Lecture 15 Moments of Inertia (II)

Moments of Inertia

$$I_x = \int_A y^2 dA \qquad\qquad I_y = \int_A x^2 dA$$

Polar Moment of Inertia

$$I_O = \int_A r^2 dA = \int_A (x^2 + y^2) dA = I_x + I_y$$

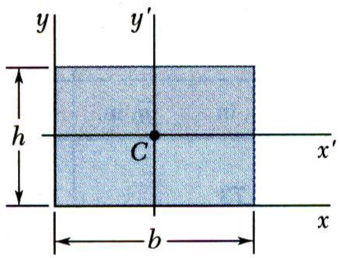
Radiu of Gyration

$$r_O = \sqrt{\frac{I_O}{A}}$$

Moments of Inertia of Composite Areas

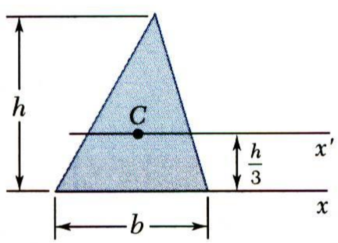
- The moment of inertia of a composite area A about a given axis is obtained by adding the moments of inertia of the component areas A_1, A_2, A_3, ... , with respect to the same axis.

Rectangle



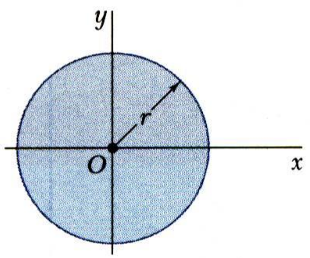
$$\bar{I}_{x'} = \frac{1}{12} bh^3$$
$$\bar{I}_{y'} = \frac{1}{12} b^3 h$$
$$I_x = \frac{1}{3} bh^3$$
$$I_y = \frac{1}{3} b^3 h$$
$$J_C = \frac{1}{12} bh(b^2 + h^2)$$

Triangle



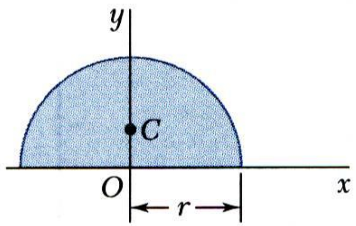
$$\bar{I}_{x'} = \frac{1}{36} bh^3$$
$$I_x = \frac{1}{12} bh^3$$

Circle



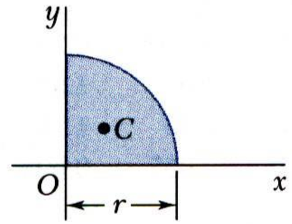
$$\bar{I}_x = \bar{I}_y = \frac{1}{4} \pi r^4$$
$$J_O = \frac{1}{2} \pi r^4$$

Semicircle



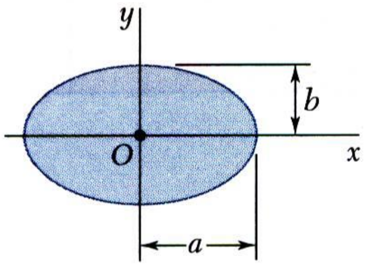
$$I_x = I_y = \frac{1}{8} \pi r^4$$
$$J_O = \frac{1}{4} \pi r^4$$

Quarter circle



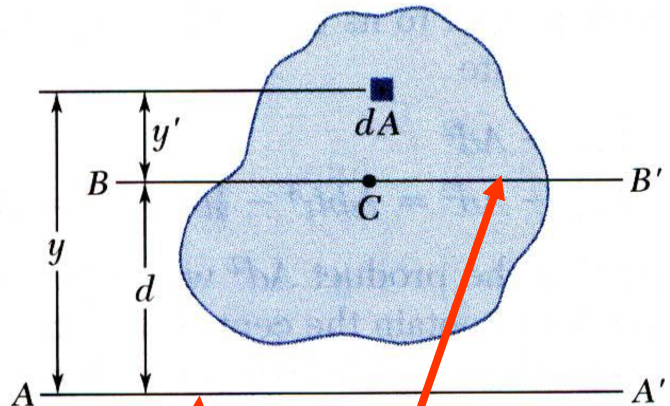
$$I_x = I_y = \frac{1}{16} \pi r^4$$
$$J_O = \frac{1}{8} \pi r^4$$

Ellipse



$$\bar{I}_x = \frac{1}{4} \pi a b^3$$
$$\bar{I}_y = \frac{1}{4} \pi a^3 b$$
$$J_O = \frac{1}{4} \pi a b(a^2 + b^2)$$

Parallel Axis Theorem (Axes that are parallel to the centroidal axis)



Centroidal axis

Desired axis

- Consider moment of inertia I of an area A with respect to the axis *AA'*

$$I = \int y^2 dA$$

- The axis *BB'* passes through the area centroid and is called a *centroidal axis.*

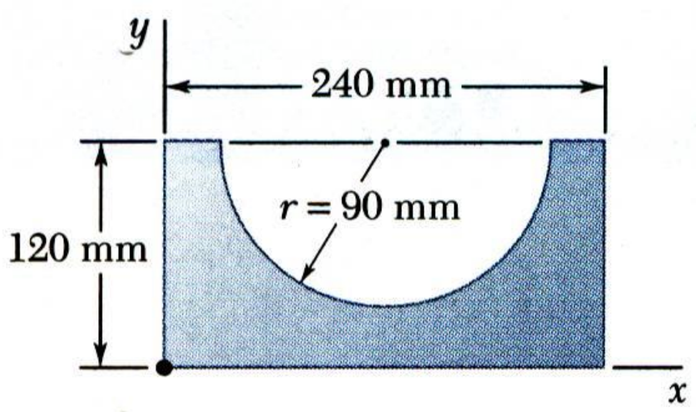
$$I = \int y^2 dA = \int (y' + d)^2 dA$$

$$= \int y'^2 dA + 2d \int y' dA + d^2 \int dA$$

$$\boxed{I = \bar{I} + Ad^2 \qquad \textit{parallel axis theorem}}$$

We usually take this as the moment of inertia w.r.t. **The centroidal axis.**

Sample Problem 9.5

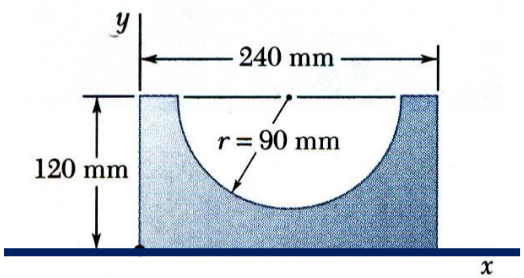


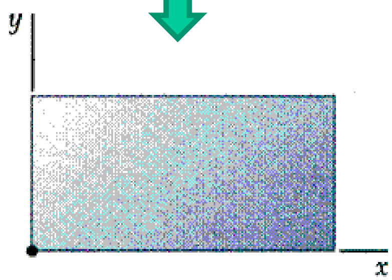
Determine the moment of inertia of the shaded area with respect to the *x* axis.

