Lecture 36 Introduction to Stability

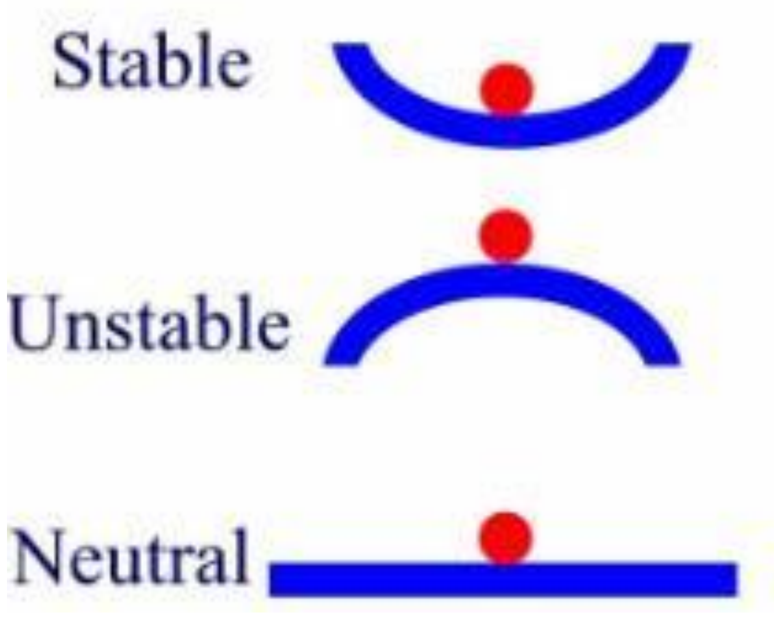
Definition (Webster's Dictionary)

Stability of a (mechanical) system is the capacity of the system to develop forces, and moments to maintain, or to restore, the system's original eqilibrium state when it is disturbed.

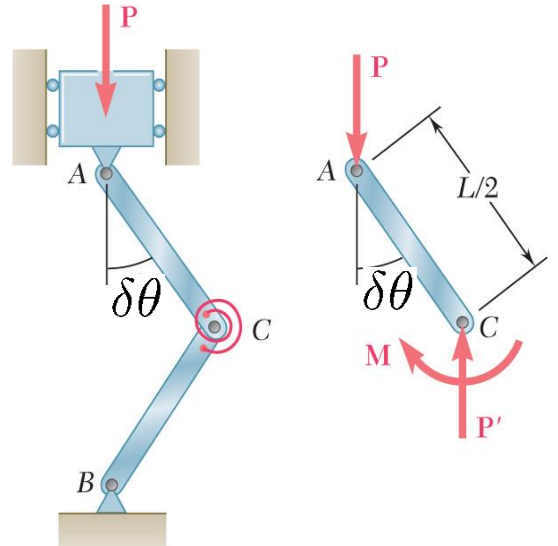
There are several elements in this definition:

- **Equilibrium State** Stabibility is referred to a system's capacity to maintain an equilibrium state.
  (**Stability of equilibrium**)
- **Disturbance** Stability is measured by a system's ability to resist disturbance;
- **Force and Moments** The system's response to disturbance;
- **Assessment of stability** Whether or not the original equilibrium is being restored.

Types of equilibrium states



Marble Analogy



(a) (b)

- Consider two rods and a torsional spring. After a small perturbation,

$$K(2\delta\theta) = \text{restoring moment}$$

$$P\frac{L}{2}\sin\delta\theta = P\frac{L}{2}\delta\theta = \text{destabilizing moment}$$

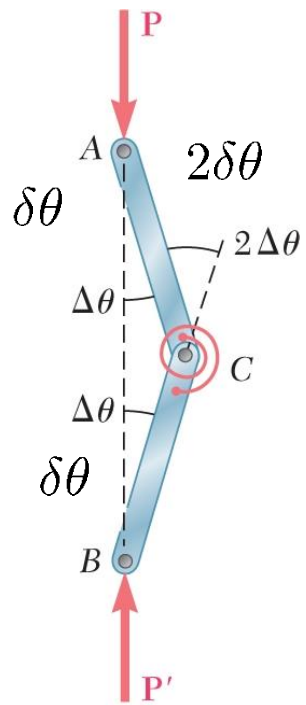
- If the restoring moment is greater than the destabilizing moment, the column is stable (tends to return to original equilibrium)

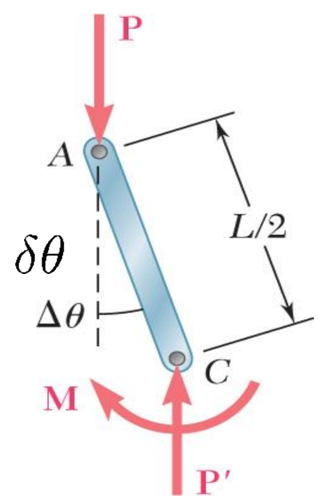
$$P\frac{L}{2}\delta\theta < K(2\delta\theta) \;\Rightarrow P < P_{cr} = \frac{4K}{L}$$

- Column is unstable (tends to deviate from the original equilibrium) if

$$P\frac{L}{2}\delta\theta > K(2\delta\theta) \;\Rightarrow P > P_{cr} = \frac{4K}{L}$$

- If the restoring moment equals the destabilizing moment, this is the critical point

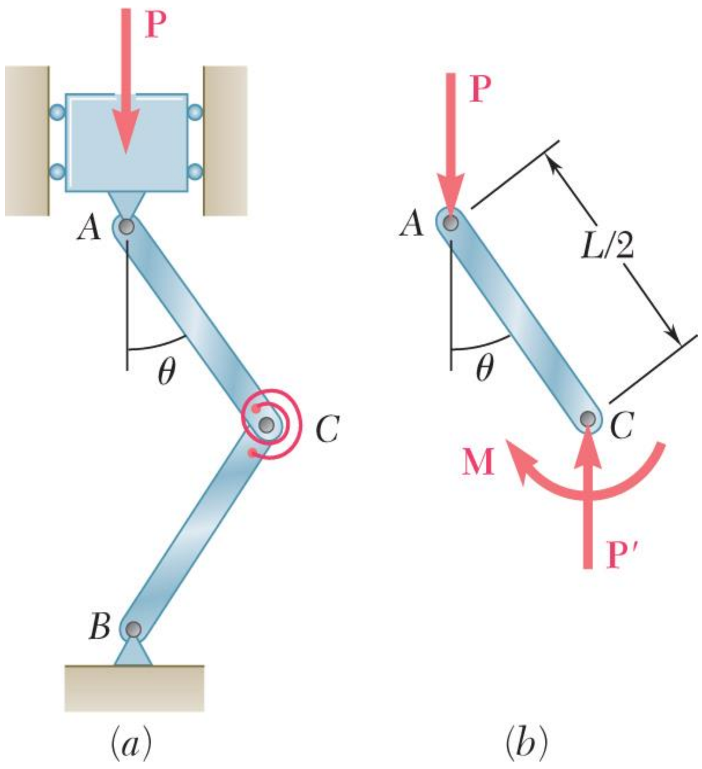




For the equilibrium position $\theta = 0$.

$$P = \frac{4K}{L} = P_{cr}$$

Other equilibrium position at finite θbut Not $\theta = 0$.



