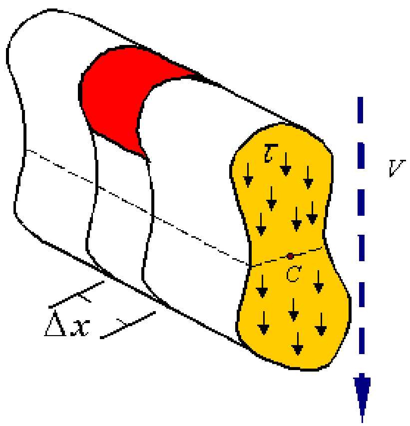
Lecture 28 Shear Stress in Beams

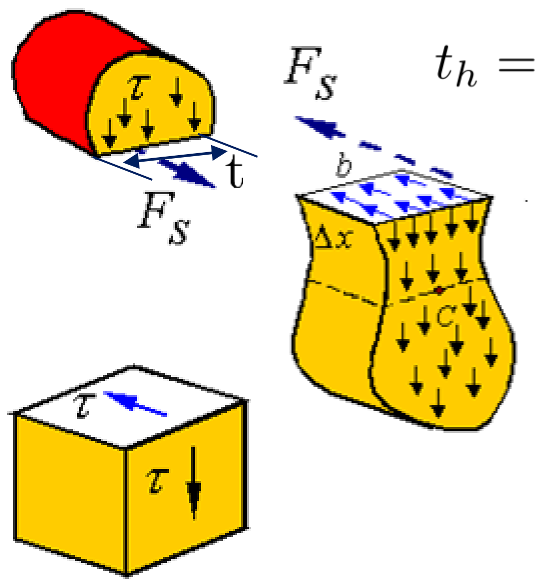
Structural member	Stress resultant	Corresponding stress
Bar	P	$\sigma = \dfrac{P}{A}$
Shaft	T	$\tau = \dfrac{T\rho}{J}$
Beam	M	$\sigma_x = -\dfrac{M_z y}{I_z}$
Beam	V	?

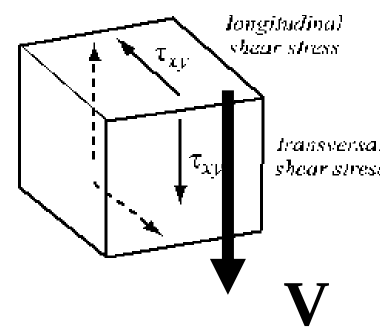
Questions:
What is the stress measure related to the shear force V ?
What is the relation between the that stress and the shear force V ?

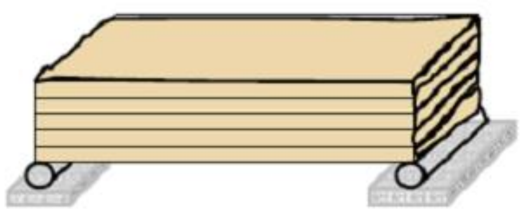
1. Vertical shear force will induce Longitudinal shear stress

$$t_h = \frac{F_s}{\Delta x t} = \tau_{yx}$$

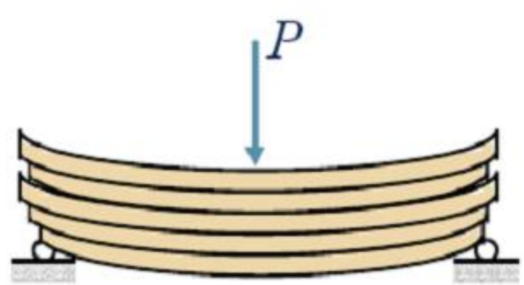








(a) Unloaded Stack of Slabs



(b) Unglued Slabs loaded

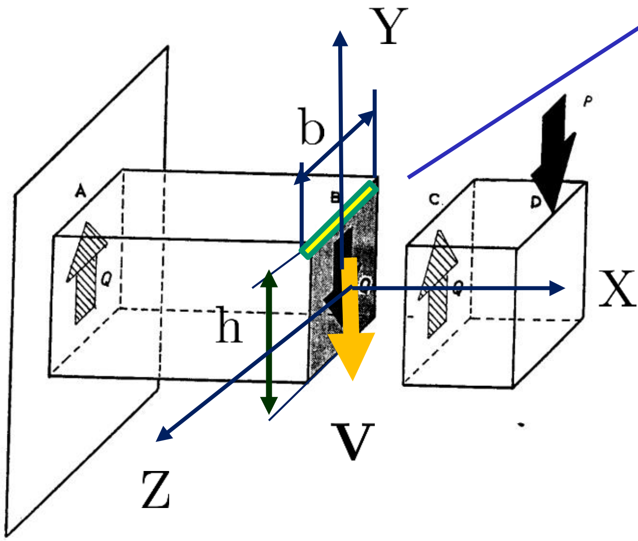
We know that $P = \int_A \sigma dA = 0$,

$M_z = -\int_A y\sigma dA$, and

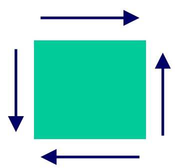
$$V = \int_A \tau dA$$

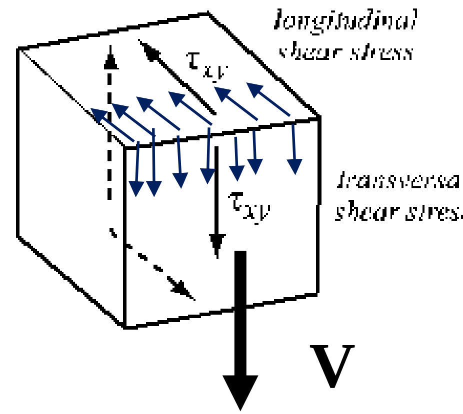
$$\text{We know} : \sigma = -\frac{My}{I},$$

but how $\tau \sim V$?

