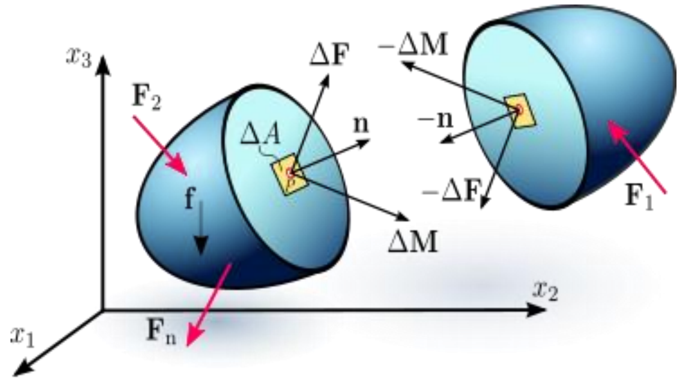
Lecture 17 Concepts of Stress and Strain



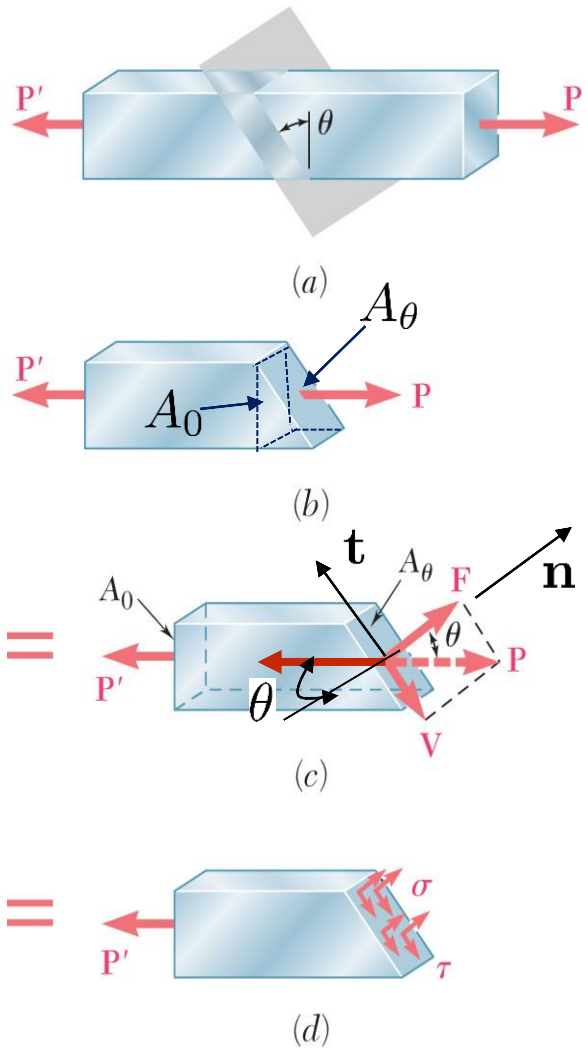
$$\mathbf{T}^{(n)} = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{F}}{\Delta A} = \sigma \mathbf{n} + \tau \mathbf{t}$$

1. Is stress a scalar?

2. Is stress a vector?

<u>3. Or is it something else?</u>

Stress on an Oblique Plane



- Resolve P into components normal and tangential to the oblique section,

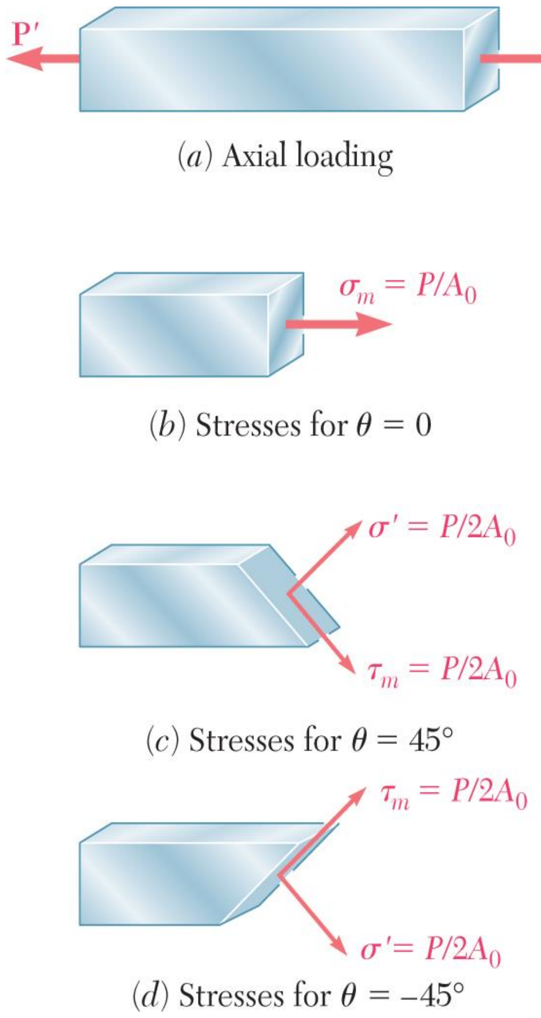
$$F = P\cos\theta, \quad V = P\sin\theta$$

- The average normal and shear stresses on the oblique plane are

$$\sigma = \frac{F}{A_\theta} = \frac{P\cos\theta}{A_0/\cos\theta} = \frac{P}{A_0}\cos^2\theta$$

$$\tau = -\frac{V}{A_\theta} = -\frac{P\sin\theta}{A_0/\cos\theta} = -\frac{P}{A_0}\sin\theta\cos\theta$$

Maximum Stresses



- Normal and shear stresses

$$\sigma = \frac{P}{A_0}\cos^2\theta, \quad \tau = -\frac{P}{A_0}\sin\theta\cos\theta$$
$$= \frac{-P}{2A}\sin 2\theta$$

- The maximum normal stress

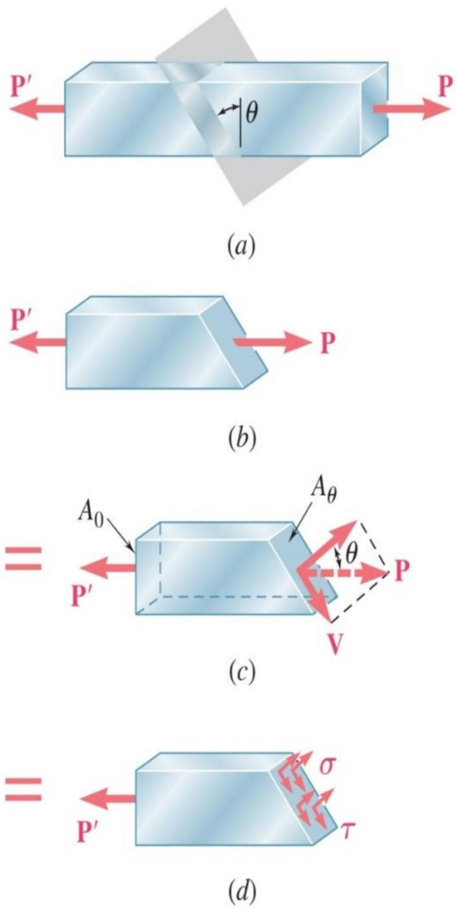
$$\sigma_m = \frac{P}{A_0}, \quad \tau' = 0, \quad \text{when} \quad \theta = 0;$$

