′** are equivalent forces.



- Moving the point of application of the force **F** to the rear bumper does not affect the motion or the other forces acting on the truck.

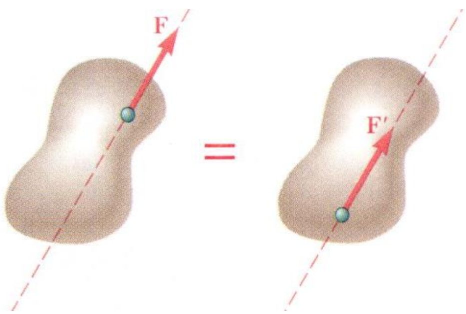


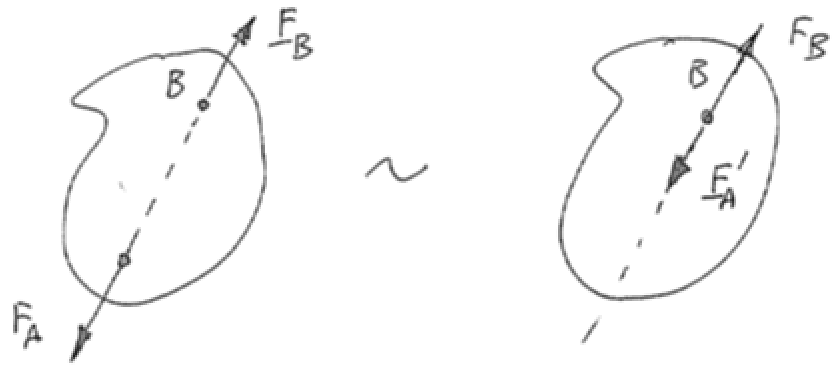
Principle of Transmissibility: Equivalent Forces

How to *move* force along its line of action ?

Principle of Transmissibility For sliding vectors

The condition of equilibrium of a rigid body will remain unchanged if a force **F** acting at a given point of rigid body is replaced by a force of the same magnitude and the same direction, but acting at a different point

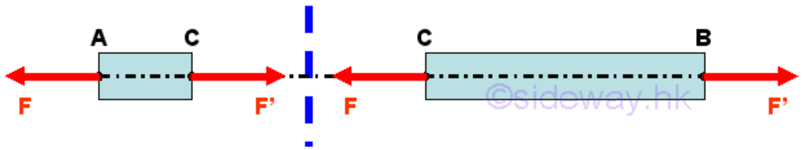




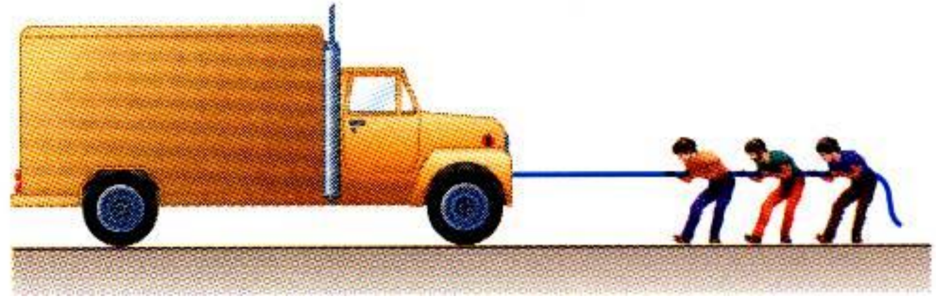
• *Principle of Transmissibility*

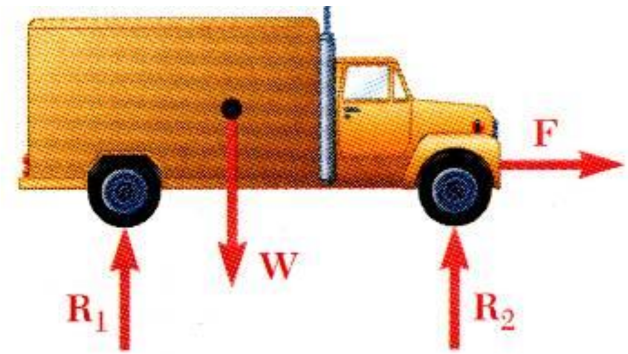
Two forces $\mathbf{F_A}$ and $\mathbf{F'_A}$ are equivalent, if they have the same magnitude, the same direction, and the same line of action,

How is this possible ?