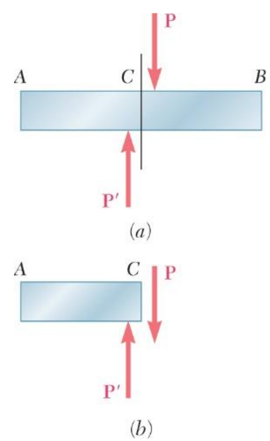


$$\tau_{xy} = \tau_{yx}$$



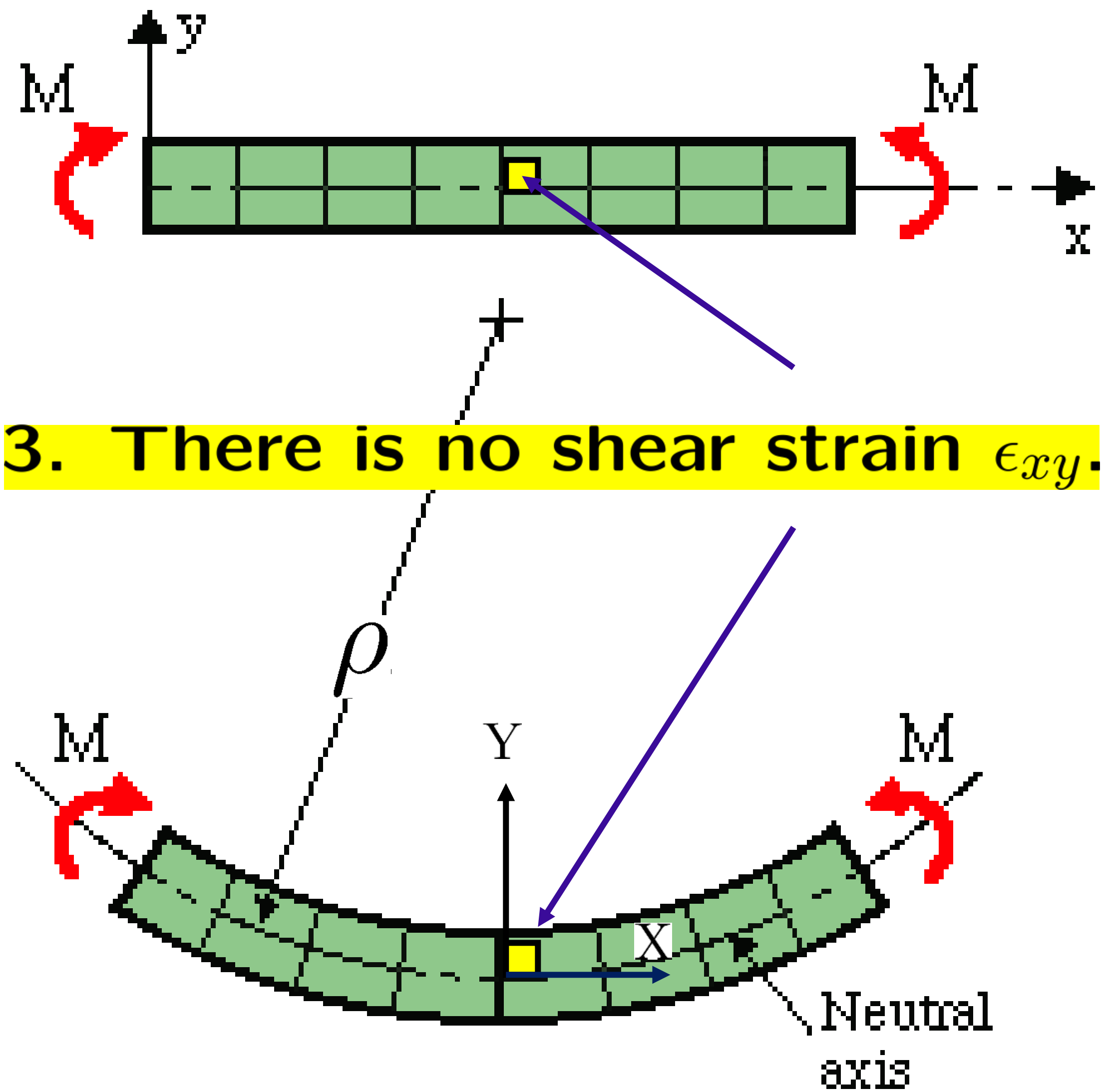


2. Can we use $\tau_{ave} = \dfrac{V}{A} = \dfrac{V}{bh}$?



Does $\tau_{\mathbf{ave}} = \tau_{\mathbf{max}}$?

Kinematic Assumptions of the Bernoulli-Euler Beam



3. There is no shear strain ϵ_{xy}.

We want to find τ_{xy} without ϵ_{xy}.

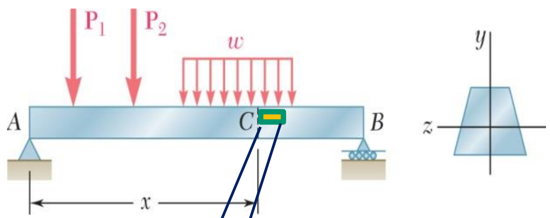
Egor Popov (1913-2001)

``In deriving the torsion and flexure formula, the same sequence of reasoning was employed. First, a strain distribution was assumed across the section, next, properties of the material were brought in to relate strain distribution with stress distribution; and finally, the equation of equilibrium were used to establish the desired relations. However, the development of the expression linking the shear force and the shear stress follows a different path. The previous procedure cannot be employed, as no simple assumption for the strain distribution due to the shear force can be made.''