- Assume that a load P is applied. After a perturbation, the system settles to a new equilibrium configuration at a finite deflection angle.

$$P\frac{L}{2}\sin\theta = K(2\theta)$$

$$\frac{PL}{4K} = \frac{P}{P_{cr}} = \frac{\theta}{\sin\theta}$$

- Noting that $sin\theta < \theta$, the assumed configuration is only possible if $P > P_{cr}$.

Stability Phase Diagram

$$\frac{P}{P_{cr}} = \frac{\theta}{\sin\theta}$$

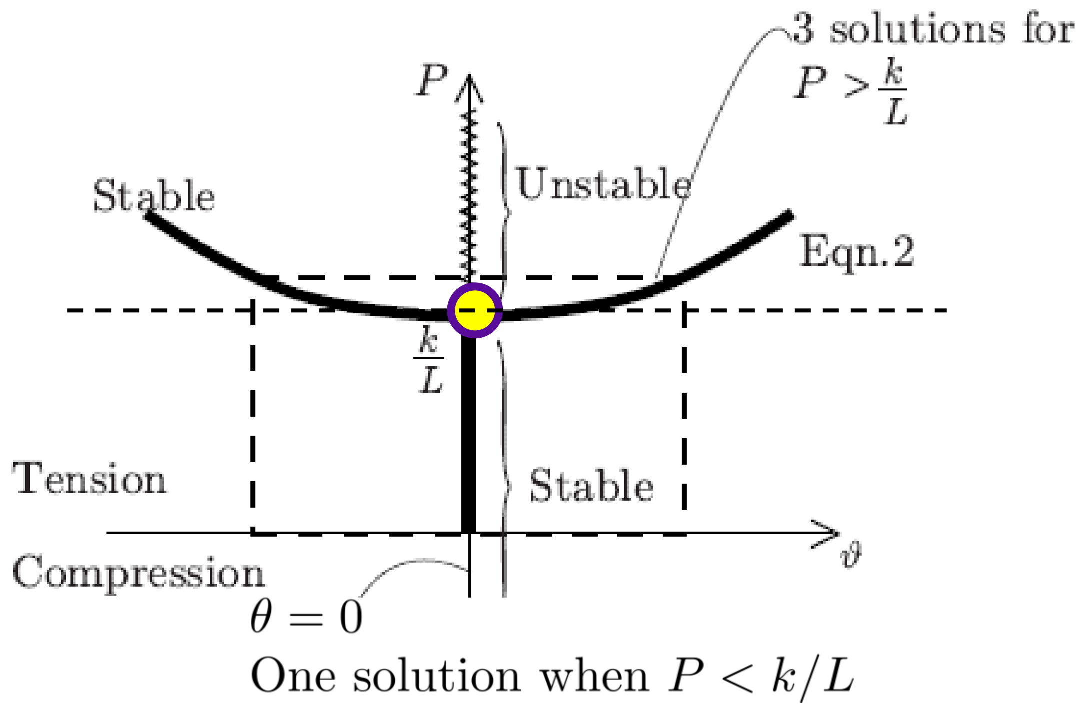
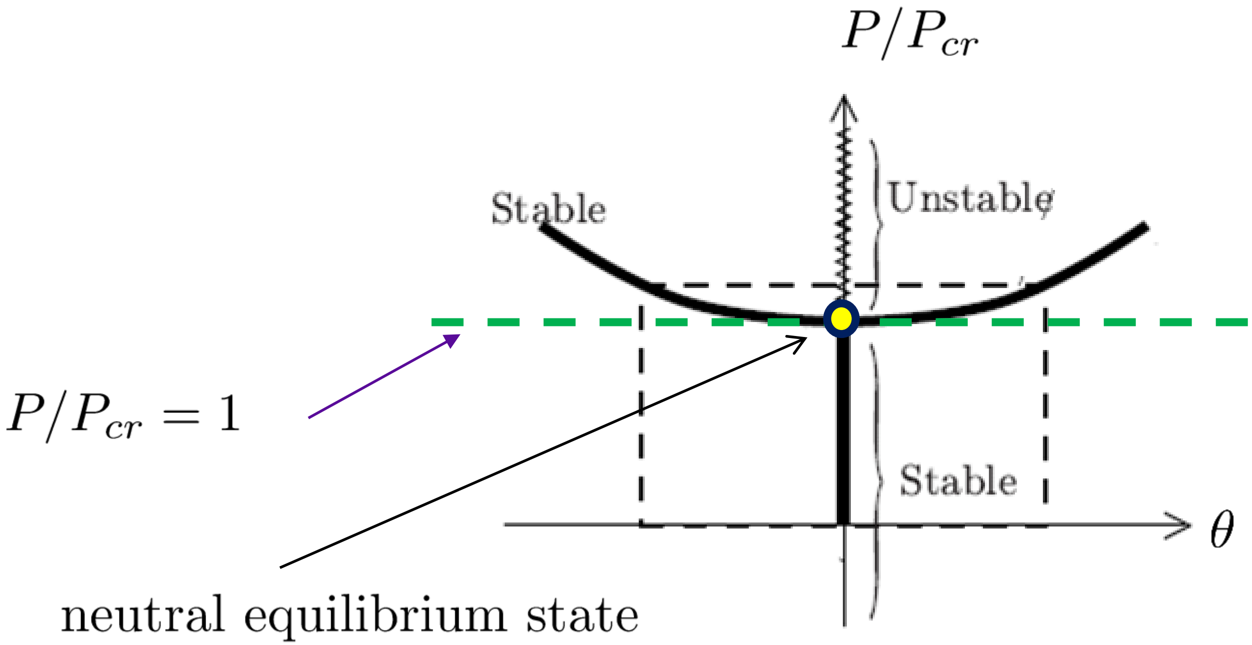