- Forces acting on rigid bodies are divided into two groups:
  - External forces
  - Internal forces

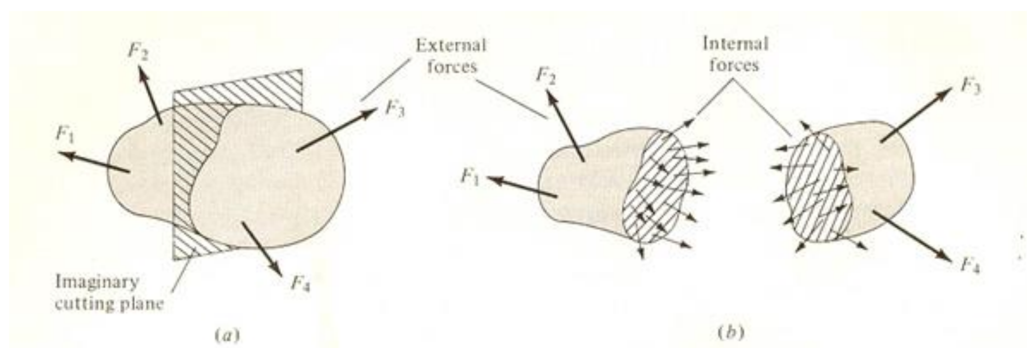




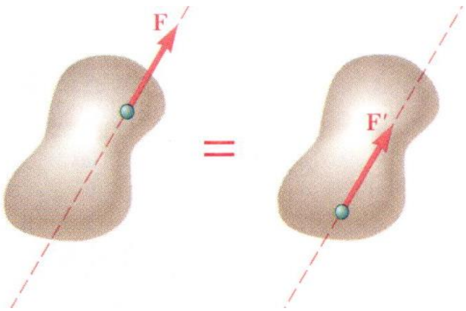
- External forces are shown in a free body diagram.

- Internal forces, such as the force between each wheel and the axle it is mounted on, would not be shown on a free body diagram of the entire truck.

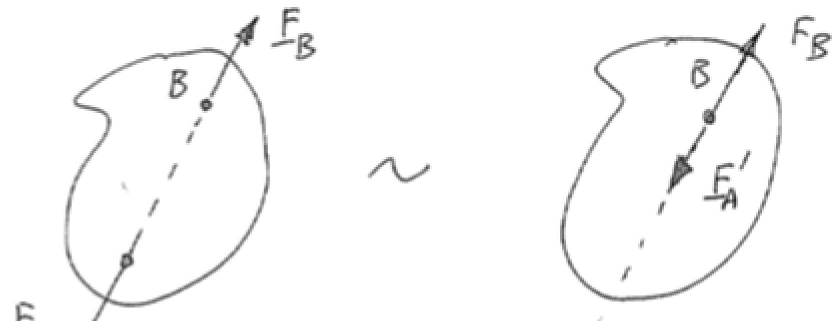
Draw Free-body Diagram of Rigid-Body



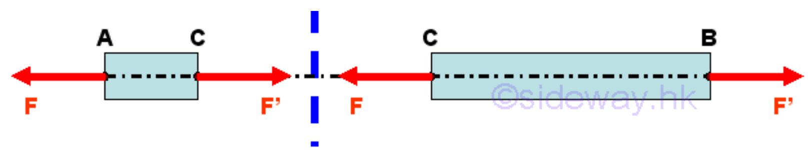
Principle of Transmissibility:
In a rigid body, conditions of equilibrium or motion are not affected by *transmitting* a force along its line of action.

