Lecture 13

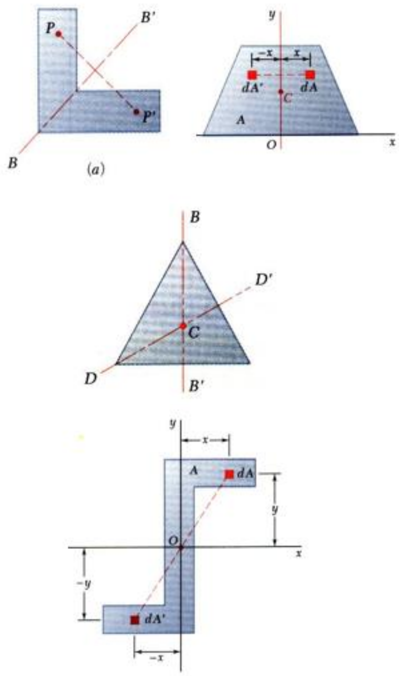
Geometric Characterization of Volume and Shape (I): Centroid and Centers of Gravity

- The earth exerts a gravitational force on each of the particles forming a body. These forces can be replaced by a single equivalent force equal to the weight of the body and applied at the *center of gravity* for the body (**without zero resultant moment).**

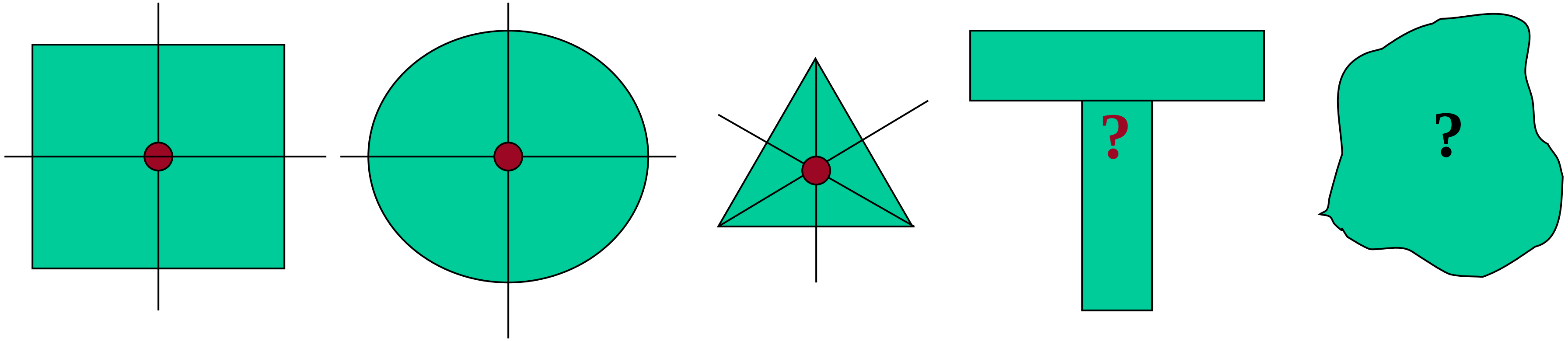
- The *centroid of a volume, area, or line* is analogous to the center of gravity of a body. The concept of the *first moment of the geometric object* is used to locate the centroid.

What is Centroid ?

Centroid is the geometrical center of a body, which is the intersection of all planes (3D) or lines (2D) that divide the object into two parts of equal amount about the plane (or line). The geometric centroid of an object coincides with its center of mass if the object has uniform density.



About these objects ?



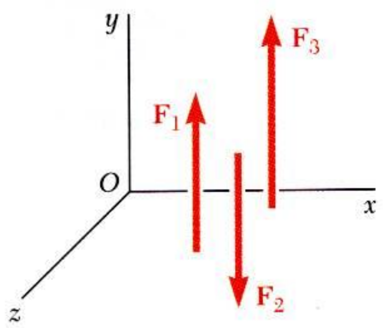
How to find the centroid of a continuum body?

Recall: The case that all the external forces are in parallel can resolve to a single resultant force with zero resultant moment.

One can always find a point O' such that

$$\mathbf{r}_{O'O} = -\mathbf{r}_{OO'}$$

$$\mathbf{R} = \sum_i \mathbf{F}_i, \quad \mathbf{M}_{O'} = \sum_i \mathbf{r}_{O'i} \times \mathbf{F}_i = 0 \ .$$



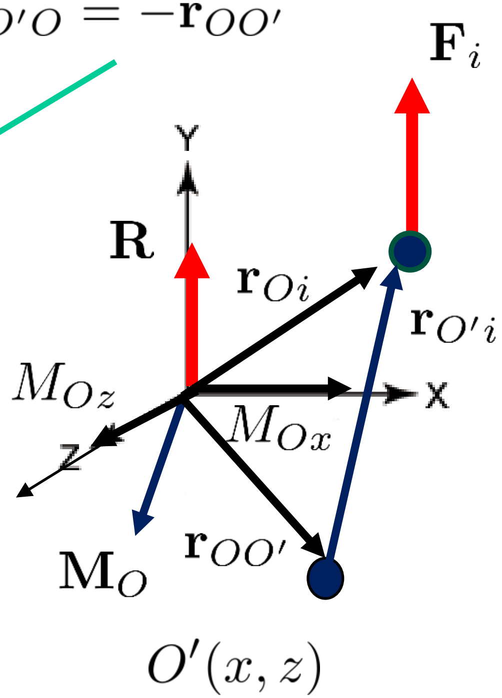
$$\mathbf{M}_{O'} = \mathbf{M}_O + \mathbf{r}_{O'O} \times \mathbf{R} = 0$$

$$\mathbf{r}_{OO'} \times \mathbf{R} = \mathbf{M}_O$$

$$\left(x\mathbf{i} + z\mathbf{k}\right) \times R\mathbf{j} = M_{Ox}\mathbf{i} + M_{Oz}\mathbf{k}$$

$$z = -\frac{M_{Ox}}{R}, \quad x = \frac{M_{Oz}}{R}$$

$$\mathbf{M}_{O'} = \sum (\mathbf{r}_{Oi} - \mathbf{r}_{OO'}) \times \mathbf{F}_i$$



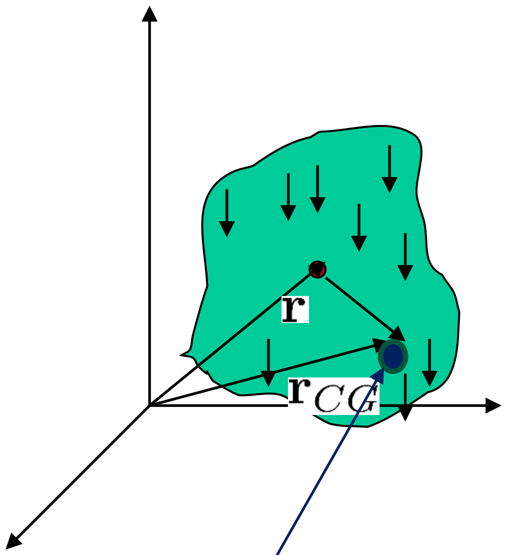
A single point !