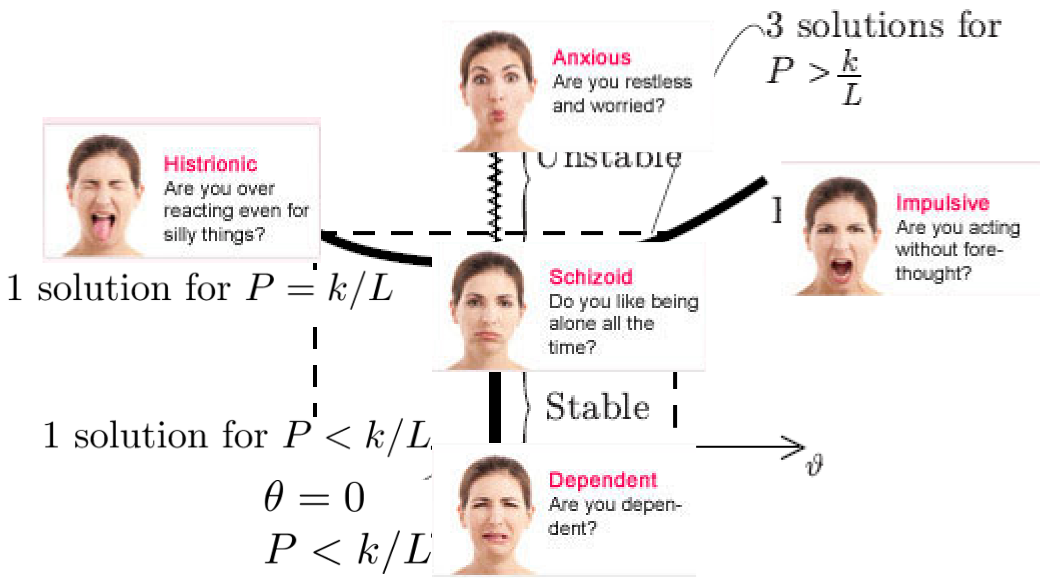
Figure 2: Solutions θ for varying loads P

Neutral equilibrium is the boundary between the stable and unstable equilibrium.



$$\textbf{For the initial equilibrium state} \begin{cases} \textbf{Stable} & P < P_{cr} \\ \textbf{Neutral} & P = P_{cr} \\ \textbf{Unstable} & P > P_{cr} \end{cases}$$

Since the neutral equilibrium state separated the state and unstate equilibrium states, the objective of the stability analysis is to find $P = P_{cr}$, i.e. the condition for neutral equlibrium state.



Solution is not unique when $P = k/L$!

Remarks

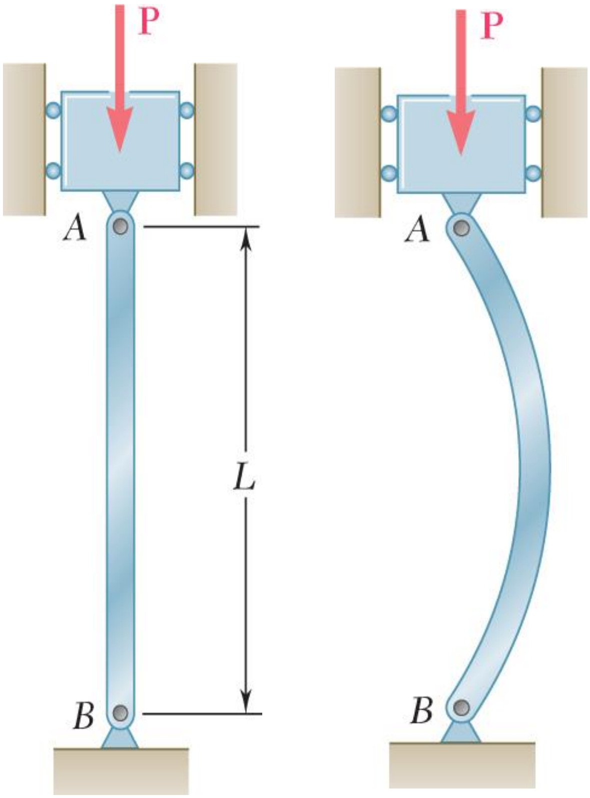
1. $\delta\theta$ is an imaginary perturbation, so it is called
"**The Virtual Displacement**".

2. For a fixed structure, when load parameter, or load vaule, changes, the structure can change its status from a stable structure to a neutral equilibrium structure, and then to a unstable structure.

3. The load parameter corresponds to the neutral equilibrium state is called "**The Critical Load**";

4. The objective of the structual stability analysis is to find the critical load for designated original equlibrium configuration.

Stability (Buckling) of Structures



- In the design of columns, cross-sectional area is selected such that

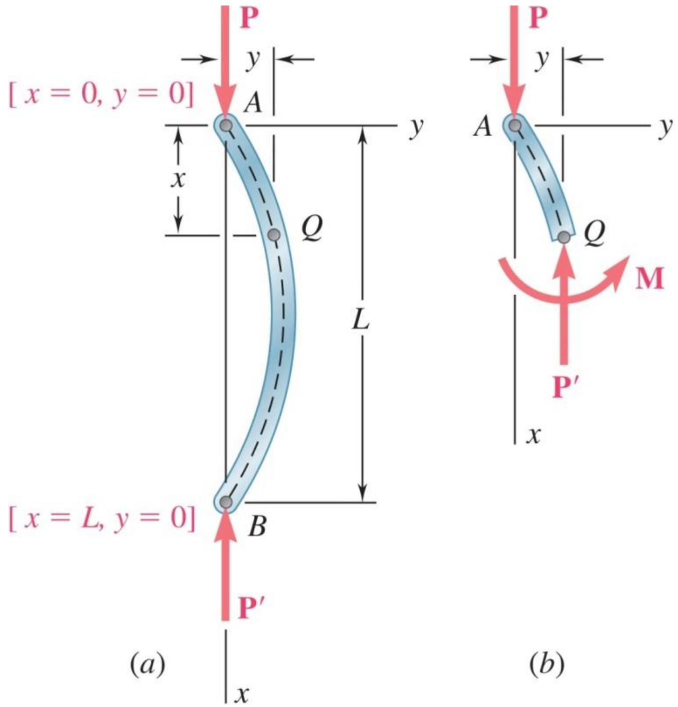
  - allowable stress is not exceeded

$$\sigma = \frac{P}{A} \leq \sigma_{all}$$

  - deformation falls within specifications

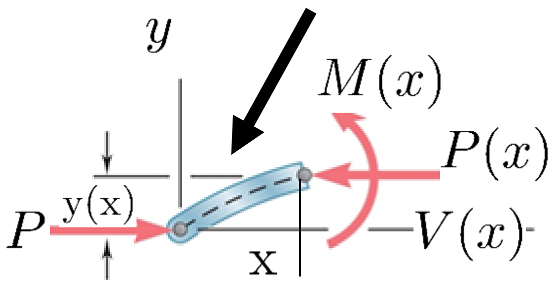
$$\delta = \frac{PL}{AE} \leq \delta_{spec}$$

- After these design calculations, may discover that the column is unstable under loading and that it suddenly becomes sharply curved or buckles.