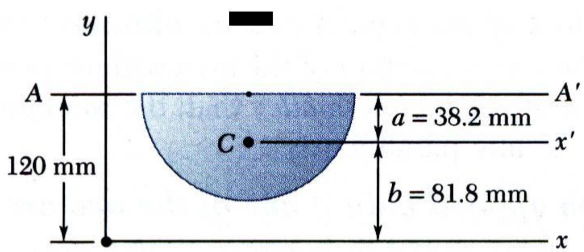
SOLUTION:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the *x* axis.
- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.

Today's Lecture Attendance Password is: Parallel Axis Theorem







Parallel Axis Theorem $I_X = \bar{I}_{X'} + d^2 A$

Centroidal Axis

<u>SOLUTION</u>:

Rectangle:

$$I_x = \tfrac{1}{3} bh^3 = \tfrac{1}{3}(240)(120) = 138.2 \times 10^6 \,\text{mm}^4$$

Half-circle: moment of inertia concerning *AA' can be found in the Table*

$$I_{AA'} = \tfrac{1}{8}\pi r^4 = \tfrac{1}{8}\pi(90)^4 = 25.76 \times 10^6 \,\text{mm}^4$$

moment of inertia with respect to *x'*,

$$\begin{aligned}\bar{I}_{x'} &= I_{AA'} - Ad^2 = 25.76 \times 10^6 - (12.72 \times 10^3)(88.8)^2 \\ &= 7.20 \times 10^6 mm^4\end{aligned}$$

$$a = \frac{4r}{3\pi} = \frac{(4)(90)}{3\pi} = 38.2 \text{ mm}$$

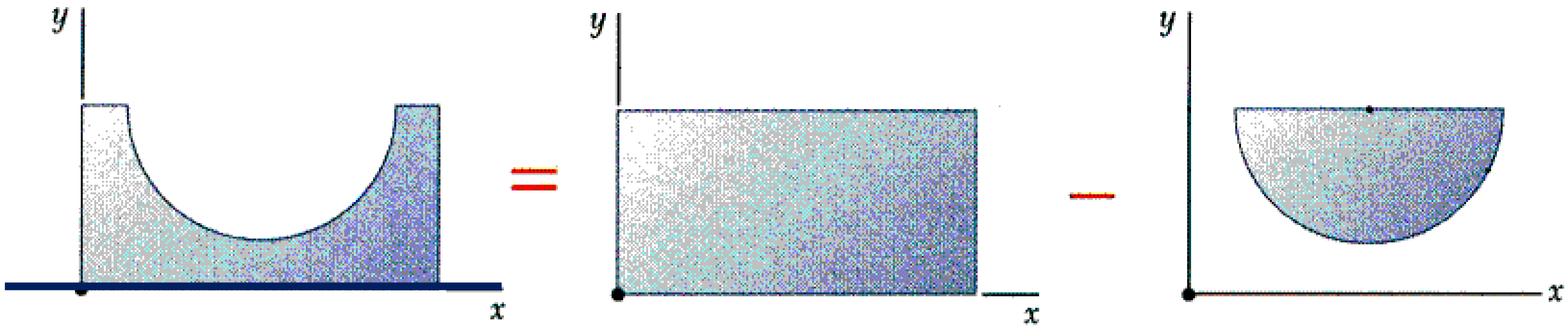
$$\text{b} = 120 - \text{a} = 81.8 \text{ mm}$$

$$A = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi(90)^2 = 12.72 \times 10^3 \text{mm}^2$$

moment of inertia with respect to *x*,

$$I_x = \bar{I}_{x'} + Ab^2 = 7.20 \times 10^6 + (12.72 \times 10^3)(81.8)^2 = 92.3 \times 10^6 \text{mm}^4$$

- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.



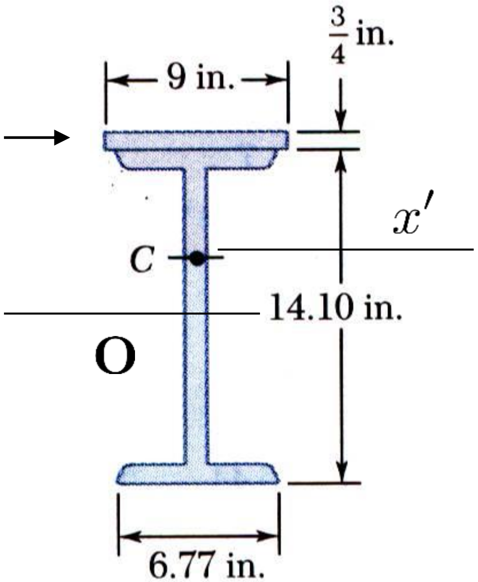
$$I_x \quad = \quad 138.2\times10^{6}\,\mathrm{mm}^{4} \quad - \quad 92.3\times10^{6}\,\mathrm{mm}^{4}$$

$$\boxed{I_x = 45.9\times10^{6}\,\mathrm{mm}^{4}}$$

Sample Problem 9.4

This is a composite shape.

$$\bar{I}_x = \frac{bh^3}{12}$$



The strength of a W14x38 rolled steel beam is increased by attaching a plate to its upper flange.

Determine the moment of inertia and radius of gyration with respect to the centroidal axis X'.

SOLUTION:

- Step1:

  Determine the location of the global centroid of the composite shape.

- Step 2:

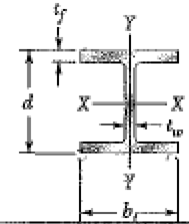
  Apply the parallel axis theorem to determine moments of inertia of the beam section and plate with respect to X'.

- Step 3:

  Calculate the radius of gyration from the moment of inertia of the composite section.