E. Popov, Mechanics of Materials, pp.415-416

Shear on the Horizontal Face of a Beam Element

$$\sigma = -\frac{My}{I}$$

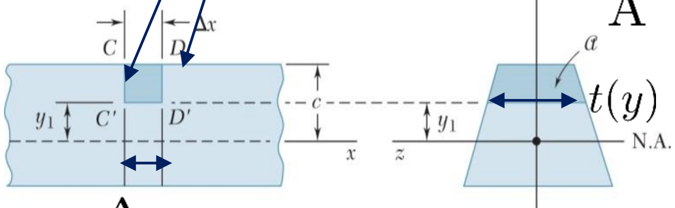


Consider a prismatic beam

$$\sum F_x = 0 = \Delta H + \int_A (\sigma_D - \sigma_C)\, dA$$

$$\sigma_x = -\frac{My}{I}$$

$$\Delta H = \frac{M_D - M_C}{I} \int_A y\, dA$$



A

$t(y)$

Δx

Define,

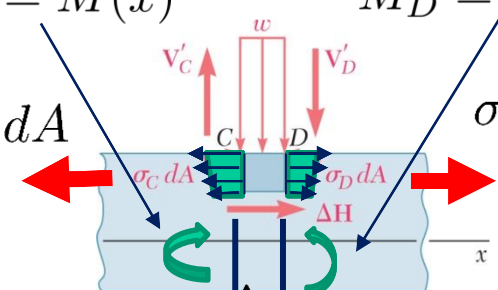
$$Q = \int_A y\, dA$$

$$M_D - M_C = \frac{dM}{dx} \Delta x = V\, \Delta x$$

$$M_C = M(x) \qquad M_D = M(x + \Delta x)$$

Substituting,

$$\sigma_C dA \qquad \sigma_D dA$$

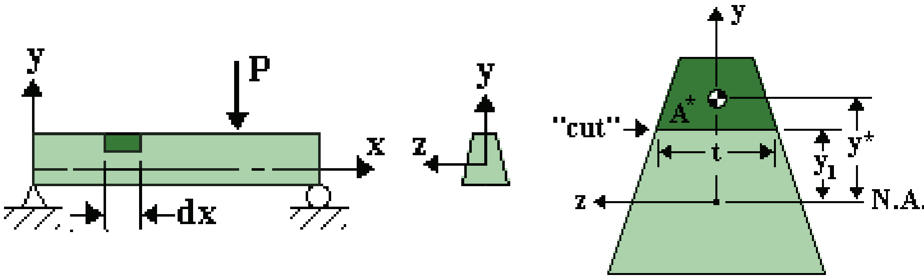


Δx

$$\Delta H = \frac{VQ}{I} \Delta x$$

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} = shear\ \ flow$$

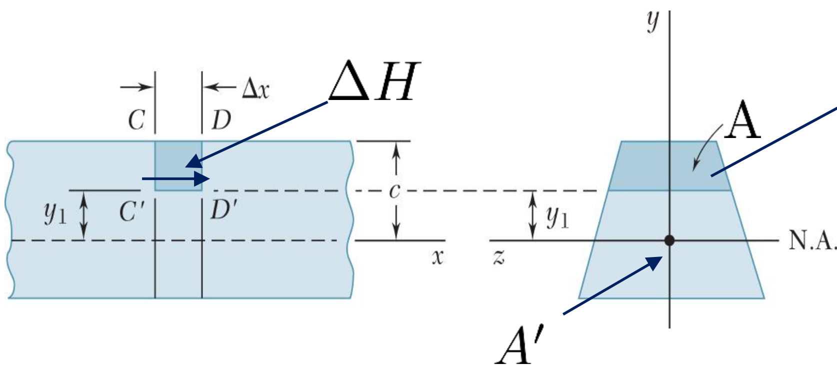
Finally, $\tau = \dfrac{\Delta H}{\Delta x\ t(y)} = \dfrac{VQ}{I_z t(y)}$