The Center of Gravity
is a point that satisfies the zero-moment condition,

$$\mathbf{M}_{CG} = \int_V (\mathbf{r} - \mathbf{r}_{CG}) \times (\rho(\mathbf{r}) g \mathbf{j}) dV = 0$$

Then

Gravity as a distributed load



$$\int_V \mathbf{r}_{CG} \times (\rho(\mathbf{r}) g \mathbf{j}) dV - \int_V \mathbf{r} \times (\rho(\mathbf{r}) g \mathbf{j}) dV = 0 \quad \rightarrow$$

$$\left\{ \int_V \mathbf{r}_{CG} \rho(\mathbf{r}) g dV - \int_V \mathbf{r} \rho(\mathbf{r}) g dV \right\} \times \mathbf{j} = 0 \ , \quad \rightarrow$$

$$\int_V \mathbf{r}_{CG} \rho(\mathbf{r}) g dV = \int_V \mathbf{r} \rho(\mathbf{r}) g dV$$

$$\boxed{\mathbf{r}_{CG} = \frac{\int_V \mathbf{r} \rho(\mathbf{r}) dV}{\int_V \rho(\mathbf{r}) dV}}$$

At this point,
the resultant moment is ZERO!

Definition of Centroid

$\mathbf{r}_C := x_C\mathbf{i} + y_C\mathbf{j} + z_C\mathbf{k}$

If $\rho(\mathbf{r}) = const.$, then

$$\boxed{\mathbf{r}_{CG} = \frac{\int_V \mathbf{r}dV}{\int_V dV} = \frac{\int_V \mathbf{r}dV}{V} =: \mathbf{r}_C}$$

$$\mathbf{r}_C = \frac{\mathbf{Q}}{V}$$

and we call,

$\mathbf{Q} = Q_{yz}\mathbf{i} + Q_{zx}\mathbf{j} + Q_{xy}\mathbf{k}$

$$\mathbf{Q} = \int_V \mathbf{r}dV, \quad \textbf{as the first moment of a volume, or shape, or line.}$$

and it has three components w.r.t. three planes,

$$Q_{yz} = \int_V xdV,\ Q_{xz} = \int_V ydV,\ \text{ and } Q_{xy} = \int_V zdV\ .$$

We can see that

$$Q_{yz} = x_C V = \bar{x}V,\ Q_{xz} = y_C V = \bar{y}V,\ \text{ and }\ Q_{xy} = z_C V = \bar{z}V\ .$$

The physical meaning of these components are the first moments w.r.t. corresponding symmetry plane.

This is the first moment

Finally,

$$\mathbf{r}_{CG} = \frac{\int_V \mathbf{r}\rho dV_r}{\int_V \rho dV_r} =: \mathbf{Q}$$

Sometimes, we replace the integral as the summation,

$$W\bar{\mathbf{r}} = \sum_i \mathbf{r}_i \rho g \Delta V_i \ .$$

where $\mathbf{r}_i$ is the position of the element ΔV_i.

For a composite object, we sometimes write it as,

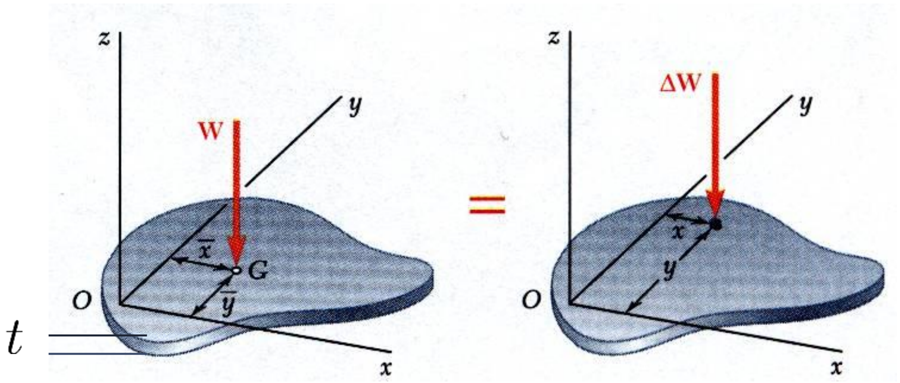
$$W\bar{\mathbf{r}} = \sum_i \mathbf{r}_i W_i \ .$$

where $\mathbf{r}_i$ is the center of mass for the i-th object, and W_i is its weight.

Center of Gravity of a 2D and 1D Body

- Center of gravity of a plate

t is the thickness of the plate



$$\Delta W = \rho g \Delta V$$

$$\bar{x}W = \int x dW$$

$$\bar{x}(\gamma A t) = \int x(\gamma t) dA$$

$$\bar{x}A = \int x dA = Q_y$$

$= $ first moment with respect to y *axis*

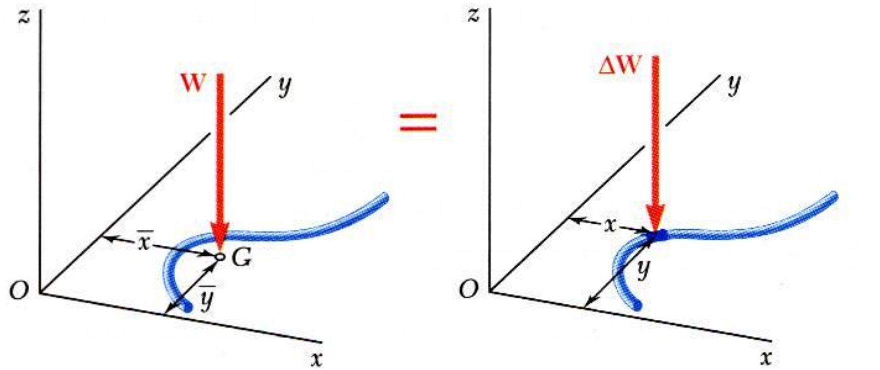
$$\bar{x} = \frac{Q_y}{A}$$

$$\bar{y}A = \int y dA = Q_x$$

$= $ first moment with respect to x axis

$$\bar{y} = \frac{Q_x}{A}$$

- Center of gravity of a wire



α is the section area
L is the length of the wire.

$$\bar{x}W = \int x\, dW$$

$$\bar{x}(\gamma L a) = \int x(\gamma a) dL$$

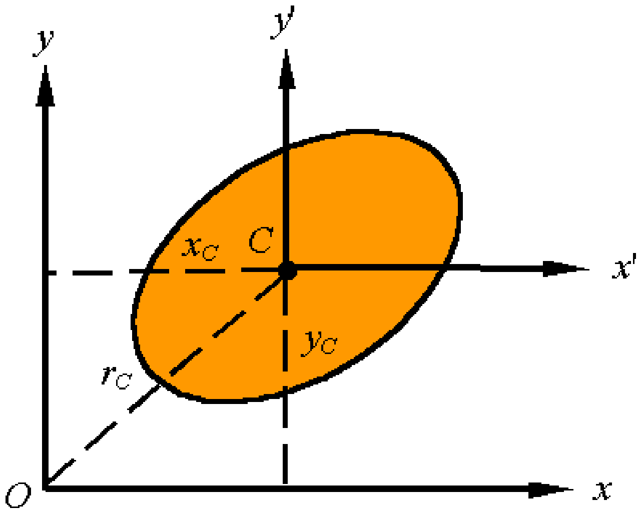
$$\bar{x}L = \int x\, dL$$

$$\bar{y}L = \int y\, dL$$

$$\bar{x} = \frac{Q_y}{L}$$

$$\bar{y} = \frac{Q_x}{L}$$

An Importan Concept: Centroidal Axes



$$X_c = \frac{\int_A x dA}{\int_A dA} \qquad Y_c = \frac{\int_A y dA}{\int_A dA}$$

Position of the centroid of a given body depends on the choice of coordinate system.

