Last Time:
We found that for the deformable two-force members:

$$(1) \quad \boxed{\Delta = fP}$$

For fixed axial force (internal)
the large the flexibility is,
the large the elongation.

$$\mathrm{f} = \frac{\mathrm{L}}{\mathrm{EA}} \longleftarrow \text{Flexibility}$$

$$(2) \quad \boxed{P = k\Delta}$$

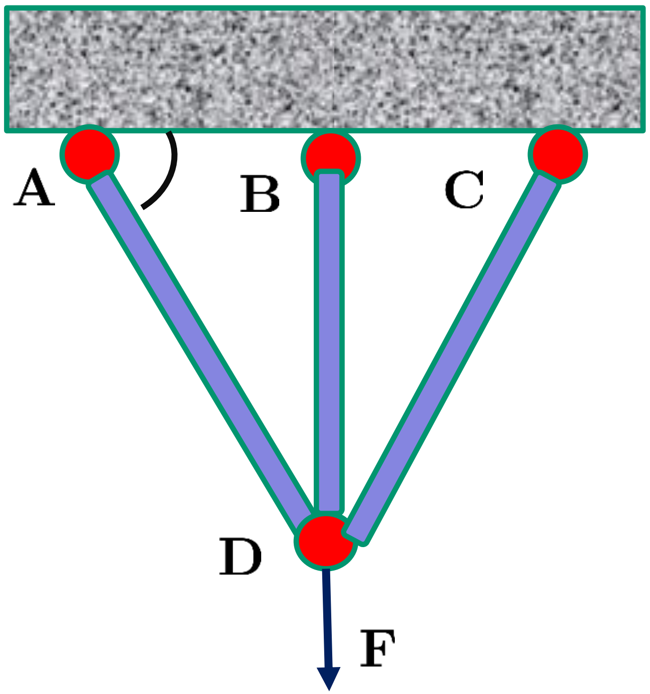
For fixed elongation,
the stiffer the bar is,
the large force it produces.

$$\mathrm{k} = \frac{\mathrm{EA}}{\mathrm{L}} \longleftarrow \text{Stiffness}$$

$$\boxed{\mathrm{k} = \frac{1}{\mathrm{f}}}$$

The reciprocal relation

Lecture 19
Statically Indeterminate Systems



$$\mathbf{m + r = 3 + 6 = 9}$$

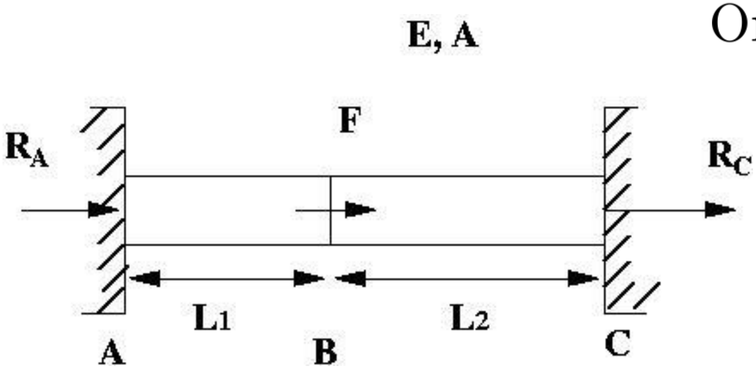
$$\mathbf{m + r > 2n = 2 \times 4 = 8}$$

Think out of the box!

How do we solve statically indeterminate problems?

1. Force method or flexibility method

Model Problem



$$f_1 = \frac{L_1}{EA}, \qquad f_2 = \frac{L_2}{EA},$$

Find the stress distribution inside the bar ?

One equation $\sum F_x = 0$, and two unknowns.

$$\mathbf{R_A} \text{ and } \mathbf{R_C} : \ \mathbf{1 < 2}$$

This is a statically indeterminate system of degree one.

Question: Is this a one bar problem or two bar problem ?

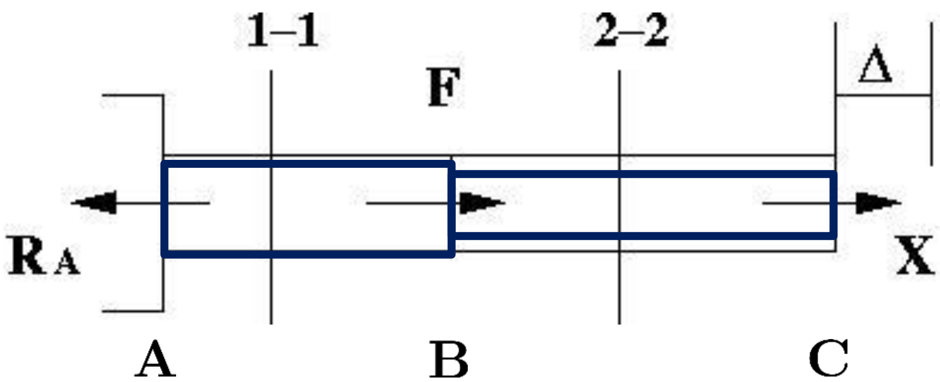
How do we solve statically indeterminate problems ?