APPENDIX B Properties of Rolled-Steel Shapes
(U.S. Customary Units)

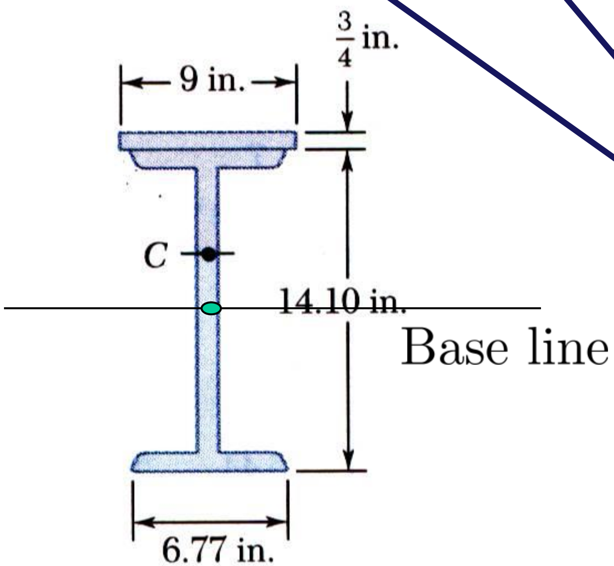
W Shapes
(Wide-Flange Shapes)



			Flange		Web Thickness	Axis X-X			Axis Y-Y		
Designation†	Area A, in^2	Depth d, in.	Width b_f, in.	Thickness t_f, in.	t_w, in.	I_x, in^4	S_x, in^3	r_x, in.	I_y, in^4	S_y, in^3	r_y, in.
W36 × 302	88.8	37.3	16.7	1.68	0.945	21100	1130	15.4	1300	156	3.82
135	39.7	35.6	12.0	0.790	0.600	7800	439	14.0	225	37.7	2.38
W33 × 201	59.2	33.7	15.7	1.15	0.715	11600	686	14.0	749	95.2	3.56
118	34.7	32.9	11.5	0.740	0.550	5900	359	13.0	187	32.6	2.32
W30 × 173	51.0	30.4	15.0	1.07	0.655	8230	541	12.7	598	79.8	3.42
99	29.1	29.7	10.50	0.670	0.520	3990	269	11.7	128	24.5	2.10
W27 × 146	43.1	27.4	14.0	0.975	0.605	5660	414	11.5	443	63.5	3.20
84	24.8	26.70	10.0	0.640	0.460	2850	213	10.7	106	21.2	2.07
W24 × 104	30.6	24.1	12.8	0.750	0.500	3100	258	10.1	259	40.7	2.91
68	20.1	23.7	8.97	0.585	0.415	1830	154	9.55	70.4	15.7	1.87
W21 × 101	29.8	21.4	12.3	0.800	0.500	2420	227	9.02	248	40.3	2.89
62	18.3	21.0	8.24	0.615	0.400	1330	127	8.54	57.5	14.0	1.77
44	13.0	20.7	6.50	0.450	0.350	843	81.6	8.06	20.7	6.37	1.26
W18 × 106	31.1	18.7	11.2	0.940	0.590	1910	204	7.84	220	39.4	2.66
76	22.3	18.2	11.0	0.680	0.425	1330	146	7.73	152	27.6	2.61
50	14.7	18.0	7.50	0.570	0.355	800	88.9	7.38	40.1	10.7	1.65
35	10.3	17.7	6.00	0.425	0.300	510	57.6	7.04	15.3	5.12	1.22
W16 × 77	22.6	16.5	10.3	0.76	0.455	1110	134	7.00	138	26.9	2.47
57	16.8	16.4	7.12	0.715	0.430	758	92.2	6.72	43.1	12.1	1.60
40	11.8	16.0	7.00	0.505	0.305	518	64.7	6.63	28.9	8.25	1.57
31	9.13	15.9	5.53	0.440	0.275	375	47.2	6.41	12.4	4.49	1.17
26	7.68	15.7	5.50	0.345	0.250	301	38.4	6.26	9.59	3.49	1.12
W14 × 370	109	17.9	16.5	2.66	1.66	5440	607	7.07	1990	241	4.27
145	42.7	14.8	15.5	1.09	0.680	1710	232	6.33	677	87.3	3.98
82	24.0	14.3	10.1	0.855	0.510	881	123	6.05	148	29.3	2.48
68	20.0	14.0	10.0	0.720	0.415	722	103	6.01	121	24.2	2.46
53	15.6	13.9	8.06	0.660	0.370	541	77.8	5.89	57.7	14.3	1.92
43	12.6	13.7	8.00	0.530	0.305	428	62.6	5.82	45.2	11.3	1.89
38	11.2	14.1	6.77	0.515	0.310	385	54.6	5.87	26.7	7.88	1.55
30	8.85	13.8	6.73	0.385	0.270	291	42.0	5.73	19.6	5.82	1.49
26	7.69	13.9	5.03	0.420	0.255	245	35.3	5.65	8.91	3.55	1.08

38	11.2	14.1	6.77	0.515	0.310	385	54.6	5.87	26.7	7.88	1.55

$9 \times 0.75 = 6.75$

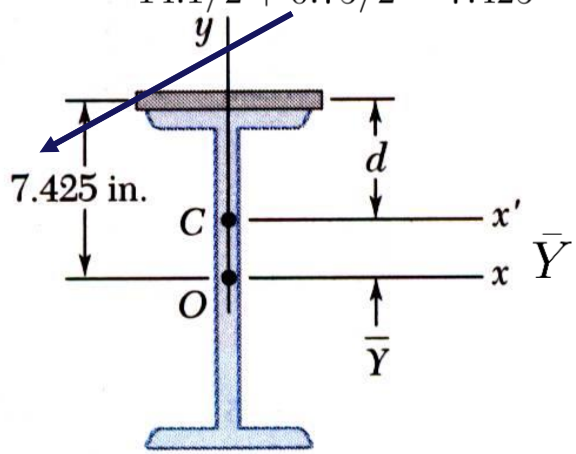


SOLUTION:

- Determine location of the centroid of composite section with respect to a coordinate system with origin at the centroid of the beam section.

Section	A, in^2	$\bar{y}$, in.	$\bar{y}A$, in^3
Plate	6.75	7.425	50.12
Beam Section	11.20	0	0
	$\sum A = 17.95$		$\sum \bar{y}A = 50.12$

$14.1/2 + 0.75/2 = 7.425$

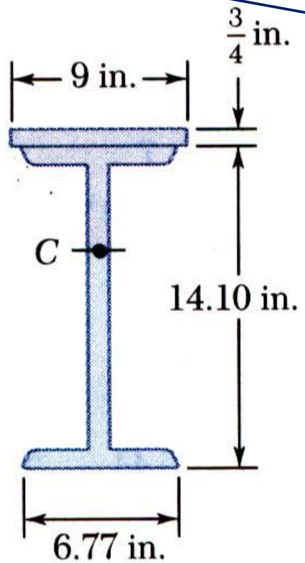


$$\bar{Y}\sum A = \sum \bar{y}A \qquad \bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{50.12\ \text{in}^3}{17.95\ \text{in}^2} = 2.792\ \text{in.}$$

