Last Time: Summary

$$\sigma_n(\theta) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta \tag{1}$$

$$\tau_n(\theta) = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \sigma_{xy}\cos 2\theta \tag{2}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + R, \qquad \tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}, \tag{*1}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - R, \quad \theta_{p2} = \theta_p + \frac{\pi}{2} \tag{*2}$$

$$\tau_n(\theta_p) = 0. \tag{*3}$$

$$\tau_{max} = R, \qquad \tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}}, \tag{*4}$$

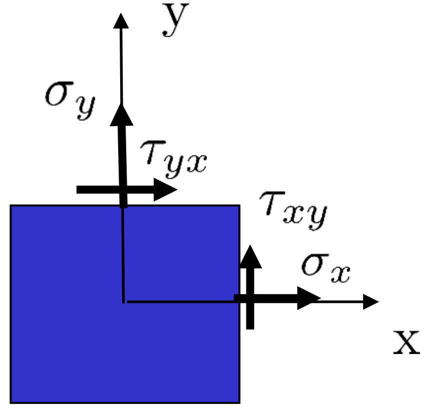
$$\tau_{min} = -R, \quad \theta_{s2} = \theta_s + \frac{\pi}{2}, \tag{*5}$$

$$\sigma_n(\theta_s) = \frac{\sigma_x + \sigma_y}{2} = \sigma_{ave}\ . \tag{*6}$$

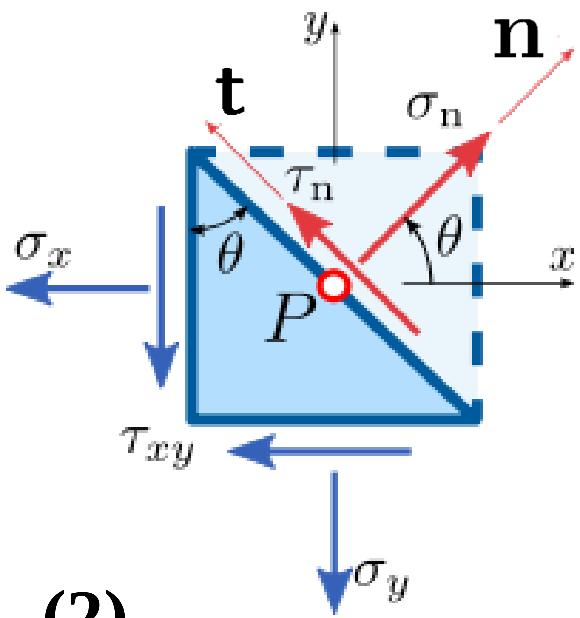
$$\theta_s = \theta_p \pm \frac{\pi}{4} \tag{*7}$$

Notethat $\sigma_1 + \sigma_2 = \sigma_x + \sigma_y$

Shear Stress Convention



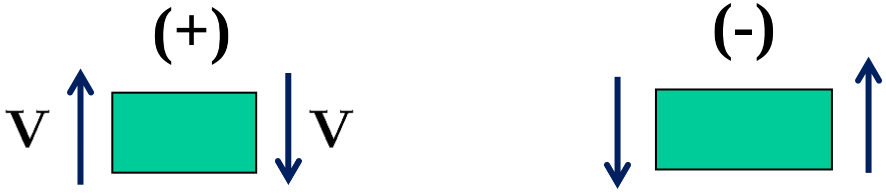
(1)

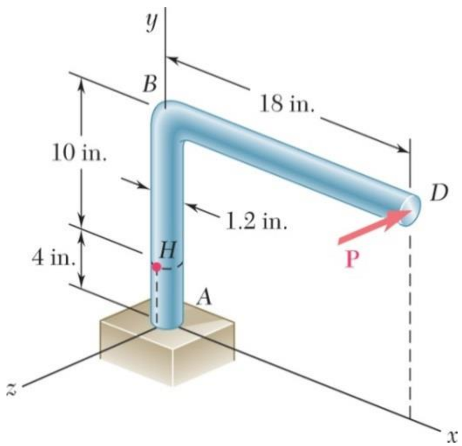


(2)

Shear force sign convention for beams

(3)





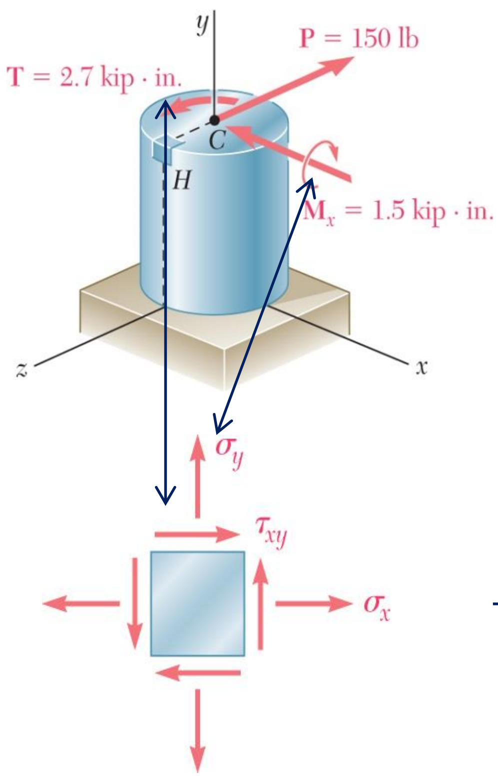
Example:
A single horizontal force *P* of 150 lb magnitude is applied to end D of lever *ABD*. **Determine (a) the normal and shearing stresses on an element at point *H* having sides parallel to the *x* and *y* axes, (b) the principal planes and principal stresses at the point *H***

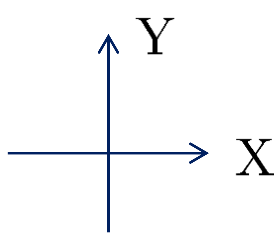
Determine an equivalent force-couple system at the center of the transverse section passing through *H*.