Horizontal Shear flow,

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} = shear \;\; flow$$

where



$$Q = \int_A y\, dA = Ay*$$

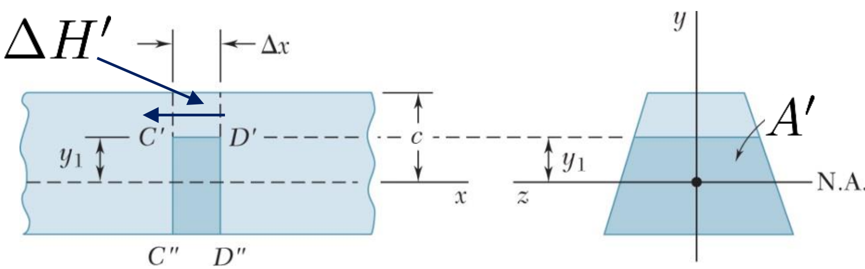
$=$ first moment of area above y_1

$$I = \int_{A+A'} y^2 dA$$

$=$ second moment of full cross section

What is: $\int_{A+A'} ydA = ?$

The same result found for the lower area



$$q' = \frac{\Delta H'}{\Delta x} = \frac{VQ'}{I} = -q$$

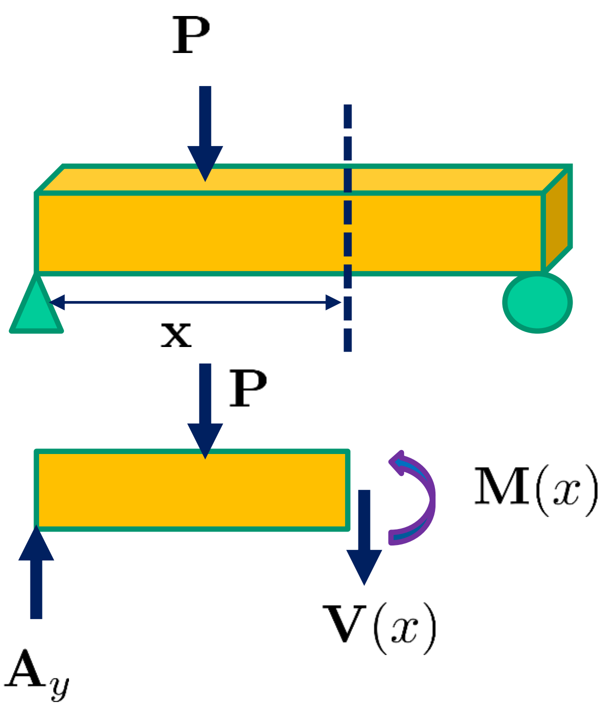
because $Q' = -Q$ and hence

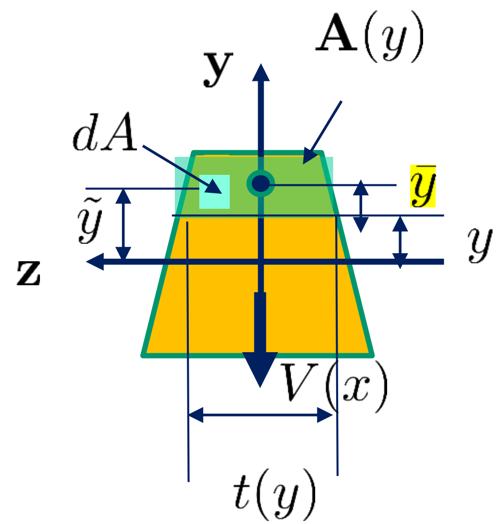
$$\Delta H' = -\Delta H \; .$$

$$Q_{total} = Q(y) + Q'(y) = 0$$

Summary: The Shear Formula

- $$\tau = \frac{\Delta H}{\Delta x\ t(y)} = \frac{V(x)Q(y)}{I_z t(y)}$$





$$Q_z(y) = \int_A \tilde{y}\, dA = \bar{y} A$$

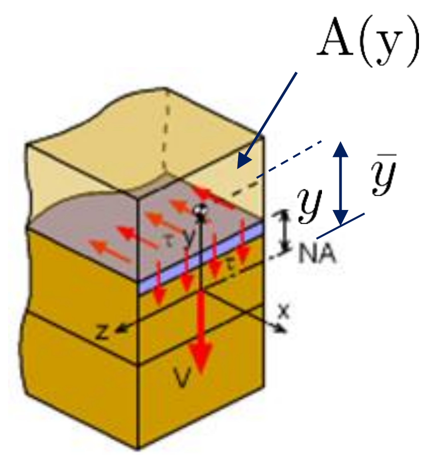
How to calculate $Q(y)$?

The first moment of area $A(y)$

$$Q(y) = \int_{A(y)} \tilde{y}\, dA = b \int_{y}^{h/2} \tilde{y}\, d\tilde{y} = \frac{b\tilde{y}^2}{2} \Big|_{y}^{h/2} = \frac{b}{2}\Big[\big(h/2\big)^2 - y^2\Big]$$

$$\tau = \frac{V(x)Q(y)}{I_z b} \;\rightarrow\; \tau_h = \frac{V(x)}{2I_z}\Big(\frac{h^2}{4} - y^2\Big) = \tau_v$$

τ_{yx} $\qquad$ τ_{xy}

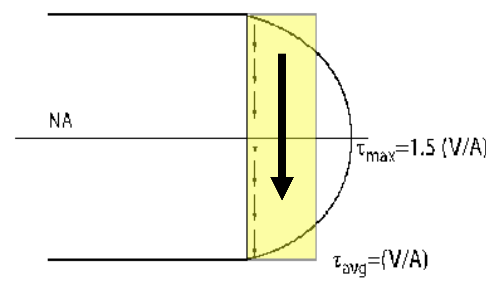


Remarks:

(1) When $y = \pm h/2, \tau(\pm h/2) = 0$;

(2) $\tau_{xy} = \dfrac{V(x)}{\big(bh^3/12\big)b}\Big(\dfrac{b}{2}\Big)\Big(\dfrac{h^2}{4} - y^2\Big) = \dfrac{3V}{2A}\Big(1 - \big(\dfrac{y}{(h/2)}\big)^2\Big)$.

Zhourawskii's solution

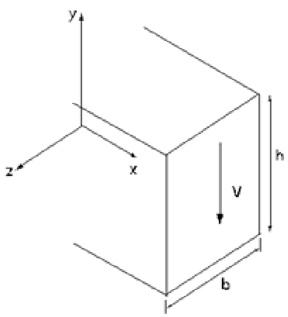


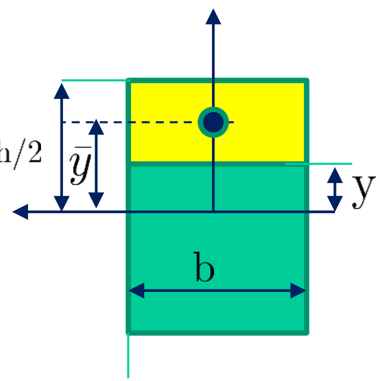
V

Shear stress profile along the depth of the beam is a parabolic function.

Shear Stresses in a Rectangular Section Beam

$$\tau = \frac{V(x)Q(y)}{I_z t} \qquad t = b; \qquad Q(y) = \frac{b}{2}\left[\left(h/2\right)^2 - y^2\right]$$

