and 11-5. The latter equations can readily be programmed for a computer. However, the graphical display of stress transformations using a Mohr's circle offers a comprehensive view of a solution and is useful in some applications. Two alternative techniques for achieving such solutions are given in what follows. The physical planes on which the transformed stresses act are clearly displayed in the first method; in the second, the derivation for stress transformation is simpler, although determining the direction of the transformed stress is a little less convenient. The choice of method is a matter of preference.

Method I The basic problem consists of constructing the circle of stress for given stresses σ_x, σ_y, and τ_{xy}, such as shown in Fig. 11-9(a), and then determining the state of stress on an *arbitrary* plane *a–a*. A procedure for determining the stresses on any inclined plane requires justification on the basis of the equations derived in Section 11-3.

According to Eq. 11-16, the center C of a Mohr's circle of stress is located on the σ axis at a distance $(\sigma_x + \sigma_y)/2$ from the origin. Point A on the circle has the coordinates (σ_x, τ_{xy}) corresponding to the *stresses acting on the right-hand face of the element* in the positive direction of the coordinate axes, Fig. 11-9(a). Point A will be referred to as the *origin of planes.* This information is sufficient to draw a circle of stress, Fig. 11-9(b).

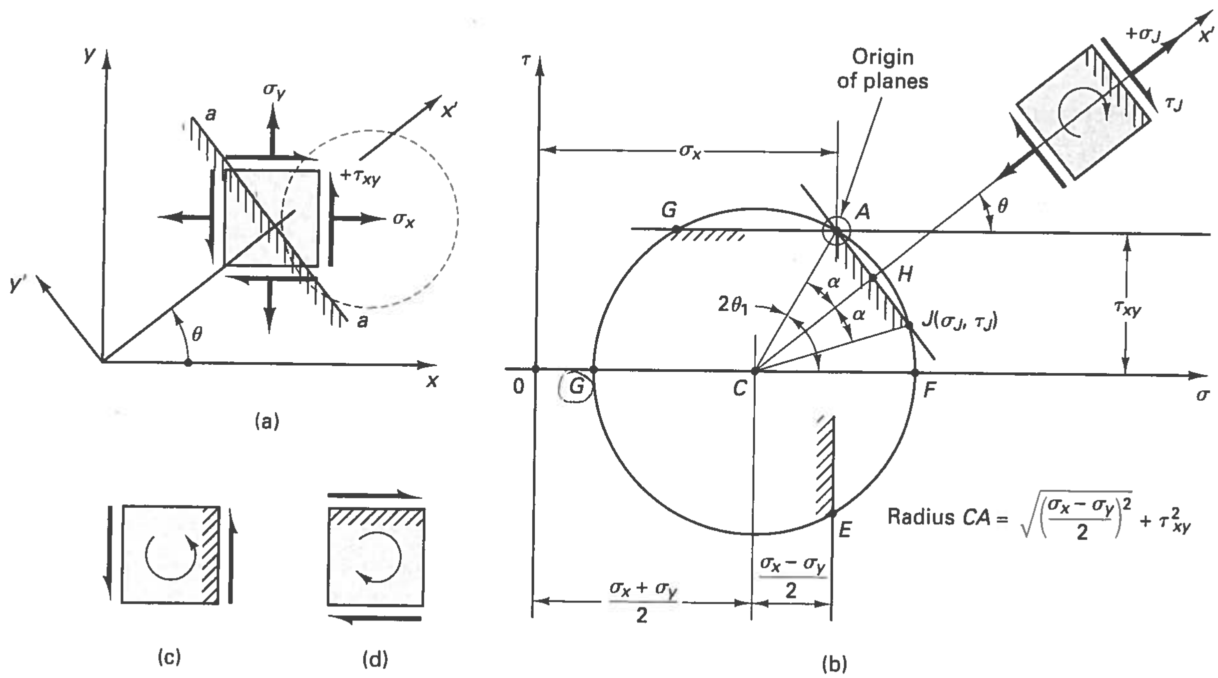

Fig. 11-9 Construction of Mohr's circle for determining stresses on an arbitrary plane.

The next step consists of drawing on the circle of stress a line through A parallel to plane a–a in the physical plane of Fig. 11-9(a). The intersection of this line at J with the stress circle gives the stresses acting on plane a–a. This requires some justification. For this purpose, the indicated geometric construction must be reviewed in detail.