38	11.2	14.1	6.77	0.515	0.310	385	54.6	5.87	26.7	7.88	1.55

$\bar{Y} = 2.792\ in \qquad A_{beam} = 11.20\ in^2 \qquad A_{plate} = 6.75\ in^2$

$$\bar{I}_x = \frac{bh^3}{12}$$



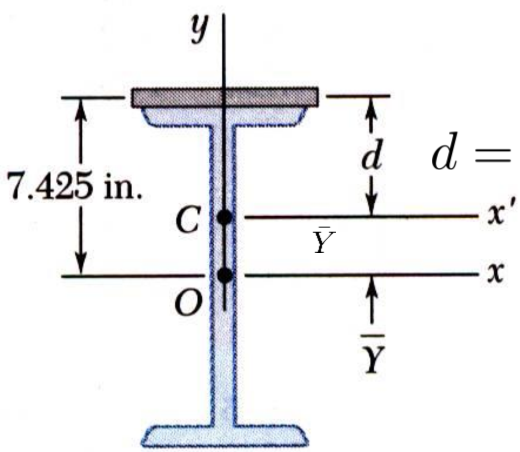
$$I_{x',\text{beam section}} = \bar{I}_x + A\bar{Y}^2 = 385 + (11.20)(2.792)^2$$

$$= 472.3\ \text{in}^4$$

$$I_{x',\text{plate}} = \bar{I}_x + Ad^2 = \tfrac{1}{12}(9)\left(\tfrac{3}{4}\right)^3 + (6.75)(7.425 - 2.792)^2$$

$$= 145.2\ \text{in}^4$$

$$I_{x'} = I_{x',\text{beam section}} + I_{x',\text{plate}} = 472.3 + 145.2$$



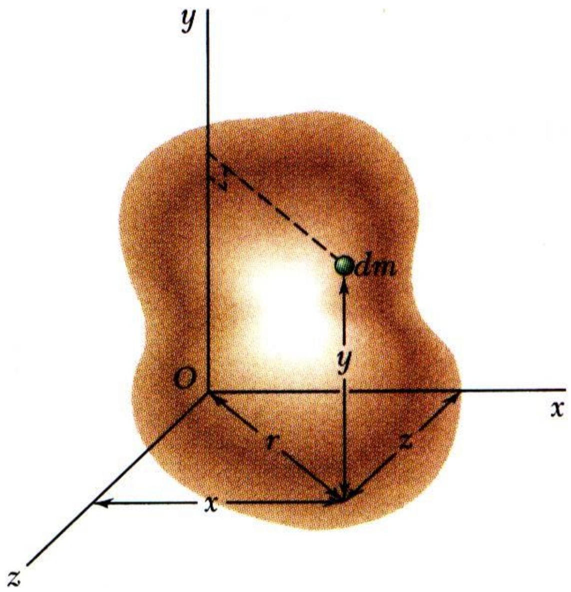
$$d = 7.425 - \bar{Y}$$

$$\boxed{I_{x'} = 618\ \text{in}^4}$$

$$k_{x'} = \sqrt{\frac{I_{x'}}{A}} = \frac{617.5\ \text{in}^4}{17.95\ \text{in}^2}$$

$$\boxed{k_{x'} = 5.87\ \text{in.}}$$

Moment of Inertia of a Mass (of 3D object)



- Moment of inertia with respect to the *y* coordinate axis is

$$I_y = \int r^2 dm = \int \left(z^2 + x^2\right) dm$$

- Similarly, for the moment of inertia with respect to the *x* and *z* axes,

$$I_x = \int \left(y^2 + z^2\right) dm$$

$$I_z = \int \left(x^2 + y^2\right) dm$$

- In SI units,

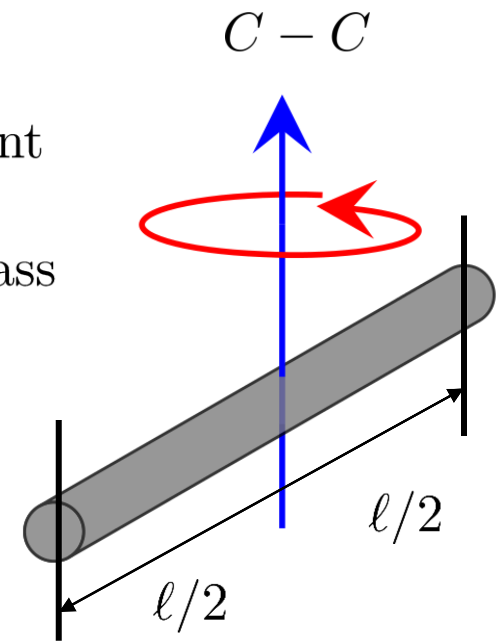
$$I = \int r^2 dm = \left(\mathrm{kg} \cdot \mathrm{m}^2\right)$$

In U.S. customary units,

$$I = \left(slug \cdot ft^2\right) = \left(\frac{lb \cdot s^2}{ft} ft^2\right) = \left(lb \cdot ft \cdot s^2\right)$$

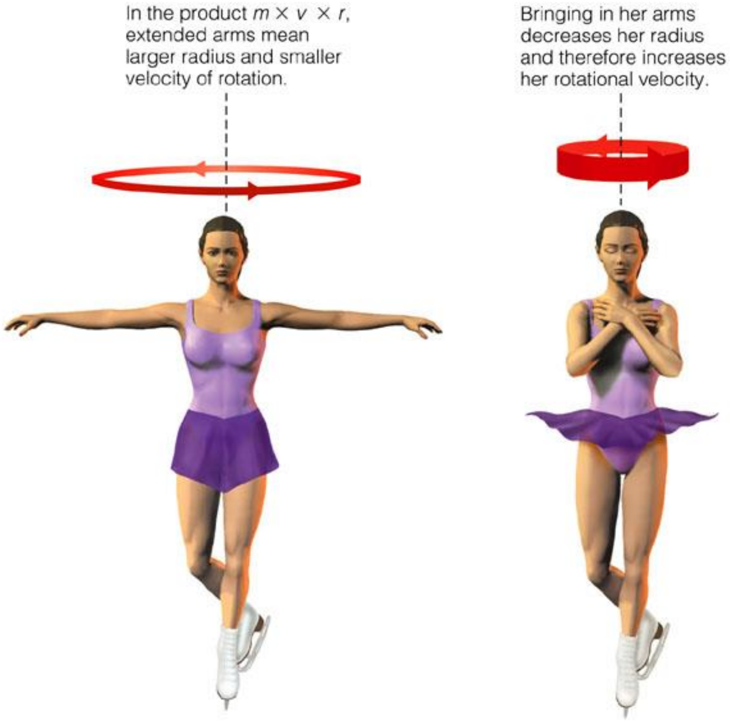
Example

Find the moment of inertia of a thin rod with constant cross section A, mass density ρ, and length ℓ about the perpendicular axis through its center of mass as shown in the figure.



$$I_C = \int_V \rho x^2 dV = \int_{-\ell/2}^{\ell/2} \rho x^2 \underbrace{A dx}_{\mathbf{dV}} = \rho A \frac{x^3}{3} \Big|_{-\ell/2}^{\ell/2} = \frac{\rho A}{3}\Big(\frac{\ell^3}{8} + \frac{\ell^3}{8}\Big) = \frac{m\ell^2}{12}$$

How to spin fast and faster ?