If we let $\mathbf{C} = \mathbf{O}$, i.e. the origin of the coordinates is the physical centroid of the body, this special coordinate is called **Centroidal Axis** for the body.

$$\text{Let} \quad \mathbf{O} = \mathbf{C} \ \rightarrow \ \mathbf{r}_C = (0, 0, 0)$$

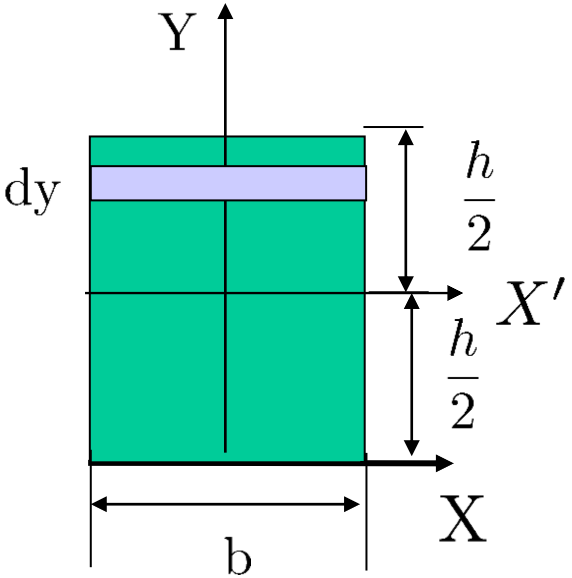
If we calculate the first moments w.r.t. $X'OY'$ coordinate system,

$$\int_A x' dA_{x'} = ?, \qquad \int_A y' dA_{y'} = ?$$

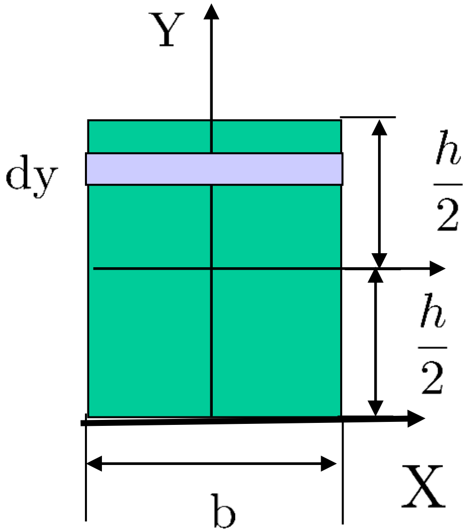
In the centroidal coordinate, all the first moments of the solid are zero.

Why ?

The first moment of a rectangular to X-axis



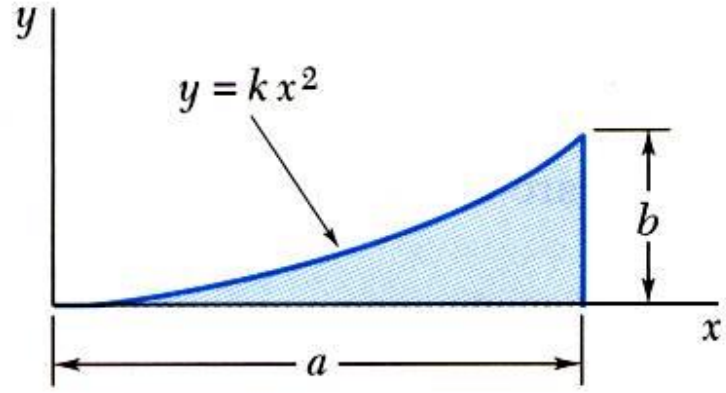
$$
\begin{aligned}
Q_x &= \int_A y dA; \quad dA = bdy \\
&= \int_0^h bydy \\
&= \frac{by^2}{2}\Big|_0^h = \frac{bh^2}{2}
\end{aligned}
$$



$$
\begin{aligned}
\bar{Q}_x &= \int_A y dA; \quad dA = bdy \\
&= \int_{-h/2}^{h/2} bydy \\
&= \frac{by^2}{2}\Big|_{-h/2}^{h/2} = \frac{b}{2}(h^2 - h^2) = 0 \ !
\end{aligned}
$$

In centroidal coordinate, all the first moments of a shape are zero.

Sample Problem 9.4

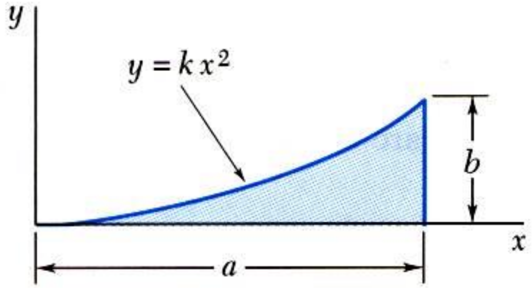


Determine by direct integration the location of the centroid of a parabolic spandrel.

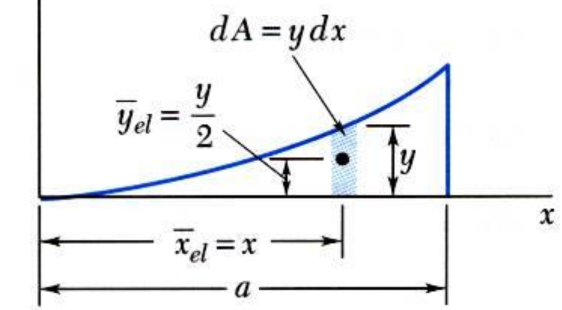
SOLUTION:

- Determine the constant k.
- Evaluate the total area.
- Using either vertical or horizontal strips, perform a single integration to find the first moments.
- Evaluate the centroid coordinates.

$$X_c = \frac{\int_A x dA}{\int_A dA} \qquad Y_c = \frac{\int_A y dA}{\int_A dA}$$



dA =y(x) dx



SOLUTION:

- Determine the constant k.

$$y = kx^2,\ b = ka^2 \ \rightarrow\ k = \frac{b}{a^2} \rightarrow y = \frac{b}{a^2}x$$