To study a new problem, one should first review the old theory.

Strategy: Reduce the statically indeterminate system to a statically determinate system first.

Force method: Solution Procedure

Step1. Release the redundant constraint to recover a statically determinate system.



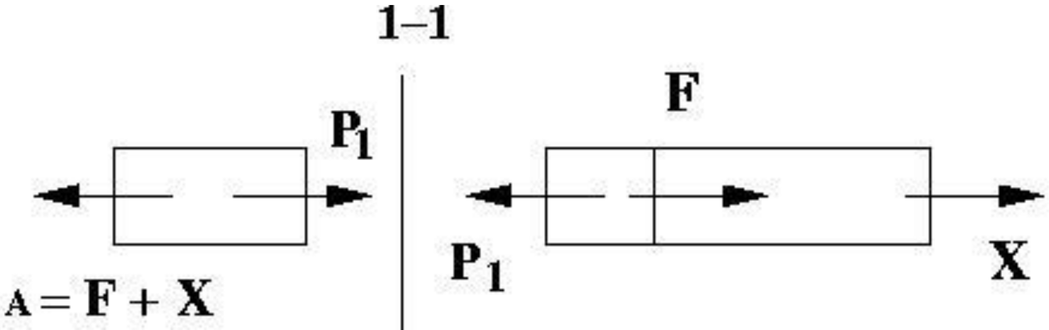
Replace the reaction force R_C with an unknown active external force X.

We release the constraint C, and now the point C has a displacement Δ. Since the system now is statically determinate, we can find the reaction by solving static equilibrium equation,

$$-R_A + F + X = 0, \;\; \Rightarrow \;\; R_A = F + X;$$

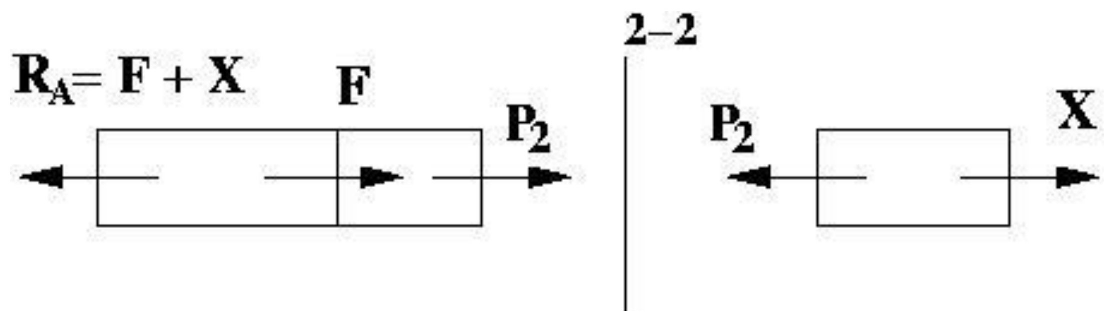
Step 2. Find internal force in terms of X and F

For section 1-1, one has



$$\sum F_x = 0, \quad \Rightarrow \quad -R_A + P_1 = 0, \quad \Rightarrow \quad \boxed{P_1 = R_A = F + X}$$

For section 2-2, one has



$$\sum F_x = 0, \qquad \Rightarrow -P_2 + X = 0, \Rightarrow \boxed{P_2 = X}$$

Then the total elongation of the system is:

$$\Delta = \Delta_1 + \Delta_2 = f_1(F + X) + f_2 X$$

Solution Procedure (Cont'd)

Step 3. Enforce the constraint:

$$\Delta = \Delta_1 + \Delta_2 = (u_B - u_A) + (u_C - u_B) = u_C - u_A = u_C = 0 \ .$$

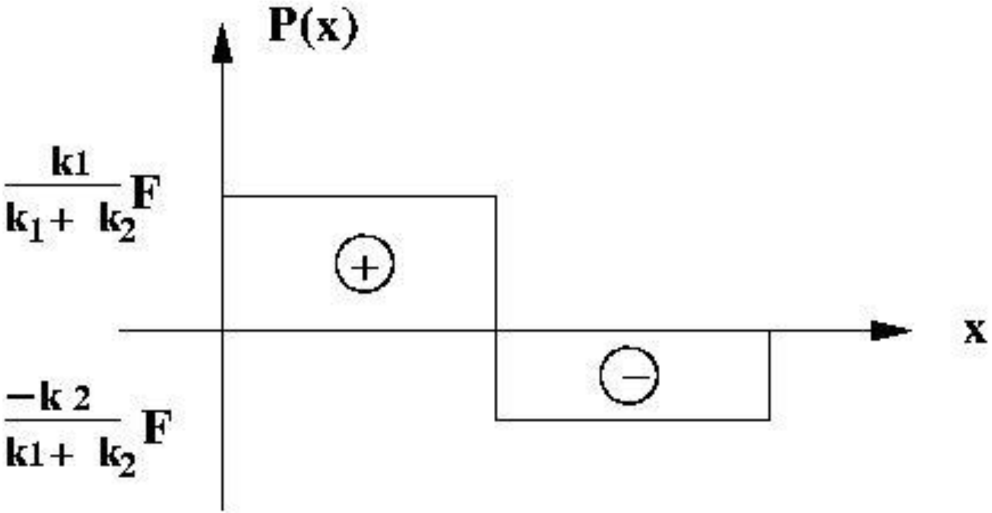
$$\text{and therefore,} \quad \Delta = \Delta_1 + \Delta_2 = f_1(F + X) + f_2 X = 0, \ (*)$$