Stability Analysis

The perturbed configuration



$$\sum F_x = 0 \;\rightarrow\; P - P(x) = 0 \;\rightarrow\; P(x) = P$$

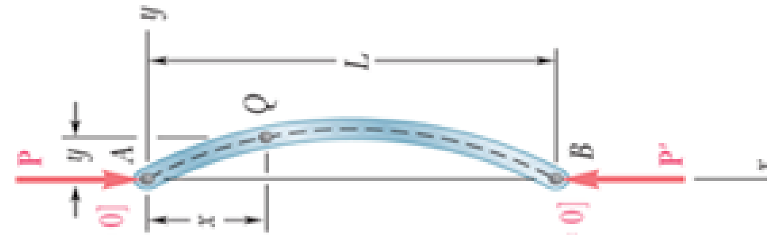
$$\sum F_y = 0 \;\rightarrow\; -V(x) = 0;$$

$$\sum M_x = 0; \quad \rightarrow\; M(x) + Py(x) = 0 \;\rightarrow\; EIy''(x) + Py(x) = 0;$$

$$y''(x) + \frac{P}{EI}y(x) = 0\ ;$$

$$\text{Let } \frac{P}{EI} = \lambda^2 \quad \rightarrow \quad y''(x) + \lambda^2 y(x) = 0; \;\rightarrow\; \quad y(x) = A\sin\lambda x + B\cos\lambda x;$$

B.C.: $y(0) = 0,\ y(L) = 0; \quad \rightarrow \quad y(x) = A \sin \lambda x + B \cos \lambda x$



(1) $y(0) = 0 \ \rightarrow \ B = 0;$

(2) $Y(L) = 0 \quad \rightarrow \quad A \sin \lambda L = 0; \qquad A \neq 0 \quad \rightarrow \quad \sin \lambda L = 0;$

What is $A = 0$?

$\sin \lambda L = 0 \quad \lambda L = n\pi, \quad n = 1, 2, \cdots\cdots$

A is arbitrary $\rightarrow y_{max} \rightarrow \infty$!

Recall $\lambda = \sqrt{\dfrac{P}{EI}} \quad \rightarrow \qquad \sqrt{\dfrac{P_n}{EI}}\ell = n\pi, \quad n = 1, 2, \cdots\cdots$

$$P_n = \frac{(n\pi)^2 EI}{L^2} \quad \rightarrow \quad (n = 1) \quad \rightarrow \qquad \boxed{P_{cr} = \frac{\pi^2 EI}{L^2}}$$

Euler's Formula

Remarks:

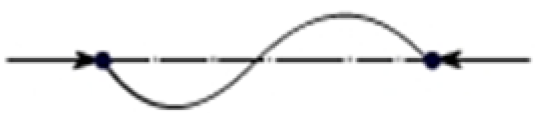
snap

(1) Buckling is referred to as the “switching” from the initial equilibrium state to another or other equlibrium states;

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0, \quad 0 < x < L$$

$$P < P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

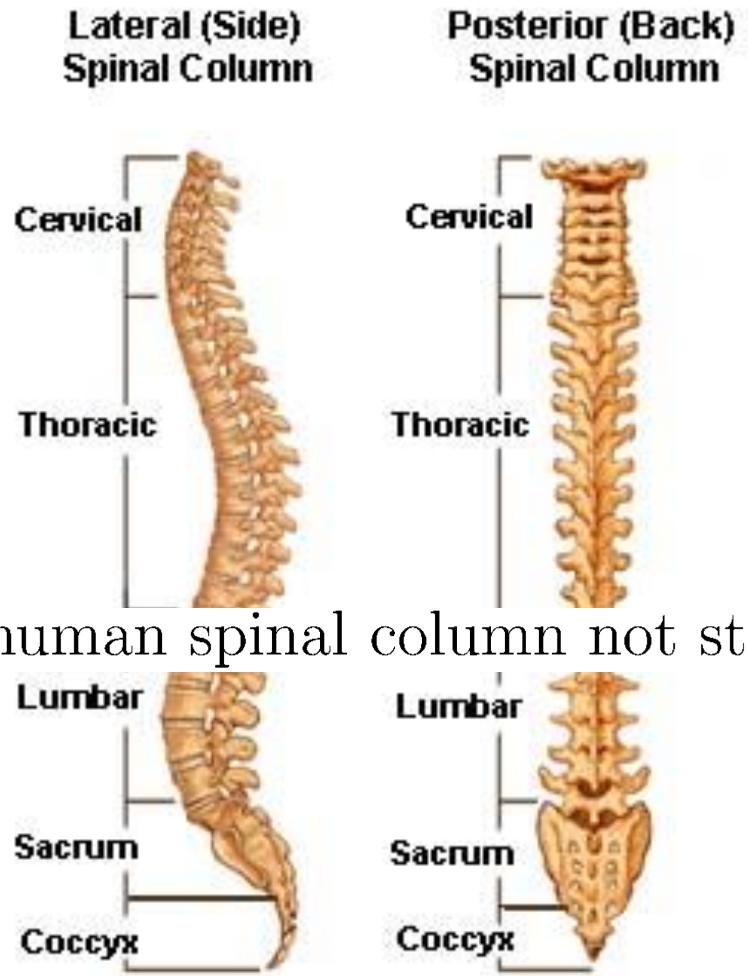


$$P_2 = \frac{4\pi^2 EI}{L^2}$$

$$P_3 = \frac{9\pi^2 EI}{L^2}$$

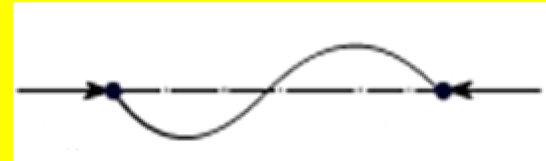
(2) The amplitude of the buckling defection A ($y(x) = A\sin\lambda x$) is indeterminate. The objective of the stability analysis is to find the critical load, not the amplitude of the deflection;

(3) When $P < P_{cr}$, the boundary condition $\sin\lambda L = 0$ cannot be satisfied, In order to satisfy the B.C., the only choice is to let $A = 0$, which represents the initial equilibrium state.