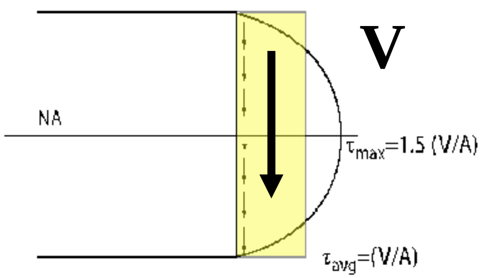




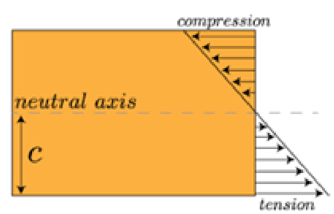
For the rectangular section beam,

$$\tau_{xy} = \frac{3V}{2A}\left(1 - \left(\frac{y}{(h/2)}\right)^2\right).$$

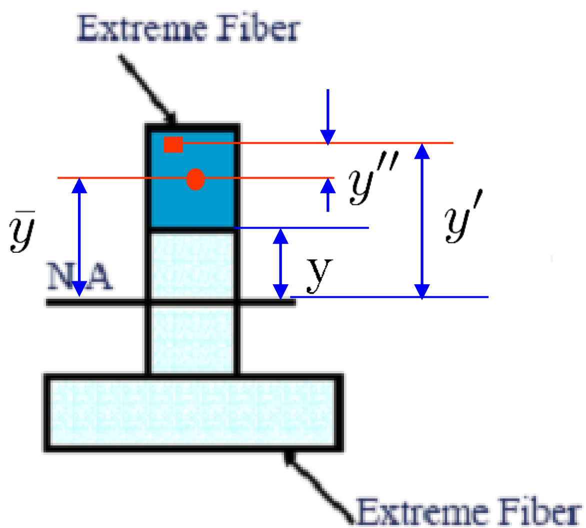
$$\tau_{max} = \tau(0) = \frac{Vh^2}{8I} = \frac{3V}{2A} = 1.5\frac{V}{A}$$



$$\boldsymbol{\tau_{\mathbf{max}} = 1.5\tau_{\mathbf{ave}}}$$



Application of Shear Formula:

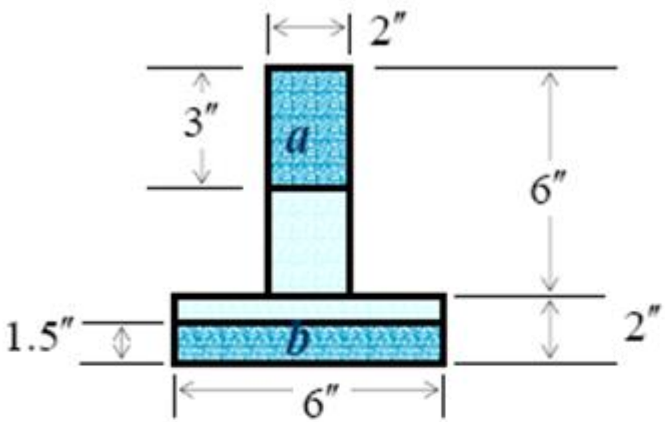


$$\tau = \frac{V(x)Q(y)}{I_z(x)t(y)} .$$

How to calculate $Q(y) = \int_A y' dA$?

Using the definition of centroid,
the first moment can be written,

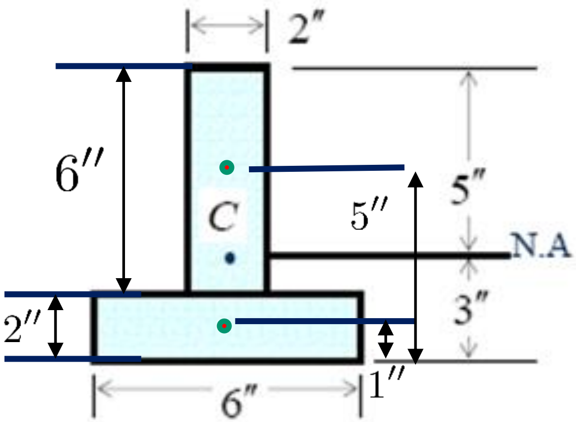
$$Q(y) \;=\; \int_A y' dA = \bar{y} A .$$



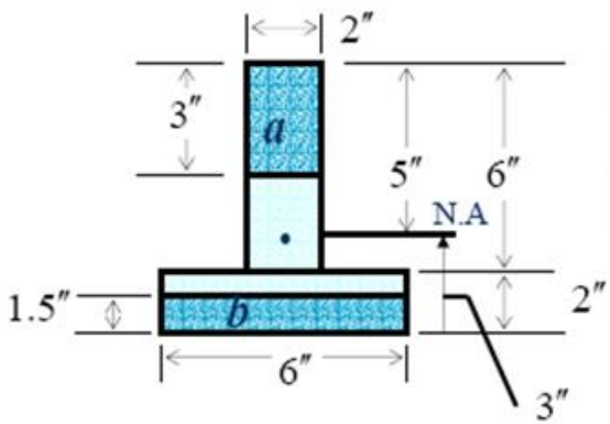
Find Q(y) for shape a **and shape** b**.**

First, we need to locate the neutral axis from the bottom edge:

$$y_C = \frac{(1)(2\times 6)+(2+3)(2\times 6)}{2\times 6+2\times 6} = \frac{72}{24} = 3'' \text{ from base}$$