From Eq. (*), we can find that

$$(f_1 + f_2)X = -f_1 F = 0, \quad \Rightarrow \quad X = -\frac{f_1 F}{f_1 + f_2} = -\frac{k_2 F}{k_1 + k_2}$$

Then we can find the internal forces,

$$P_1 = F + X = F - \frac{k_2 F}{k_1 + k_2} = \frac{k_1 F}{k_1 + k_2}; \quad P_2 = X = -\frac{k_2 F}{k_1 + k_2};$$



Or

$$P_1 = \frac{f_2 F}{f_1 + f_2}, \quad and$$

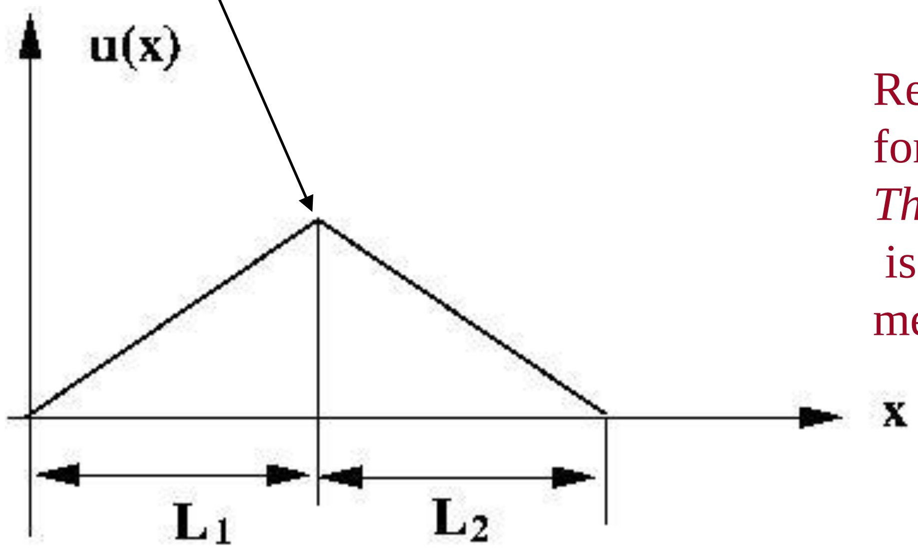
$$P_2 = -\frac{f_1 F}{f_1 + f_2}$$

Solution Procedure (Cont'd)

Step 4. Displacement diagram:

Since displacement field inside a two-force member is a linear function, one only needs to find the elongation at point B, which is

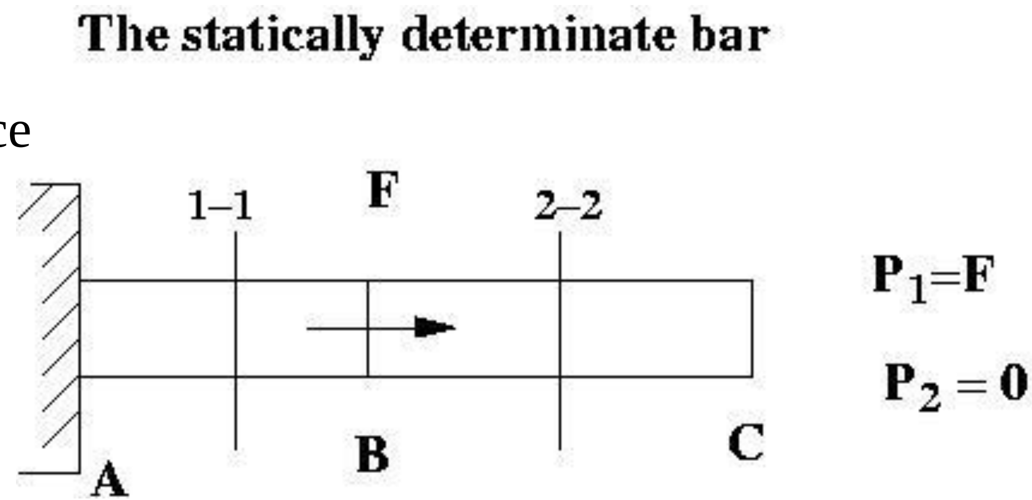
$$\Delta_1 = u_B - \mathrm{u_A} = \mathrm{u_B} = f_1 P_1 = \frac{f_1 f_2}{f_1 + f_2} F$$