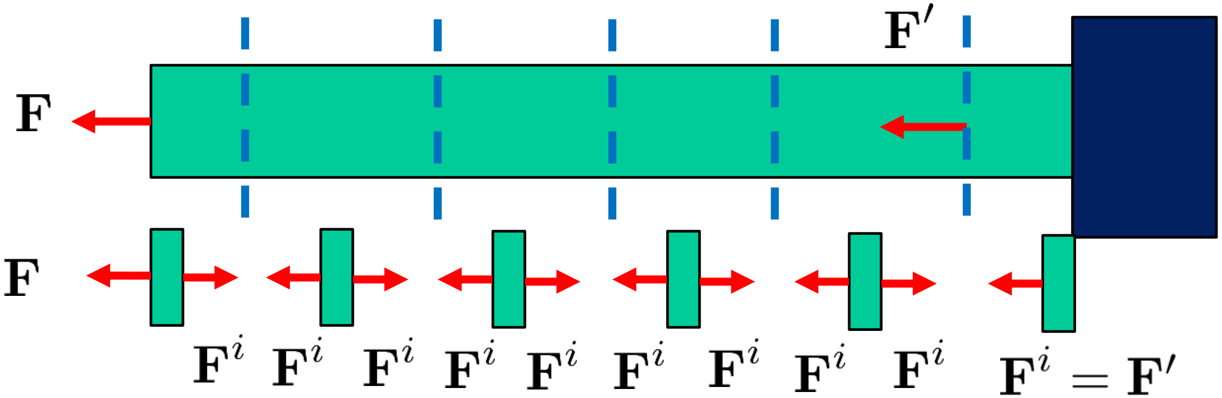


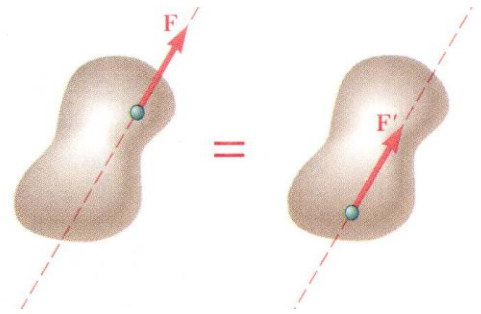
- *Principle of Transmissibility*



Internal forces always come as a pair.



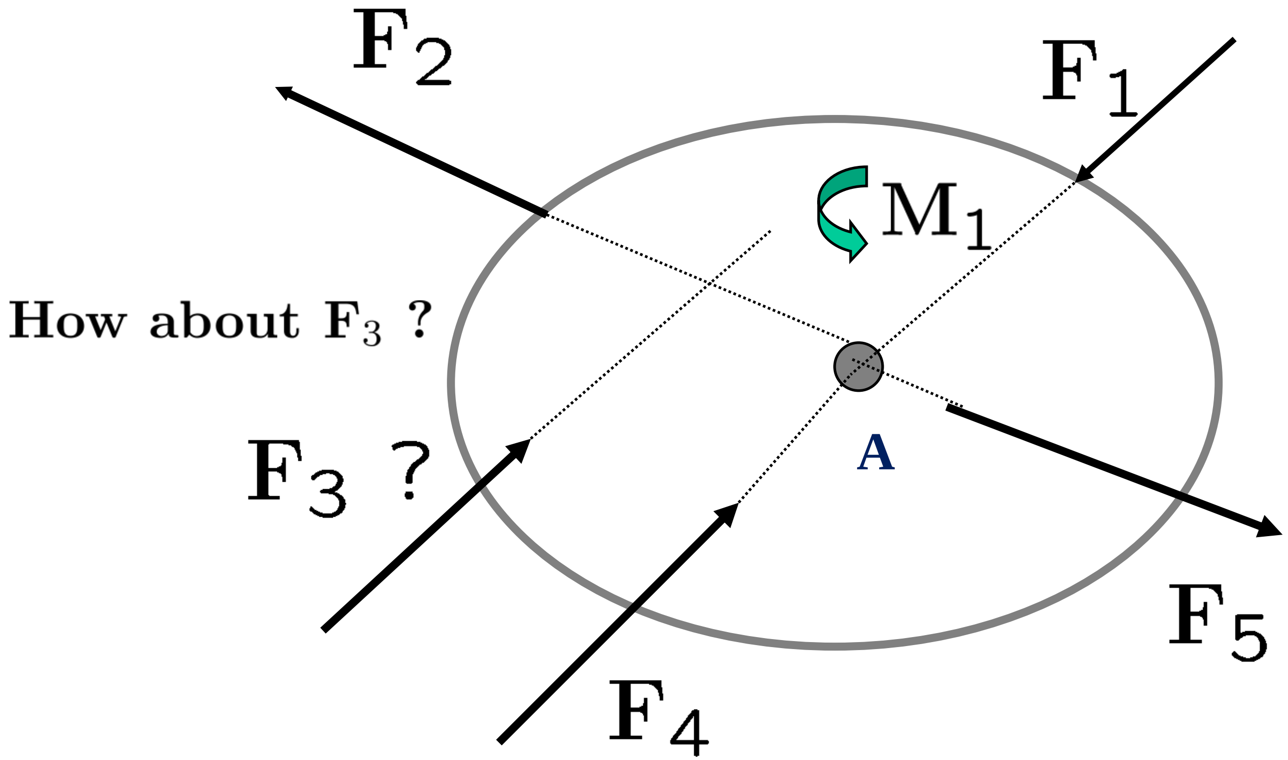




- *Principle of Transmissibility*

Remarks: In rigid body, a force can be moved along its line of action without affecting its equilibrium state.

