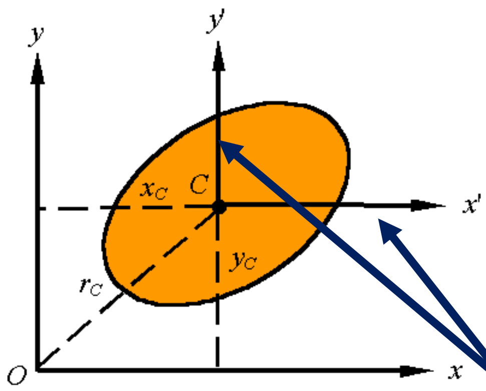
Summary



$$X_c = \frac{\int_A x dA}{\int_A dA}$$

$$Y_c = \frac{\int_A y dA}{\int_A dA}$$

(1) First moments Q_x and Q_y:

(2) Position of the centroid

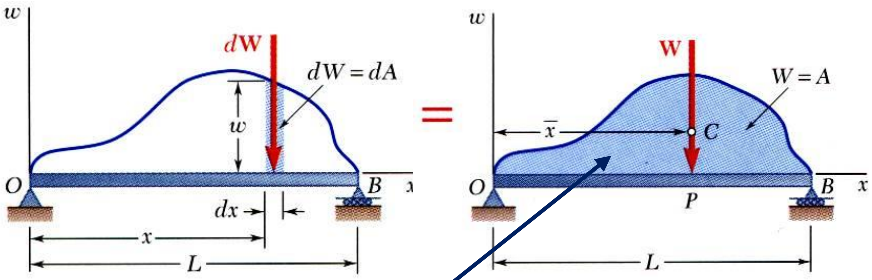
(3) The centroidal axes

(4) The centroid for composite shapes

Global Centriod

$$\bar{X} = \frac{\sum_i \bar{x}_i A_i}{\sum_i A_i}, \quad \text{and} \quad \bar{Y} = \frac{\sum_i \bar{y}_i A_i}{\sum_i A_i};$$

Application: How to simplify distributed Loads on beams ?



$$\mathbf{M}_{CG} = \sum_{i=1}^{N} (\mathbf{r}_i - \bar{\mathbf{r}}) W_i = 0$$

$$\bar{X} \sum_i A_i = \sum_i \bar{x}_i A_i$$

$$dW = w dx$$

$$W = \int_0^L w dx \Leftrightarrow \int dA = A$$

- A vertical distributed load is represented by plotting the load per unit length, *w* (N/m) . The total load is equal to the area under the load curve.

first moment

$$\bar{X}$$

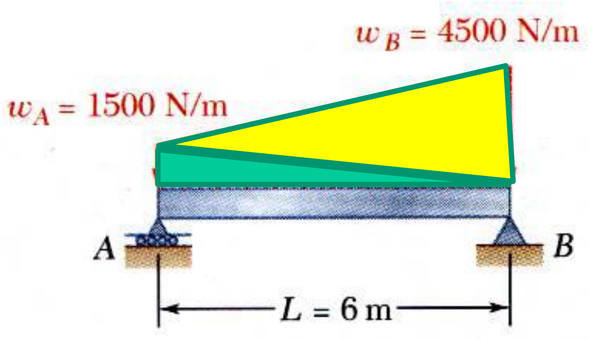
$$(OC)W = \int x dW$$

$$(OC)A = \int_0^L x dA = \bar{x} A$$

- A distributed load can be replace by a concentrated load with a magnitude equal to the area under the load curve and a line of action passing through the area centroid.

$$\int (x - \bar{x}) w(x) dx = 0 \quad \rightarrow \quad \bar{x} = \frac{\int_0^L x w(x) dx}{\int_0^L w(x) dx}$$

Sample Problem 9.9



Determine the equivalent concentrated load and the reactions at the supports.

SOLUTION:

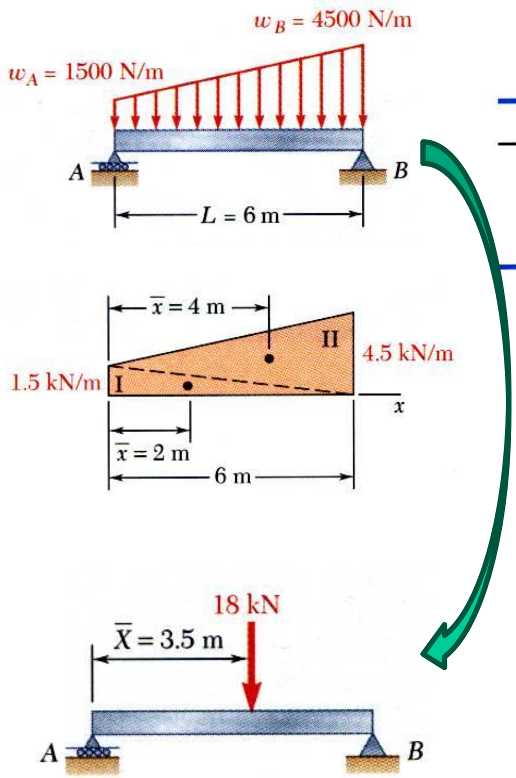
- The magnitude of the concentrated load is equal to the total load or the area under the curve.
- The line of action of the concentrated load passes through the centroid of the area under the curve.
- Determine the support reactions by summing moments about the beam ends.