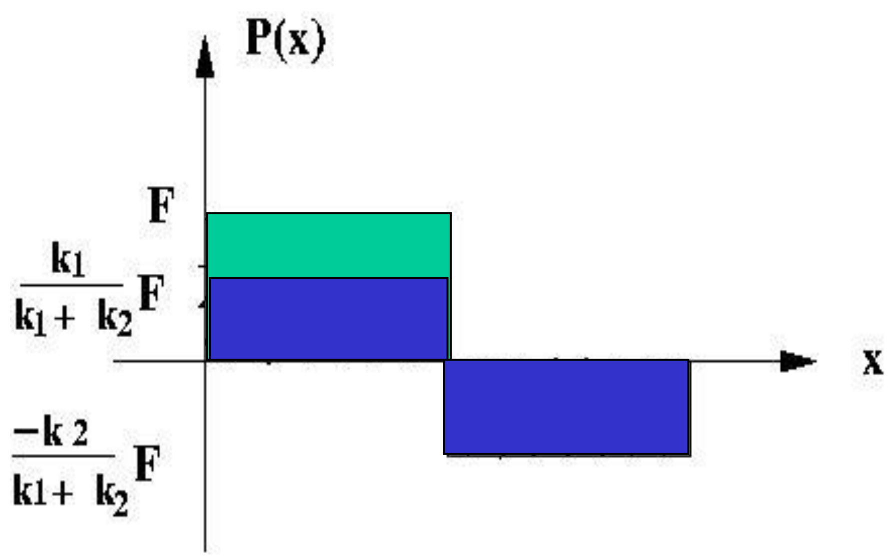
Remark: Since the unknowns are forces, the method is called *The force method*, and since the elongation is expressed in terms of flexibility, this method is also called the *Flexibility* Method.

Why statically indeterminate system ?

Consider a statically determinate bar and compare the internal force distribution with that of the statically indeterminate bar,



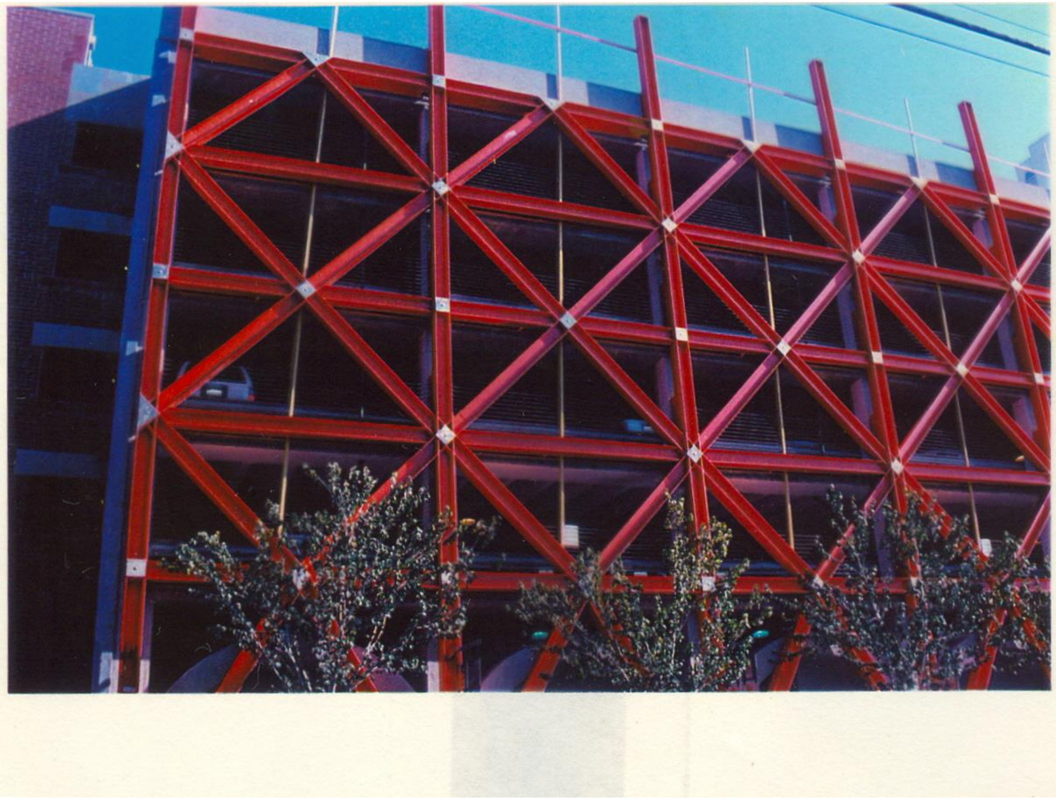
$$
\begin{aligned}
P_1 &= \frac{k_1 F}{k_1 + k_2}, \quad \text{and} \\
P_2 &= -\frac{k_2 F}{k_1 + k_2}
\end{aligned}
$$