$$P = 150\,\text{lb}$$

$$T = (150\,\text{lb})(18\,\text{in}) = 2.7\,\text{kip}\cdot\text{in}$$

$$M_x = (150\,\text{lb})(10\,\text{in}) = 1.5\,\text{kip}\cdot\text{in}$$





- Evaluate the normal and shearing stresses at *H*.

$$\sigma_y = +\frac{Mc}{I} = +\frac{(1.5\,\text{kip}\cdot\text{in})(0.6\,\text{in})}{\frac{1}{4}\pi(0.6\,\text{in})^4}$$

$$\tau_{xy} = +\frac{Tc}{J} = +\frac{(2.7\,\text{kip}\cdot\text{in})(0.6\,\text{in})}{\frac{1}{2}\pi(0.6\,\text{in})^4}$$

$$\boxed{\sigma_x = 0 \quad \sigma_y = +8.84\,\text{ksi} \quad \tau_y = +7.96\,\text{ksi}}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(7.96)}{0 - 8.84} = -1.8$$

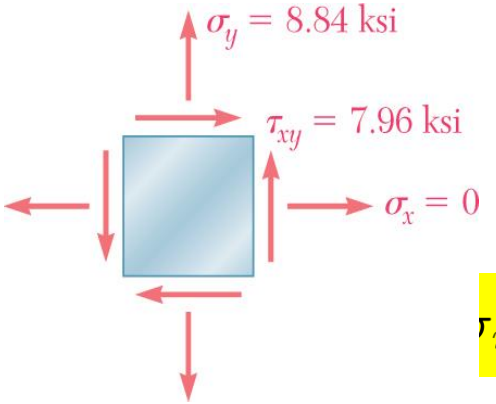
$$2\theta_p = -61.0°, 119°$$

You don't know which one is θ_{p1}.

$$2\theta_{p2} = 2\theta_{p1} + 180^o$$

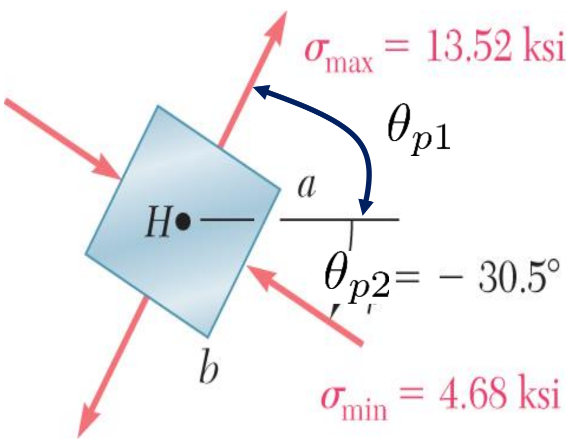
$$\boxed{\theta_p = -30.5°,\ 59.5°}$$

This is θ_{p1}.

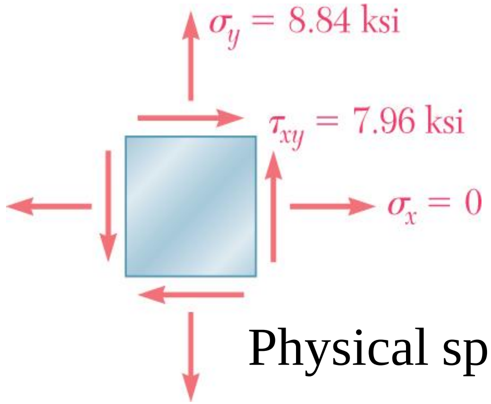


$$\sigma_n(\theta) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta \qquad (1)$$

Physical space



$$\boxed{\begin{aligned}\sigma_{max} &= +13.52\,\text{ksi}\\ \sigma_{min} &= -4.68\,\text{ksi}\end{aligned}}$$



Physical space

- Determine the maximum shear stress planes and calculate the maximum shear stresses.

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}} = \frac{8.84/2}{7.96} = 0.5556$$

$$2\theta_s = 29.06^o, -150.04^o$$

You don't know which one is θ_{s1}.

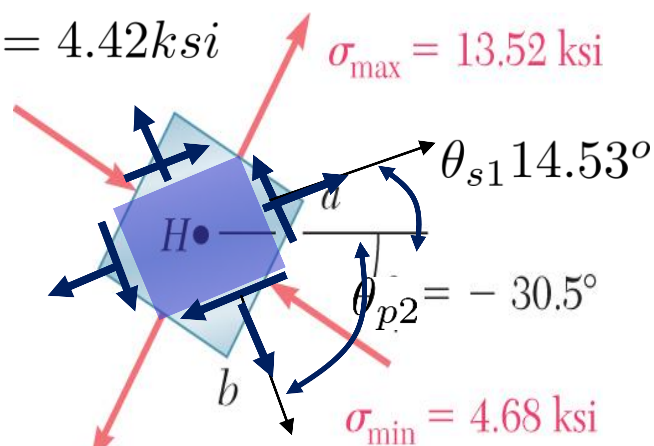
$$\theta_s = 14.53^o, -75.02^o$$

This is θ_{s1}.