$$\bar{x} = \frac{\sum_i \bar{x}_i A_i}{\sum_i A_i}$$

SOLUTION:

$$A_I = \frac{1}{2} \times 1.5 \times 6 = 4.5$$

$$A_{II} = \frac{1}{2} \times 4.5 \times 6 = 13.5$$

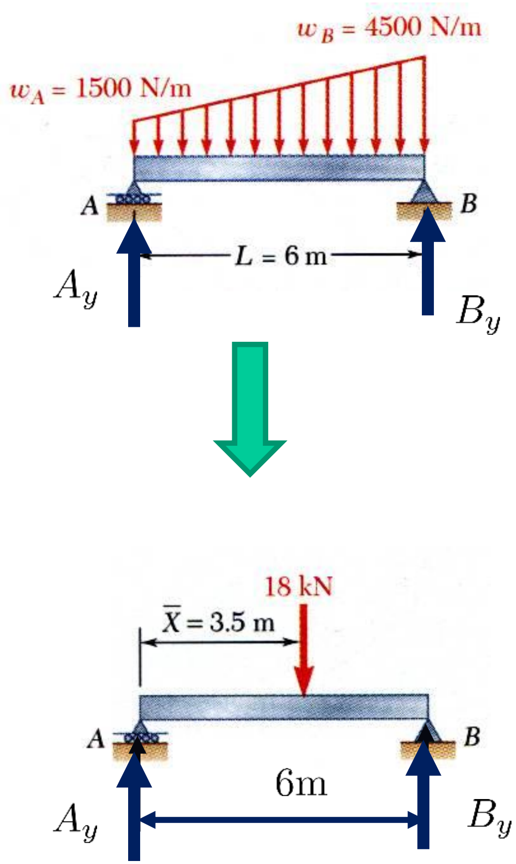


Component	A, kN	$\bar{x}$, m	$\bar{x}A$, kN·m
Triangle I	4.5	2	9
Triangle II	13.5	4	54
	$\Sigma A = 18.0$		$\Sigma \bar{x} A = 63$

$$\boxed{F = 18.0\ \text{kN}}$$

$$\bar{X} = \frac{63\ \text{kN} \cdot \text{m}}{18\ \text{kN}}$$

$$\boxed{\bar{X} = 3.5\ \text{m}}$$



- Determine the support reactions by summing moments about the beam ends.

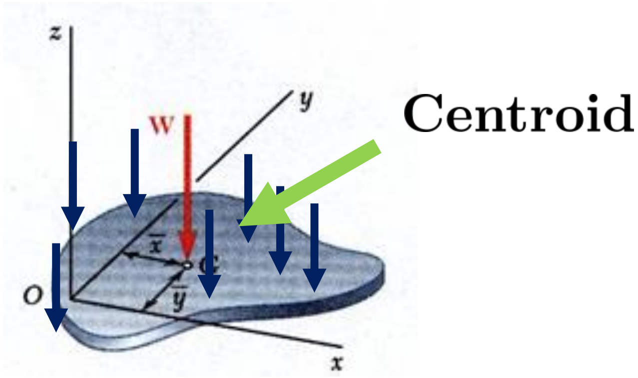
$$\sum M_A = 0: \quad B_y(6\,\mathrm{m}) - (18\,\mathrm{kN})(3.5\,\mathrm{m}) = 0$$

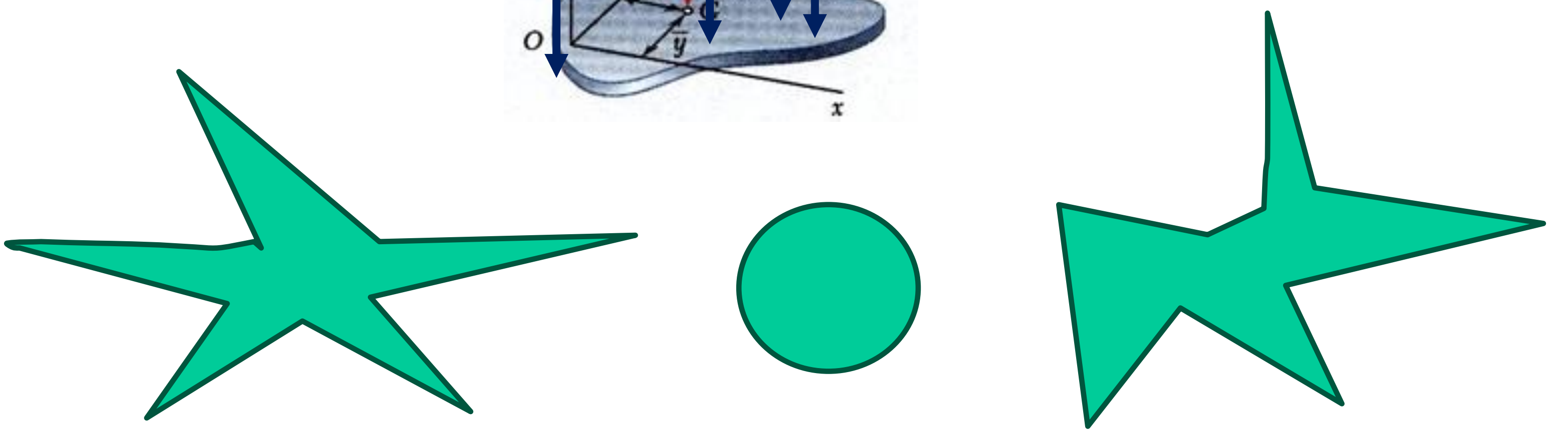
$$\boxed{B_y = 10.5\,\mathrm{kN}}$$

$$\sum M_B = 0: \quad -A_y(6\,\mathrm{m}) + (18\,\mathrm{kN})(6\,\mathrm{m} - 3.5\,\mathrm{m}) = 0$$