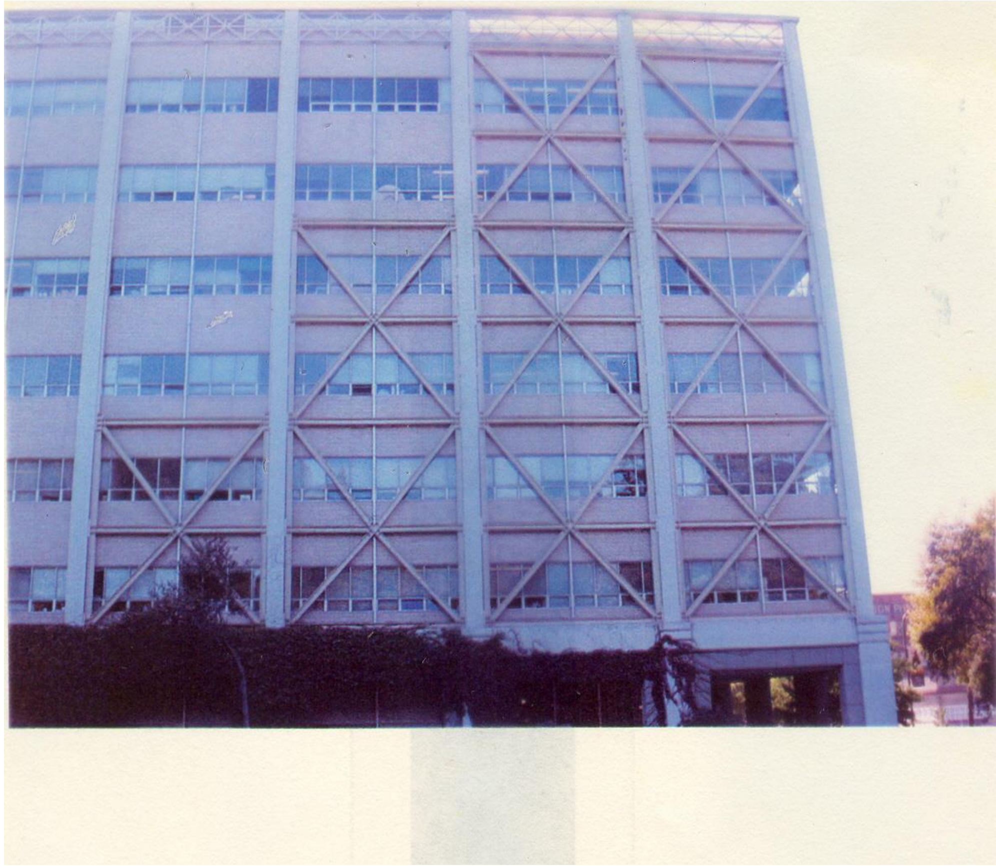
$$P^{Sid}_{max} < P^{Sd}_{max}$$

$$\sigma^{Sid}_{max} < \sigma^{Sd}_{max}$$

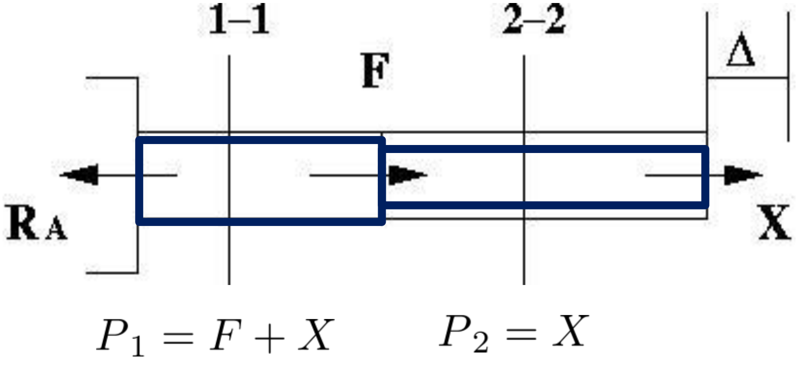
Statically Indeterminate Structures



Statically Indeterminate Structures



Statically Indeterminate Structures



$$\Delta = \Delta_1 + \Delta_2 = f_1(F + X) + f_2 X = 0;$$

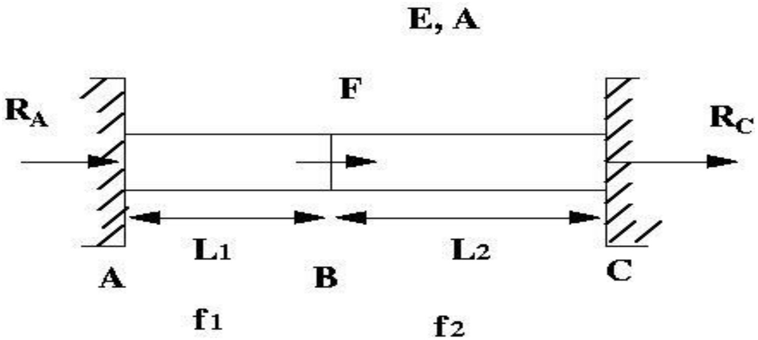
Compatibility condition

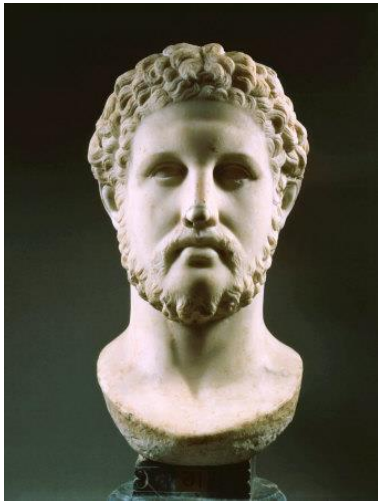
Remark: **The method** is called as the **force method** or **flexibility method.**

Today's Lecture Attendance Password is: Force Method