$$\tau_{max,min} = \pm\sqrt{((\sigma_x - \sigma_y)/2)^2 + \tau_{xy}^2}$$

$$= \pm\sqrt{(8.84/2)^2 + 7.96^2} = \pm 9.07ksi$$

$\sigma_{ave} = 4.42ksi$

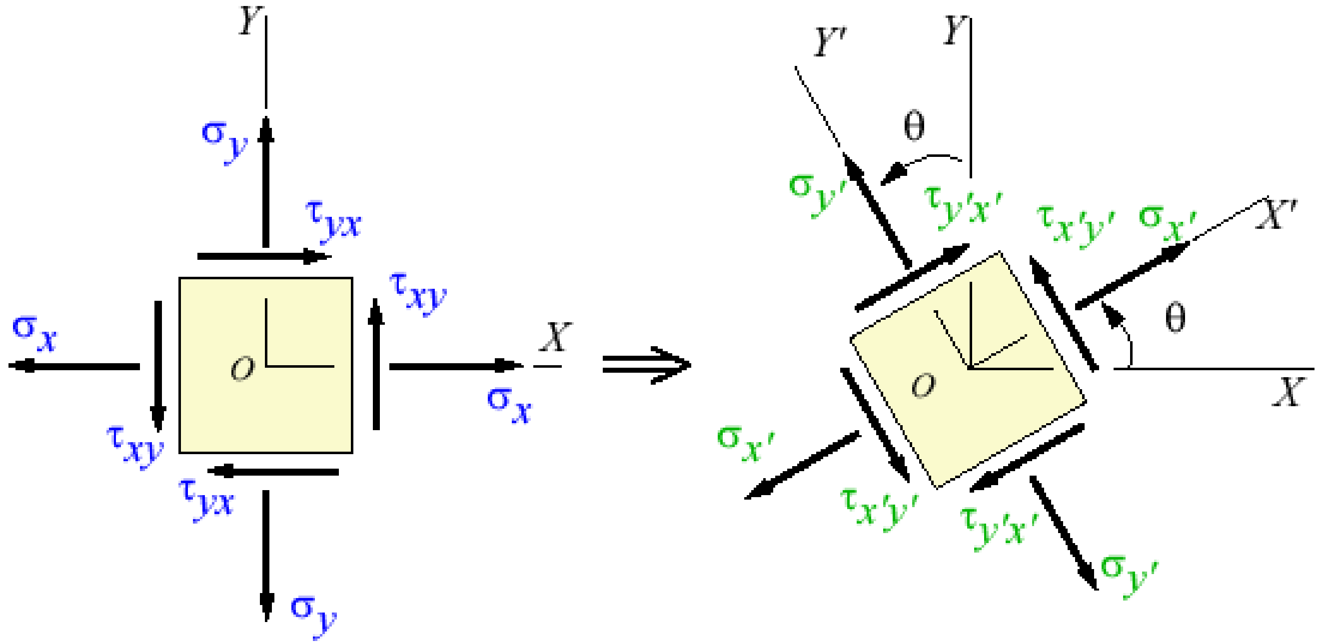


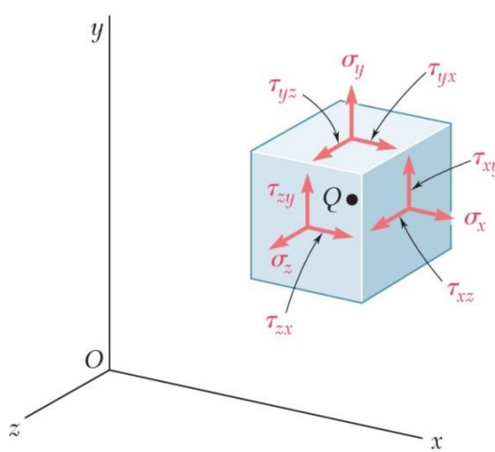
$$\tau_n(\theta) = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \sigma_{xy} \cos 2\theta$$

$$\tau_{max} = 9.07ksi$$

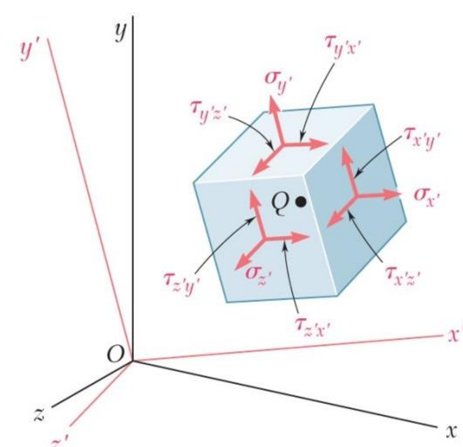
$\theta_{s2} = -75.02^o$

Lecture 34 Mohr's Circle (I)



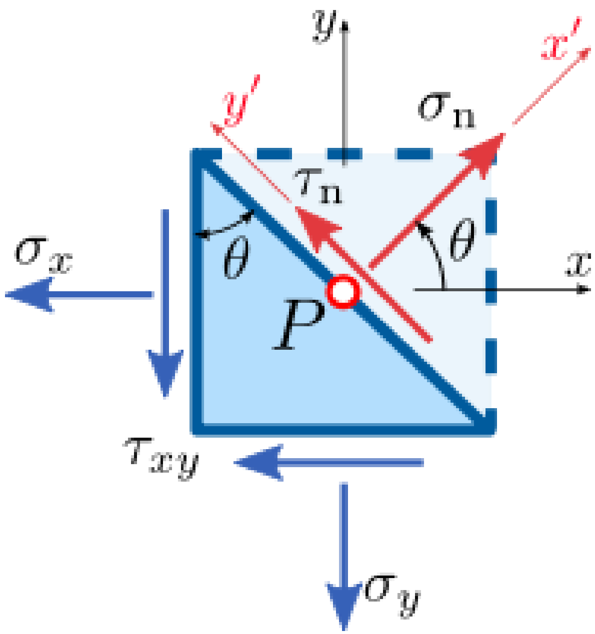


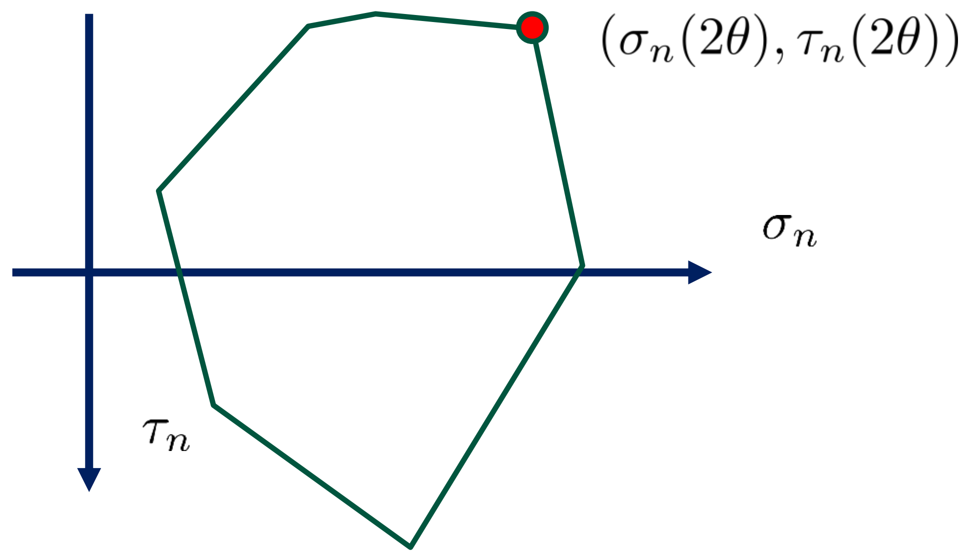
(a)