$$\boxed{A_y = 7.5\,\mathrm{kN}}$$

Lecture 14
Distributed Forces (II): Moments of Inertia

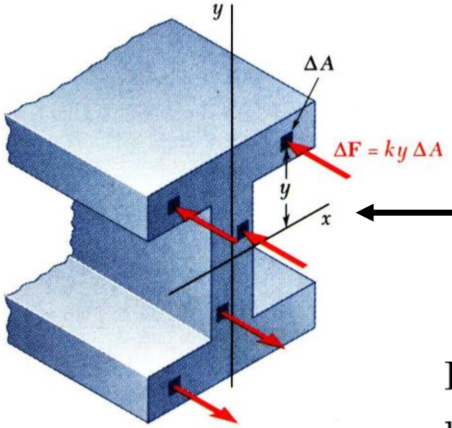


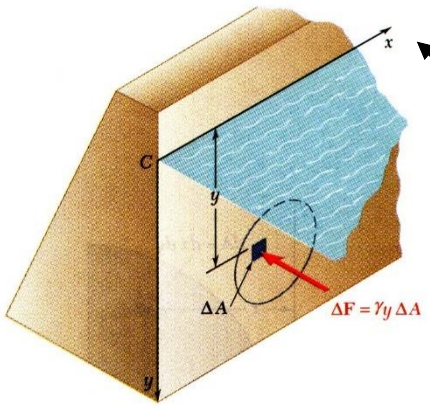


Oftentimes, we would like to know how mass is distributed over a continuum body or the volume or shape of a continuum body. This is because the mechanical moments generated by the distributed force depend on the span of a continuum body.

Moment of Inertia of an Area

When the force distributions are linearly proportional to the distance to an axis.





Force per unit area

$$\Delta\vec{F} = ky\Delta A$$

$$R = k\int y\,dA = 0$$

$$M = k\int y^2 dA$$

$$\Delta F = p\Delta A = \gamma y\Delta A$$

$$R = \gamma\int y\,dA$$

$$M_x = \gamma\int y^2 dA$$

$$\mathbf{\Delta F} = -ky\mathbf{k}\Delta A$$

$$M_x = \int_A (y\mathbf{j}) \times \mathbf{\Delta F} = -M$$

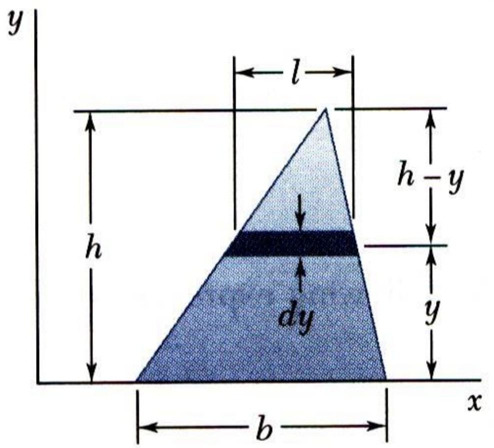
$$\int y\,dA = Q_x = \text{first moment}$$

$$\boxed{\int y^2 dA = \text{second moment}}$$

$$\mathbf{\Delta F} = -\gamma y\mathbf{k}\Delta A$$

$$M_x = \int_A (-y\mathbf{j}) \times \mathbf{\Delta F}$$

Sample Problem 9.1



Determine the moment of inertia of a triangle with respect to its base.

SOLUTION:

- A differential strip parallel to the *x* axis is chosen for *dA*.

$$dI_x = y^2 dA \qquad dA = \ell(y)dy$$

- For similar triangles,

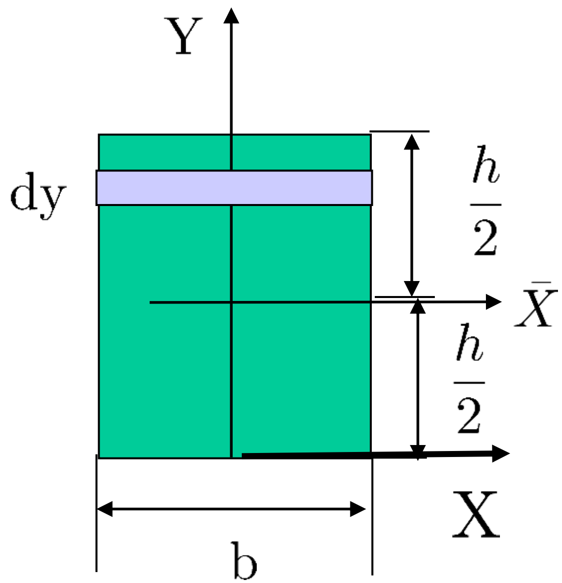
$$\frac{l}{b} = \frac{h-y}{h} \qquad l = b\frac{h-y}{h} \qquad dA = b\frac{h-y}{h}dy$$

- Integrating dI_x from $y = 0$ to $y = h$,

$$I_x = \int y^2 dA = \int_0^h y^2 b \frac{h-y}{h} dy = \frac{b}{h}\int_0^h \left(hy^2 - y^3\right)dy$$

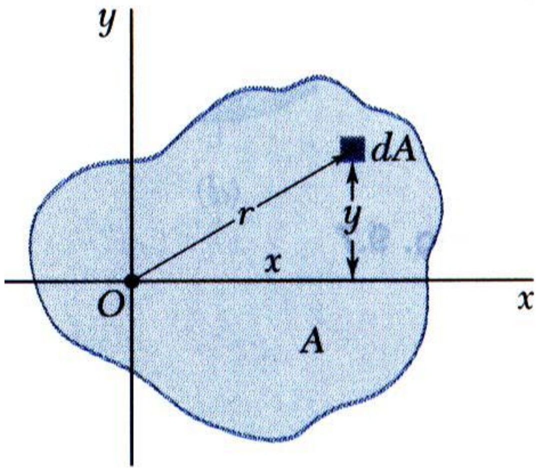
$$= \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h$$

$$\boxed{I_x = \frac{bh^3}{12}}$$



$$
\begin{aligned}
I_{\bar{x}} &= \int_A y^2 dA; \quad dA = bdy \\
&= \int_0^h by^2 dy \\
&= \frac{by^3}{3}\Big|_0^h = \frac{bh^3}{3}
\end{aligned}
$$