However, it does affect its internal force state. It is just we don't care the internal force state of a rigid body (model).



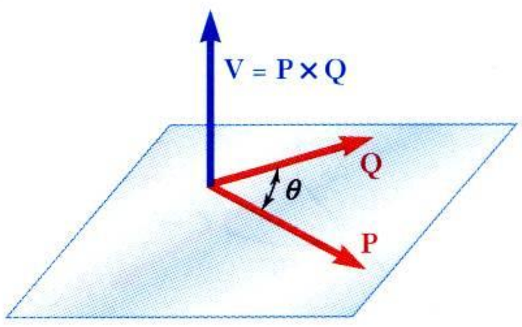
We need to move $\mathbf{F}_3$ parallel to its line of action ! while maintaining its static equilibrium.

- First, we need to learn some new statics concepts, including:
  - moment of a force about a point
  - moment of a force about an axis
  - moment due to a couple

Vector Products

- Concept of the moment of a force about a point requires the understanding of the *vector product* or *cross product*.

- Vector product of two vectors **P** and **Q** is defined as the vector **V** which satisfies the following conditions:
  1. Line of action of **V** is perpendicular to plane containing **P** and **Q**.
  2. Magnitude of **V** is $V = PQ\sin\theta$
  3. Direction of **V** is obtained from the right-hand rule.



(a)

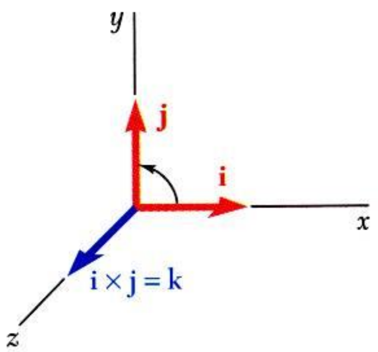
(b)

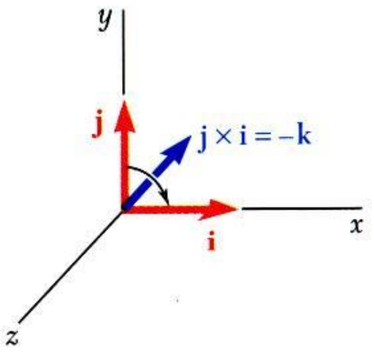
- Vector products:
  - are not commutative; however, $\boldsymbol{Q}\times\boldsymbol{P} = -(\boldsymbol{P}\times\boldsymbol{Q})$
  - are distributive, $\boldsymbol{P}\times(\boldsymbol{Q}_1+\boldsymbol{Q}_2) = \boldsymbol{P}\times\boldsymbol{Q}_1 + \boldsymbol{P}\times\boldsymbol{Q}_2$
  - are not associative, $(\boldsymbol{P}\times\boldsymbol{Q})\times\boldsymbol{S} \neq \boldsymbol{P}\times(\boldsymbol{Q}\times\boldsymbol{S})$

Vector Products: Rectangular Components

- Vector products of Cartesian unit vectors:

$$
\begin{array}{lll}
\vec{i}\times\vec{i} = 0 & \vec{j}\times\vec{i} = -\vec{k} & \vec{k}\times\vec{i} = \vec{j} \\
\vec{i}\times\vec{j} = \vec{k} & \vec{j}\times\vec{j} = 0 & \vec{k}\times\vec{j} = -\vec{i} \\
\vec{i}\times\vec{k} = -\vec{j} & \vec{j}\times\vec{k} = \vec{i} & \vec{k}\times\vec{k} = 0
\end{array}
$$





- Vector products in terms of rectangular coordinates

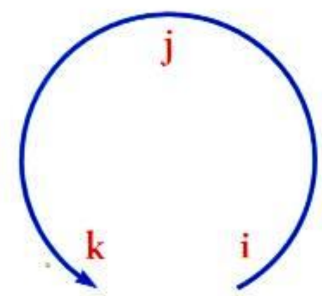
$$
\vec{V} = \left(P_x\vec{i} + P_y\vec{j} + P_z\vec{k}\right)\times\left(Q_x\vec{i} + Q_y\vec{j} + Q_z\vec{k}\right)
$$

$$
= \left(P_yQ_z - P_zQ_y\right)\vec{i} + \left(P_zQ_x - P_xQ_z\right)\vec{j}
$$

$$
+ \left(P_xQ_y - P_yQ_x\right)\vec{k}
$$

$$
= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}
$$

$$
\mathbf{i} = \mathbf{e}_1 \qquad \mathbf{j} = \mathbf{e}_2 \qquad \mathbf{k} = \mathbf{e}_3
$$