(b)

Let's come back to the Cauchy's stress transform formula again:





$$\sigma_n(\theta) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta \tag{1}$$

$$\tau_n(\theta) = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \sigma_{xy}\cos 2\theta \tag{2}$$

Is there a way that we can see these solutions for $0 \le 2\theta \le 2\pi$?

$$\sigma_n(\theta) - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \sigma_{xy} \sin 2\theta \tag{1}$$

$$\tau_n(\theta) = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \sigma_{xy} \cos 2\theta \tag{2}$$

Let:

$$X = \sigma_n(\theta), \; Y = \tau_n(\theta), \; c = \frac{\sigma_x + \sigma_y}{2}, \; b = \frac{\sigma_x - \sigma_y}{2}, \; a = \tau_{xy}, \; R = \sqrt{b^2 + a^2}$$

$$\begin{cases} X - c & = & b \cos 2\theta + a \sin 2\theta \\ Y & = & -b \sin 2\theta + a \cos 2\theta \end{cases}$$

Square the both sides of the above equations,

$$\begin{aligned} (X - c)^2 &= b^2 \cos^2 2\theta + 2ab \cos 2\theta \sin 2\theta + a^2 \sin^2 2\theta \\ Y^2 &= b^2 \sin^2 2\theta - 2ab \cos 2\theta \sin 2\theta + a^2 \cos^2 2\theta \ , \end{aligned}$$

Square the both sides of the above equations,

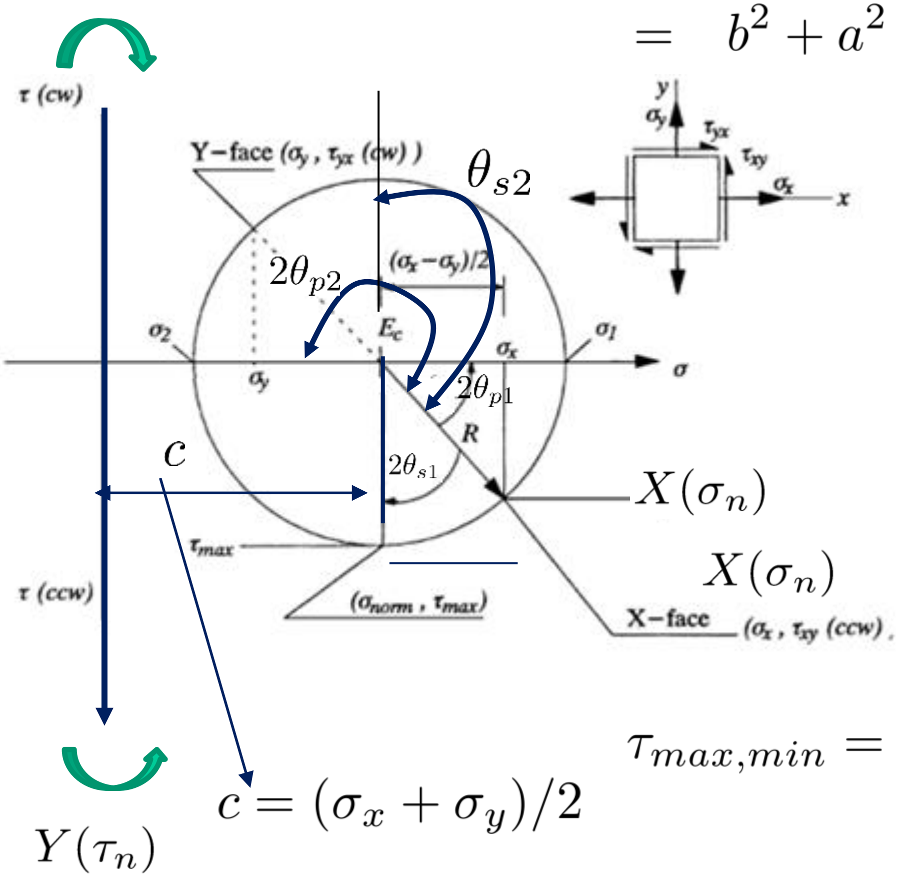
$$
\begin{aligned}
(X-c)^2 &= b^2\cos^2 2\theta + 2ab\cos 2\theta\sin 2\theta + a^2\sin^2 2\theta \\
Y^2 &= b^2\sin^2 2\theta - 2ab\cos 2\theta\sin 2\theta + a^2\cos^2 2\theta\ ,
\end{aligned}
$$

Add the two equations together,

$$
\begin{aligned}
(X-c)^2 + Y^2 &= b^2(\cos^2 2\theta + \sin^2 2\theta) + a^2(\cos^2 2\theta + \sin^2 2\theta) \\
&= b^2 + a^2 = R^2
\end{aligned}
$$

What kind equation this is ?

It is a circle in stress space, which is called Mohr's circle.