Variation in a theme: 2. Superposition method

Solution strategy:
divide and conquer (divida et impera)

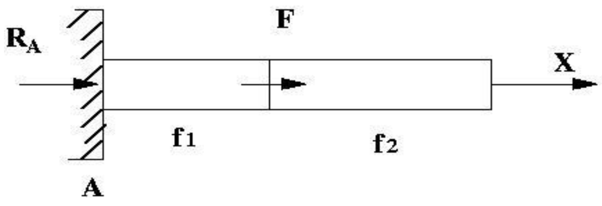


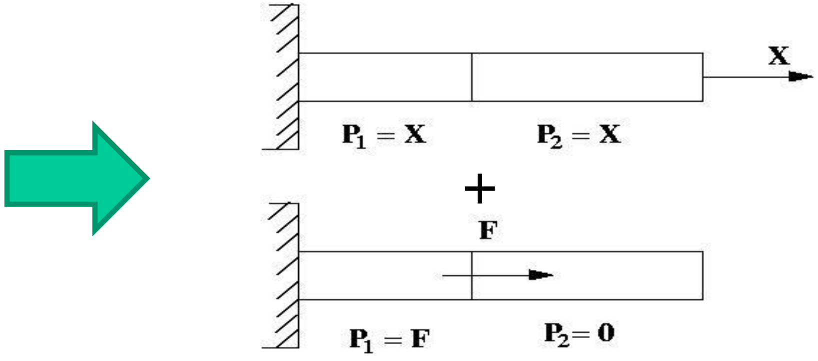


Philip of Macedonia

Step 2. Divide the system into two sub-systems
And each sub-system has only one external force

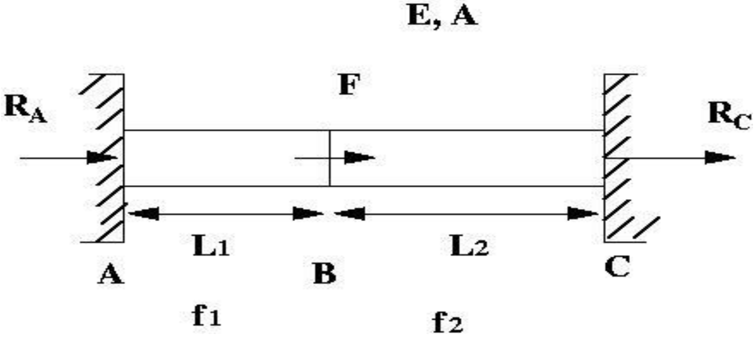
Step 1. Release the constraint C





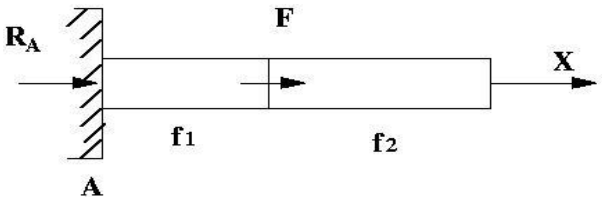
Step 3. Adding two sub-systems together (superposition) and enforce the constraint

Model Problem

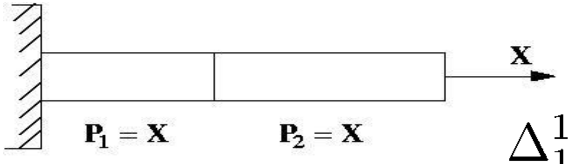


[Solution]