- Evaluate the total area.

$$A = \int dA = \int_0^a y dx = \frac{b}{a^2}\frac{x^3}{3}\Big|_0^a = \frac{ab}{3}$$

$$Q_x = \int \bar{y}_{el} dA = \int_0^a \frac{y}{2} y dx = \int_0^a \frac{1}{2}\left(\frac{b}{a^2}x^2\right)^2 dx$$

$$= \left[\frac{b^2}{a^4}\frac{x^5}{10}\right]_0^a = \frac{ab^2}{10}$$

$$Q_y = \int \bar{x}_{el} dA = \int_0^a x y dx = \int_0^a x\left(\frac{b}{a^2}x^2\right) dx$$

$$= \left[\frac{b}{a^2}\frac{x^4}{4}\right]_0^a = \frac{a^2 b}{4}$$

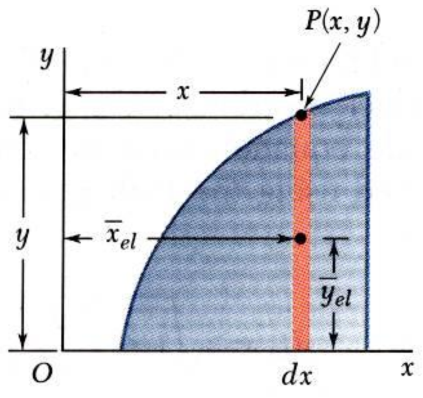
$$X_c = \frac{Q_y}{A} = \frac{3a}{4}$$

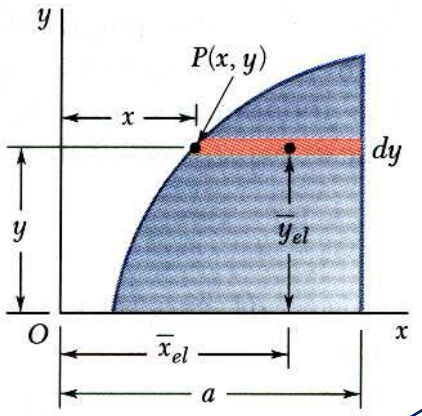
$$Y_c = \frac{Q_x}{A} = \frac{3b}{10}$$

Determination of Centroids by Integration

$$\bar{x}A = \int x dA = \iint x\, dxdy = \int \bar{x}_{el}\, dA$$
$$\bar{y}A = \int y dA = \iint y\, dxdy = \int \bar{y}_{el}\, dA$$

- Double integration to find the first moment may be avoided by defining *dA* as a thin rectangle or strip.





$$x_{el} = x + (a - x)/2 = (a + x)/2$$

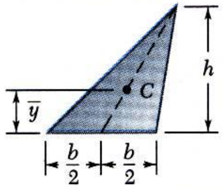
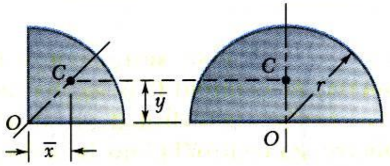
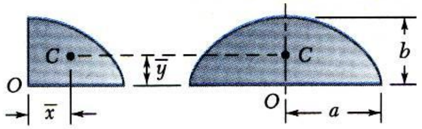
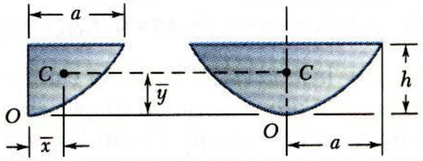
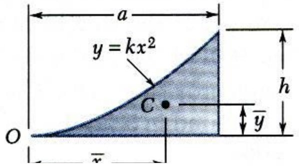
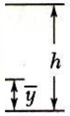
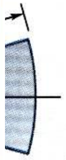
$$\bar{x}A = \int \bar{x}_{el}\, dA$$
$$= \int x\big(y(x)dx\big)$$
$$\bar{y}A = \int \bar{y}_{el}\, dA$$
$$= \int \frac{y(x)}{2}\big(y(x)dx\big)$$

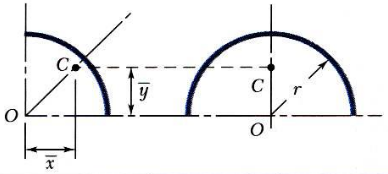
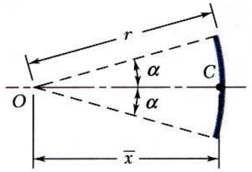
$$\bar{x}A = \int \bar{x}_{el}\, dA$$
$$= \int \frac{a+x}{2}\big[(a-x)dy\big]$$
$$\bar{y}A = \int \bar{y}_{el}\, dA$$
$$= \int y(x)\big[(a-x)dy\big]$$

$$x = x(y)$$

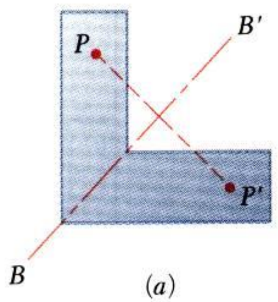
Centroids of Common Shapes of Areas

Remember them, because some of them will be in million dollar questions.

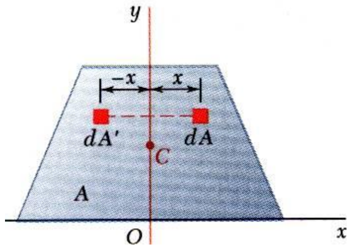
Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
		$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2

Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r\sin\alpha}{\alpha}$	0	$2\alpha r$

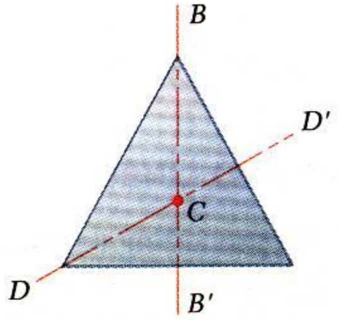
Today’s Lecture Attendance Password: Centroidal Axis



(a)

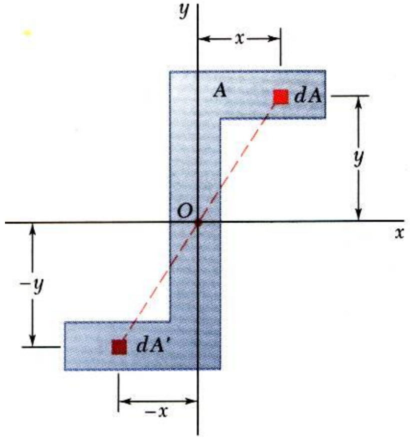


- The first moment of an area with respect to a line of symmetry is zero.