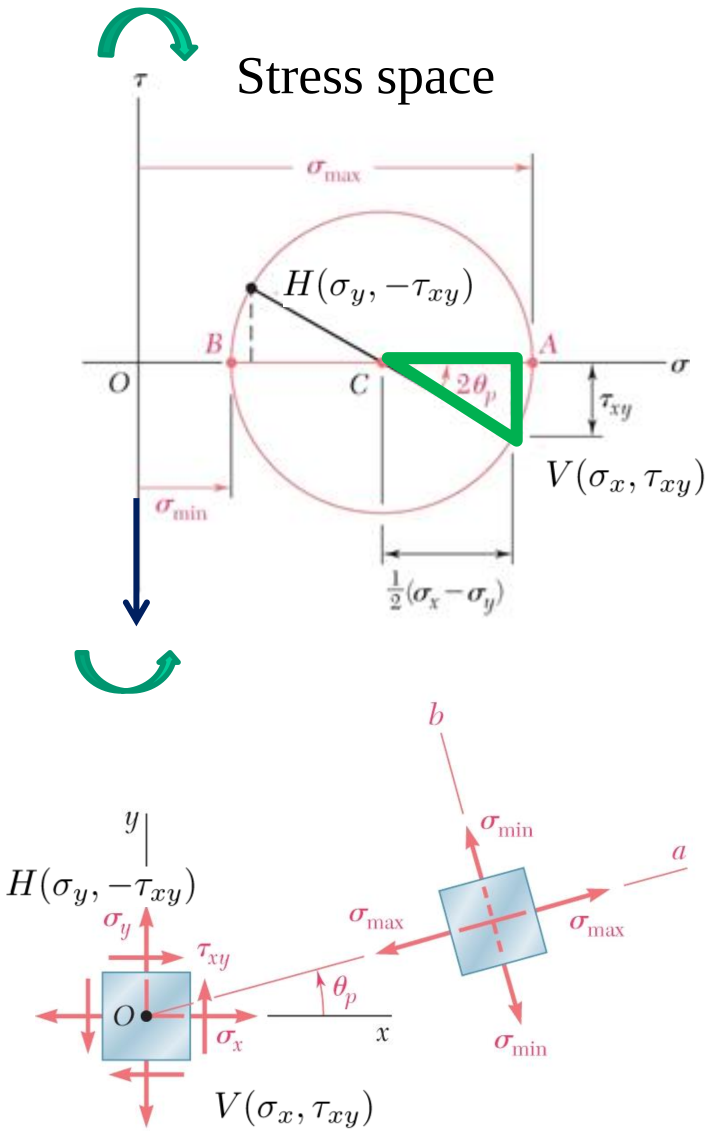
$$\tau_{max,min} = \pm R$$

$$\sigma_1 = c + R = \frac{\sigma_x + \sigma_y}{2} + R$$

$$\sigma_2 = c - R = \frac{\sigma_x + \sigma_y}{2} - R$$

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y = 2c$$

Stress space



Physical space

- The previous equations are combined to yield parametric equations for a circle,

$$(\sigma_n(\theta) - \sigma_{ave})^2 + \tau_n^2(\theta) = R^2$$

where

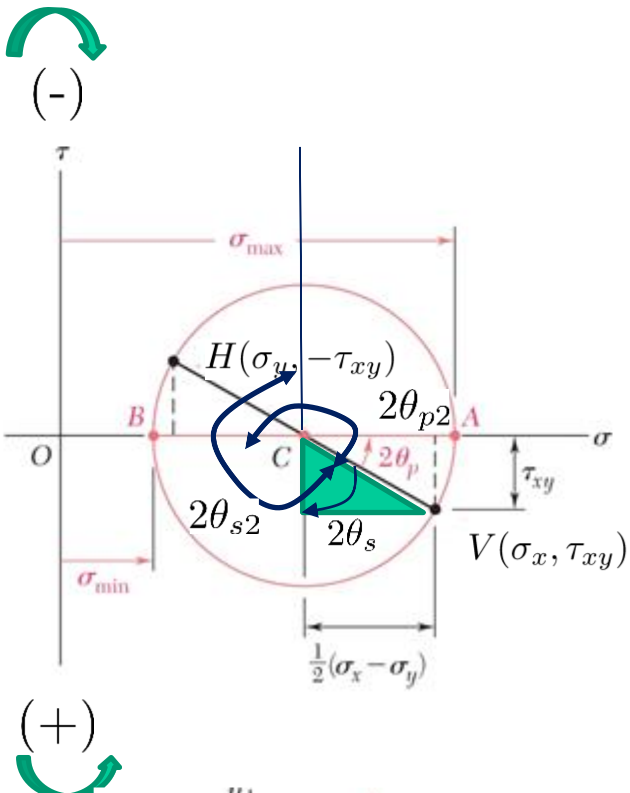
$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- *Principal stresses* occur on the *principal planes of stress* with zero shearing stresses.

$$\sigma_{\text{max,min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Note : defines two angles separated by 90°



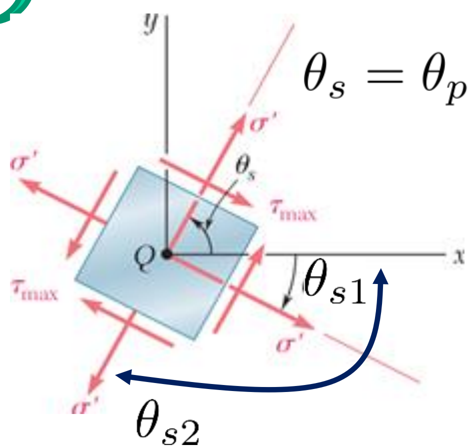
$$2\theta_{s2} = 2\theta_{s1} + \pi \;\rightarrow\; \theta_{s2} = \theta_{s1} + \pi/2$$

$$2\theta_{p2} = 2\theta_{p1} + \pi \;\rightarrow\; \theta_{p2} = \theta_{p1} + \pi/2$$

Maximum shearing stress occurs for $\sigma_{x'} = \sigma_{ave}$

$$\tau_{\max} = R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\theta_s = \theta_p \pm 45^o$$



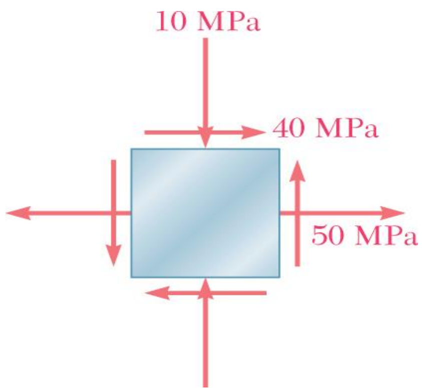
$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