- The maximum shear stress

$$\tau_m = \frac{P}{A_0}\sin 45^o \cos 45^o = \frac{P}{2A_0},$$

$$\sigma' = \frac{P}{2A_0}, \quad \text{at} \quad \theta = -45^o;$$

- Normal and shearing stresses on an oblique plane



$$\sigma_\theta = \left(\frac{P}{A}\right)\cos^2\theta \qquad \tau_\theta = -\left(\frac{P}{2A}\right)\sin 2\theta$$

Question:

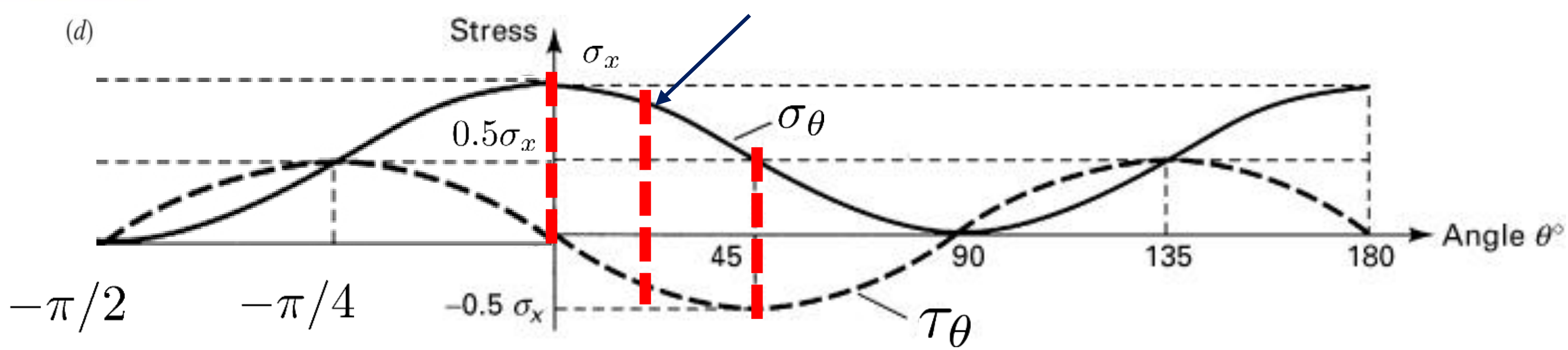
How do we quantify the stress state at a material point ?

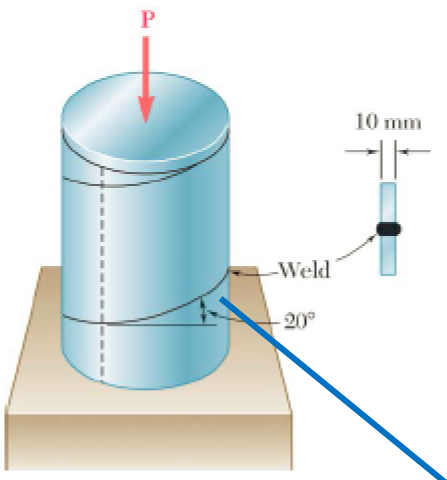
We have infinite many combinations of normal and shear stresses.

$$\boldsymbol{\sigma} = [\sigma(0), \tau(0)] = \sigma_x[1, 0], \quad \sigma_x = P/A$$

$$\boldsymbol{\sigma} = [\sigma(\pi/4), \tau(\pi/4)] = 0.5\sigma_x[1, -1]$$

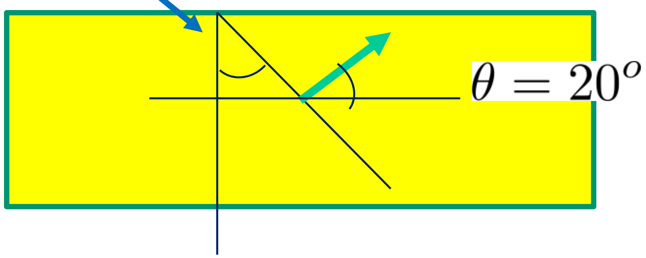
$$\boldsymbol{\sigma} = [\sigma(\theta), \tau(\theta)]$$





PROBLEM 8.31

A steel pipe of 400-mm outer diameter is fabricated from 10-mm thick plate by welding along a helix that forms an angle of 20° with a plane perpendicular to the axis of the pipe. Knowing that a 300-kN axial force **P** is applied to the pipe, determine the normal and shearing stresses in directions respectively normal and tangential to the weld.



$$\sigma = \frac{P}{A_0}\cos^2\theta = \frac{(-300\times 10^3)}{12.25522\times 10^{-3}}\cos^2 20^o = -21.621\times 10^6 P_a$$

$$\tau = -\frac{P}{2A_0}\sin 2\theta = \frac{-(-300)\times 10^3\sin 40^o}{((2)(12.25522\times 10^{-3})} = 7.8695\times 10^6 P_a$$

This problem is rated-R

How can we define the stress state at one point ?

1. Can we use a scalar to represent the stress state ?

2. Can we use A vector to represent the stress state ?

The potato dilemma

Cauchy Stress Tensor

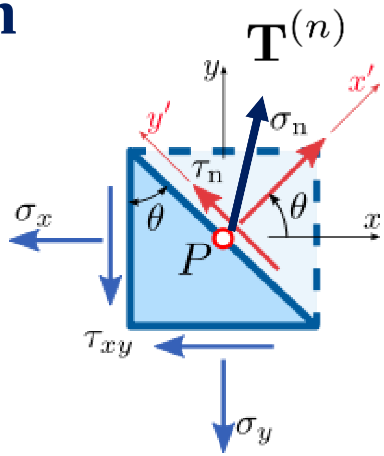
Cauchy found that one can use the traction vectors on three independent planes to represent the traction vector on any arbitrary plane. Hence, in practice, we usually use the traction components on three mutually perpendicular planes to represent the stress state at one material point.

Augstin Louis Cauchy (1787-1857)

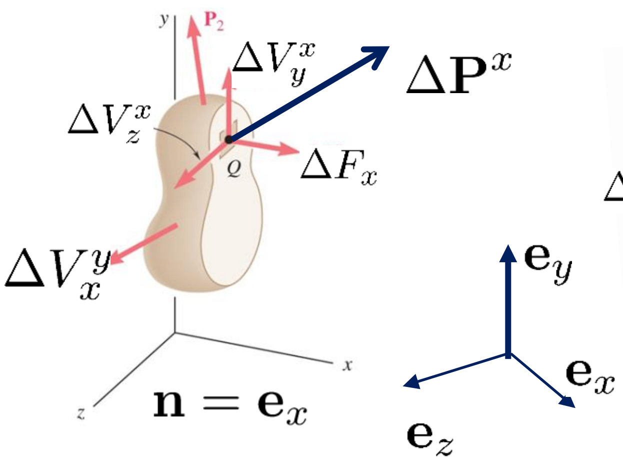
``Man dies, but his work lives."