Why is human spinal column not straight ?

$$P < P_{cr} = \frac{\pi^2 EI}{L^2}$$

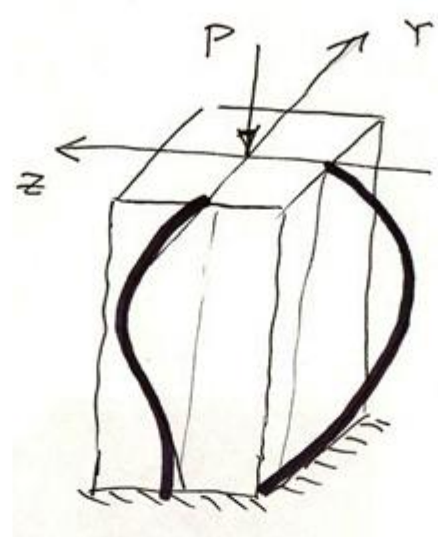


$$P < P_{cr} = \frac{4\pi^2 EI}{L^2}$$

$$P_{cr} = \frac{9\pi^2 EI}{L^2}$$

Human Spinal Column

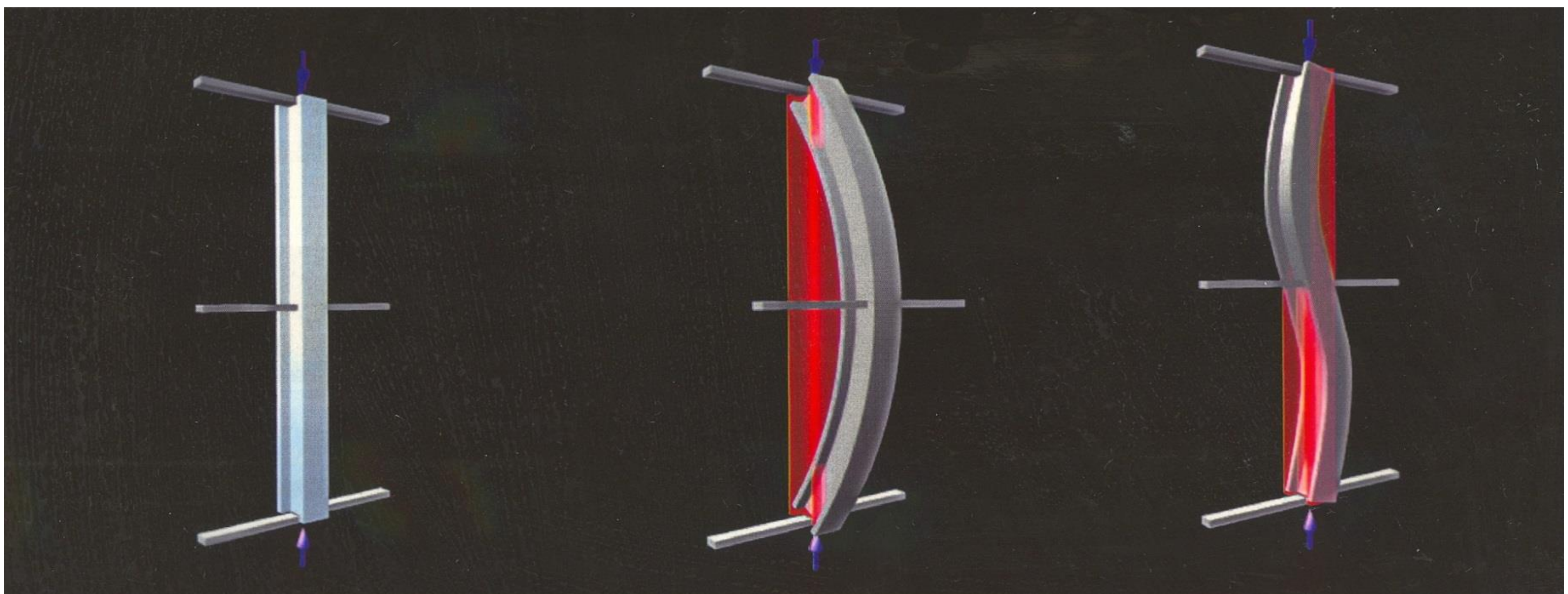
(4) In Euler formula: $P_{cr} = \dfrac{\pi^2 EI}{L^2}$. What is I ?



For a beam with rectangular cross section, we have two Is i.e. $(I_z, and\ I_y)$.
The I used in the Euler formula should be

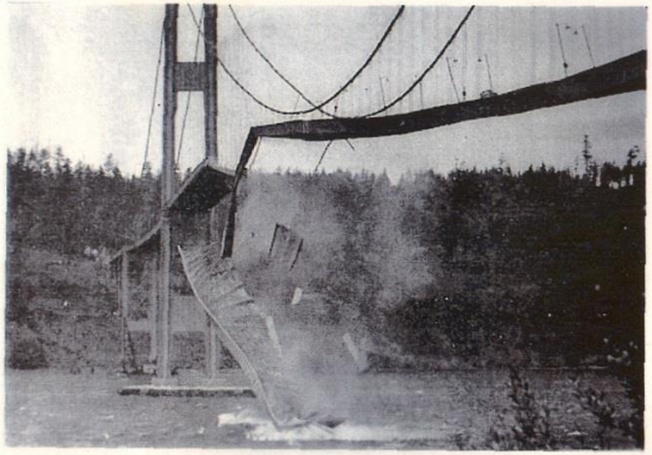
$$I = min\{I_z, I_y\},$$

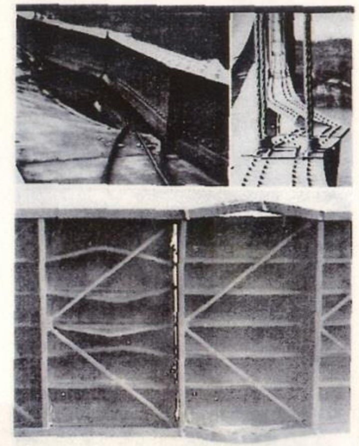
i.e. $P_{cr} = \dfrac{\pi^2 EI_{min}}{L^2}$.



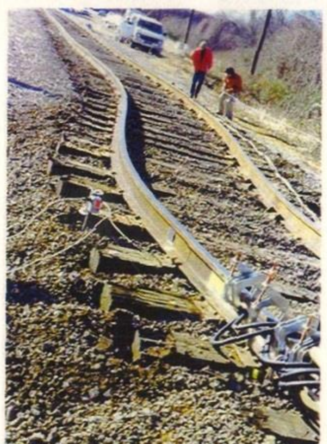
Beam Buckling with Respect Different Axes