- If an area possesses a line of symmetry, its centroid lies on that axis

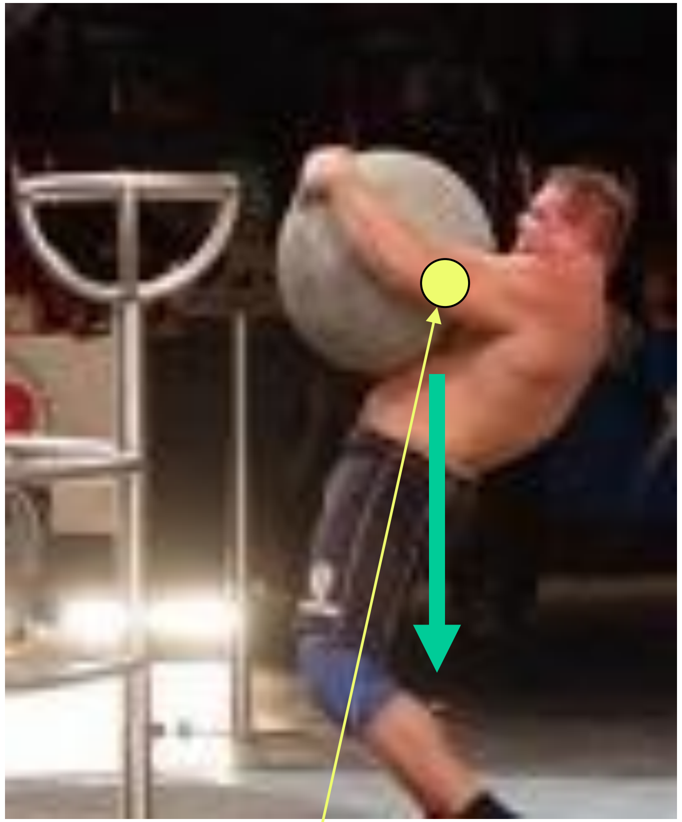
- If an area possesses two lines of symmetry, its centroid lies at their intersection.



- An area is symmetric with respect to a center *O* if for every element *dA* at (*x*,*y*) there exists an area *dA’* of equal area at (–*x*, –*y*).

- The centroid of the area coincides with the center of symmetry.

Is this magic or physics ?



Can you move the Center of Gravity?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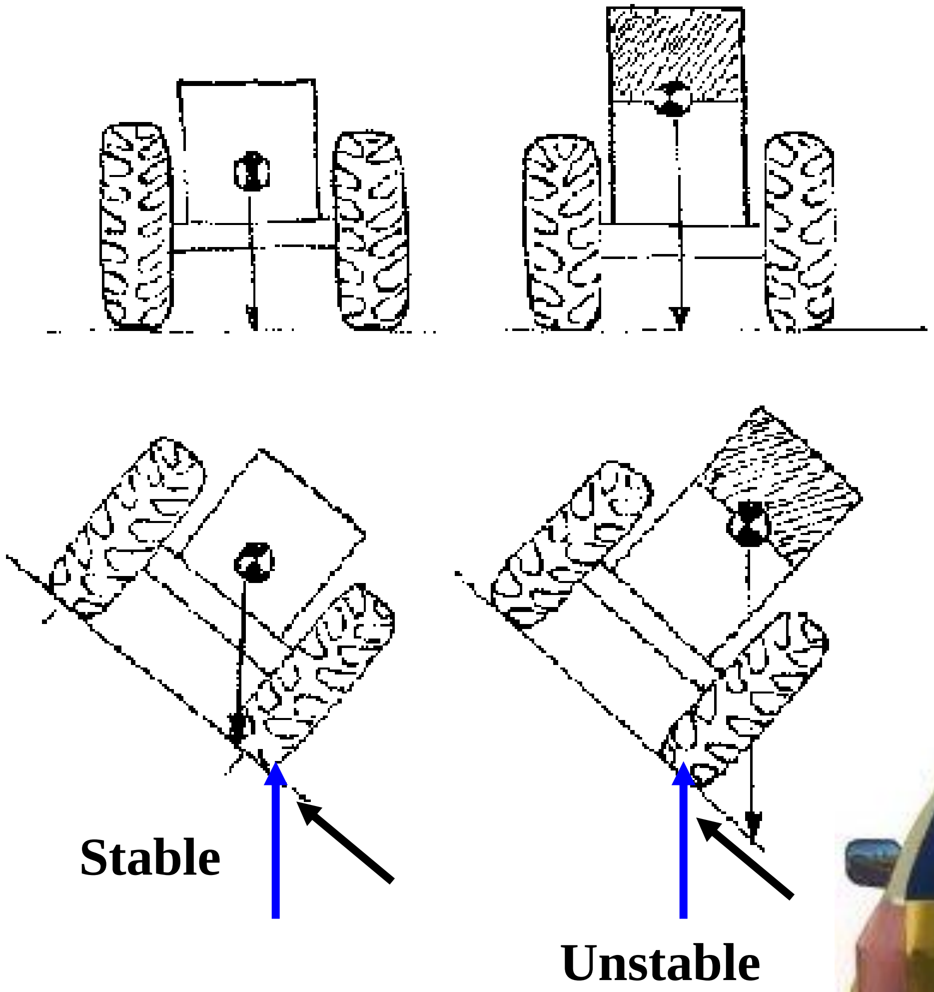
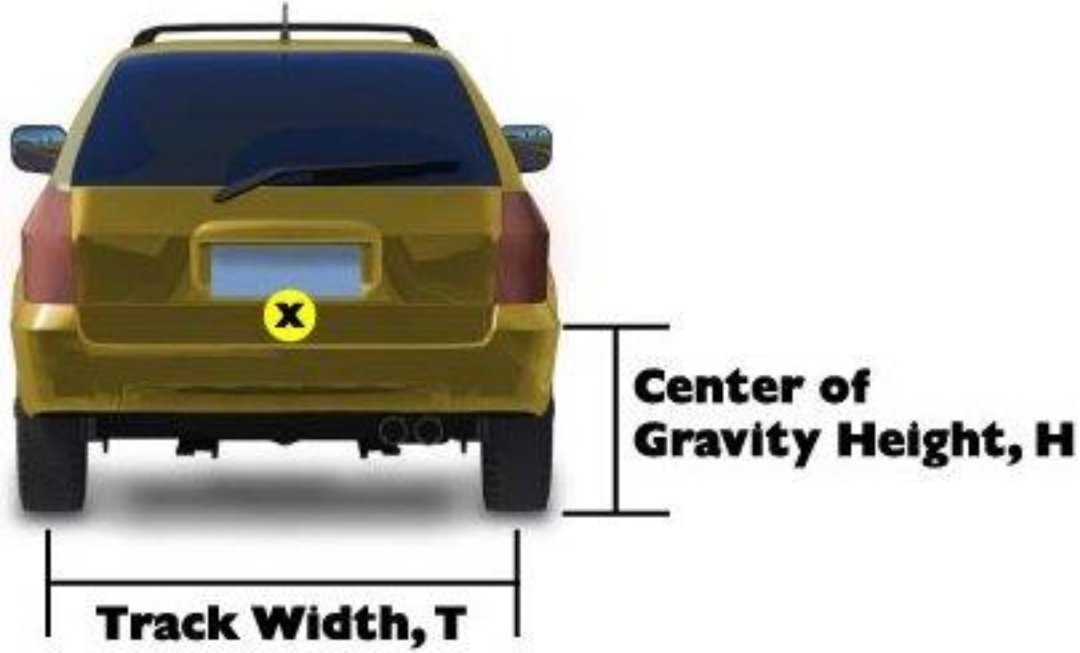
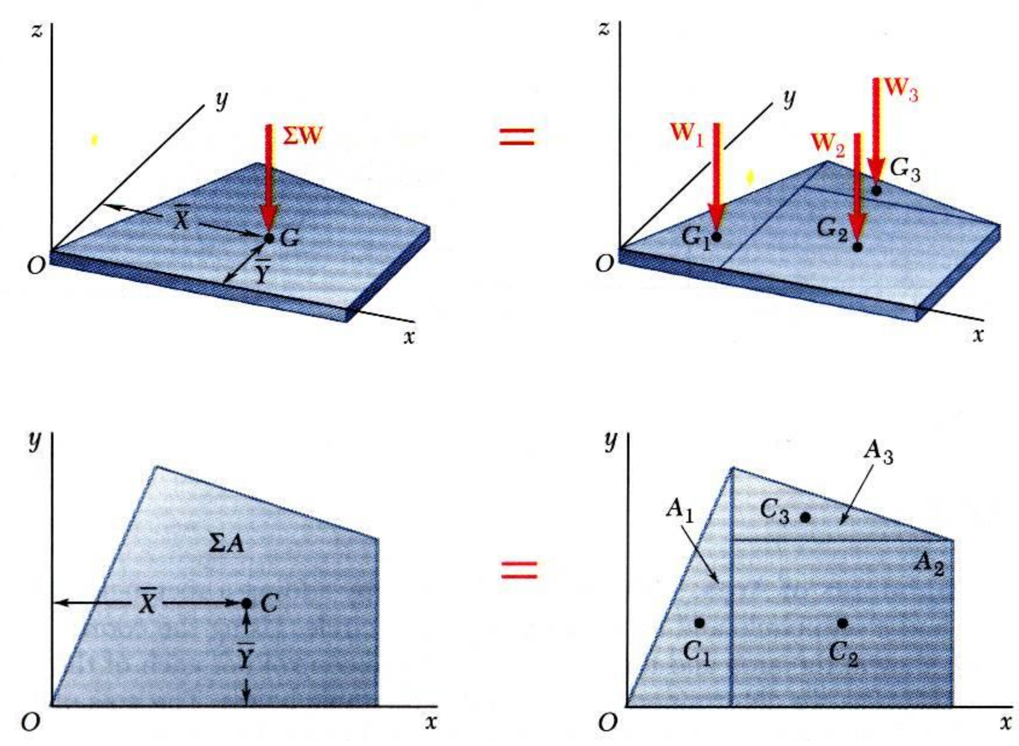
If we treat the car as A rigid body,