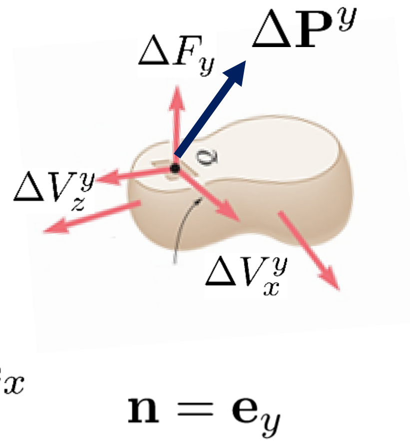
Cauchy Theorem

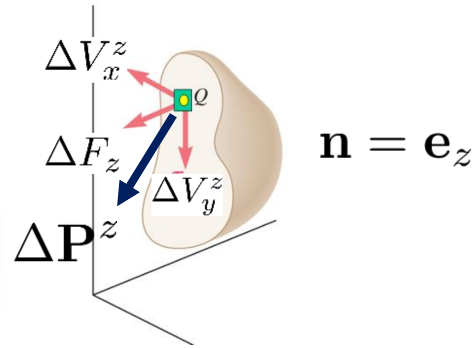
$$\begin{bmatrix} T_x^{(n)} \\ T_y^{(n)} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix}$$



$$\Delta\mathbf{P}^x = \Delta F_x \mathbf{e}_x + \Delta V_y^x \mathbf{e}_y + \Delta V_z^x \mathbf{e}_z \qquad \Delta\mathbf{P}^y = \Delta V_x^y \mathbf{e}_x + \Delta F_y^y \mathbf{e}_y + \Delta V_z^y \mathbf{e}_z$$







$$\Delta\mathbf{P}^z = \Delta V_x^z \mathbf{e}_x + \Delta V_y^z \mathbf{e}_y + \Delta F_z^z \mathbf{e}_z$$

$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A}$$

$$\tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta V_y^x}{\Delta A}$$

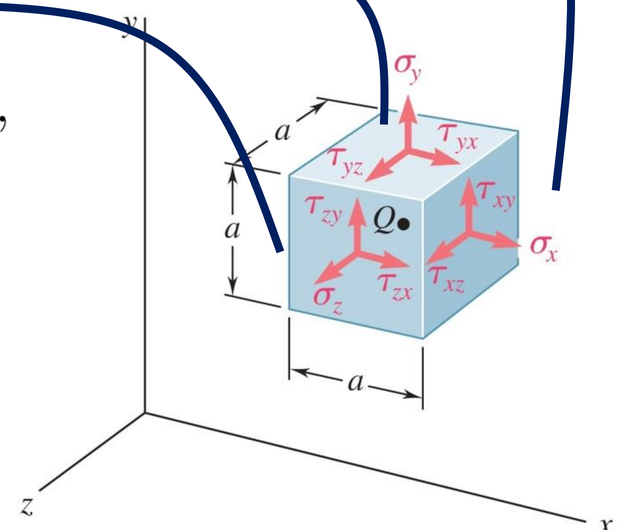
$$\tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta V_z^x}{\Delta A}$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \boxed{\sigma_{xx} \quad \tau_{xy} \quad \tau_{xz}} \\ \tau_{yx} \quad \sigma_{yy} \quad \tau_{yz} \\ \tau_{zx} \quad \tau_{zy} \quad \sigma_{zz} \end{bmatrix}$$

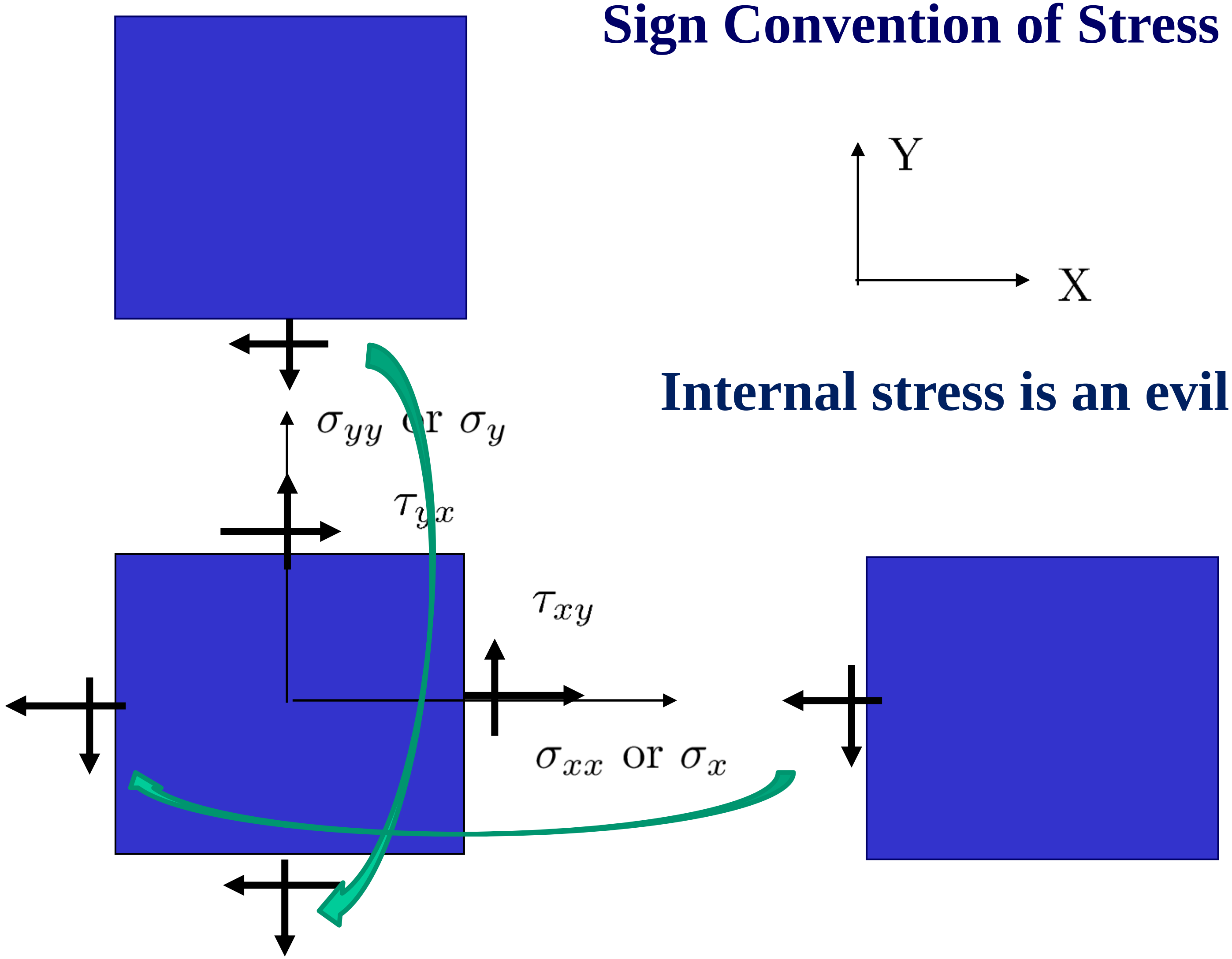
Stress Tensor

In two-dimensional space,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix}$$



Sign Convention of Stress Tensor



Internal stress is an evil twin!

If the normal of a surface is along the positive direction of an axis, the positive direction of the stress components on the surface is the positive direction of the axis.

State of Stress

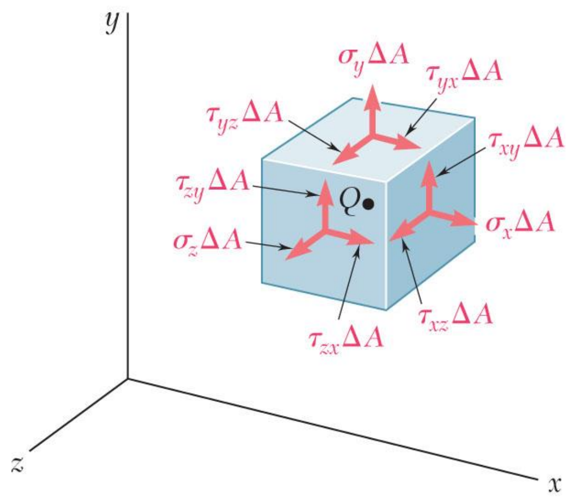
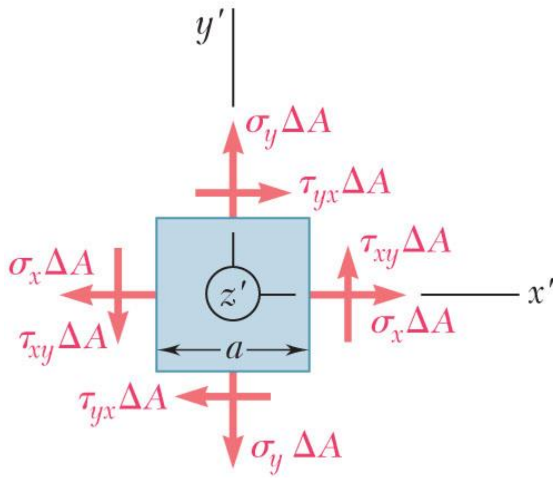


Fig. 8.33 Positive stress components at point Q.