Note : defines two angles separated by 90^{o} and offset from θ_p by 45^{o}

$$\sigma' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

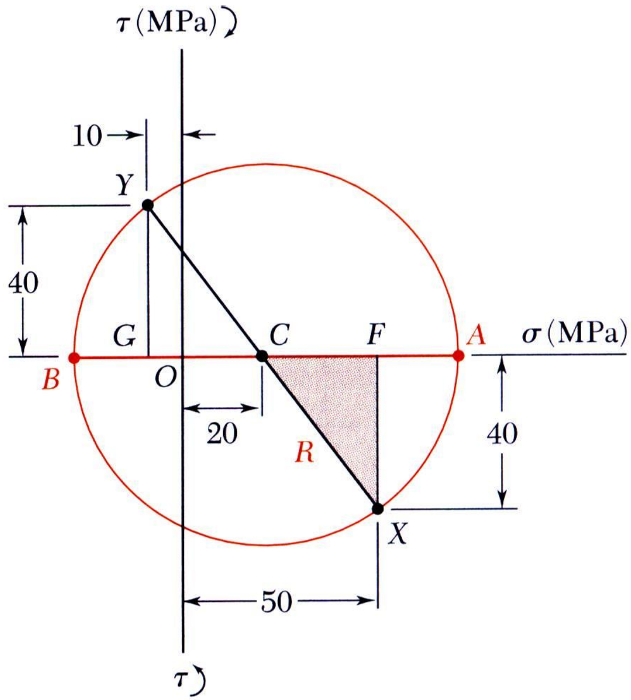
Example 1

$$V \to X \text{ and } H \to Y$$



$$X : (50, 40)$$

$$Y : (-10, -40)$$



For the state of plane stress shown,
(a) construct Mohr's circle, determine (b) the principal planes, (c) the principal stresses, (d) the maximum shearing stress and the corresponding normal stress.

$$CF = (\sigma_x - \sigma_y)/2$$

$$FX = \tau_{xy}$$

SOLUTION:

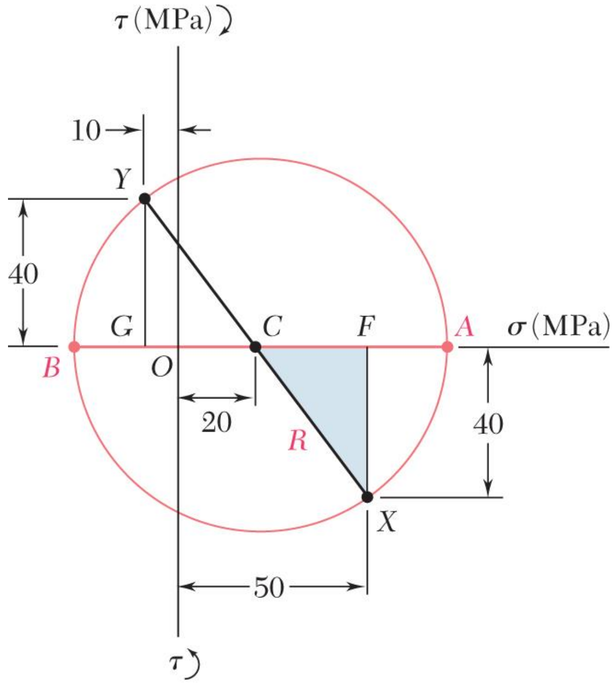
- Construction of Mohr's circle

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{(50)+(-10)}{2} = 20\,\text{MPa}$$

$$CF = 50 - 20 = 30\,\text{MPa} \qquad FX = 40\,\text{MPa}$$

$$R = CX = \sqrt{(30)^2 + (40)^2} = 50\,\text{MPa}$$

$$OC = \sigma_{ave} \qquad CA = BC = R$$



- Principal planes and stresses

$$\sigma_{\max} = OA = OC + CA = 20 + 50$$

$$\boxed{\sigma_{\max} = 70\,\text{MPa}}$$

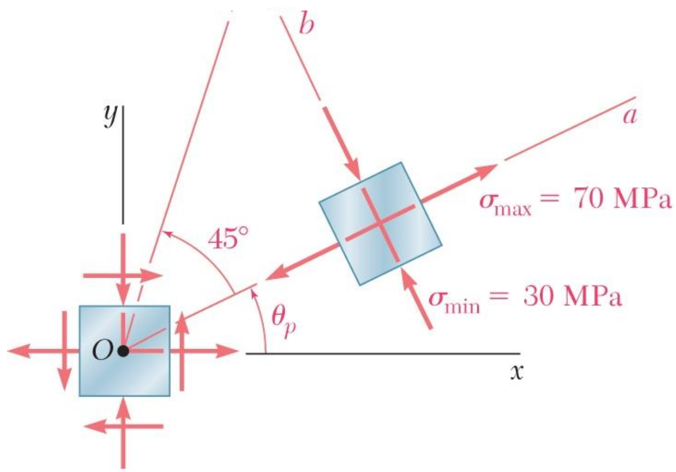
$$\sigma_{\min} = OB = OC - BC = 20 - 50$$

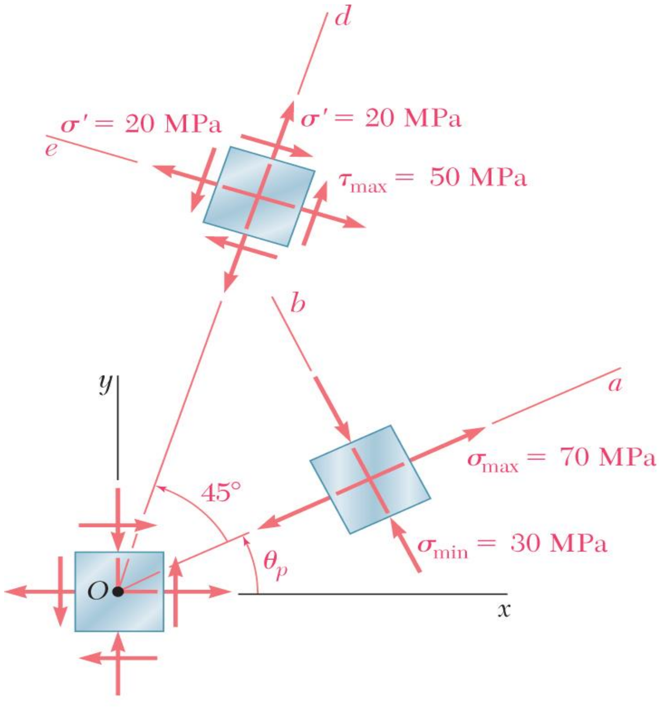
$$\boxed{\sigma_{\min} = -30\,\text{MPa}}$$