$$Q(y) = \int_A y'\,dA = \bar{y}A\ .$$

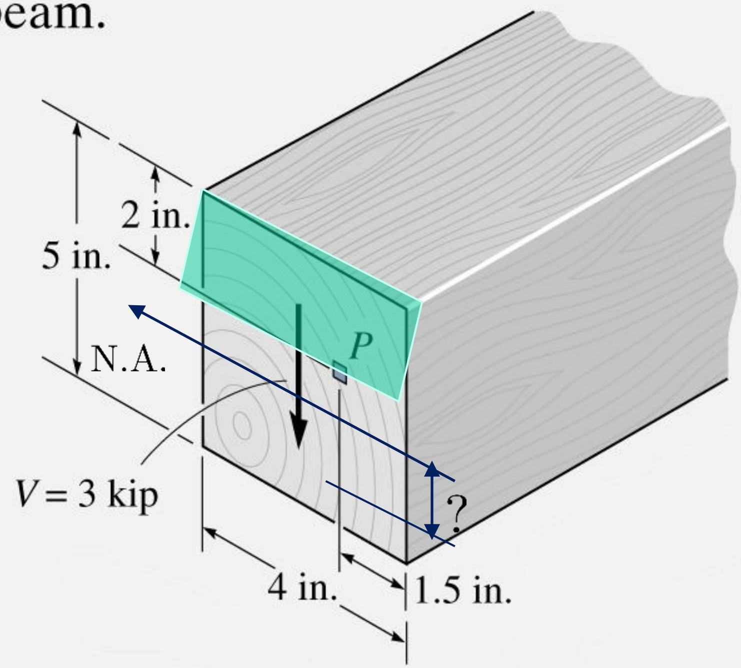


$$Q_a = (5-1.5)[3\times 2] = 21\ \text{in}^3$$

$$Q_b = \left(3-\frac{1.5}{2}\right)[1.5\times 6] = 20.25\ \text{in}^3$$

EXAMPLE 7-1

The beam shown in Fig. 7–10*a* is made of wood and is subjected to a resultant internal vertical shear force of $V = 3$ kip. (*a*) Determine the shear stress in the beam at point *P*, and (*b*) compute the maximum shear stress in the beam.



(a)

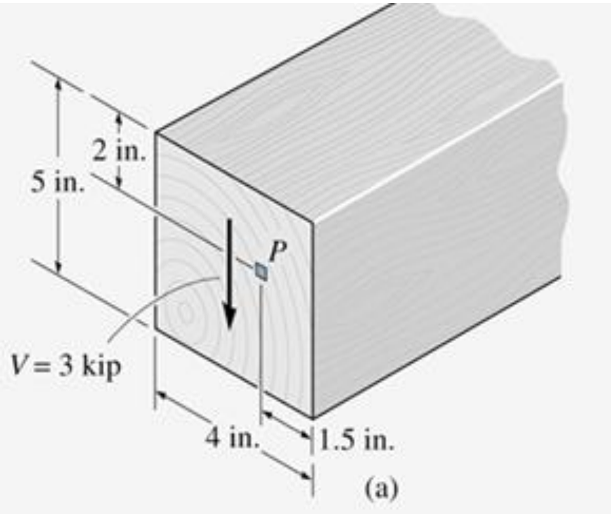
Where does the maximum shear stress occur ?

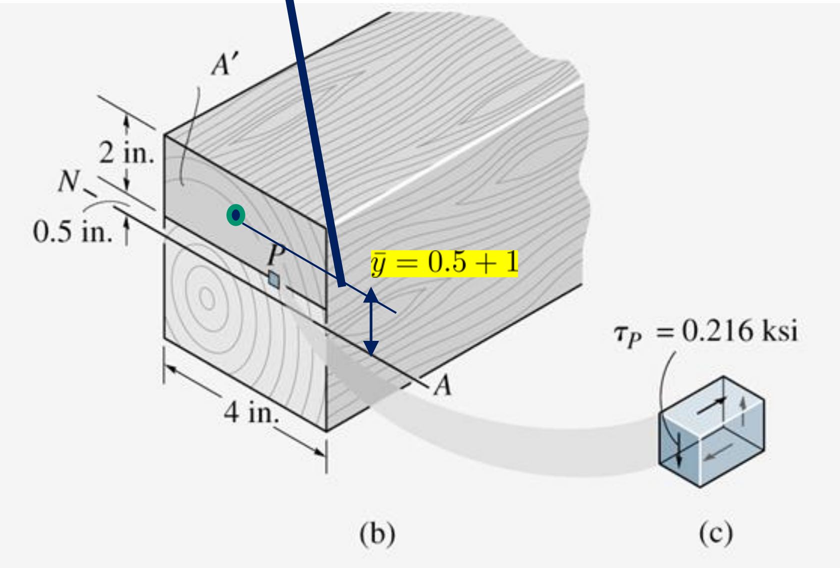
SOLUTION

Part (a)

$$I = \frac{1}{12}bh^3 = \frac{1}{12}(4\text{ in.})(5\text{ in.})^3 = 41.7\text{ in}^4$$

$$Q = \bar{y}'A' = \left[0.5\text{ in.} + \frac{1}{2}(2\text{ in.})\right](2\text{ in.})(4\text{ in.}) = 12\text{ in}^3$$



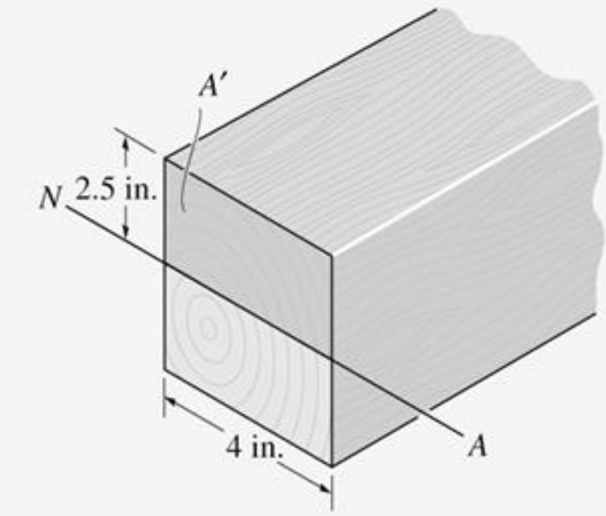


Shear Stress. The shear force at the section is $V = 3$ kip. Applying the shear formula, we have

$$\tau_P = \frac{VQ}{It} = \frac{(3\text{ kip})(12\text{ in}^3)}{(41.7\text{ in}^4)(4\text{ in.})} = 0.216\text{ ksi}$$

Part (b)

$$Q = \bar{y}'A' = \left[\frac{2.5\text{ in.}}{2}\right](4\text{ in.})(2.5\text{ in.}) = 12.5\text{ in}^3$$