- The combination of forces generated by the stresses must satisfy the conditions for equilibrium:

$$\boxed{\sum F_x = \sum F_y} = \sum F_z = 0$$

$$\sum M_{x'} = \sum M_{y'} = \boxed{\sum M_{z'} = 0}$$

- Consider the moments about the z axis:

$$\sum M_{z'} = 0 = \left(\tau_{xy}\Delta A\right)a - \left(\tau_{yx}\Delta A\right)a$$

$$\tau_{xy} = \tau_{yx}$$

similarly, $\tau_{yz} = \tau_{zy}$ and $\tau_{yz} = \tau_{zy}$

- Only six components of stress are required to define the complete state of stress.

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Symmetric Tensor

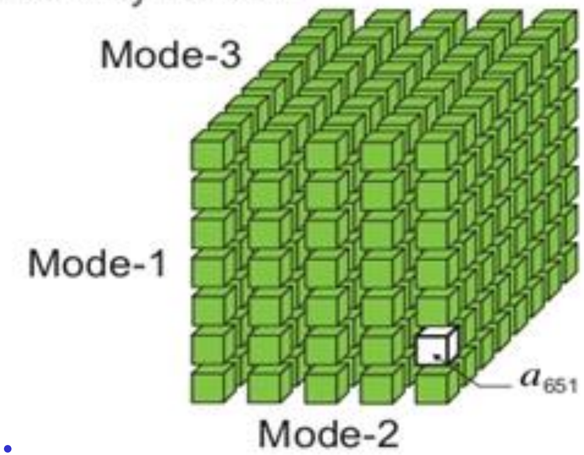
Recall: What is vector ?

Vector has three scalar components

Now we need three vectors to describe stress:
In mathematics, we call such quantity as a tensor.
more precisely as the second order tensor.

Third-order Tensor

A tensor is a multidimensional array
E.g., three-way tensor:



In general, we can call everything as tensor:

Scalar: the zero-order tensor, 3^0
Vector: the first-order tensor, 3^1
The second order tensor, 3^2

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

………

In general, one can have a n-th order tensor.

Do we have other quantity that is a tensor ?

$$\mathbf{I} = \begin{bmatrix} \int_A x^2 dA & \int_A xy dA \\ \int_A xy dA & \int_A y^2 dA \end{bmatrix}$$

The second order tensor is an isomorphism of the matrix.

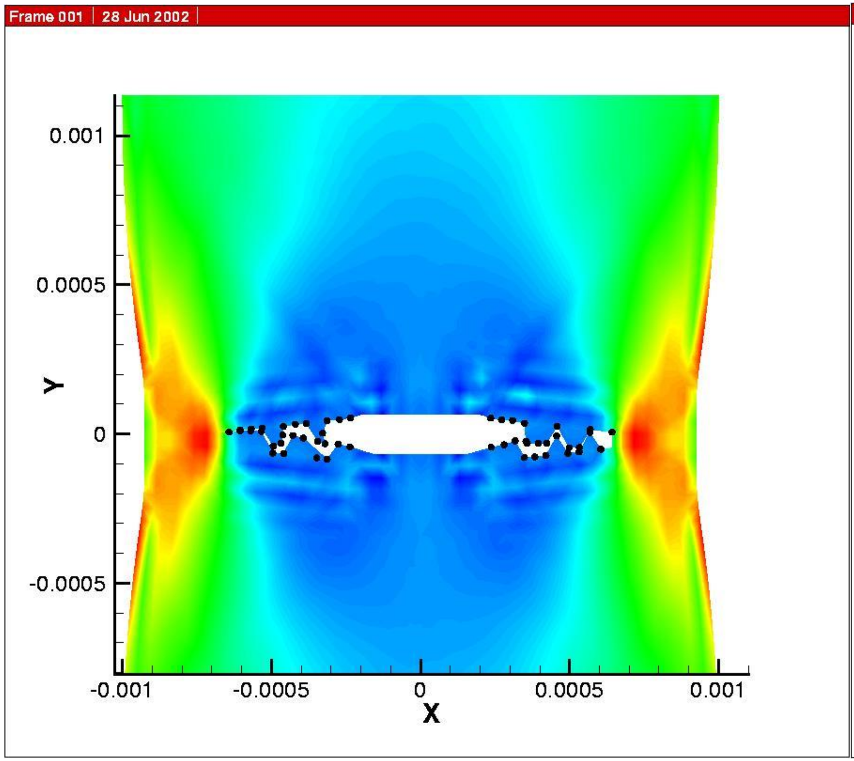
$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

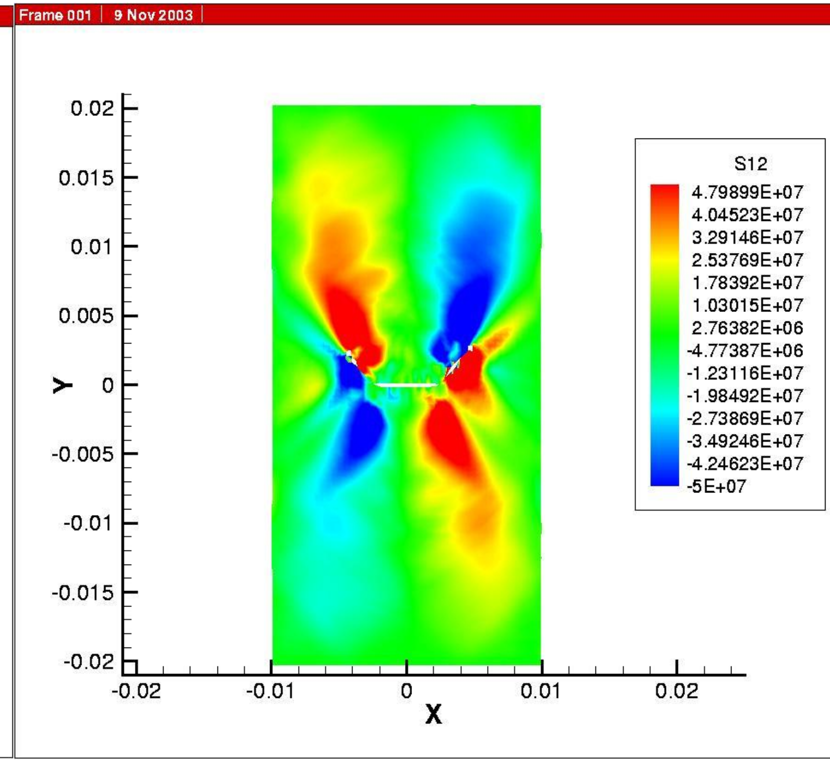
What is matrix ?Matrix is a linear transformation.