$$\tan 2\theta_p = \frac{FX}{CP} = \frac{40}{30}$$

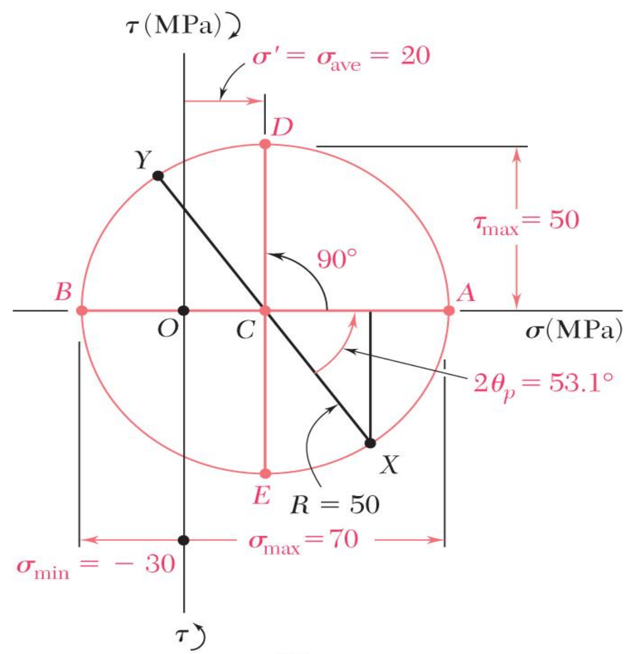
$$2\theta_p = 53.1^\circ$$

$$\boxed{\theta_p = 26.6^\circ}$$





(a)



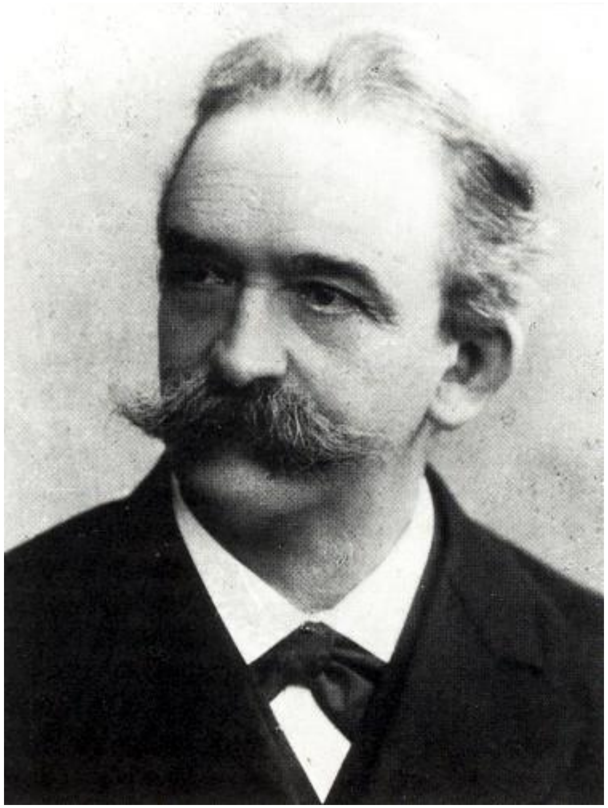
(b)

- Maximum shear stress

$$\theta_s = \theta_p + 45^\circ \qquad \tau_{\max} = R \qquad \sigma' = \sigma_{ave}$$

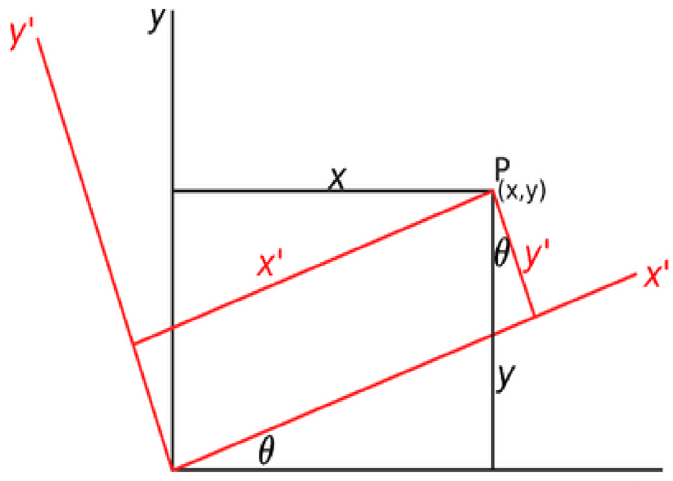
$$\boxed{\theta_s = 71.6^\circ} \qquad \boxed{\tau_{\max} = 50\,\text{MPa}} \qquad \boxed{\sigma' = 20\,\text{MPa}}$$

- *Christian Otto Mohr (October 8, 1835 - October 2, 1918) was a German civil engineer, one of the most celebrated of the nineteenth century.*
- *Starting in 1855, his early working life was spent in railroad engineering for the Hanover and Oldenburg state railways, designing some famous bridges and making some of the earliest uses of steel trusses.*