Right-hand Rule

Today’s lecture password:

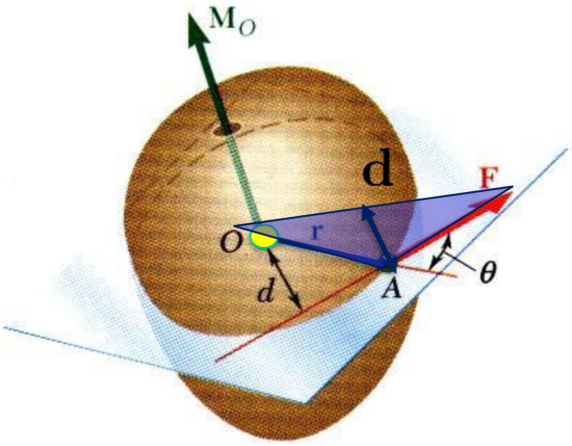
Vector Products

Moment of a Force About a Point

- The *moment* of **F** about *O* is defined as

$$\boldsymbol{M_O} = \boldsymbol{r} \times \boldsymbol{F}$$

$\mathbf{r}$ is $\mathbf{r}_{OA}$!



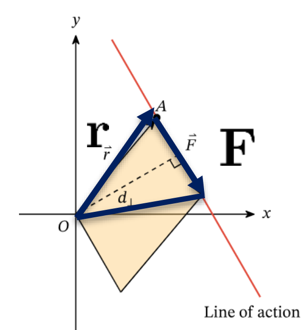
(a)

- The moment vector $\mathbf{M}_O$ is perpendicular to the plane containing *O* and the force **F**.

(b)

- Magnitude of $\mathbf{M}_O$, $M_O = rF\sin\theta = Fd$, measures the tendency of the force to cause rotation of the body about an axis along $\mathbf{M}_O$. The sense of the moment may be determined by the right-hand rule.

- Any force **F′** that has the same magnitude and direction as **F** is *equivalent* if it also has the same line of action and therefore, produces the same moment.

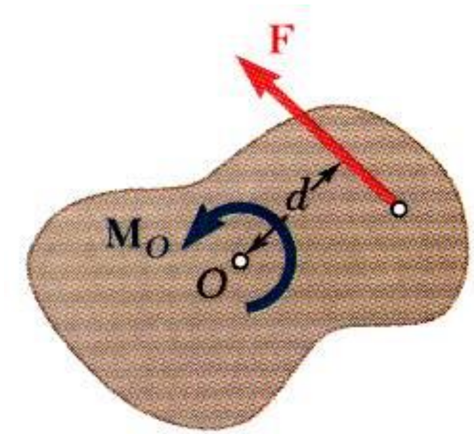


$$M_O = Frsin\theta$$

Moment of a Force About a Point: Special Case

- *Two-dimensional structures* have length and breadth but negligible depth and are subjected to forces contained only in the plane of the structure.

- The plane of the structure contains the point *O* and the force **F**. $\mathbf{M}_O$, the moment of the force about *O*, is perpendicular to the plane.

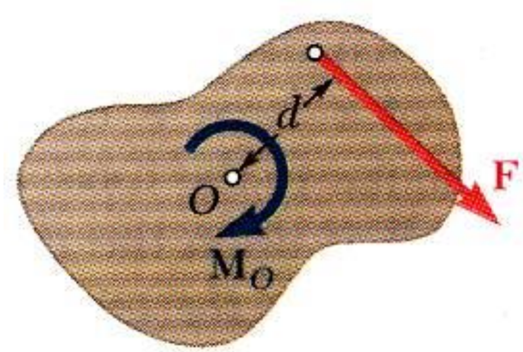


(*a*) $\boldsymbol{M}_O = +Fd$

Sign Convention

- If the force tends to rotate the structure counterclockwise, the sense of the moment vector is out of the plane of the structure and the magnitude of the moment is positive.