$$\begin{bmatrix} T_x^{(n)} \\ T_y^{(n)} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix}$$

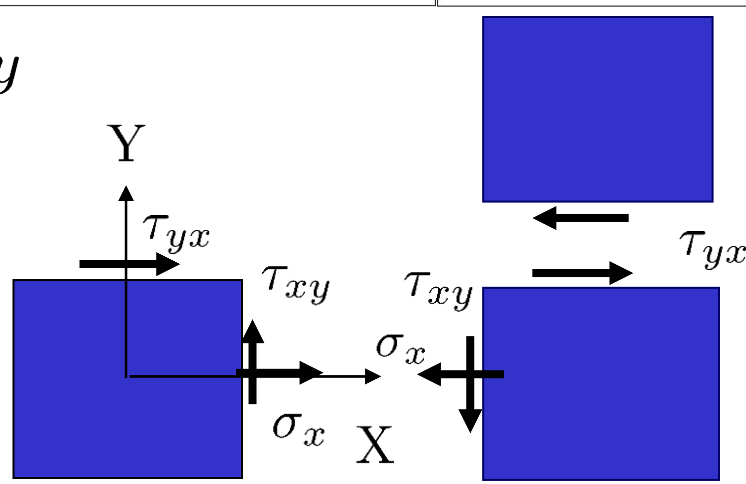
In general, tensor is a multi-array (multi-linear) representation form that can store data and information in various dimensions.

Stress Fields





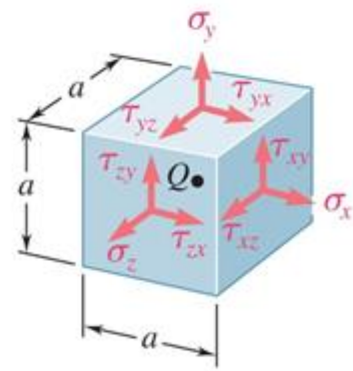
σ_y τ_{xy}



Three Takeaways

1. Equilibrium is balance of forces NOT stresses !

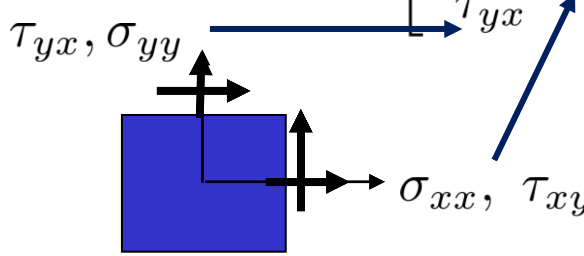
2. The first step of Stress Analysis is:



In two-dimensional space,

3. Stress is a Tensor !

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix}$$



Today’s Lecture Attendance Passcode is: Stress Tensor

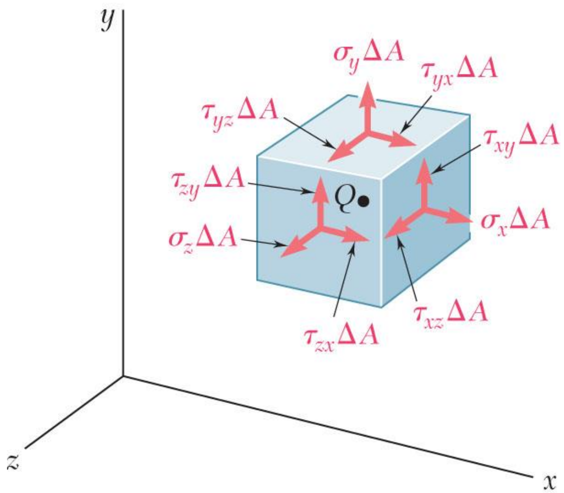


Fig. 8.33 Positive stress components at point *Q*.

- Consider the moments about the *z-axis*:

$$\sum M_{z'} = 0 = \left(\tau_{xy}\Delta A\right)a - \left(\tau_{yx}\Delta A\right)a$$

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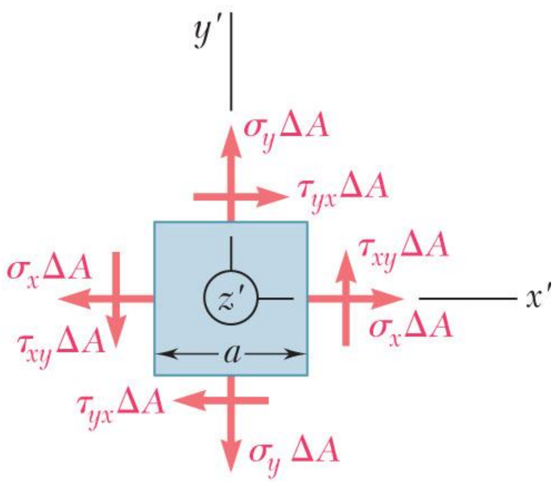


- Only six components of stress are required to define the complete state of stress.

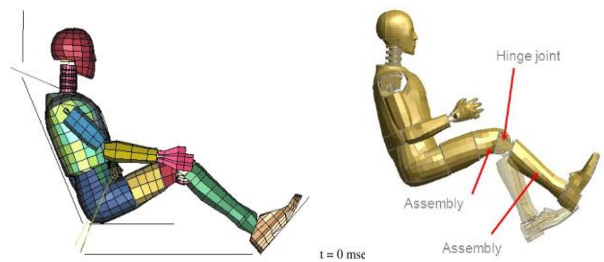
$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \; ?$$

Cauchy stress is a Symmetric Tensor

Lecture 17 Deformable Solid Model: Equilibrium, Kinematics, and Material Responses

Physical reality

Particle model



Rigid-body model

Braced Relaxed

Deformable solid model

Essentially, all models are wrong, but some are useful.

George E. P. Box

II. What is Strain ?

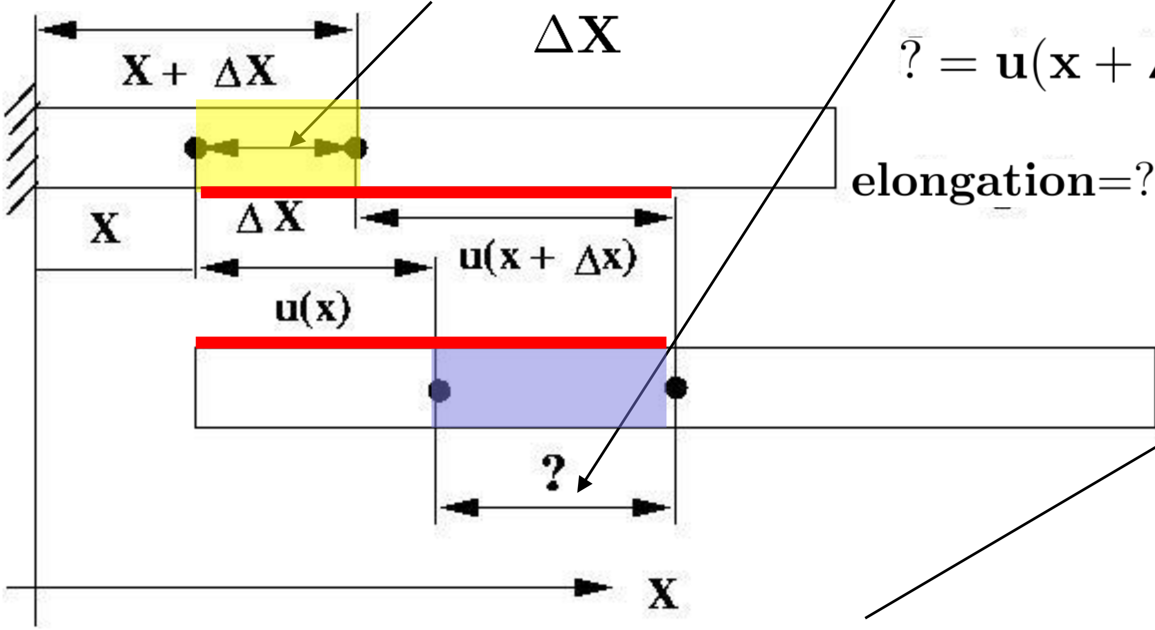
$$\underline{\mathbf{u}(\mathbf{x}) + ?} = \underline{\mathbf{u}(\mathbf{x} + \Delta\mathbf{x}) + \Delta\mathbf{X}}$$