According to the previous derivation shown in Fig. 11-8(c), angle ACF in Fig. 11-9(b) is equal to $2\theta_1$. Further, since line CH is drawn perpendicular to line AJ, angle ACJ is bisected, and $\alpha = 2\theta_1 - \theta$. Hence, angle JCF is $\theta - \alpha = 2\theta - 2\theta_1$, and it remains to be shown that the coordinates of point J define the stresses acting on inclined plane a–a. For this purpose, one notes from Fig. 11-9(b) that if R is the radius of a circle, $R \cos 2\theta_1 = (\sigma_x - \sigma_y)/2$ and $R \sin 2\theta_1 = \tau_{xy}$. Then, forming expressions for the normal and shear stresses at J based on the construction of the circle in Fig. 11-9(b) and making use of trigonometric identities for double angles, one has

$$\begin{aligned}\sigma_J &= \frac{\sigma_x + \sigma_y}{2} + R\cos(2\theta - 2\theta_1)\\ &= \frac{\sigma_x + \sigma_y}{2} + R\,(\cos 2\theta \cos 2\theta_1 + \sin 2\theta \sin 2\theta_1) \qquad (11\text{-}18)\\ &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\end{aligned}$$

and

$$\begin{aligned}\tau_J &= R\sin(2\theta - 2\theta_1) = R\sin 2\theta\cos 2\theta_1 - R\cos 2\theta \sin 2\theta_1\\ &= +\frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta \qquad (11\text{-}19)\end{aligned}$$

Except for the sign of τ_J, the last expressions are identical to Eqs. 11-1 and 11-2 and, therefore, define the stresses acting on the element shown in the upper right quadrant of Fig. 11-9(b). The hatched side of this element is parallel to line AJ on the stress circle, which is parallel to line a–a in Fig. 11-9(a). However, since the sign of τ_J is opposite to that in the basic transformation, Eq. 11-2, a special rule for the direction of shear stress has to be introduced.

For this purpose, consider the initial data for the element shown in Fig. 11-9(a), where all stresses are shown with positive sense. By isolating the shear stresses acting on the vertical faces, Fig. 11-9(c), it can be seen that *these stresses alone* cause a *counterclockwise* couple. By considering lines emanating from the origin of planes A, for the first case, Fig. 11-9(c), the circle is intersected at E, whereas for the second case, Fig. 11-9(d), it is intersected at G. This can be generalized into a rule: If the point of intersection of a line emanating from the origin of planes A intersects the circle *above* the σ axis, the shear stresses on the opposite sides of an element cause a *clockwise* couple. Conversely, if the point of intersection lies *below* the σ axis, the shear stresses on the opposite sides cause a *counterclockwise*