Centroid of Composite Areas (Volume)



For a composite solid,

$$\mathbf{M}_{CG} = \sum_{i=1}^{N} (\mathbf{r}_i - \bar{\mathbf{r}}) \times (-W_i \mathbf{k}) = 0$$

- Composite plates

$$\bar{X} \sum_i W_i = \sum_i \bar{x}_i W_i$$

$$\bar{Y} \sum_i W_i = \sum_i \bar{y}_i W_i$$

- Composite area

$$\bar{X} \sum_i A_i = \sum_i \bar{x}_i A_i$$

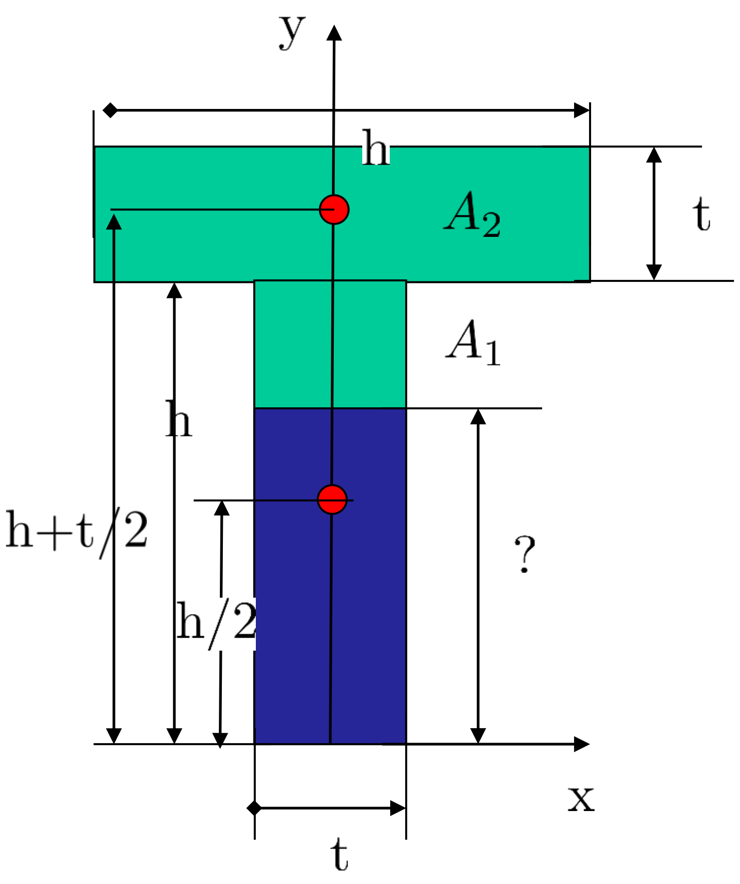
$$\bar{Y} \sum_i A_i = \sum_i \bar{y}_i A_i$$

Global Centroid

$$\bar{X} = \frac{\sum_i \bar{x}_i W_i}{\sum_i W_i}, \quad \text{and} \quad \bar{Y} = \frac{\sum_i \bar{y}_i W_i}{\sum_i W_i};$$

$$\bar{X} = \frac{\sum_i \bar{x}_i A_i}{\sum_i A_i}, \quad \text{and} \quad \bar{Y} = \frac{\sum_i \bar{y}_i A_i}{\sum_i A_i};$$

Find the centroid of the following T-section.



This slide is **uncut.**
Viewer Discretion is advised.
This problem is rated **R** (reviewable).