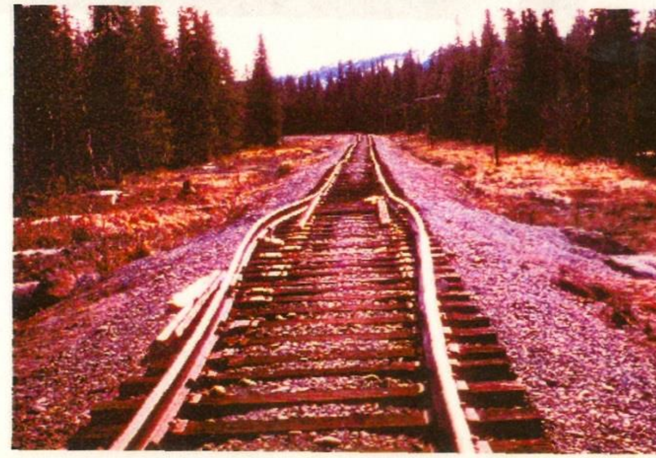
Buckling and Collapse of Tacoma Bridge

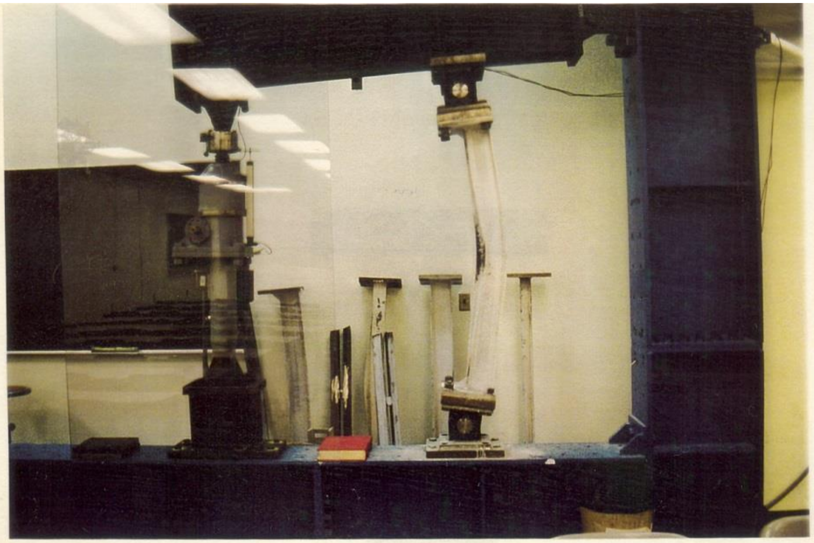


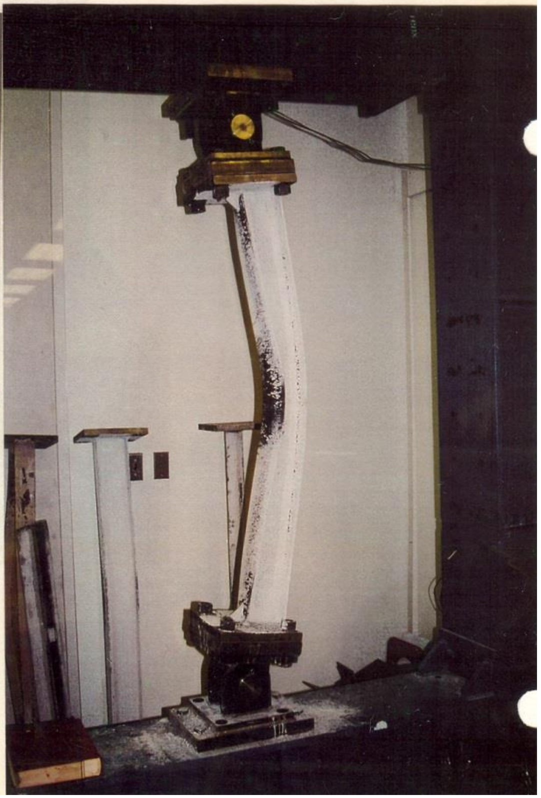


Buckling of Railroads



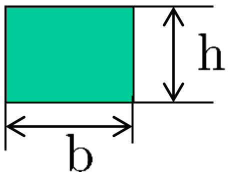




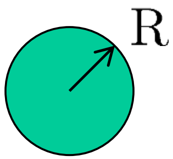


(5) We can write $I = Ar^2$ where A is the area of the section, and r is the radius of gyration.

Examples

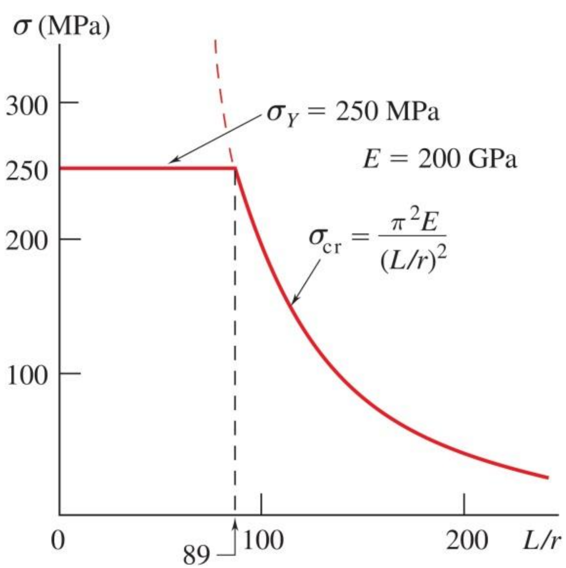
(a) 

$$I_z = \frac{bh^3}{12} = (bh)\frac{h^2}{12} = Ar^2 \quad \rightarrow \quad r = \frac{h}{\sqrt{12}};$$

(b) 

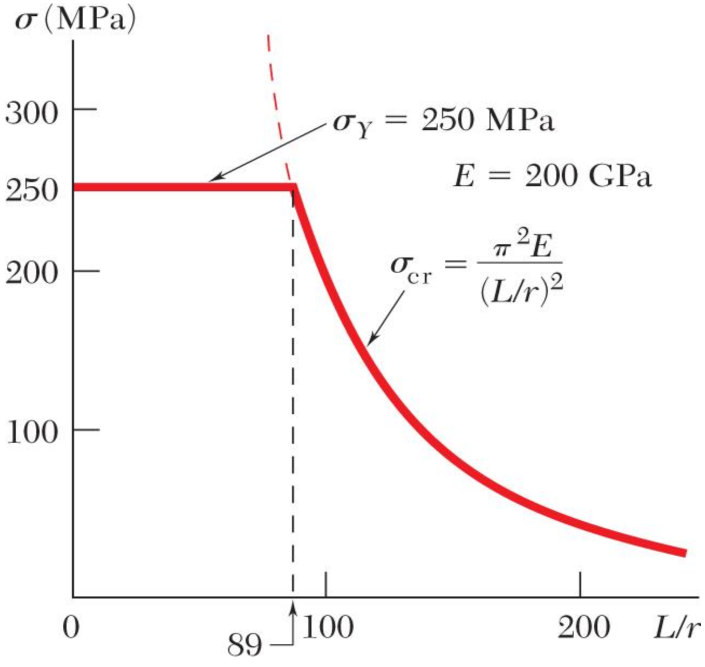
$$I_z = \frac{\pi R^4}{4} = \pi R^2 \frac{R^2}{4} = Ar^2 \quad \rightarrow \quad r = \frac{R}{2};$$

We can then find the critical stress as: $\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E(Ar^2)}{L^2 A} = \frac{\pi^2 E}{(L/r)^2}$, where L/r is the slenderness ratio of the column.



$$\sigma_{cr} \propto E, \frac{1}{(L/r)^2}$$

Therefore:
The shorter the column ($L/r \rightarrow 1$), the larger σ_{cr};
The slender the column ($L/r >> 1$), the smaller σ_{cr}.