Polar Moment of Inertia



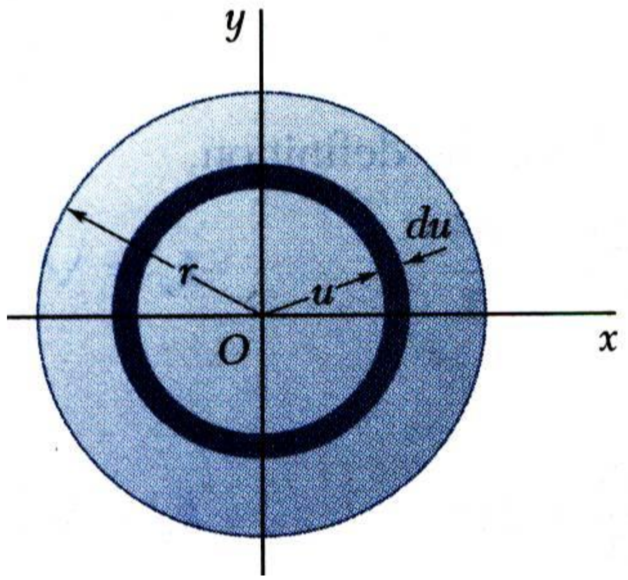
- The *polar moment of inertia* is an important parameter in problems involving torsion of cylindrical shafts and rotations of slabs.

$$J_0 = \int r^2 dA$$

- The polar moment of inertia is related to the rectangular moments of inertia,

$$J_0 = \int r^2 dA = \int \left(x^2 + y^2\right) dA = \int x^2 dA + \int y^2 dA$$
$$= I_y + I_x$$

Sample Problem 9.2



a) Determine the centroidal polar moment of inertia of a circular area by direct integration.

b) Using the result of part *a*, determine the moment of inertia of a circular area with respect to a diameter.

$$dJ_O = u^2 dA \qquad dA = 2\pi u\, du$$

$$J_O = \int dJ_O = \int_0^r u^2 (2\pi u\, du) = 2\pi \int_0^r u^3 du$$

$$\boxed{J_O = \frac{\pi}{2} r^4}$$

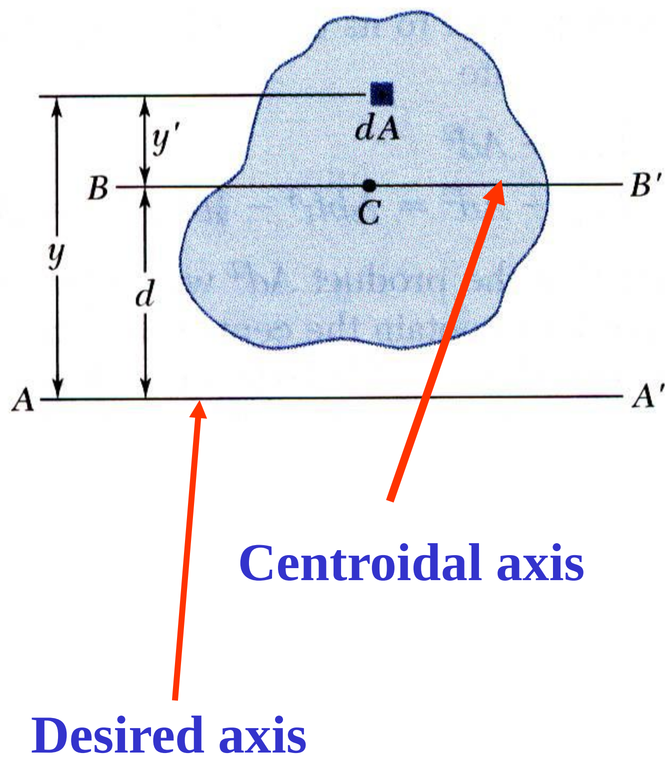
- From symmetry, $I_x = I_y$,

$$J_O = I_x + I_y = 2I_x \qquad \frac{\pi}{2} r^4 = 2I_x$$

$$k_x = r/2$$

$$\boxed{I_{diameter} = I_x = \frac{\pi}{4} r^4}$$

Parallel Axis Theorem (Axes that are parallel to the centroidal axis)



- Consider moment of inertia *I* of an area *A* with respect to the axis *AA'*

$$I = \int y^2 dA$$

- The axis *BB'* passes through the area centroid and is called a *centroidal axis.*

$$I = \int y^2 dA = \int (y' + d)^2 dA$$
$$= \int y'^2 dA + 2d \int y' dA + d^2 \int dA$$

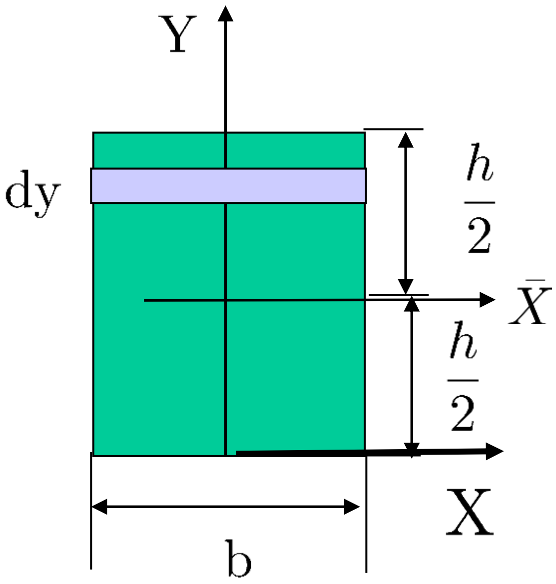
$$\boxed{I = \bar{I} + Ad^2 \quad \textit{parallel axis theorem}}$$

We usually take this as the moment of inertia w.r.t. **The centroidal axis.**

The moment of inertia with respect to am axis that is parallel to The centroidal axis is equal to

$$I_x = \bar{I}_{\bar{x}} + d^2 A$$

$$\boxed{I_x = \frac{1}{3} b h^3}$$



$$\begin{aligned}
\bar{I}_{\bar{x}} &= \int_A y^2 dA; \quad dA = b dy \\
&= \int_{-h/2}^{h/2} b y^2 dy \\
&= \frac{b y^3}{3} \Big|_{-h/2}^{h/2} \\
&= \frac{b}{3}\left(\frac{h^3}{8} - \frac{-h^3}{8}\right) = \frac{b h^3}{12}
\end{aligned}$$