$$K = \frac{1}{2}\int_V \rho \mathbf{v}^2 dV = \frac{1}{2}\int_V \rho \mathbf{r}^2 \omega^2 dV = \frac{\rho J}{2}\omega^2$$

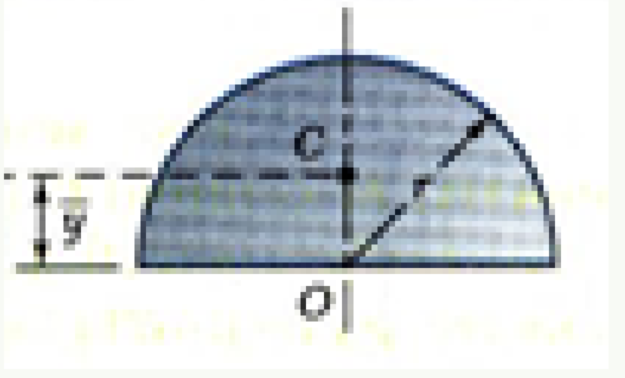
[redacted]

Divers reducing their moments of inertia to increase their rates of rotation.

C30/ME85 Million Dollar Questions

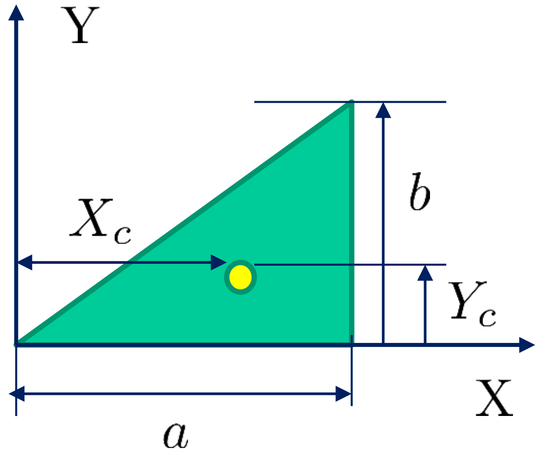
Q1. The center of gravity of a semi-circle area lies at a distance of ___ from its base measured along the vertical radius.

(A) 1.8r/3
(B) 3r/4π
(C) 4r/ 3π
(D) 3r/ 8



Ans: (C)

Q2. What is the centeriod for the following shape ?



A. $(X_c, Y_c) = (a/3, b/3)$;

B. $(X_c, Y_c) = (2a/3, b/3)$;

C. $(X_c, Y_c) = (2a/3, 2b/3)$;

D. $(X_c, Y_c) = (a/3, 2b/3)$.

Ans: (B)

Q3. The semi-circle shown in the fiure has the radius r, and area $A = \frac{r^2\pi}{2}$, and the moment of inertia with respect to the axis AA' is $I_{AA'} = \pi r^4/8$. The centroidal axis is x'-axis. Which of the following is the moment of inertia w.r.t. x-axis for the semi-circle shown in the figure:

$$(a)\ I_x = \frac{\pi r^4}{8} + (a+b)^2 A;$$

Ans: (c)

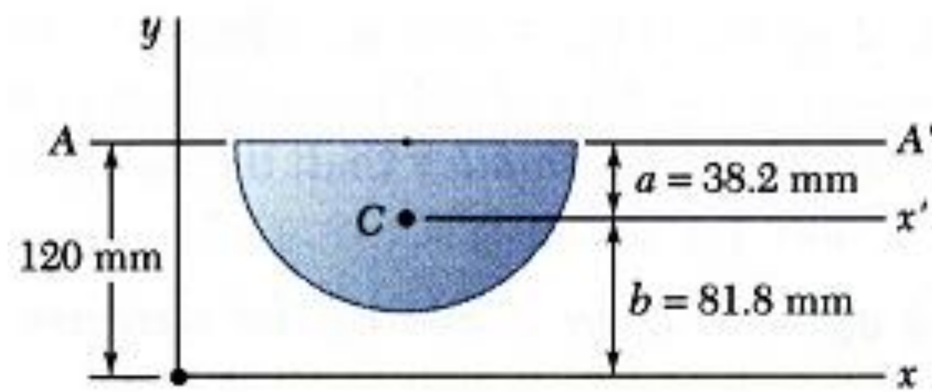
$$(b)\ I_x = \frac{\pi r^4}{8} + b^2 A;$$

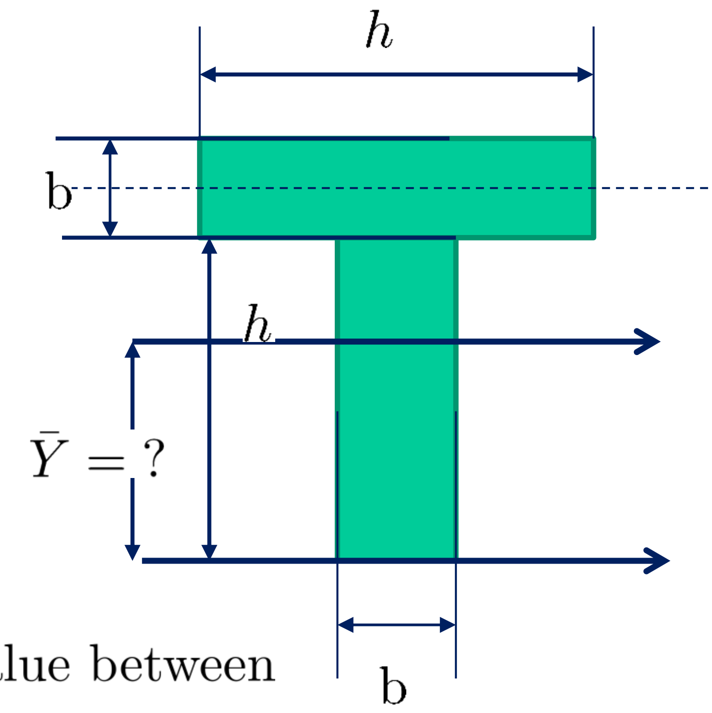
$$(c)\ I_x = \frac{\pi r^4}{8} - a^2 A + b^2 A;$$

$$(d)\ I_x = \frac{\pi r^4}{8} + (a-b)^2 A;$$

$$(e)\ I_x = \frac{\pi r^4}{8} + (a^2+b^2) A\ .$$

(d) *None of the above.*





Ans: (b)

Q4: Which of the following is the distance value between the centroid axis and the baseline?