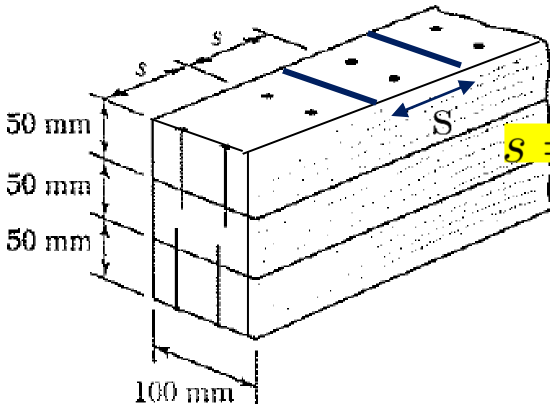


Shear Stress. Applying the shear formula yields

$$\tau_{\max} = \frac{VQ}{It} = \frac{(3\text{ kip})(12.5\text{ in}^3)}{(41.7\text{ in}^4)(4\text{ in.})} = 0.225\text{ ksi}$$

Note that this is equivalent to

$$\tau_{\max} = 1.5\frac{V}{A} = 1.5\frac{3\text{ kip}}{(4\text{ in.})(5\text{ in.})} = 0.225\text{ ksi}$$



$s = \Delta x$

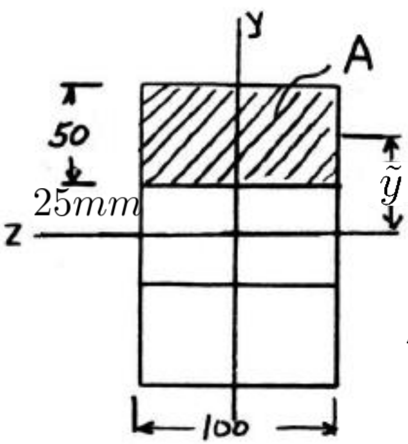
Fig. P13.1

13.1 Three full-size 50 × 100-mm boards are nailed together to form a beam that is subjected to a vertical shear of 1500 N. Knowing that the allowable shearing force in each nail is 400 N, determine the largest longitudinal spacing s that can be used between each pair of nails.

Nails deliver horizontal shear force, which is horizontal shear flow × the longitudinal spacing.

We need to find the shear flow first: $q = \dfrac{VQ}{I_z} = \dfrac{\Delta H}{\Delta x}$;

$$V = 1500N$$



$$I_z = \frac{bh^3}{12} = \frac{(100)(150)^3}{12} = 28.125 \times 10^6 mm^4$$

$$Q(y) = y_c A = (50)(50 \times 100) = 250 \times 10^{-6} m^3$$

$$\tilde{y} = 50mm$$

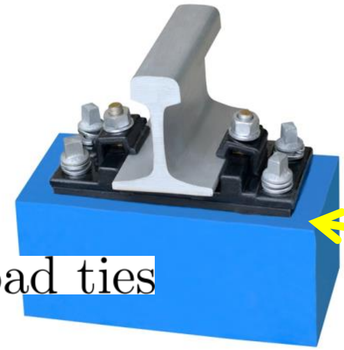
Hence $$q = \frac{(1500)(250)}{28.125} = 13.333 \times 10^3 N/m$$

$$\Delta H$$

$$qs = 2F_{nail} = 2 \times 400 = 800N \quad \rightarrow \quad s = \frac{800}{13333} = 60 \times 10^{-3} m$$

$s = \Delta x$

The maximum nail spacing is 60 mm.



Dmitrii Ivanovich Zhuravskii (1821-1891)

$$\tau = \frac{VQ}{It} \; .$$

Continuous Welded Rail

12.27 Determine (*a*) the distance *a* for which the maximum absolute value of the bending moment in the beam is as small as possible, (*b*) the corresponding maximum normal stress due to bending. (See hint for Prob. 12.25.)

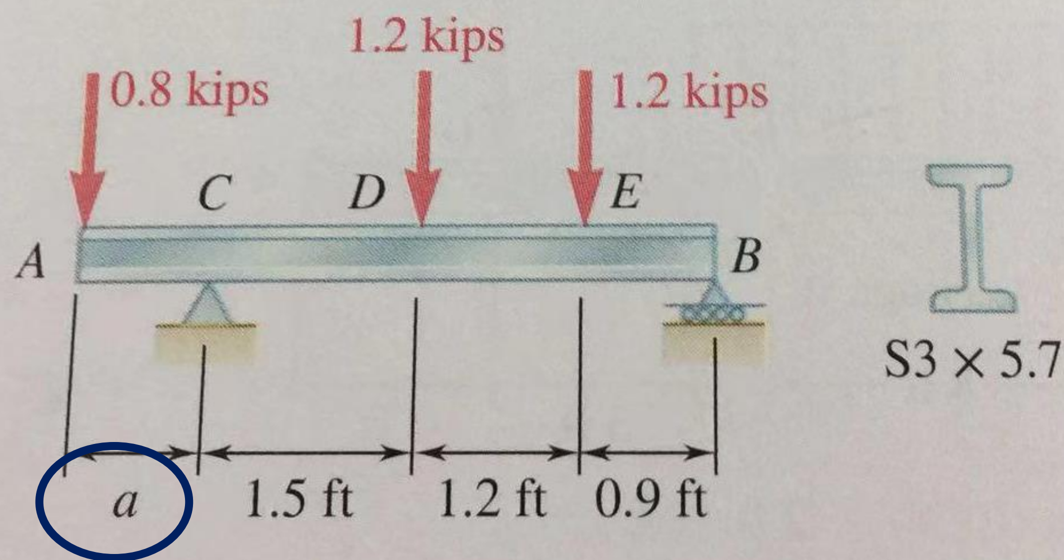
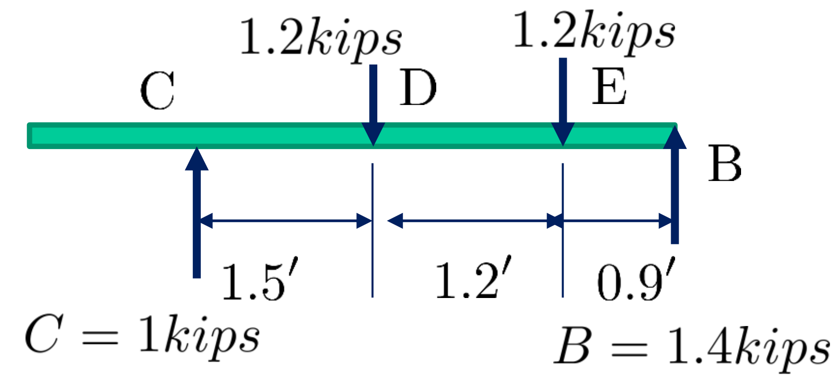
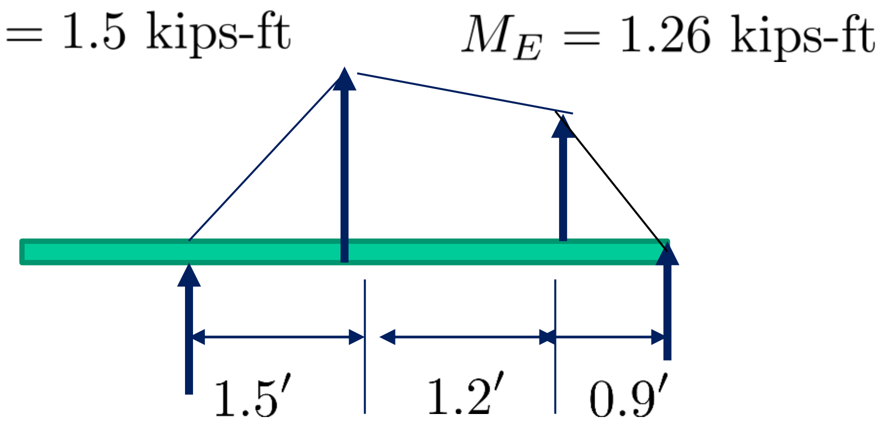