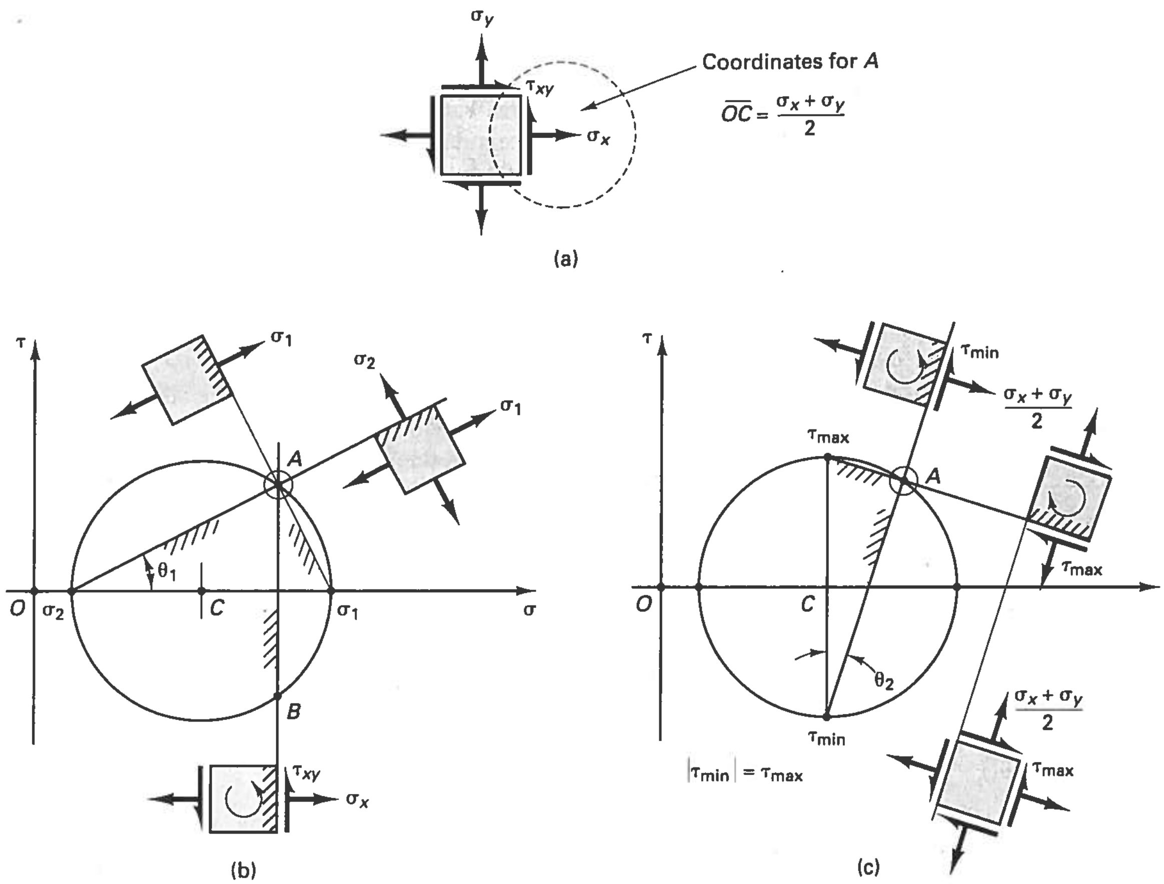

Fig. 11-10 Determining principal normal and maximal shear stresses.

couple. According to this rule, the shear stresses at J in Fig. 11-9(b) act with a clockwise sense.

This general procedure is illustrated for two particularly important cases. For the data given in Fig. 11-10(a), the principal stresses are found in Fig. 11-10(b), and the maximum shear stresses are found in Fig. 11-10(c). For the first case, it is known that the extreme values on the abscissa, σ_1 and σ_2, give the principal stresses. Connecting these points with the origin of planes A locates the planes on which these stresses act. Angle θ_1 can be determined by trigonometry. Either one of the two solutions is sufficient to obtain the complete solution shown on the element on the right.

The magnitudes of the maximum absolute shear stresses are known to be given by the radius of the Mohr's circle. As shown in Fig. 11-10(c), these stresses are located above and below C. Connecting these points with the origin of planes A determines the planes on which these stresses act. The corresponding elements are shown in the upper two diagrams of the elements, where the associated mean normal stresses are also indicated. Either one of these solutions with the aid of equilibrium concepts is sufficient for the complete solution shown on the bottom element in the figure.

Method 2 The state of stress in the xy coordinate system is shown in Fig. 11-11(a). The origin for these coordinates is arbitrarily chosen at the center of the infinitesimal element. The objective is to transform the given stresses to those in the rotated set of $x'y'$ axes as shown in Figs. 11-11(a) and (b) by using Mohr's circle.