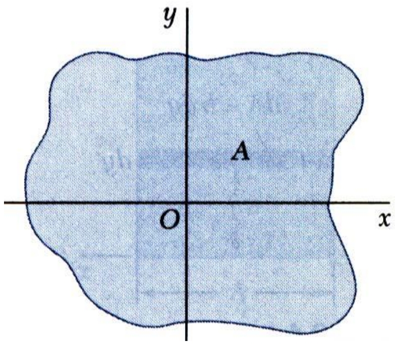
$$I_x = \bar{I}_x + (h/2)^2 A = \frac{b h^3}{12} + \frac{h^2}{4}(bh) = \frac{b h^3}{3}$$

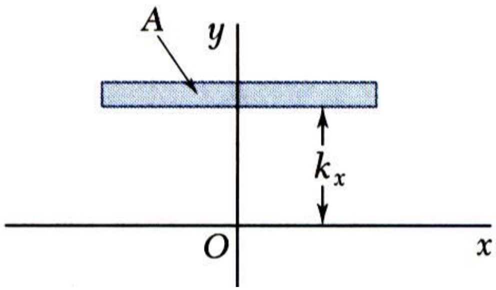
Radius of Gyration

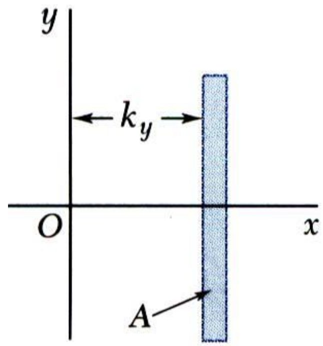
$k_x =$ *radius of gyration* with respect to the x axis

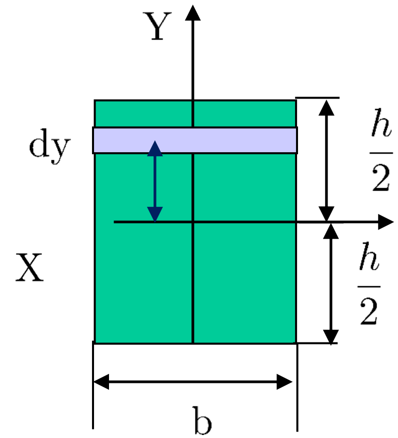
$$I_x = k_x^2 A \qquad k_x = \sqrt{\frac{I_x}{A}}$$

$$r_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{bh^3/12}{bh}} = \frac{h}{2\sqrt{3}}$$







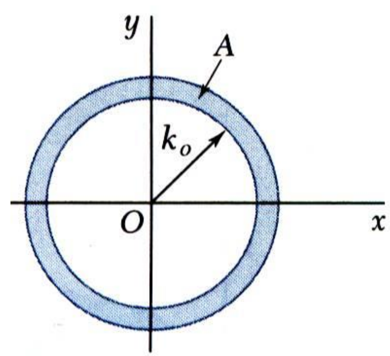


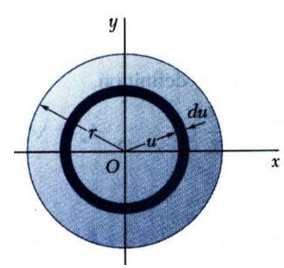
Radius of gyration is NOT radius

$$I_y = k_y^2 A \qquad k_y = \sqrt{\frac{I_y}{A}}$$

$$J_O = k_O^2 A \qquad k_O = \sqrt{\frac{J_O}{A}}$$

$$k_O^2 = k_x^2 + k_y^2$$





$$I_x = \frac{\pi r^4}{4}$$

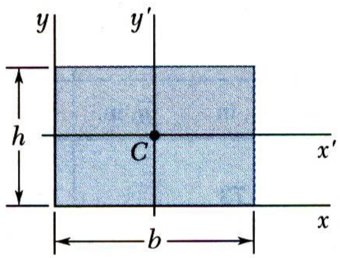
$$k_x = r/2$$

$$k_o = \sqrt{\frac{J_O}{A}} = \sqrt{\frac{\pi r^4/2}{\pi r^2}} = r/\sqrt{2}$$

Moments of Inertia of Composite Areas

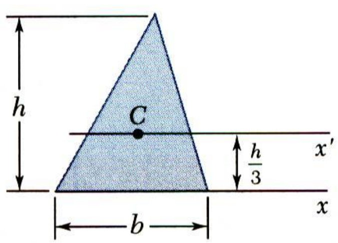
- The moment of inertia of a composite area A about a given axis is obtained by adding the moments of inertia of the component areas A_1, A_2, A_3, ... , with respect to the same axis.

Rectangle



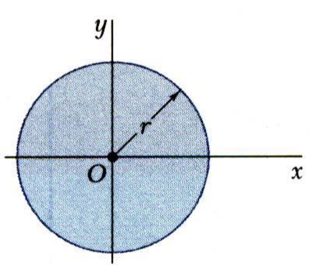
$$\bar{I}_{x'} = \tfrac{1}{12} bh^3$$
$$\bar{I}_{y'} = \tfrac{1}{12} b^3h$$
$$I_x = \tfrac{1}{3} bh^3$$
$$I_y = \tfrac{1}{3} b^3h$$
$$J_C = \tfrac{1}{12} bh(b^2 + h^2)$$

Triangle



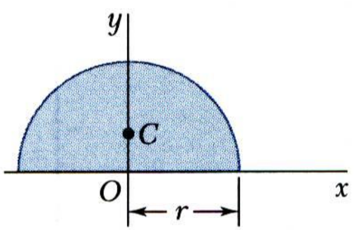
$$\bar{I}_{x'} = \tfrac{1}{36} bh^3$$
$$I_x = \tfrac{1}{12} bh^3$$