- The value of stress corresponding to the critical load,

$$P > P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$\sigma = \frac{P}{A} > \sigma_{cr} = \frac{P_{cr}}{A}$$

$$\sigma_{cr} = \frac{\pi^2 E\left(Ar^2\right)}{L^2 A}$$

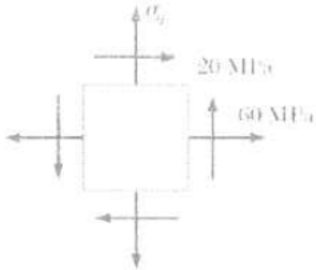
$$= \frac{\pi^2 E}{(L/r)^2} = critical\ stress$$

$$\frac{L}{r} = slenderness\ ratio$$

- Preceding analysis is limited to centric loadings.

Do NOT try this at home !

[Throat against spear: Martial art or Mechanics?]



PROBLEM 14.40 Find σ_y?

$\sigma_x = 60MP_a$

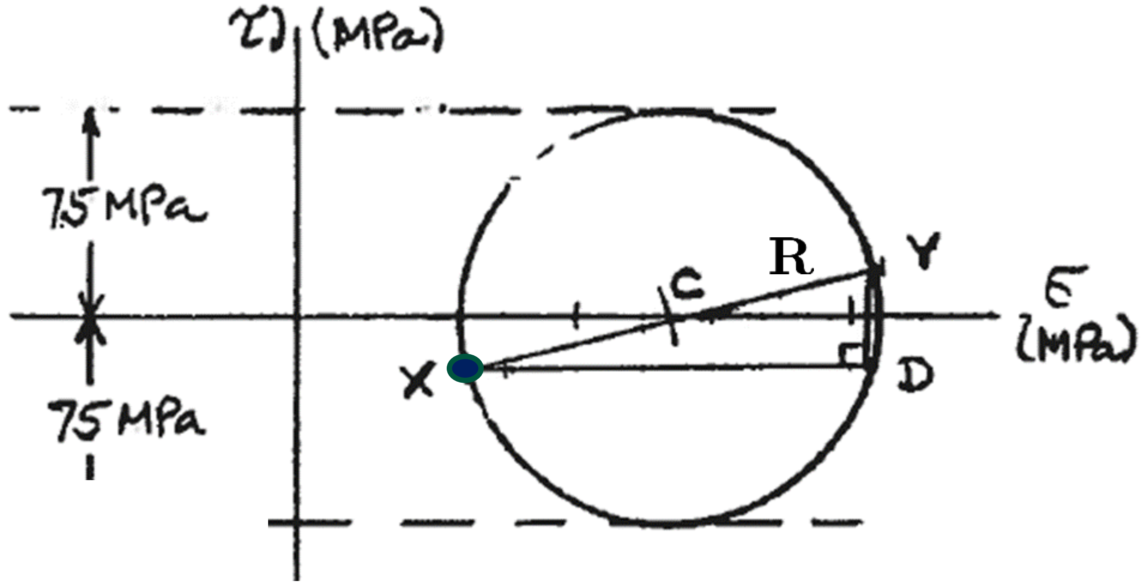
$\tau_{xy} = 20MP_a$

Solve Prob. 14.24, using Mohr's circle.

PROBLEM 14.24 For the state of plane stress shown, determine the largest value of σ_y for which the maximum in-plane shearing stress is equal to or less than 75 MPa.

$\tau_{max} = 75MP_a$

$\sigma_x = 60$ MPa, $\sigma_y = ?$, $\tau_{xy} = 20$ MPa



$$\tau_{max} = R = 75 \text{ MPa}$$

$$\overline{XY} = 2R = 150 \text{ MPa}$$

$$\overline{DY} = (2)(\tau_{xy}) = 40 \text{ MPa}$$

$$\overline{XD} = \sqrt{\overline{XY}^2 - \overline{DY}^2} = \sqrt{150^2 - 40^2} = 144.6 \text{ MPa}$$

$$\sigma_y = \sigma_x + \overline{XD} = 60 + 144.6$$

$$\sigma_y = 205 \text{ MPa} \blacktriangleleft$$

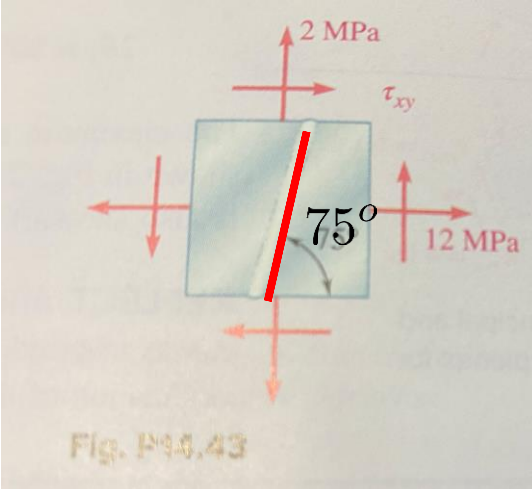


Fig. P14.43

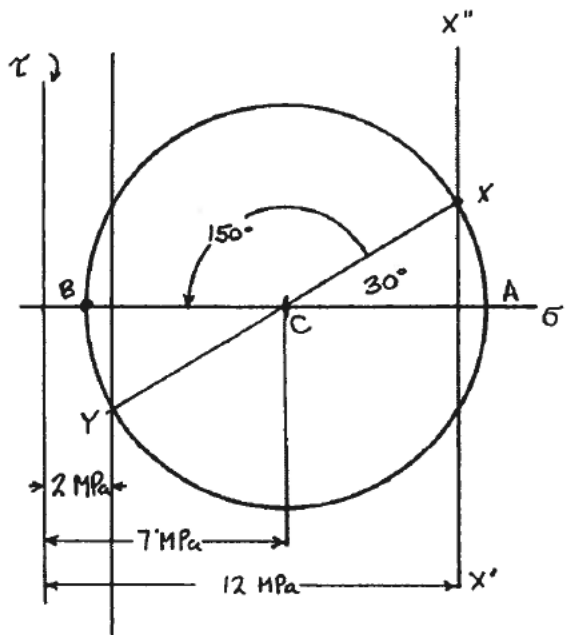
PROBLEM 14.43

For the state of plane stress shown, use Mohr's circle to determine (*a*) the value of τ_{xy} for which the in-plane shearing stress parallel to the weld is zero, (*b*) the corresponding principal stresses.