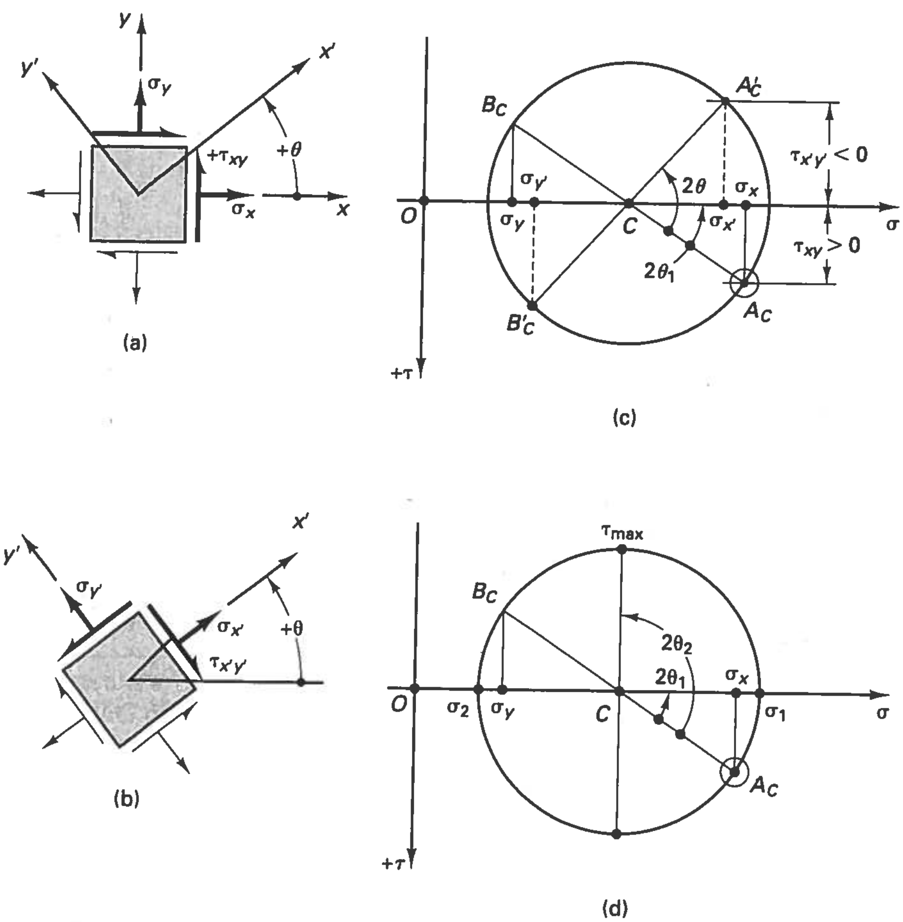

Fig. 11-11 Alternative construction of Mohr's circle of stress. Stresses on arbitrary and principal normal and shear planes are shown in (c) and (d), respectively.

As before, the center C of the Mohr's circle is located at $(\sigma_x + \sigma_y)/2$. Again, the right-hand face of the element defines σ_x and τ_{xy} used to locate a point on the circle. However,

if $\tau_{xy} > 0$, it is plotted *downward* at σ_x, and
if $\tau_{xy} < 0$, it is plotted *upward* at σ_x.

This, in effect, amounts to directing the positive τ axis downward, and is so shown in Fig. 11-11(c). The coordinates of σ_x and τ_{xy} locate the governing point A_C on the circle. This point corresponds to point A in the earlier construction; see Fig. 11-8. However, because of the opposite directions of the positive τ axes, whereas points A and A_C are related, they are not the same. Point B_C, conjugate to point A_C, can be located on the circle as shown in Fig. 11-11(c). The double angle 2θ follows from geometry.

Next the diameter $A_C B_C$ is rotated through an angle 2θ in the *same sense* that the x' axis is rotated through the angle θ with respect to the x axis. Then the new point A'_C determines the stresses $\sigma_{x'}$ and $\tau_{x'y'}$ on the right-hand face of the element in Fig. 11-11(b). Note that for the case shown, the shear stress $\tau_{x'y'}$ is negative, since at $\sigma_{x'}$ it is above the σ axis. Similar considerations apply to the conjugate point B'_C defining the stresses on the plane normal to the y' axis.

The expressions for $\sigma_{x'}$ and $\tau_{x'y'}$ can be formulated from the construction of the Mohr's circle shown in Fig. 11-11(c) using Eqs. 11-16 and 11-17. After simplifications, these relations, except for the sign of $\tau_{x'y'}$, reduce to the basic stress transformation relations, Eqs. 11-1 and 11-2. Hence this construction of the Mohr's circle is justified. For proof, modify Eqs. 11-18 and 11-19.

The procedure for determining the principal normal stress is shown in Fig. 11-11(d). After drawing a Mohr's circle, the principal stresses σ_1 and σ_2 are known. The required rotation θ_1 of the axes in the direction of these stresses is obtained by calculating the double angle $2\theta_1$ from the diagram. Similarly, the principal shear stresses are given by the coordinates of the points on a circle at their extreme values on the τ axis. The required rotation θ_2 of these axes is obtained by calculating the double angle $2\theta_2$ from the diagram.

Method 1 is used in the two examples that follow.

Example 11-3

Given the state of stress shown in Fig. 11-12(a), transform it (a) into the principal stresses, and (b) into the maximum shear stresses and the associated normal stresses. Show the results for both cases on properly oriented elements. Use Method I.