Circle



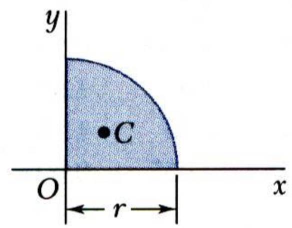
$$\bar{I}_x = \bar{I}_y = \tfrac{1}{4}\pi r^4$$
$$J_O = \tfrac{1}{2}\pi r^4$$

Semicircle



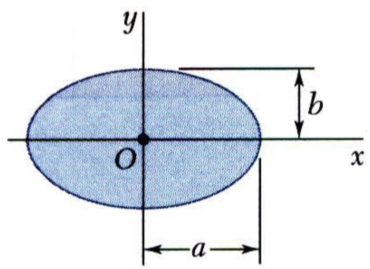
$$I_x = I_y = \tfrac{1}{8}\pi r^4$$
$$J_O = \tfrac{1}{4}\pi r^4$$

Quarter circle



$$I_x = I_y = \tfrac{1}{16}\pi r^4$$
$$J_O = \tfrac{1}{8}\pi r^4$$

Ellipse



$$\bar{I}_x = \tfrac{1}{4}\pi ab^3$$
$$\bar{I}_y = \tfrac{1}{4}\pi a^3b$$
$$J_O = \tfrac{1}{4}\pi ab(a^2 + b^2)$$

Summary

(2) Moment of Inertia:

$$I_x = \int_A y^2 dA \; ; \quad I_y = \int_A x^2 dA$$

(3) Polar Moment of Inertia:

$$I_\rho = \int_A (x^2 + y^2) dA = \int_A r^2 dA$$

Second moment

Radiu of Gyration

$$r_O = \sqrt{\frac{I_O}{A}}$$

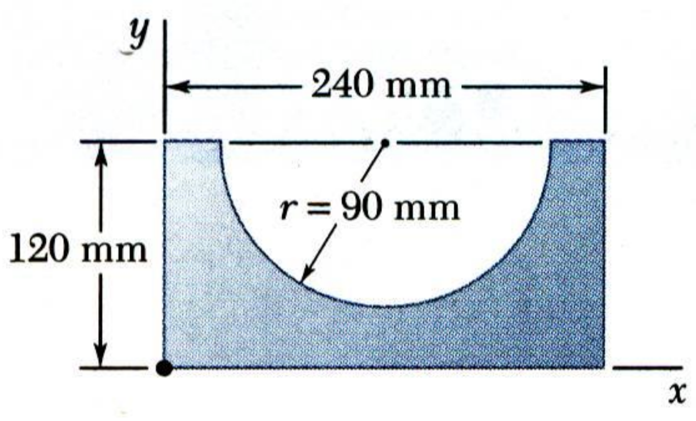
(4) Parallel Axis Theorem:

$$I_x = \bar{I}_x + dA$$

where $\bar{I}_x$ is the moment of inertia with respect to the centroidal axis, and d is the distance between the x-axis and the centroidal axis x'.

Today's Lecture Attendance Password is: Moment of Inertia

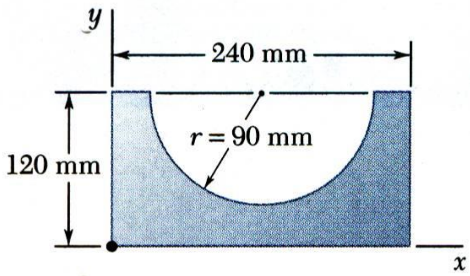
Sample Problem 9.5

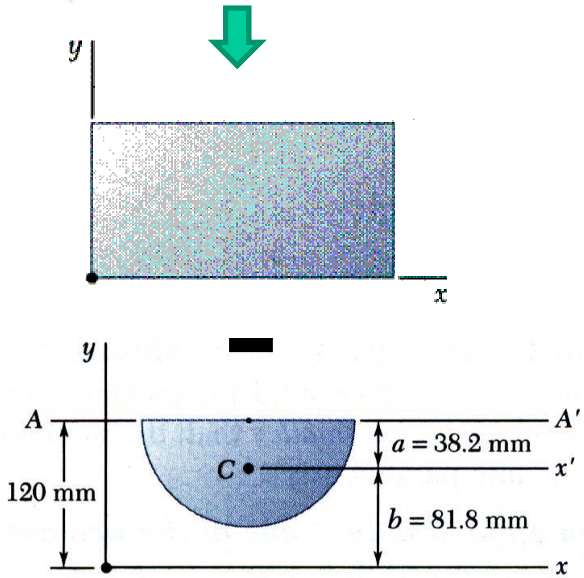


Determine the moment of inertia of the shaded area with respect to the *x* axis.

SOLUTION:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the *x* axis.
- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.





Parallel Axis Theorem $I_X = \bar{I}_{X'} + d^2 A$

Centroidal Axis

$$a = \frac{4r}{3\pi} = \frac{(4)(90)}{3\pi} = 38.2 \text{ mm}$$

$$\text{b} = 120 - \text{a} = 81.8 \text{ mm}$$

$$A = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi(90)^2$$
$$= 12.72 \times 10^3 \text{mm}^2$$

<u>SOLUTION</u>:

Rectangle:

$$I_x = \tfrac{1}{3}bh^3 = \tfrac{1}{3}(240)(120) = 138.2 \times 10^6 \text{mm}^4$$

Half-circle:

moment of inertia with respect to *AA'*,

$$I_{AA'} = \tfrac{1}{8}\pi r^4 = \tfrac{1}{8}\pi(90)^4 = 25.76 \times 10^6 \text{mm}^4$$