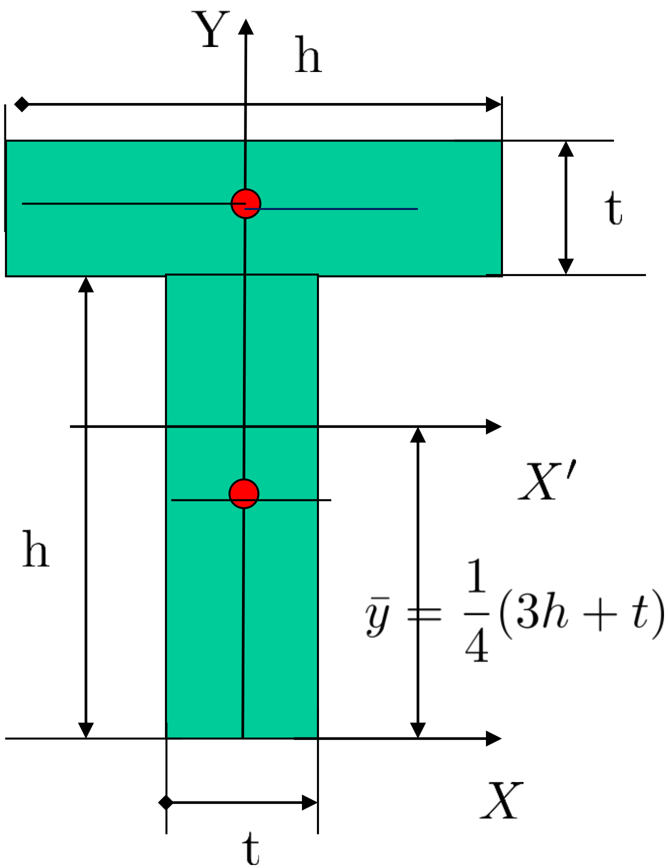
(a) $\bar{Y} = \frac{1}{2}(h + b)$;

(b) $\bar{Y} = \frac{1}{4}(3h + b)$;

(c) $\bar{Y} = \frac{1}{2}(h + 2b)$;

(d) $\bar{Y} = \frac{1}{4}(2h + b)$.

Q5. What is the moment of inertia $I_{X'}$ for the following T-section with respect to the centroidal axis X'?



(**A**) $I_{X'} = \frac{1}{12}(hb^3 + h^3 b)$

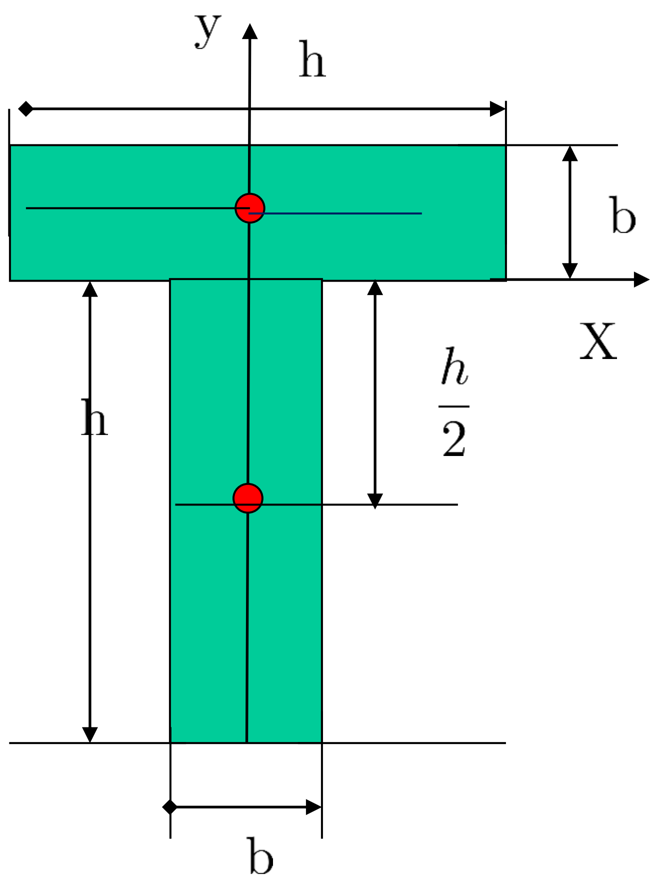
(**B**) $I_{X'} = \frac{1}{12}(h^2 b^2 + h^3 b)$

(**C**) $I_{X'} = \frac{1}{12}(h^3 b + hb^3) + \frac{1}{8}(b + h)^2 bh$

(**D**) $I_{X'} = \frac{1}{12}(h^3 b + hb^3 + (b + h)^2 bh)$

Ans. (C)

Q6. What is the moment of inertia I_X for the following T-section with respect to the axis X?



$$(\mathbf{A})\quad I_X = \frac{1}{12}(h^3b + hb^3)$$

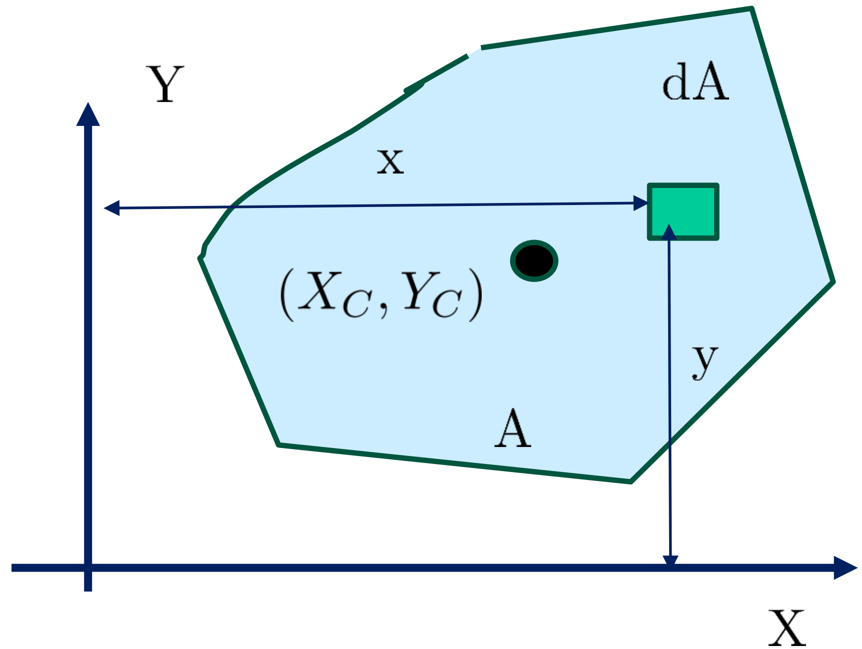
$$(\mathbf{B})\quad I_X = \frac{1}{12}(h^3b + hb^3) + \frac{(b^2 + h^2)}{4}bh$$

$$(\mathbf{C})\quad I_X = \frac{1}{12}(h^3b + hb^3) + \frac{1}{8}(b + h)^2 bh$$

$$(\mathbf{D})\quad I_X = \frac{1}{12}(h^2b^2 + h^4) + \frac{(h^2 + b^2)}{2}hb$$

Ans. (B)

Q7. Consider a 2D shape shown in the figure. What is the first moment of the shape Q_X?