$$\sigma_x = 12MP_a \qquad \tau_{xy} = ?$$
$$\sigma_y = 2MP_a$$

Point X of Mohr's circle must lie on $X'X''$ so that $\sigma_x = 12$ MPa. Likewise, point Y lies on line $Y'Y''$ so that $\sigma_y = 2$ MPa. The coordinates of C are

$$\frac{2+12}{2},\quad 0 = (7\text{ MPa}, 0).$$



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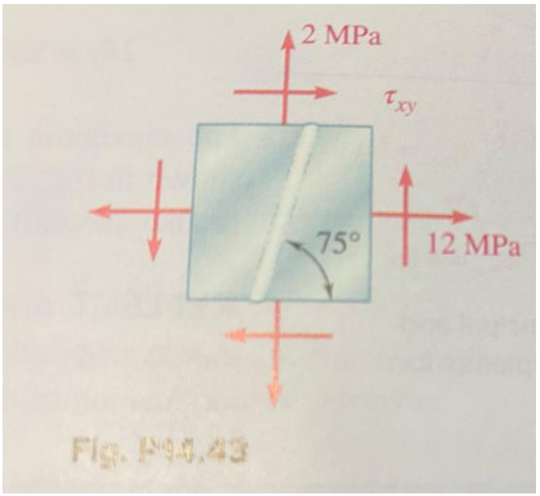
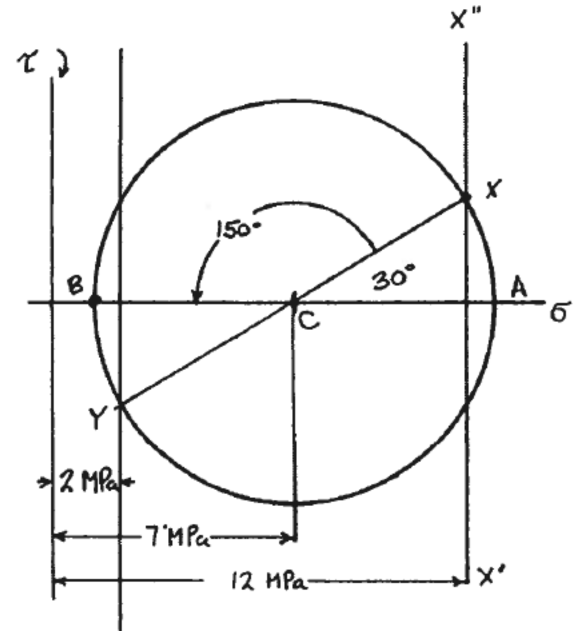




Counterclockwise rotation through 150° brings line CX to CB, where $\tau = 0$.

$$R = \frac{\sigma_x - \sigma_y}{2}\sec 30° = \frac{12-2}{2}\sec 30° = 5.7735\text{ MPa}$$

(a) $$\tau_{xy} = -\frac{\sigma_x - \sigma_y}{2}\tan 30°$$

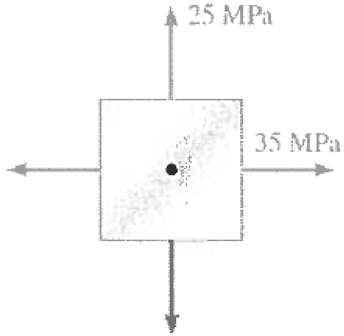
$$= -\frac{12-2}{2}\tan 30°$$ $\tau_{xy} = -2.89$ MPa ◀

(b) $\sigma_a = \sigma_{\text{ave}} + R = 7 + 5.7735$ $\sigma_a = 12.77$ MPa ◀

$\sigma_b = \sigma_{\text{ave}} - R = 7 - 5.7735$ $\sigma_b = 1.226$ MPa ◀

PROBLEM 14.46

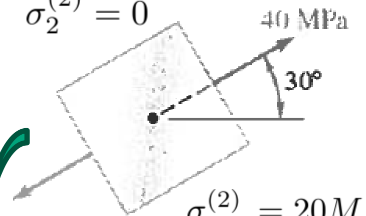
Determine the principal planes and the principal stresses for the state of plane stress resulting from the superposition of the two states of stress shown.



+

$\sigma_1^{(2)} = 40MP_a$

$\sigma_2^{(2)} = 0$



$\sigma_{ave}^{(2)} = 20MP_a$

$R^{(2)} = \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2} = 20MP_a$

$\sigma_x^{(1)} = 35MP_a$

$\sigma_y^{(1)} = 25MP_a$

$\tau_{xy}^{(1)} = 0$

Mohr's circle for 2nd stress state:

$\sigma_x = 20 + 20\cos 60°$

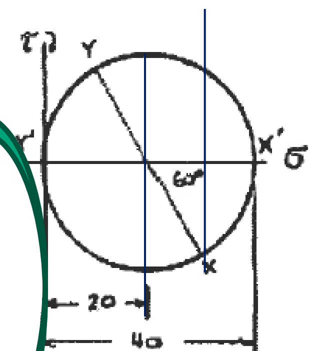
$= 30$ MPa

$\sigma_y = 20 - 20\cos 60°$

$= 10$ MPa

$\tau_{xy} = 20 \sin 60°$

$= 17.32$ MPa

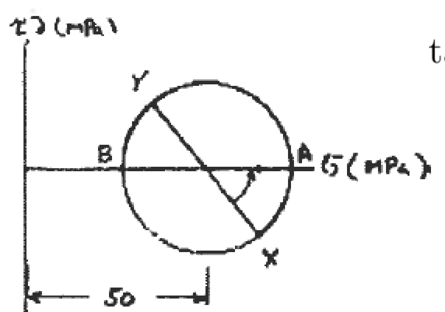


Resultant stresses:

$\sigma_x = 35 + 30 = 65$ MPa

$\sigma_y = 25 + 10 = 35$ MPa

$\tau_{xy} = 0 - 17.32 = 17.32$ MPa



$$\sigma_{ave} = \frac{1}{2}(\sigma_x + \sigma_y) = \frac{1}{2}(65 + 35) = 50\ MP_a$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{(2 \times 17.32}{65 - 35} = 1.13547$$

$$2\theta_p = 49.11^o \qquad \theta_p = 24.6^o, \quad \theta_{p2} = 114.6^o$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 22.91\ MP_a$$

$$\sigma_1 = \sigma_{ave} + R \qquad \sigma_1 = 72.91\ MP_a$$


Happy
Weekend!