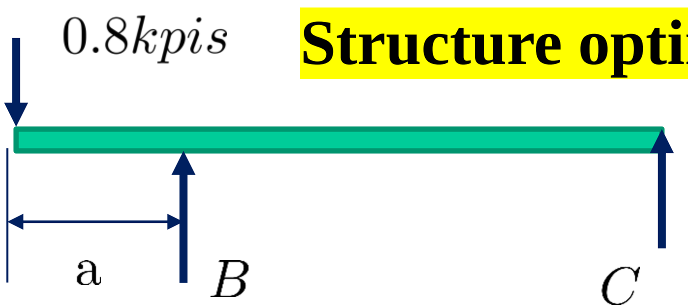
Fig. P12.27

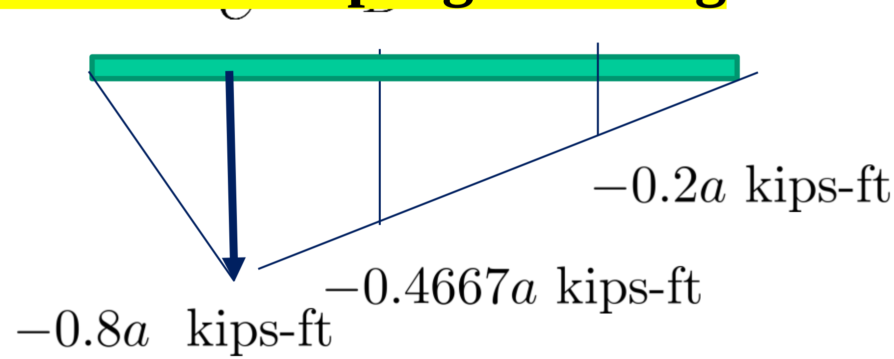
Hint: Draw the bending-moment diagram and equate the absolute values of the largest positive and negative bending moments obtained.

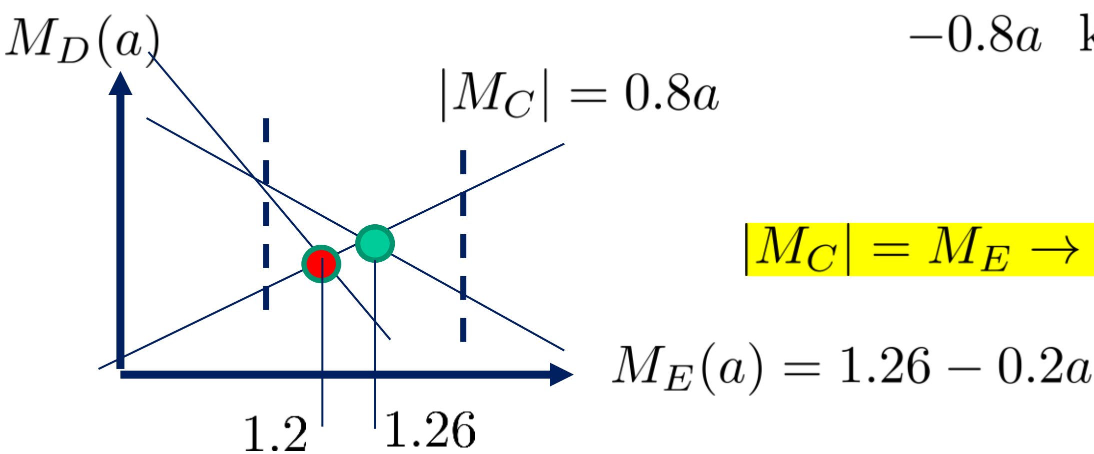




Structure optimization via linear programming







$$M_E = 1.26 - 0.2a$$

$$M_D = 1.5 - 0.4667a$$

$$|M_C| = M_E \rightarrow 1.26 - 0.2a = 0.8a \rightarrow a = 1.26ft$$

$$|M_C| = M_D \rightarrow 1.5 - 0.4667a = 0.8a \rightarrow a = 1.20ft$$