- If the force tends to rotate the structure clockwise, the sense of the moment vector is into the plane of the structure and the magnitude of the moment is negative.



(*b*) $\boldsymbol{M}_O = -Fd$

Recall:

The moment of **F** about *O*:

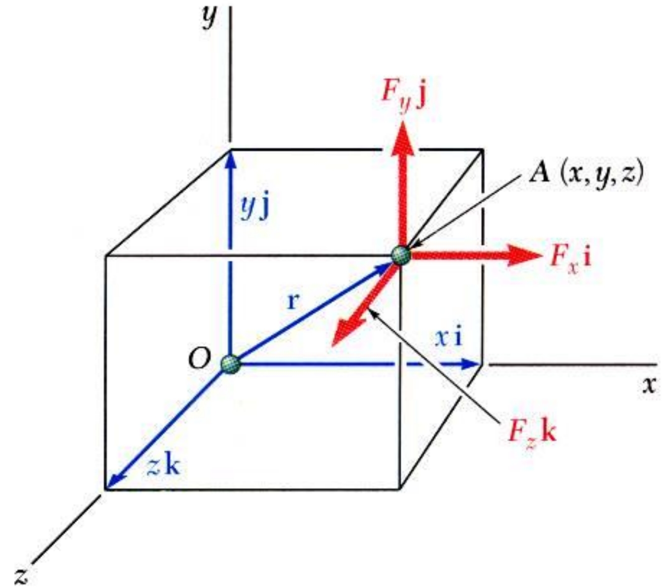
$$\vec{M}_O = \vec{r} \times \vec{F}, \qquad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

$$\vec{M}_O = M_x\vec{i} + M_y\vec{j} + M_z\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(yF_z - zF_y \right)\vec{i} + \left(zF_x - xF_z \right)\vec{j} + \left(xF_y - yF_x \right)\vec{k}$$



The components of $\vec{M}_O$, M_x, M_y, and M_z, represent the moments about the *x*, *y*, and *z* axis, respectively.

Rectangular Components of the Moment of a Force

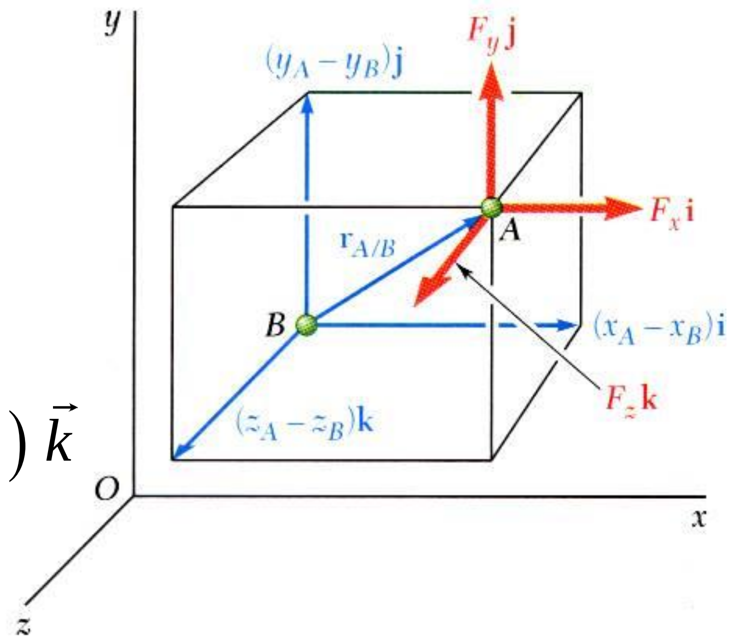
The moment of ***F*** about *B*,

$$\vec{M}_B = \vec{r}_{A/B} \times \vec{F} = \mathbf{r}_{BA} \times \mathbf{F}_A$$

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$

$$= (x_A - x_B)\,\vec{i} + (y_A - y_B)\,\vec{j} + (z_A - z_B)\,\vec{k}$$

$$\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$$



$$\vec{M}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ (x_A - x_B) & (y_A - y_B) & (z_A - z_B) \\ F_x & F_y & F_z \end{vmatrix}$$