SOLUTION

To construct Mohr's circle of stress, the following quantities are required:

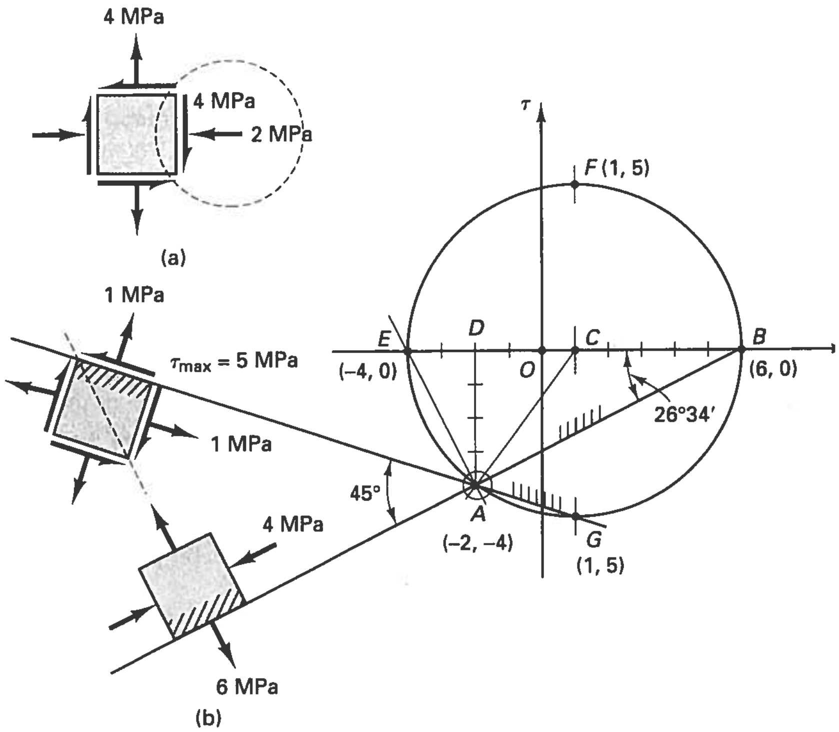

Fig. 11-12

1. Center of circle on the σ axis: $(-2 + 4)/2 = +1\text{MPa}$
2. Origin of planes A from data on the right face of element: $(-2, -4)$ MPa
3. Radius of circle: $CA = \sqrt{CD^2 + DA^2} = 5$ MPa

After drawing the circle, one obtains $\sigma_1 = +6$ MPa, $\sigma_2 = -4$ MPa, and $\tau_{max} = 5$ MPa.

Line AB on the stress circle locates the principal plane for $\sigma_1 = 6$ MPa. The angle θ_1 is 26°34′, since $\tan \theta_1 = AD/DB = 4/8 = 0.5$. The other principal stress, $\sigma_2 = -4$ MPa, acts at right angle to the aforementioned plane. These results are shown on a properly oriented element.

Line AG on the circle at 45° with the principal planes determines the planes for maximum shear, $\tau_{max} = 5$ MPa, and the associated mean normal stress $\sigma' = 1$ MPa. The latter stress corresponds to σ at the circle center. Complete results are shown on a properly oriented element.

It is worthy to note that the directions of the principal stresses can be anticipated and can be used in calculations as a check. A suitable inspection procedure is shown in Fig. 11-13. To begin with, it is known that tensile stresses of equal magnitude to the shear stress develop along a diagonal, as shown in Fig. 11-6. Therefore, the maximum tensile stress σ_1, which is the

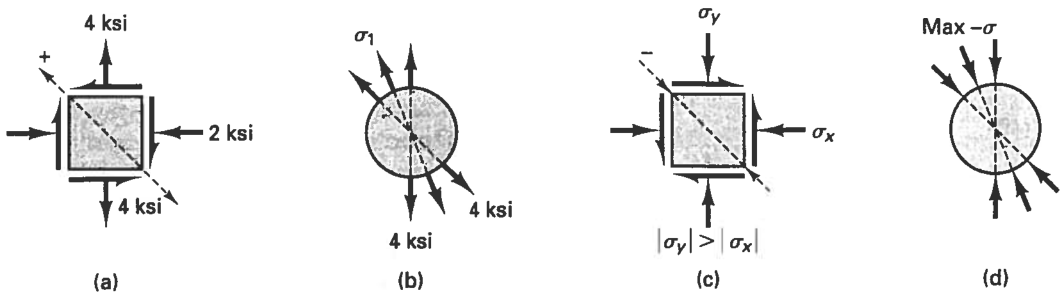

Fig. 11-13 Qualitative estimates of directions for principal stresses.

result of all stresses, must act as shown in Fig. 11-13(b). Situations with compressive stresses can be treated similarly, Fig. 11-13(d).