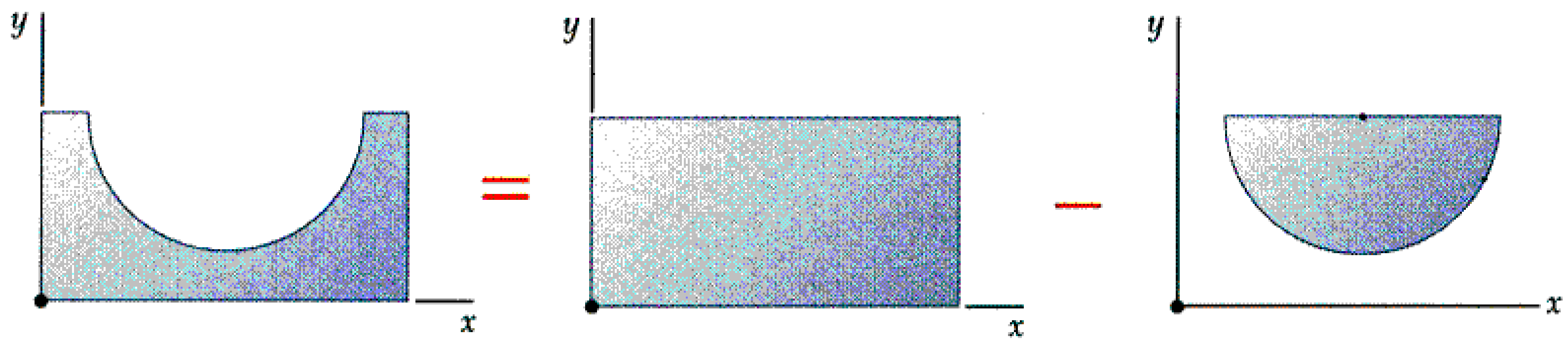
moment of inertia with respect to *x'*,

$$\begin{aligned} \bar{I}_{x'} &= I_{AA'} - Ad^2 = 25.76 \times 10^6 - (12.72 \times 10^3)(88.8)^2 \\ &= 7.20 \times 10^6 mm^4 \end{aligned}$$

moment of inertia with respect to *x*,

$$I_x = \bar{I}_{x'} + Ab^2 = 7.20 \times 10^6 + \left(12.72 \times 10^3\right)(81.8)^2$$
$$= 92.3 \times 10^6 \text{mm}^4$$

- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.



$$I_x \quad = \quad 138.2\times10^6\,\text{mm}^4 \quad - \quad 92.3\times10^6\,\text{mm}^4$$

$$\boxed{I_x = 45.9\times10^6\,\text{mm}^4}$$

Why the name: Moment of Inertia ?

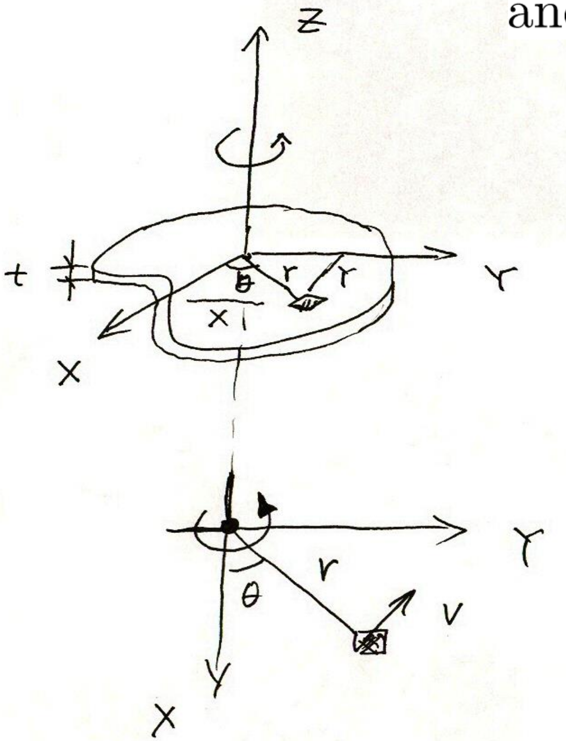
Why do we call the second moment of an object as the moment of inertia ?

What is inertia ? In rectilinear motion,

$$M\mathbf{a} = \mathbf{F}$$

Mass or inertia

Consider a thin plate rotating around z-axis, and we assume the mass density $\rho = const.$



For infinitesimal volume of mass: $\rho \underbrace{tdA}_{dV} = dm$

Its velocity is: $v = r\dot{\theta}$,
and its acceleration is

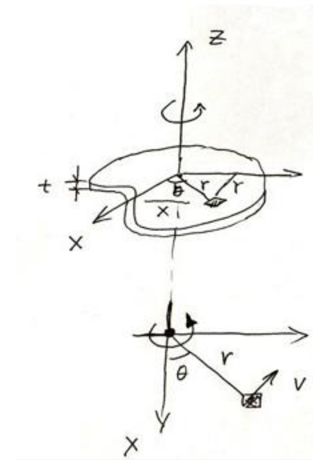
$$a = r\ddot{\theta} \ .$$

Suppose that the force acting on the infinitesimal element is dF,

$$adm = dF \ .$$

$$\mathbf{a}dm = d\mathbf{F}, \qquad dm = \rho t dA$$

Consider the moment equation around z-axis,



$$\mathbf{r} \times \mathbf{a}dm = \mathbf{r} \times d\mathbf{F} \;\rightarrow\; \int_A \mathbf{r} \times \mathbf{a}dm = \int_A \mathbf{r} \times d\mathbf{F}$$

$$\int_A r(r\ddot{\theta})dm = \int_A r^2 \ddot{\theta}\rho t dA = M_z$$

$$t\rho\ddot{\theta}\boxed{\int_A r^2 dA} = t\rho\ddot{\theta}J_O = M_z, \;\rightarrow\; t\rho J_O\ddot{\theta} = M_z, \;\leftarrow\; M\mathbf{a} = \mathbf{F}$$

Polar moment of inertia

The polar moment of inertia of the thin plate is,

$$J_O = \int_A r^2 dA = (x^2 + y^2)dA$$

$J_O = \int_A \rho^2 dA$ is the moment of inertia force of $\rho\ddot{\theta}$.