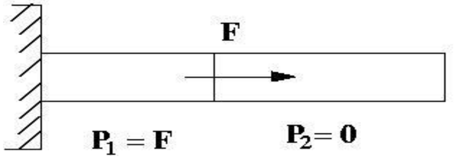
Step 1. Release the constraint C



Step 2. Divide the system into two sub-systems
And each sub-system has only one external force



$$\Delta_1^1 = f_1 X,\ \Delta_2^1 = f_2 X, \Rightarrow \Delta^1 = \Delta_1^1 + \Delta_2^1 = (f_1 + f_2)X$$

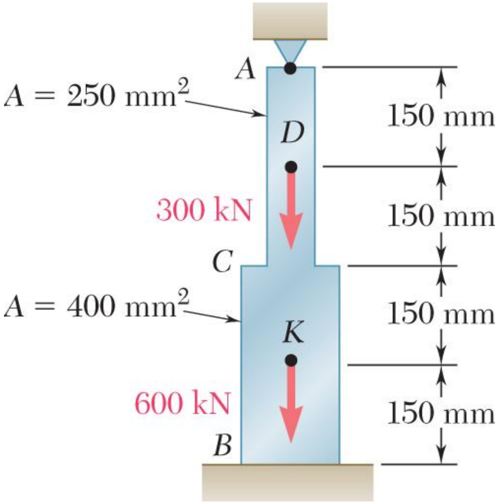


$$\Delta_1^2 = f_1 F,\ \Delta_2^2 = 0, \Rightarrow \Delta^2 = \Delta_1^2 + \Delta_2^2 = f_1 F$$

Step 3. Adding two sub-systems together (superposition) and enforce the constraint

$$\Delta^{total} = \Delta^1 + \Delta^2 = (f_1 + f_2)X + f_1 F = 0; \quad \Rightarrow \quad X = \frac{-f_1 F}{f_1 + f_2}$$

Example 9.4



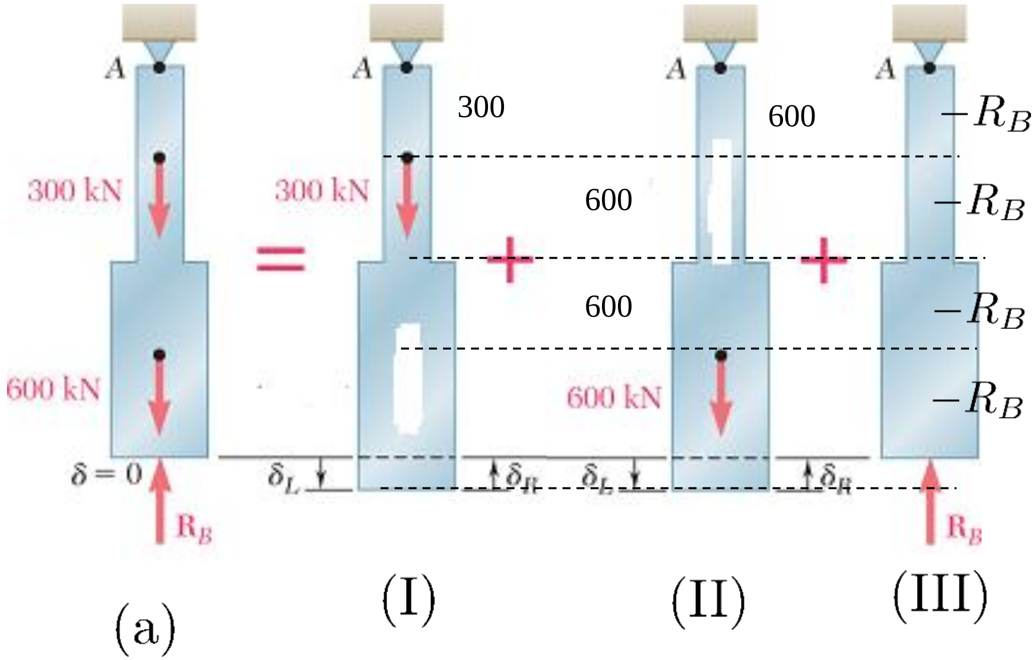
Determine the reactions at *A* and *B* for the steel bar and loading shown, assuming a close fit at both supports before the loads are applied.

SOLUTION:

Step 1. Remove the constraint at B. Add the reaction $\mathbf{R}_B$;

Step 2. Split the original system into three simple systems.