Global Centriod

$$\bar{X} = \frac{\sum_i \bar{x}_i A_i}{\sum_i A_i}, \quad \text{and} \quad \bar{Y} = \frac{\sum_i \bar{y}_i A_i}{\sum_i A_i};$$

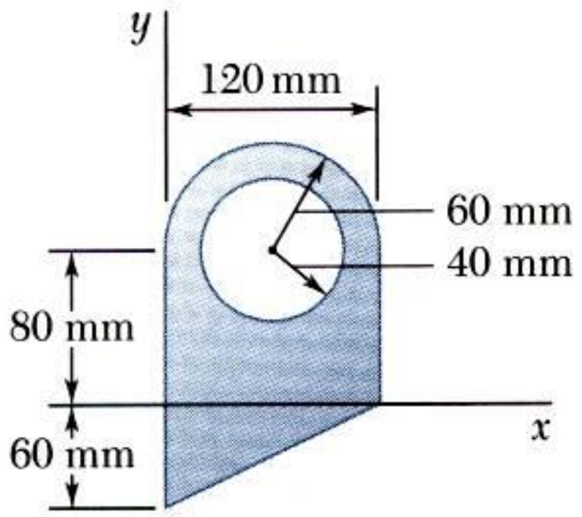
What is $\bar{X}$?

$$A_1 = th,\, A_2 = th$$

$$\bar{y}_1 = \frac{h}{2},\ \bar{y}_2 = h + \frac{t}{2}$$

$$\bar{Y} = \frac{1}{2ht}\left(\frac{h}{2}(ht) + (h + \frac{t}{2})(ht)\right) = \frac{1}{4}(3h + 2t)$$

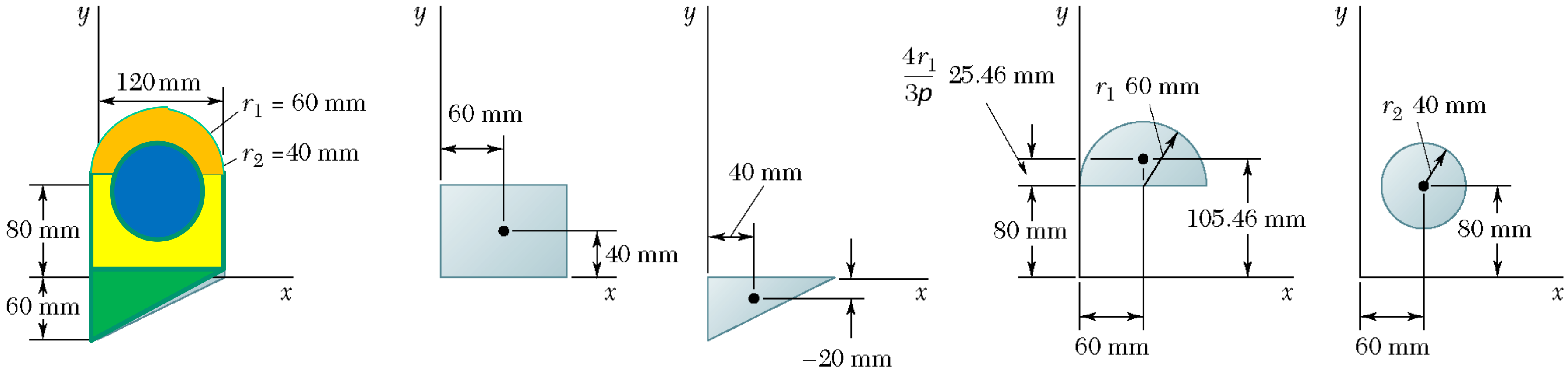
Sample Problem



For the plane area shown, determine the first moments with respect to the *x* and *y* axes and the location of the centroid.

<u>SOLUTION</u>:

- Divide the area into a triangle, rectangle, and semicircle with a circular cutout.
- Calculate the first moments of each area with respect to the axes.
- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.
- Compute the coordinates of the area centroid by dividing the first moments by the total area.



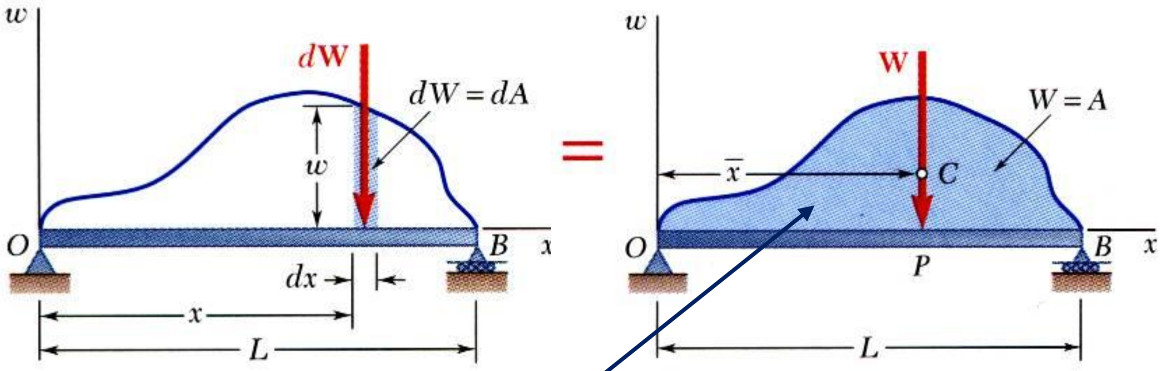
Component	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	-20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$

- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.

$$\boxed{\begin{aligned} Q_x &= +506.2 \times 10^3 \text{ mm}^3 \\ Q_y &= +757.7 \times 10^3 \text{ mm}^3 \end{aligned}}$$

$$\bar{x} = \frac{Q_y}{A} = \frac{757.7}{13.828} = 54.79 \qquad \bar{y} = \frac{Q_x}{A} = \frac{506.2}{13.828} = 36.6$$

Application: How to simplify distributed Loads on beams ?



$$\mathbf{M}_{CG} = \sum_{i=1}^{N} (\mathbf{r}_i - \bar{\mathbf{r}}) W_i = 0$$

$$\bar{X} \sum_i A_i = \sum_i \bar{x}_i A_i$$

$$dW = w dx$$

$$W = \int_0^L w dx \Leftrightarrow \int dA = A$$

- A vertical distributed load is represented by plotting the load per unit length, *w* (N/m) . The total load is equal to the area under the load curve.

- A distributed load can be replace by a concentrated load with a magnitude equal to the area under the load curve and a line of action passing through the area centroid.

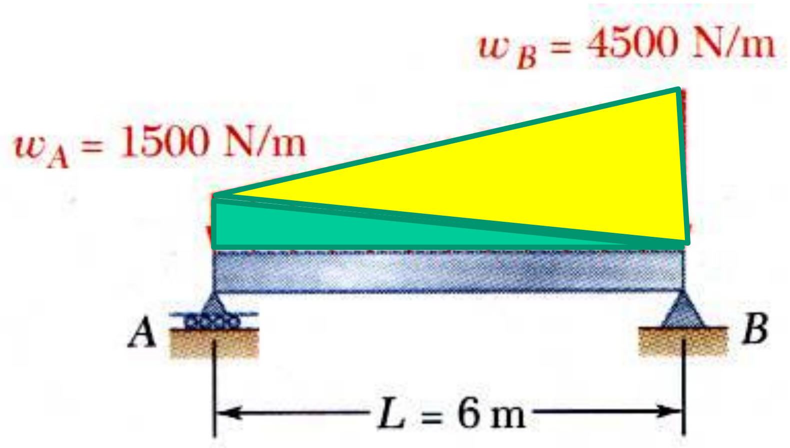
first moment

$$\underbrace{(OC)}_{\bar{X}} W = \int x dW$$

$$(OC)A = \int_0^L x dA = \bar{x}A$$

$$\int (x - \bar{x}) w(x) dx = 0 \quad \rightarrow \quad \bar{x} = \frac{\int_0^L x w(x) dx}{\int_0^L w(x) dx}$$