We can define,

$$I_{z,mass} = \rho t \int_A (x^2 + y^2) dA = \int_A (x^2 + y^2) dm, \text{ and}$$

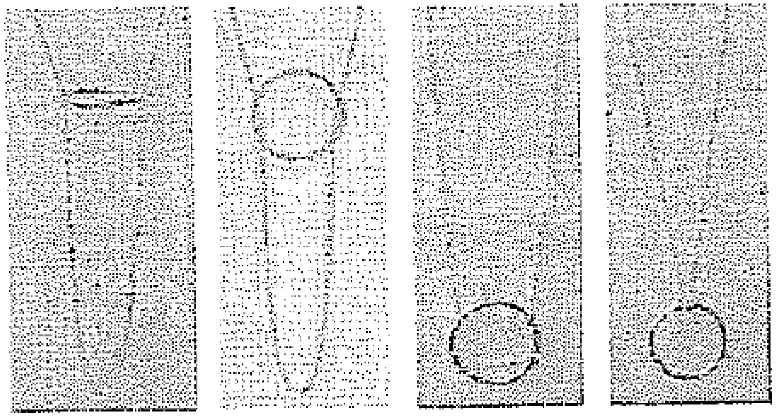
Moment of Inertia $\longrightarrow$ $I_{z,mass}\ddot{\theta} = M_z$

The moment of inertia, otherwise known as the angular mass or rotational inertia, of a rigid body determines the torque needed for a desired angular acceleration about a rotational axis.

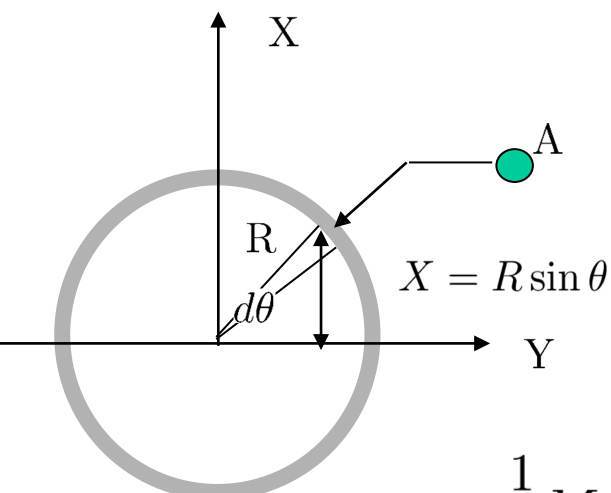
The moment of inertia of body with the shape of the cross-section is **the second moment of this area** about the z-axis perpendicular to the cross-section, weighted by its density.

We can define the second moment of the area A as,

$$I_z = \int_A (x^2 + y^2) dA$$

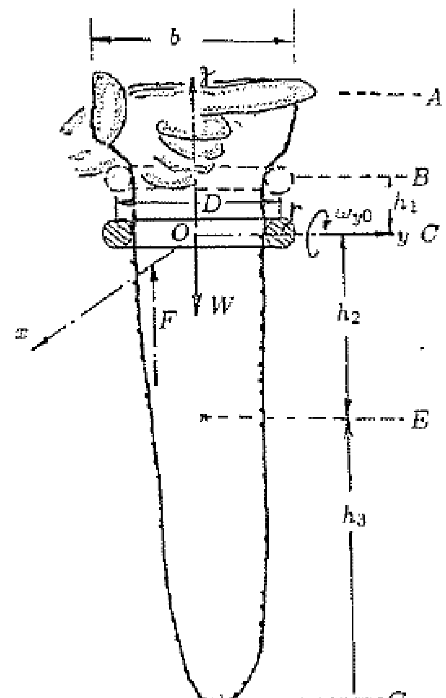


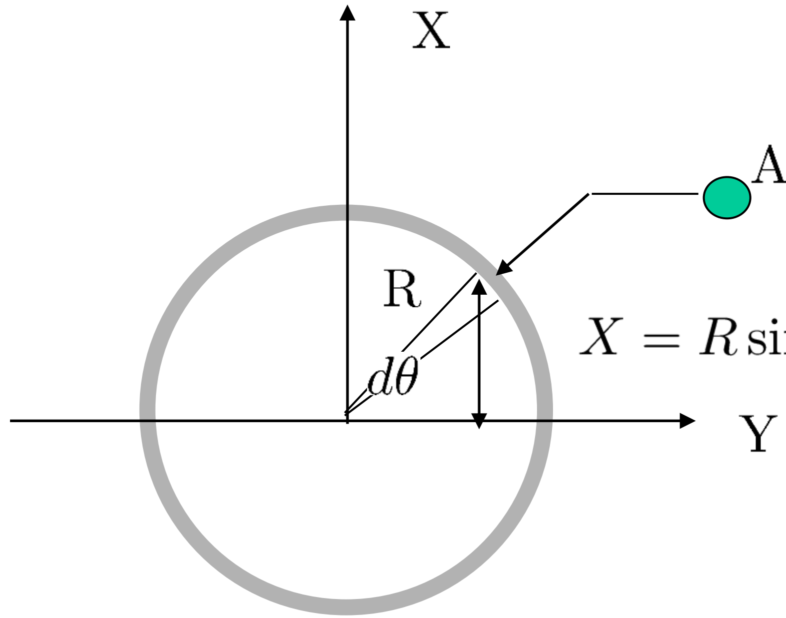
(a) (b) (c) (d)



Magic Ring Trick

$$\frac{1}{2}Mv^2 = \frac{1}{2}\rho J(\dot{\theta}_y)^2 = \frac{1}{2}\rho J(\omega)^2$$





$$dV = ARd\theta$$

$$X = R\sin\theta$$

$$\begin{aligned} J_Y &= \int_V X^2 dV = 2\int_0^\pi X^2 ARd\theta = 2A\int_0^\pi R^3 \sin^2\theta d\theta \\ &= R^3 A\pi \end{aligned}$$

$$\frac{1}{2}Mv^2 = \frac{1}{2}\rho J_Y \omega^2$$

This is for your Thanksgiving Party Performance!

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