Superposition Method



$$A_1 = A_2 = 400 \times 10^{-6} m^2;$$

$$A_3 = A_4 = 250 \times 10^{-6} m^2$$

$$L_1 = L_2 = L_3 = L_4 = 0.150m$$

$$f_1 = f_2 = 375/E$$
$$f_3 = f_4 = 600/E$$

$$\Delta_I = f_4(300)$$

$$\Delta_{II} = f_2(600) + f_3(600) + f_4(600)$$

$$\Delta_{III} = -f_1 R_B - f_2 R_B - f_3 R_B - f_4 R_B$$

$$\Delta = \Delta_I + \Delta_{II} + \Delta_{III} = 0 \ \rightarrow \ 2(f_1 + f_3)R_B = 300 f_3 + 600(f_1 + 2f_3)$$

$$R_B = \frac{1}{f_1 + f_3}(300 f_1 + 450 f_3) = 577(kN)$$

Remarks:

1. The advantages of the superposition method is: one does not need to draw free-body diagram;
2. Applicability of superposition method as a general methodology:

Linear : $(X+Y)^1 = X^1 + Y^1$

Nonlinear: $(X+Y)^2 \neq X^2 + Y^2$

$$(X+Y)^4 \neq X^4 + Y^4$$

$$\sin(X+Y) \neq \sin X + \sin Y$$

Caveat: Superposition method is only applicable to linear problems !

Summary on Statically indeterminate Systems

1. Remove redundant constraints and add the corresponding reaction forces.

2. Solve the problem as a static determinant problem.

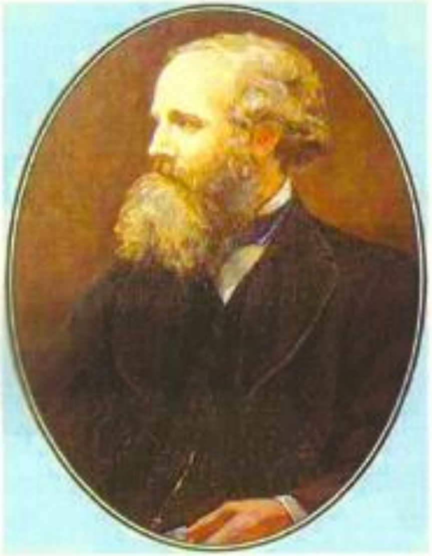
3. Enforce the displacement/deformation constraint conditions and determine the reaction forces.

4. When the problem is linear, one can use superposition method.

5. The maximum stress in a statically indeterminate structure is always smaller than that in the corresponding statically determinate structure.

6. The internal force distribution depends on the stiffness of the member.

The person who invented force method

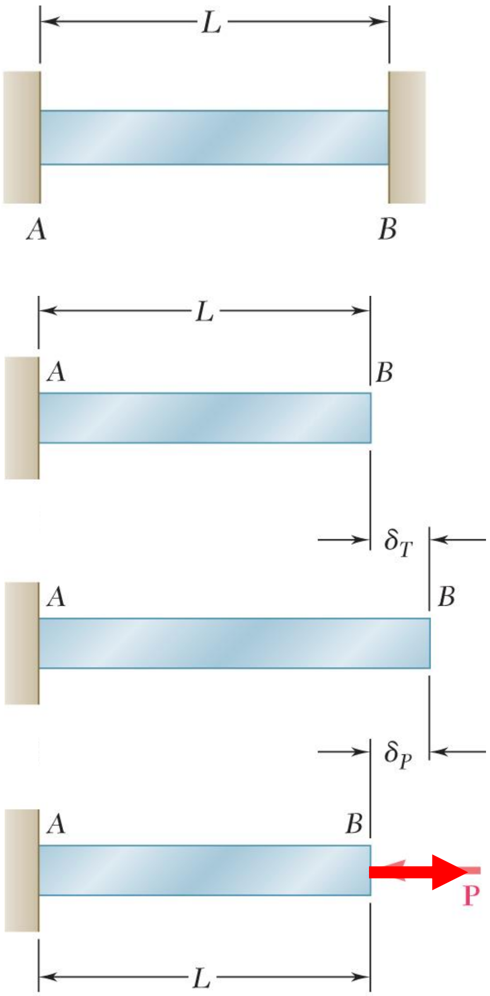


James Clerk Maxwell (13 June 1831 – 5 November 1879)

Scottish mathematician and physicist, is known for his contribution in electromagnetics theory, (the celebrated Maxwell equations), and his contribution in statistical physics (Maxwellian distribution).

Today's Lecture Attendance Password is: Force Method

Thermal Stresses



- A temperature change results in a change in length or *thermal strain*. There is no stress associated with the thermal strain unless the elongation is restrained by the supports.

- Treat the additional support as redundant and apply the principle of superposition.

$$\delta_T = \alpha(\Delta T)L \qquad \delta_P = \frac{PL}{AE}$$

$\alpha =$ thermal expansion coef.

$$\delta = \delta_T + \delta_P = 0$$

$$\alpha(\Delta T)L + \frac{PL}{AE} = 0$$

$$P = -AE\alpha(\Delta T)$$

$$\sigma = \frac{P}{A} = -E\alpha(\Delta T)$$


Erick©
Happy
Weekend!