Sample Problem 9.9

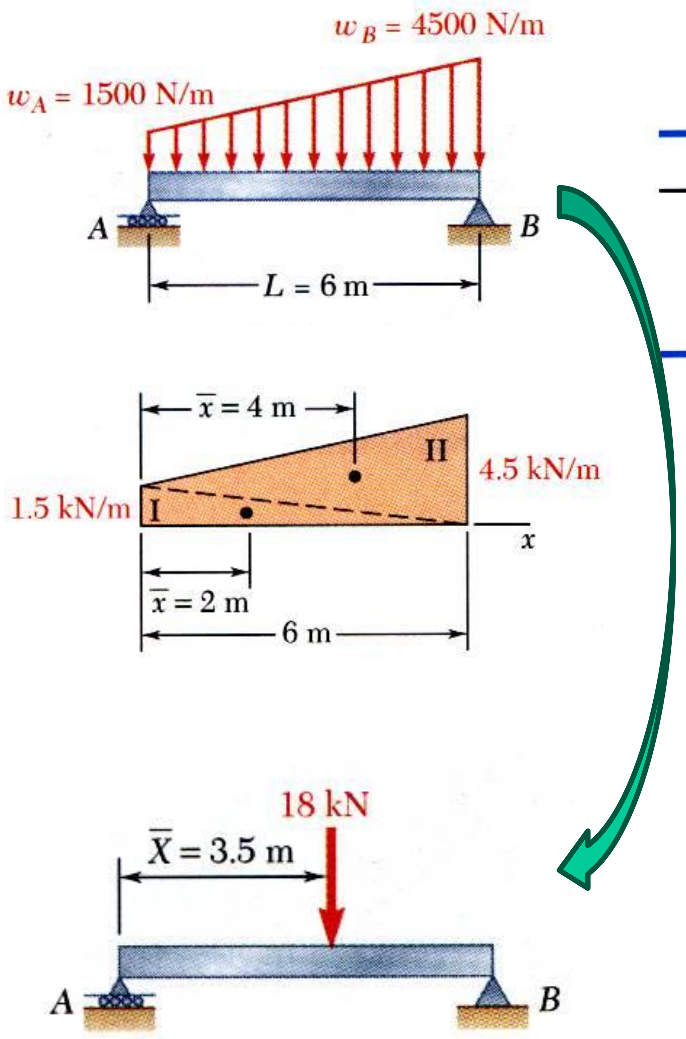


Determine the equivalent concentrated load and the reactions at the supports.

SOLUTION:

- The magnitude of the concentrated load is equal to the total load or the area under the curve.
- The line of action of the concentrated load passes through the centroid of the area under the curve.
- Determine the support reactions by summing moments about the beam ends.

$$\bar{x} = \frac{\sum_i \bar{x}_i A_i}{\sum_i A_i}$$



SOLUTION:

$$A_I = \frac{1}{2} \times 1.5 \times 6 = 4.5$$

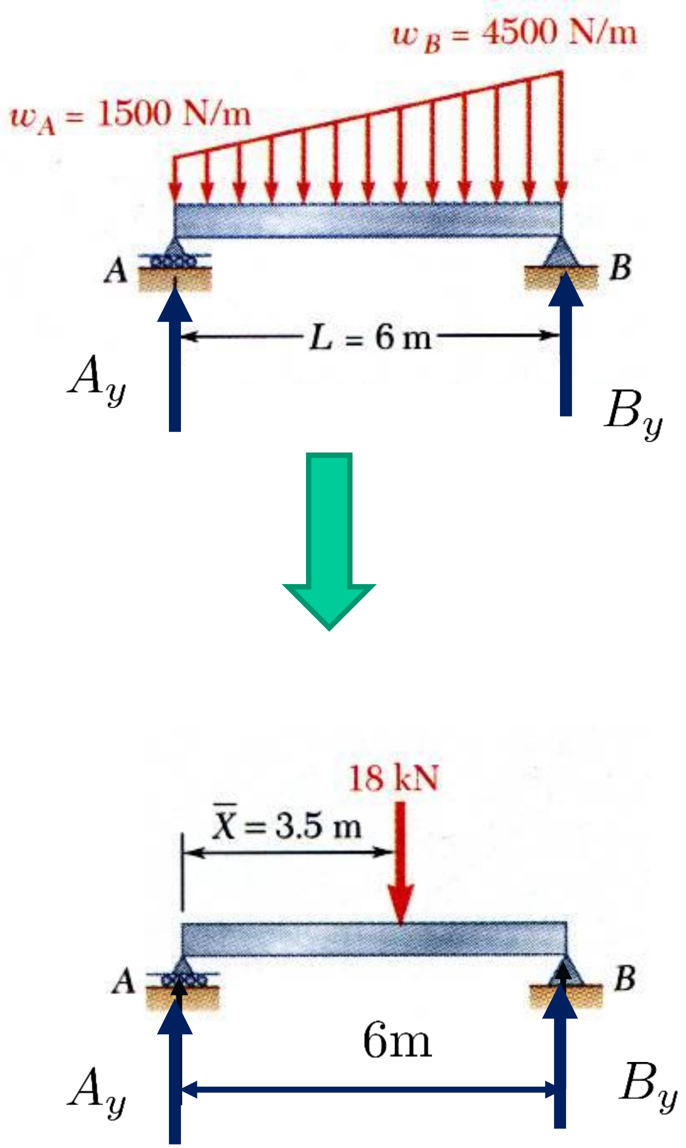
$$A_{II} = \frac{1}{2} \times 4.5 \times 6 = 13.5$$

Component	A, kN	$\bar{x}$, m	$\bar{x}A$, kN·m
Triangle I	4.5	2	9
Triangle II	13.5	4	54
	$\Sigma A = 18.0$		$\Sigma \bar{x} A = 63$

$$\boxed{F = 18.0\ \text{kN}}$$

$$\bar{X} = \frac{63\ \text{kN}\cdot\text{m}}{18\ \text{kN}}$$

$$\boxed{\bar{X} = 3.5\ \text{m}}$$



- Determine the support reactions by summing moments about the beam ends.

$$\sum M_A = 0: \quad B_y(6\,\mathrm{m}) - (18\,\mathrm{kN})(3.5\,\mathrm{m}) = 0$$

$$\boxed{B_y = 10.5\,\mathrm{kN}}$$

$$\sum M_B = 0: \quad -A_y(6\,\mathrm{m}) + (18\,\mathrm{kN})(6\,\mathrm{m} - 3.5\,\mathrm{m}) = 0$$

$$\boxed{A_y = 7.5\,\mathrm{kN}}$$