Example 11-4

Using Mohr's circle, transform the stresses shown in Fig. 11-14(a) into stresses acting on the plane at an angle of $22\frac{1}{2}°$ with the vertical axis. Use Method I.

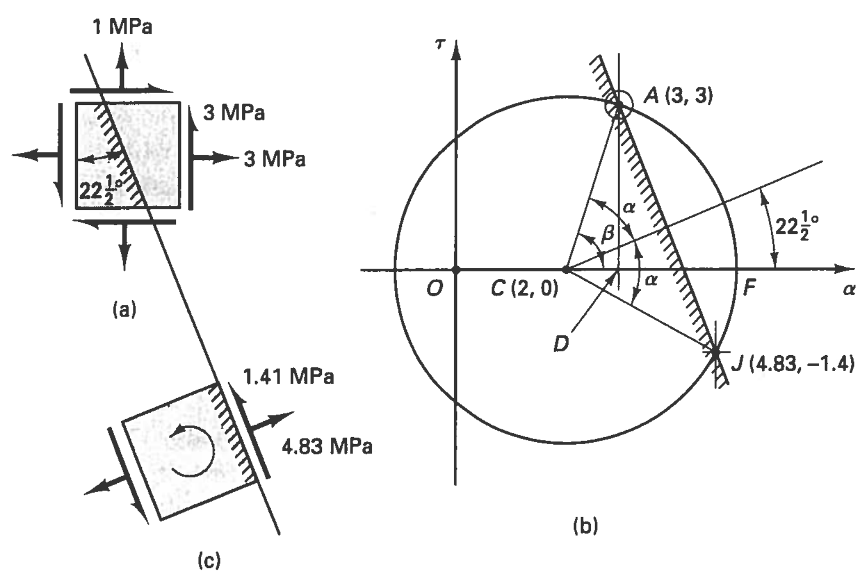

Fig. 11-14

SOLUTION

For this case, the center of the circle is at $(1 + 3)/2 = +2$ MPa on the σ axis. The origin of planes A is at (3,3), and the radius $R = \sqrt{1^2 + 3^2} = 3.16$ MPa. By using these data, a stress circle is plotted in Fig. 11-14(b) on which an inclined line at 22.5° locating point J is drawn.

Angle β is 71.57°, since $\tan \beta = AD/CD = 3$. A normal to AJ forms an angle of 22.5° with the σ axis. Therefore, $\alpha = 71.57° - 22.5° = 49.07°$, and angle FCJ is $\alpha - 22.5° = 26.57°$. This locates J on the circle. Hence, $\sigma_J = 2 + R\cos(-26.57°) = 2 + 3.16(0.894) = 4.83$ MPa, and $\tau_J = R \sin(-26.57°) = 3.16(-0.447) = -1.41$ MPa.

These results are shown on a properly oriented element in Fig. 11-14(c). Since τ_J is negative, the shear stresses are shown acting counterclockwise.

Again it should be remarked that the equations for stress transformation are identical in form to the equations for determining the principal axes and moments of inertia of areas (Section 9-3). Therefore, Mohr's circle can be constructed for finding these equations.[4]

11-8. Principal Stresses for a General State of Stress[5]

Traditionally, in an introductory text on solid mechanics, attention is largely confined to stresses in two dimensions. Since, however, the physical elements studied are always three dimensional, for completeness, it is desirable to consider the consequences of three dimensionality on stress transformations. The concepts developed in this section have an impact on the discussion that follows in this chapter, as well as on some issues considered in the next chapter.

Consider a general state of stress and define an infinitesimal tetrahedron[6] as shown in Fig. 11-15(a). Instead of considering an inclined plane in the xy coordinate system, as before for a wedge, the unknown stresses are sought on an arbitrary oblique plane ABC in the three-dimensional xyz coordinate system. A set of known stresses on the other three faces of the mutually perpendicular planes of the tetrahedron is given. These stresses are the same as shown earlier in Fig. 1-3(a).

A unit normal **n** to the oblique plane defines its orientation. This unit vector is identified by its direction cosines l, m, and n, where $\cos \alpha = l$, $\cos \beta = m$, and $\cos \gamma = n$. The meaning of these quantities is illustrated in

[4]See J. L. Meriam and L. G. Kraige, *Statics*, 2nd ed. (New York: Wiley, 1986).

[5]This section is more advanced and can be omitted.

[6]A tetrahedron was first introduced in the study of stress transformations by the great French mathematician A. L. Cauchy in the 1820s.