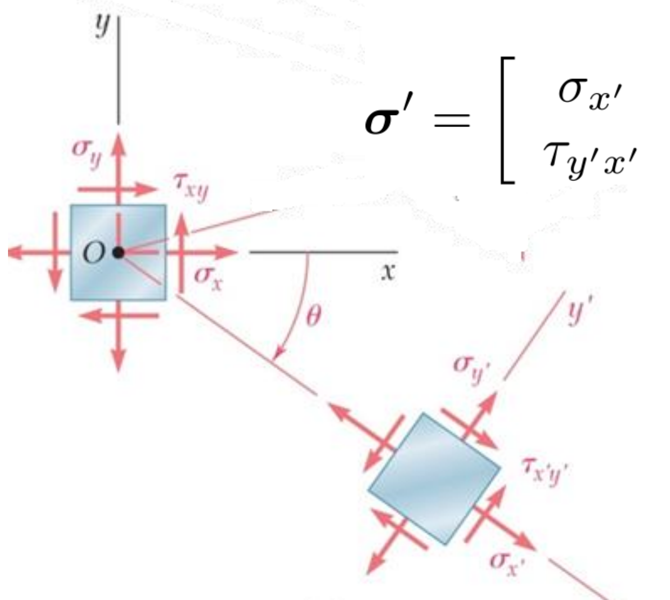
Christian Otto Mohr
(October 8, 1835 – October 2, 1918)

Variation in a theme: Stress Transformation

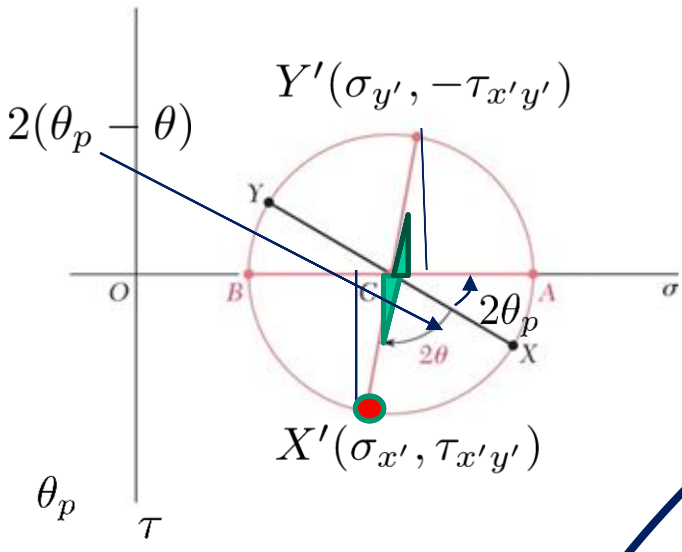


Now we ask:
If we use a different coordinate system, say $X'OY'$, which rotates an angle θ from the original coordinates, XOY, would the Cauchy stress change ?
i.e.

$$\boldsymbol{\sigma}' = \begin{bmatrix} \sigma_{x'} & \tau_{x'y'} \\ \tau_{y'x'} & \sigma_{y'} \end{bmatrix} =? = \boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

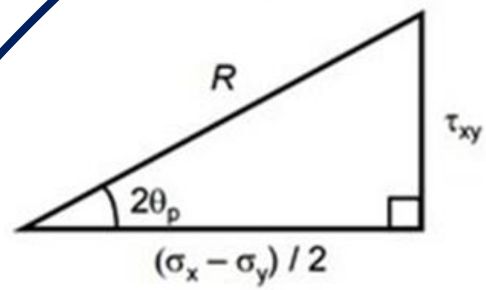


(a)



- For the state of stress at an angle θ with respect to the *xy* axes, construct a new diameter *X'Y'* at an angle 2θ with respect to *XY*.

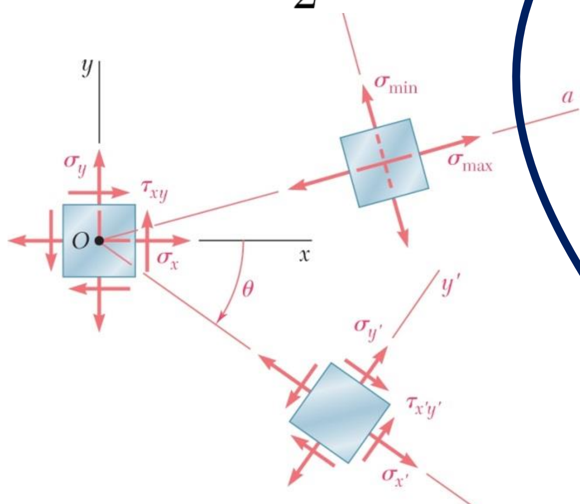
$$
\begin{aligned}
\sigma_{x'} &= c - R\cos(\pi - 2(\theta_p - \theta)) \\
\sigma_{y'} &= c + R\cos(\pi - 2(\theta_p - \theta)) \\
\tau_{x'y'} &= R\sin(\pi - 2(\theta_p - \theta))
\end{aligned}
$$



$$\sin 2\theta_p = \frac{\tau_{xy}}{R}$$

$$\cos 2\theta_p = \frac{\sigma_x - \sigma_y}{2R}$$

$$c = \frac{\sigma_x + \sigma_y}{2}$$



(a)

$$\cos(\pi - 2(\theta_p - \theta)) = -\cos 2(\theta_p - \theta)) = -(\cos 2\theta_p \cos 2\theta + \sin 2\theta_p \sin 2\theta)$$

$$\sin(\pi - 2(\theta_p - \theta)) = -\sin 2(\theta_p - \theta)) = -(\cos 2\theta_p \sin 2\theta - \sin 2\theta_p \cos 2\theta)$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (T1)$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \qquad (T2)$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (T3)$$

Stress Transformation Formula

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (T1)$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \qquad (T2)$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (T3)$$

Stress Invariants

However, $\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y = \sigma_1 + \sigma_2 = I_1$

Consider $\sigma_{x'} - \sigma_{y'} = (\sigma_x - \sigma_y)\cos 2\theta + 2\tau_{xy}\sin 2\theta$. Thus

$$\frac{\sigma_{x'} - \sigma_{y'}}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta,$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\rightarrow \quad \left(\frac{\sigma_{x'} - \sigma_{y'}}{2}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 = \left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 = R^2 = I_2$$

where $I_1 = \sigma_x + \sigma_y$ and $I_2 = R^2$ are called as stress invariants.

$$\text{Recall :} \quad \sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm R = \frac{1}{2} I_1 \pm R$$

and $\tau_{max,min} = \pm R$. These will never change with the coordinates.

What are invaraints of a stress tensor ?