(A) $Q_X = 0;$

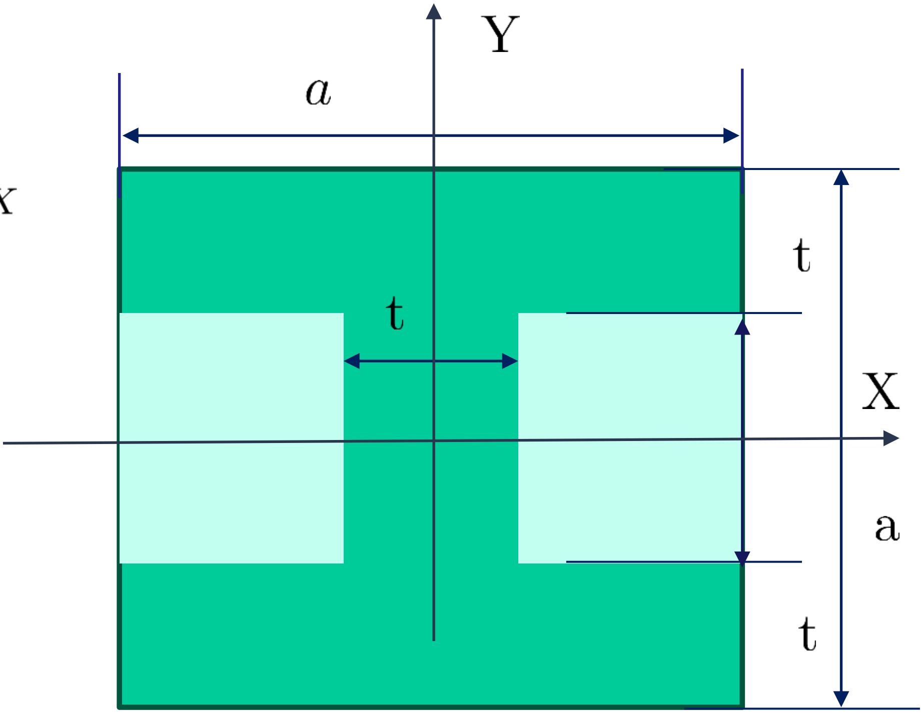
(B) $Q_X = \int_A x dA$

(C) $Q_X = Y_C A$

(D) $Q_X = X_C A$

Ans. (C)

Q8. What is the moment of inertia I_X of the shape I-section?



(**A**) $I_X = \dfrac{1}{12}a^4$

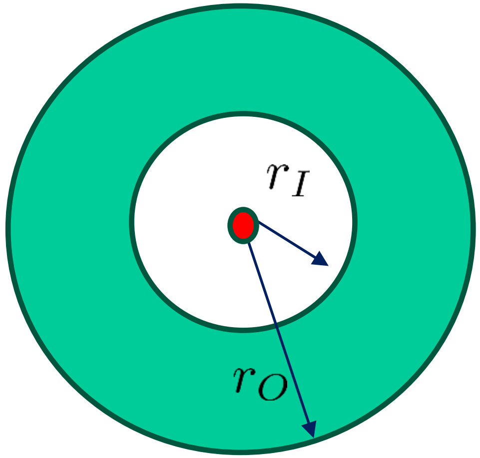
(**B**) $I_X = \dfrac{1}{6}at^3 + \dfrac{1}{12}t(a-2t)^3$

(**C**) $I_X = \dfrac{1}{12}a^4 + \dfrac{1}{12}(a-t)(a-2t)^3$

(**D**) $I_X = \dfrac{1}{12}a^4 - \dfrac{1}{12}(a-t)(a-2t)^3$

Ans: (D)

Q9. What is the polar moment of inertia J_O of this hollow circular section?



(**A**) $J_O = \dfrac{\pi}{2} r_O^4$

(**B**) $J_O = \dfrac{\pi}{4} r_O^4$

(**C**) $J_O = \dfrac{\pi}{2}(r_O^4 - r_I^4)$

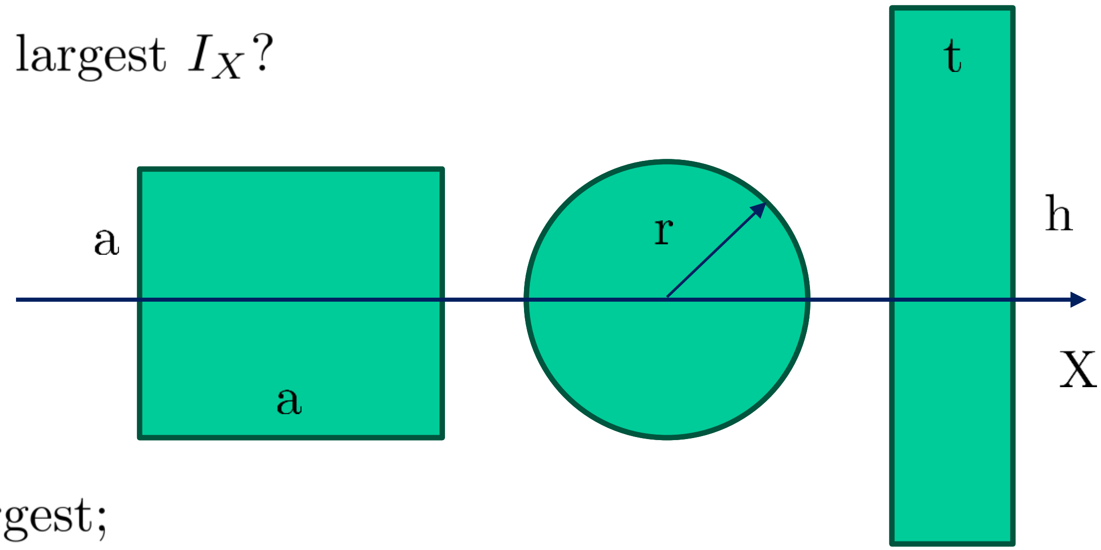
(**D**) $J_O = \dfrac{\pi}{4}(r_O^4 - r_I^4)$

Ans: (C)

Q10. Assume that the following shapes have the same area:

$$a^2 = \pi r^2 = th \text{ , and } t = a/4; \; h = 4a$$

Which section area has the largest I_X?