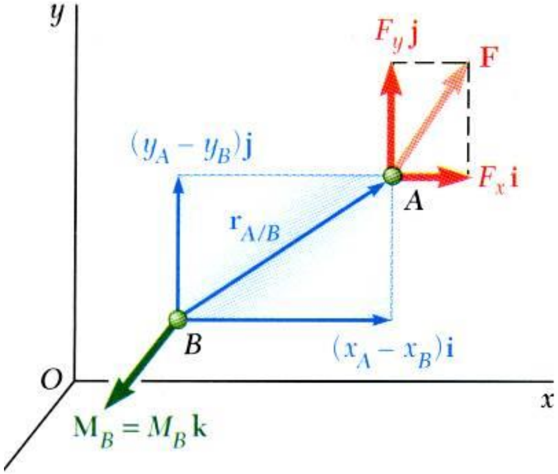
For two-dimensional structures,

$$\vec{M}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ (x_A - x_B) & (y_A - y_B) & (z_A - z_B) \\ F_x & F_y & F_z \end{vmatrix}$$

$$\vec{M}_B = \left[(x_A - x_B)\, F_y - (y_A - y_B)\, F_z \right] \vec{k}$$

$$M_B = M_Z$$

$$= (x_A - x_B)\, F_y - (y_A - y_B)\, \boxed{F_x}$$

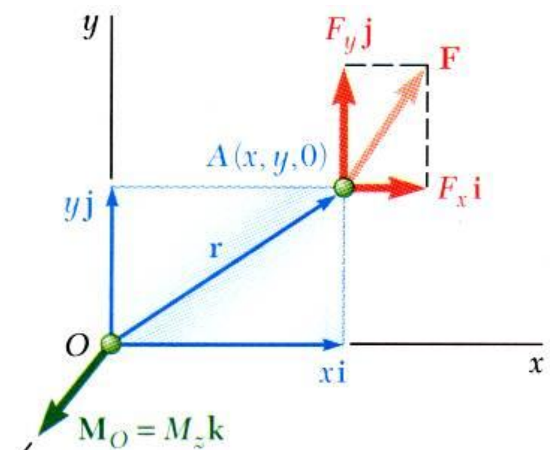


Special Case : $\mathbf{x}_O = 0, \quad \mathbf{x}_A = \mathbf{x}$

$$\vec{M}_O = \left(xF_y - \boxed{yF_x} \right) \vec{k}$$

$$M_O = M_Z$$

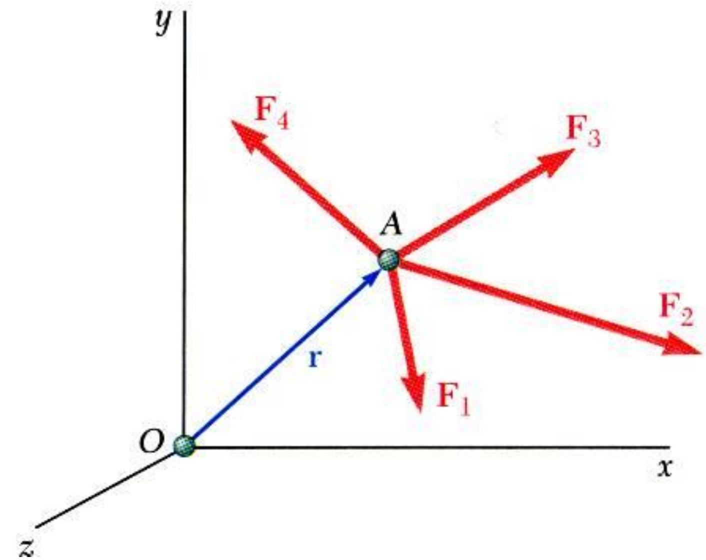
$$= xF_y - \boxed{yF_x}$$



Varignon's Theorem

- The moment about a given point O of the resultant of several concurrent forces is equal to the sum of the moments of the various forces about the same point O.

$$\vec{r} \times \left(\vec{F}_1 + \vec{F}_2 + \cdots\right) = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \cdots$$

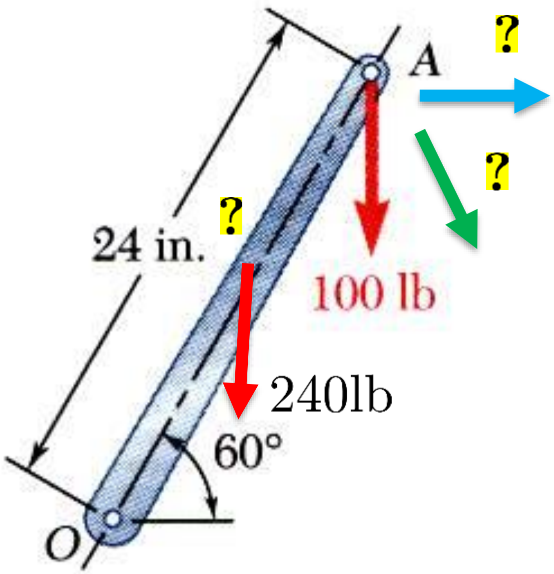


- Varignon's Theorem makes it possible to replace the direct determination of the moment of a force **F** by the moments of two or more component forces of **F**.