Definition 1: DISPLACEMENT is the change of position of a material point during a motion

undeformed length

deformed length $=?$

$$? = \mathbf{u}(\mathbf{x} + \boldsymbol{\Delta}\mathbf{x}) + \boldsymbol{\Delta}\mathbf{X} - \mathbf{u}(\mathbf{x})$$



$$\textbf{elongation} = ? - \boldsymbol{\Delta}\mathbf{X} = \mathbf{u}(\mathbf{x} + \boldsymbol{\Delta}\mathbf{x}) - \mathbf{u}(\mathbf{x})$$

Definition 2: DEFORMATION is the relative displacement

1. If $u(X + \Delta X) = u(X)$, there is no deformation, and the bar undergoes rigid-body motion;

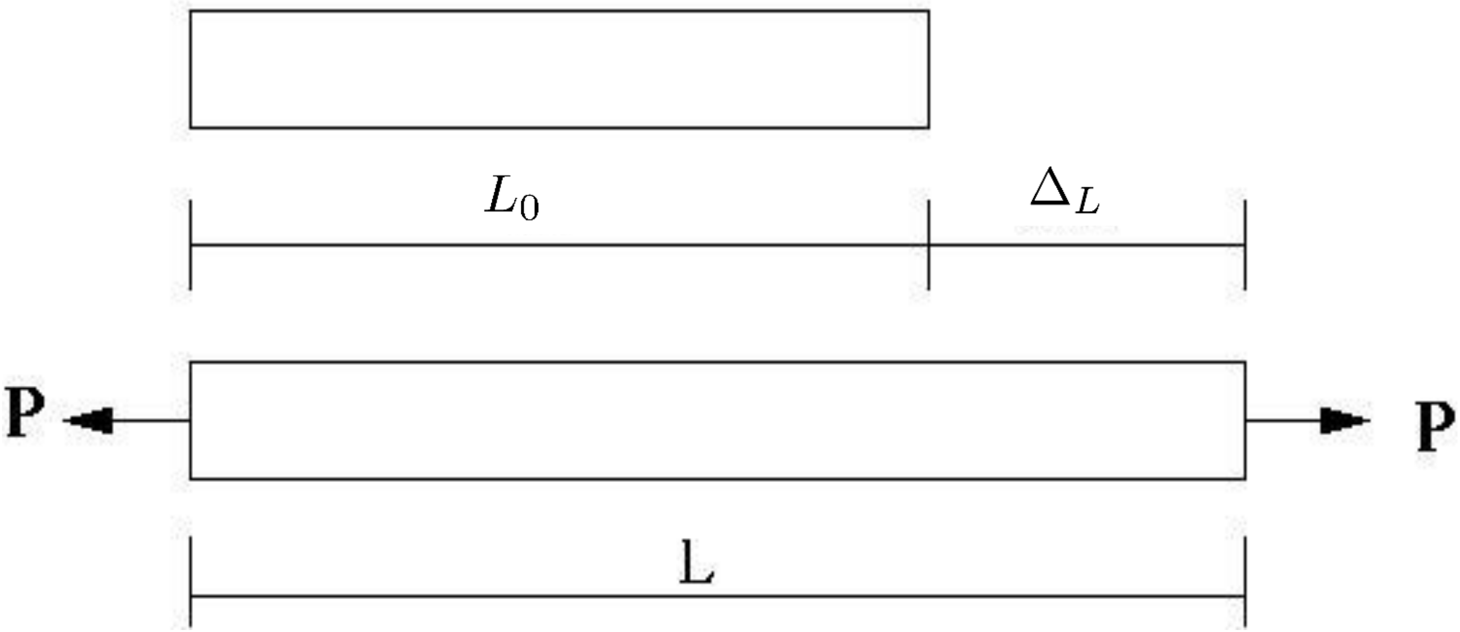
2. If $u(X + \Delta X) \neq u(X)$, the infinitesimal segment either elongates or shortens.