(A) I_X for square is the largest;

(B) I_X for rectangular section is the largest;

(C) Square and circular sections have the same I_X;

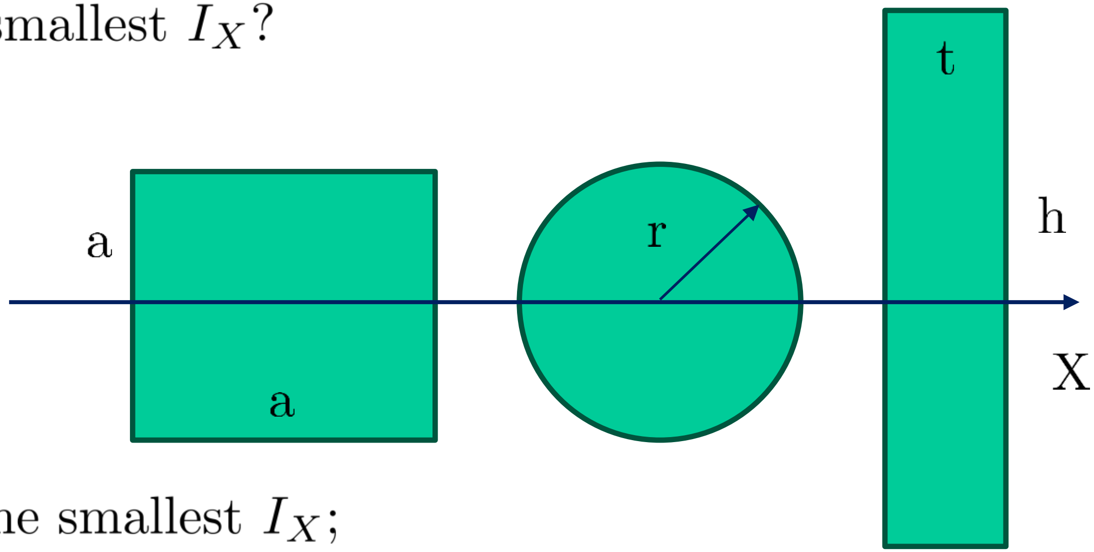
(D) The circular section has the largest I_X.

Ans: (B)

Q11. Assume that the following shapes have the same area:

$$a^2 = \pi r^2 = th \ , \ \text{and} \ \ t = a/4; \ h = 4a$$

Which section area has the smallest I_X?



(A) The square section is the smallest I_X;

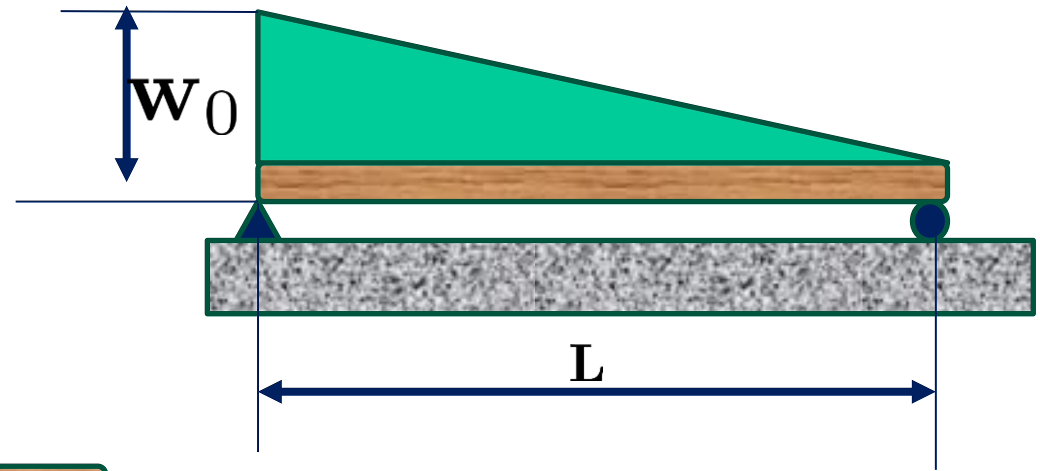
(B) I_X for rectangular section is the smallest;

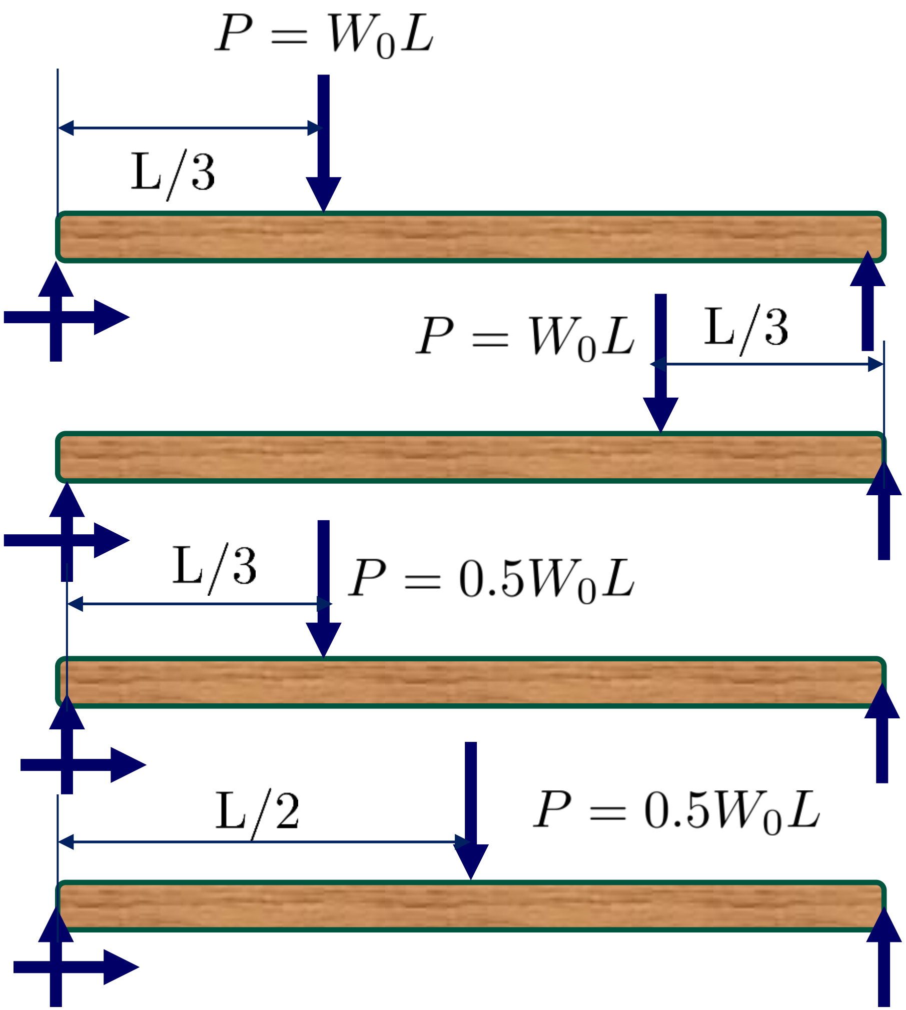
(C) Square and rectangular sections have the same I_X;

Ans (D)

(D) The circular section has the smallest I_X.

Q12 Which of the following is the correct free-body diagram





(A)

(B)

(C)

(D)

Ans (C)