$$\mathbf{r} \times \sum_i \mathbf{F}_i = \sum_i \mathbf{r} \times \mathbf{F}_i$$

Pierre Varignon
(1654-1722)

Sample Problem 3.1



A 100-lb vertical force is applied to the end of a lever that is attached to a shaft (not shown) at *O*.

Determine:

a) the moment about *O*,

b) the horizontal force at *A* that creates the same moment,

c) the smallest force at A that produces the same moment,

d) the location for a 240-lb vertical force to produce the same moment,

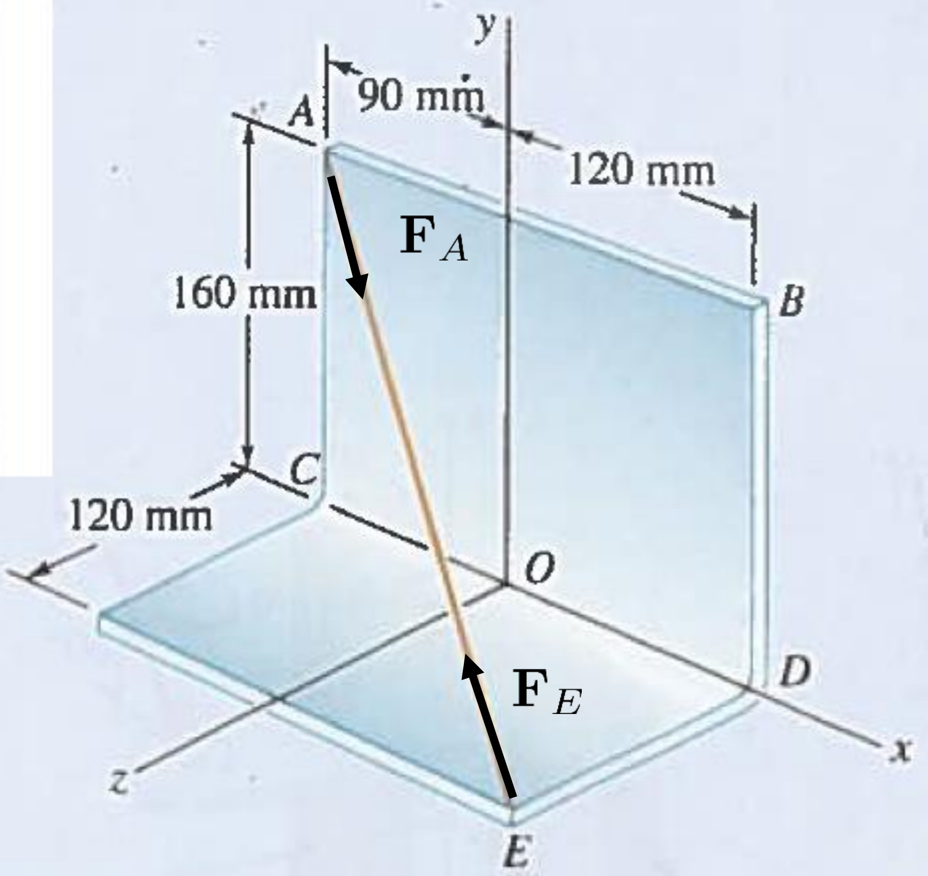


Fig. P3.16

Find

$$\mathbf{M}_O = \mathbf{r}_{OA} \times \mathbf{F}_A \ .$$

$$\mathbf{M}_O = \mathbf{r}_{OE} \times \mathbf{F}_E$$

O: (0, 0, 0)

A: (-90, 160, 0)

E: (120, 0, 120)

$$\mathbf{r}_{AE} = 210\mathbf{i} - 160\mathbf{j} + 120\mathbf{k}$$

3.16 The wire AE is stretched between the corners A and E of a bent plate. Knowing that the tension in the wire is 435 N, determine the moment about O of the force exerted by the wire (*a*) on corner A, (*b*) on corner E.