Definition of average strain

Normal Strain is a measure of relative deformation

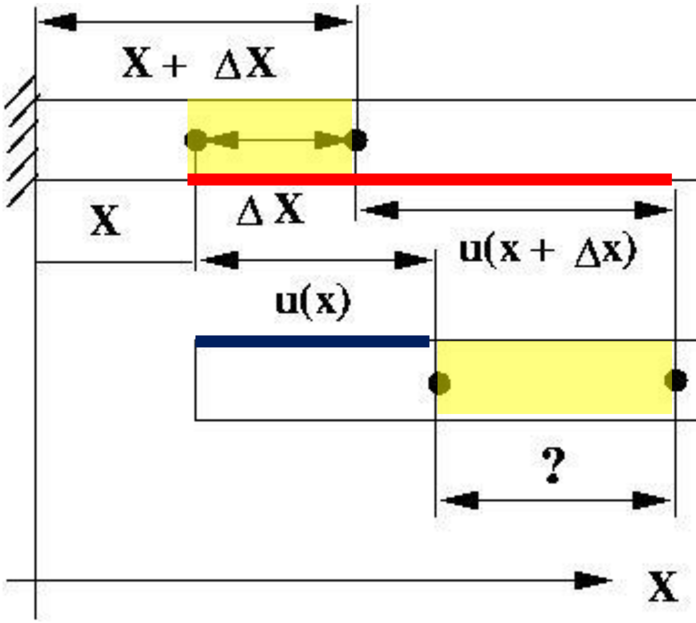
Average strain

$$\epsilon = \frac{\Delta_L}{L_0}$$



Mathematics Definition of local strain

Normal Strain is a measure of relative deformation

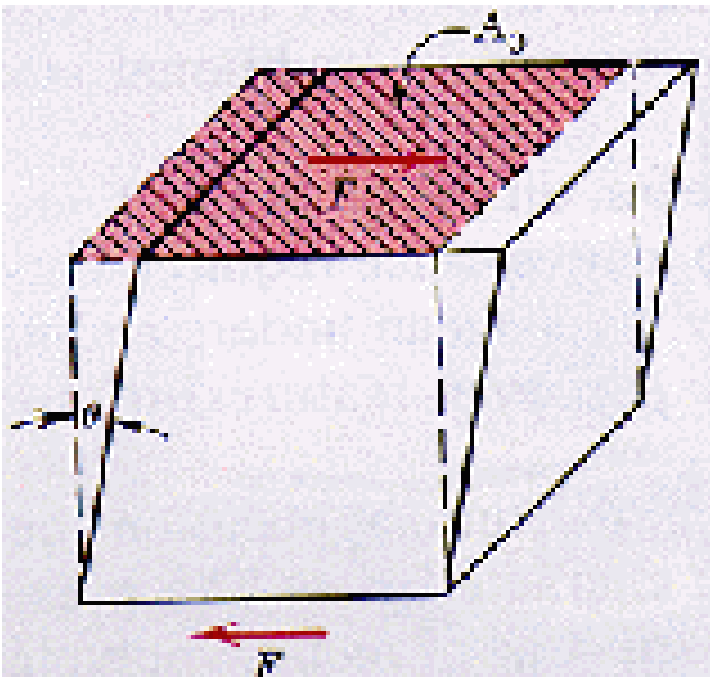


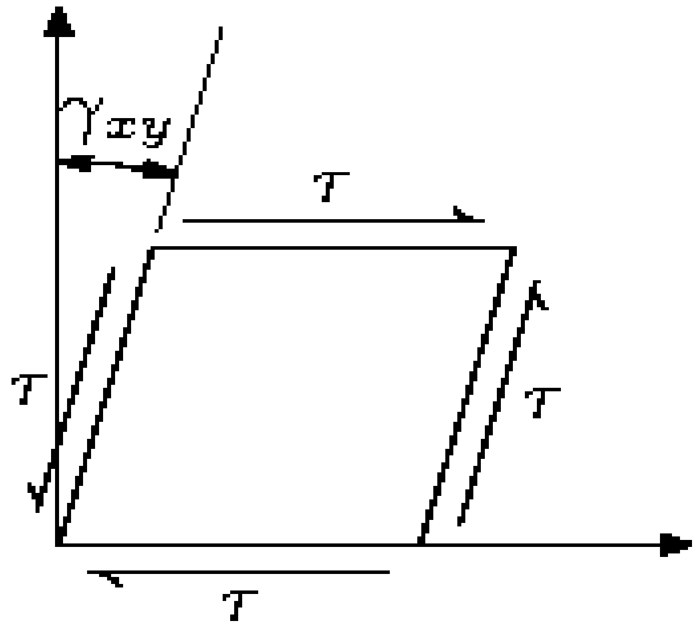
$$? = \mathbf{u}(\mathbf{x} + \mathbf{\Delta x}) + \mathbf{\Delta X} - \mathbf{u}(\mathbf{x})$$

$$\mathbf{elongation} = ? - \mathbf{\Delta x}$$
$$= \mathbf{\Delta u} = \mathbf{u}(\mathbf{x} + \mathbf{\Delta x}) - \mathbf{u}(\mathbf{x})$$

$$\epsilon_{\mathbf{x}} = \lim_{\mathbf{\Delta x} \to \mathbf{0}} \frac{\mathbf{u}(\mathbf{x} + \mathbf{\Delta x}) - \mathbf{u}(\mathbf{x})}{\mathbf{\Delta x}} = \frac{\mathbf{du}}{\mathbf{dx}}$$

Definition of Shear Strain





$$\gamma = \tan\theta \cong \theta$$

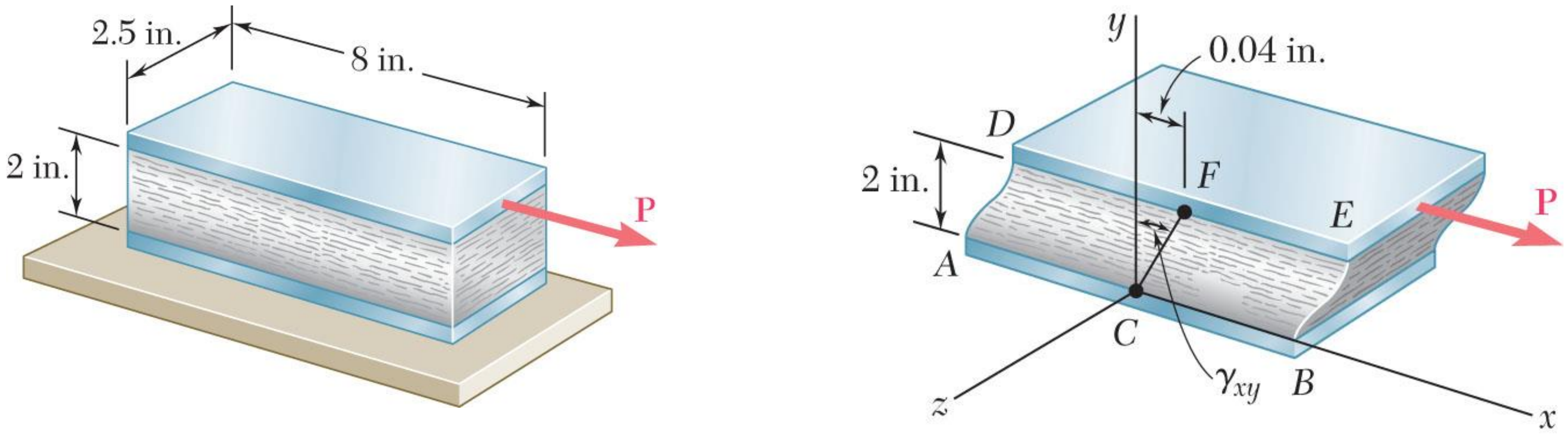
1. Shear Strain is the change of a right angle;
2. Shear Strain is a measure of change of shape

Shear deformation in the crust of earth

Original

Shear strain $= 0.57735$

How to calculate shear strain ?



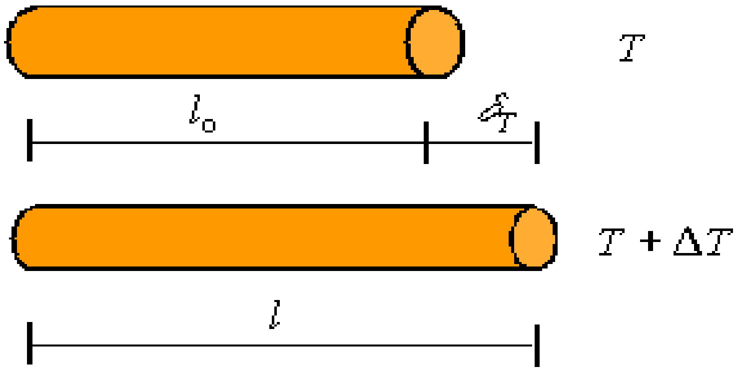
SOLUTION:

- Determine the average shear strain of the block.

$$\gamma_{xy} \approx \tan \gamma_{xy} = \frac{0.04\,\text{in.}}{2\,\text{in.}} \qquad \gamma_{xy} = 0.020\,\text{rad}$$

Thermal strain

Linear coefficient of thermal expansion:

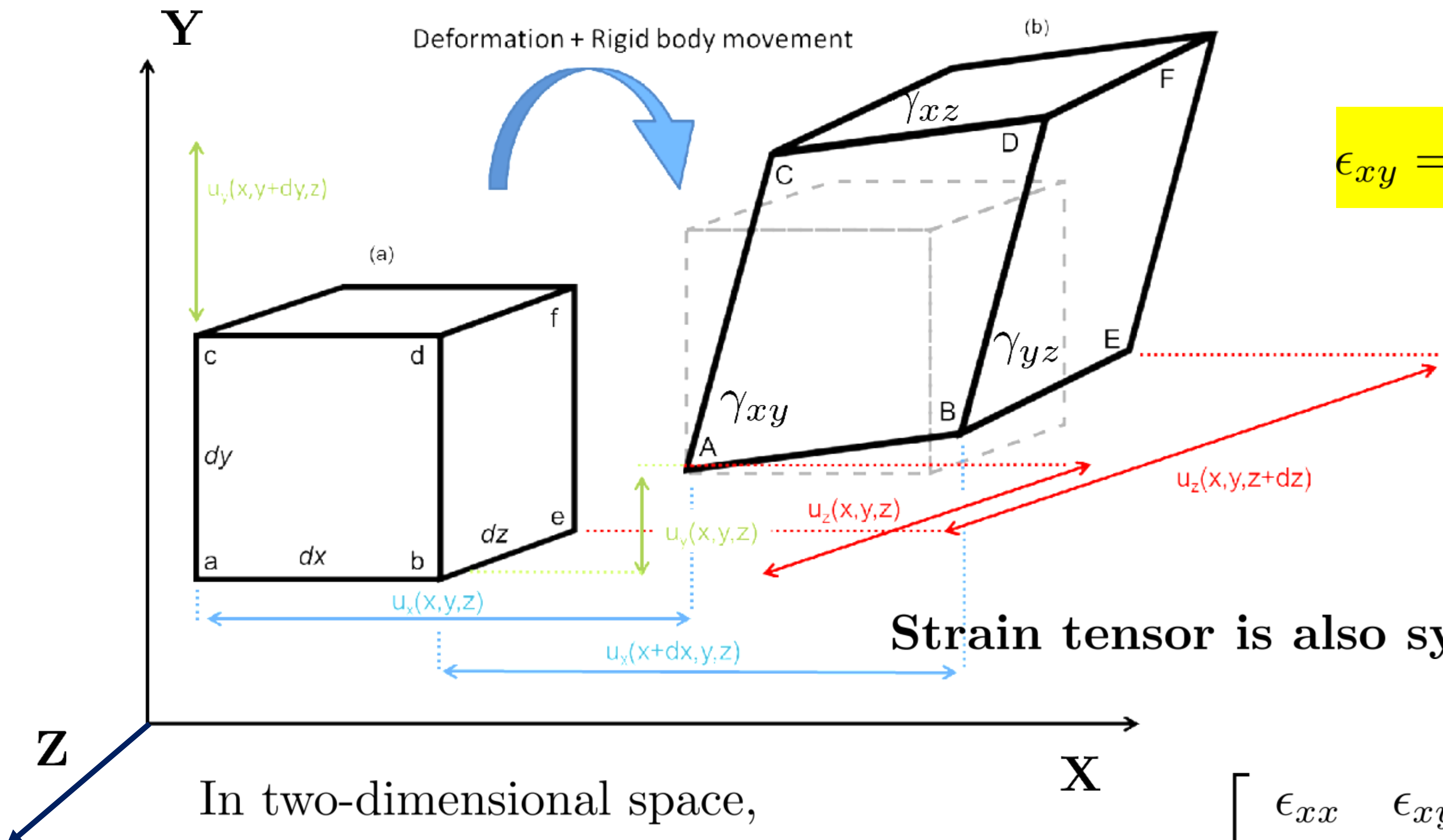


$$\varepsilon_T = \frac{\delta_T}{L_0} = \alpha \Delta T,$$

α *is the linear coefficient of thermal expansion* $\quad m/(m^oC)$

$m/(m^oF)$

Strain is also a tensor!



$$\epsilon_{xy} = \frac{1}{2}\gamma_{xy}$$

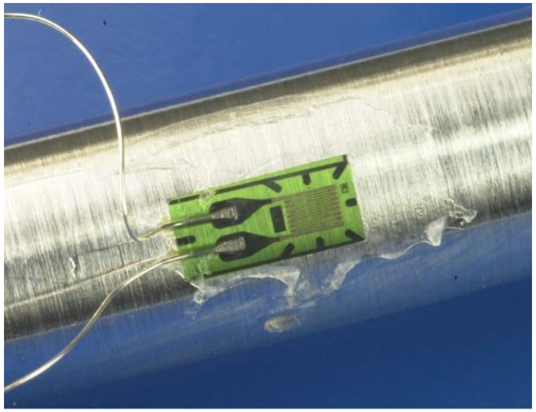
Strain tensor is also symmetric!

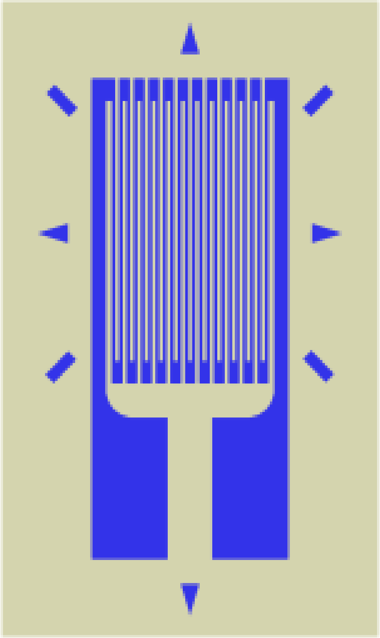
In two-dimensional space,

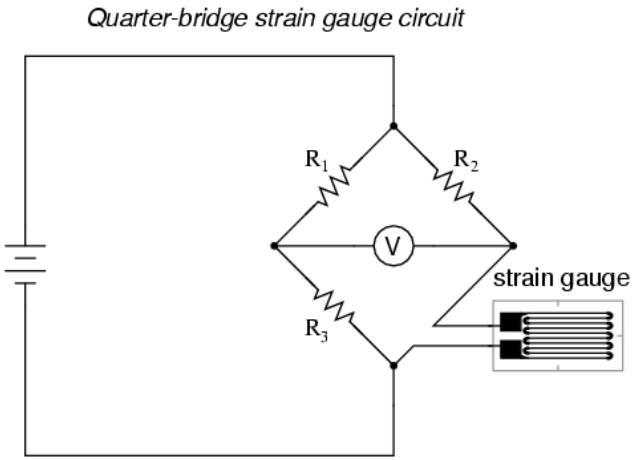
$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_{yy} \end{bmatrix}$$

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

How to measure strain ?

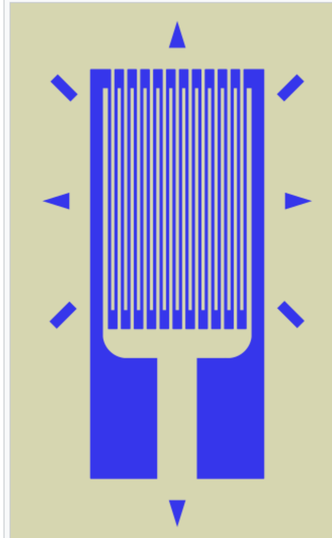






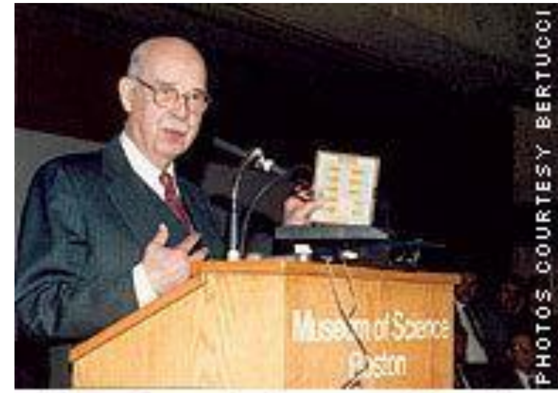
Ed Simmons and Peter K. Stein at the 61st Shock & Vibration Symposium, October 1990, Pasadena CA, where Ed was a featured and honored guest. Ed was wearing the Renaissance uniform which he often wore in his later years. Yes he was a genious and he was different! Photo by P. Stein

Edward E. Simmons, Jr. (1911–2004)



Typical foil strain gauge; the blue region is conductive and resistance is measured from one large blue pad to the other. The gauge is far more sensitive to strain in the vertical direction than in the horizontal direction. The markings outside the active area help to align the gauge during installation.

1938



Arthur Ruge wins Inventor of the Year Award in 1986.

Arthur Claude Ruge (1905—2000)