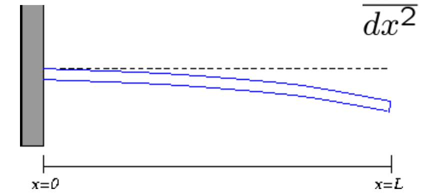
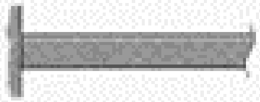
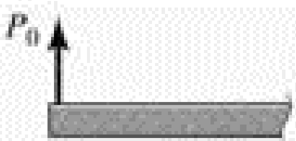
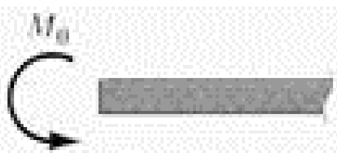
Lecture 31 Beam Deflection

Boundary Conditions

$$\frac{d^2}{dx^2}\left(EI\frac{d^2}{dx^2}y(x)\right) = -w(x):$$



Type	Symbol*
Fixed End	 X=0
Simple Support	
Free End	
Concentrated Force	
Concentrated Couple	

$$\mathbf{1}:\ y(0) = 0,\ \theta(0) = y'(0) = 0;$$

$$\mathbf{2}:\ y(0) = 0,\ \kappa(0) = M(0) = EIy''(0) = 0;$$

$$\mathbf{3}:\ M(0) = EIy''(0) = 0,\ V(0) = EIy'''(0) = 0;$$

$$\mathbf{4}:\ M(0) = EIy''(0) = 0,\ V(0) = EIy'''(0) = P_0;$$

$$\mathbf{5}:\ M(0) = EIy''(0) = -M_0,\ V(0) = EIy'''(0) = 0;$$

Equation of the Elastic Curve (II) $EIy^{(iv)}(x) = -w(x)$:

$$(1) V(x) = EIy'''(x) = \int_0^x -w(t_1)dt_1 + C_1, \text{ so } C_1 = V(0) = EIy'''(0);$$

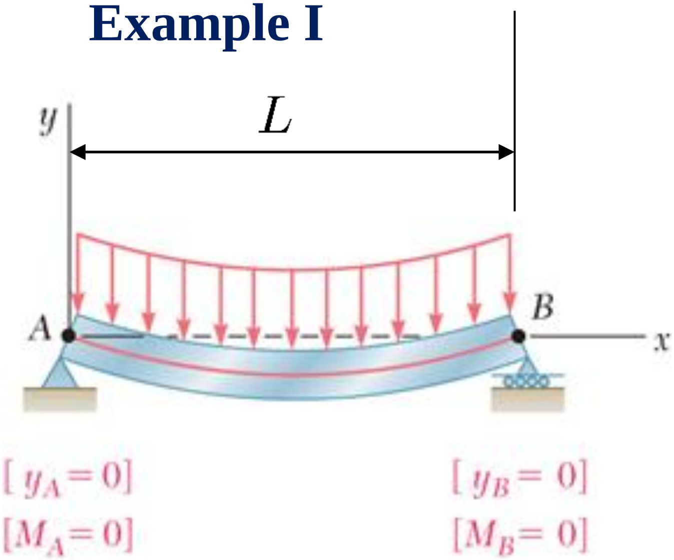
$$(2) \quad M(x) = EIy''(x) = \int_0^x \int_0^{t_2} -w(t_1)dt_1 dt_2 + C_1 x + C_2,$$
$$\text{so} \quad C_2 = M(0) = EIy''(0);$$

$$(3) \quad EI\theta = EIy'(x) = \int_0^x \int_0^{t_3} \int_0^{t_2} -w(t_1)dt_1 dt_2 dt_3 + C_1 \frac{x^2}{2} + C_2 x + C_3,$$
$$\text{so} \quad C_3 = EI\theta(0);$$

$$(4) \quad EIy(x) = \int_0^x \int_0^{t_4} \int_0^{t_3} \int_0^{t_2} -w(t_1)dt_1 dt_2 dt_3 dt_4 + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2}$$
$$+C_3 x + C_4, \qquad \text{so} \quad C_4 = EIy(0);$$

Remark: In a given problem, one cannot know all four boundary conditions at one end. One can only find two boundary conditions at a given end.

Example I



$$w(x) = w_0 \qquad EIy^{(iv)} = -w_0$$

[Solution]

$$EIy''' = -w_0 x + C_1$$

$$EIy'' = -\frac{w_0 x^2}{2} + C_1 x + C_2, \quad (C_2 = M(0) = 0)$$

Based on $M(L) = 0$,

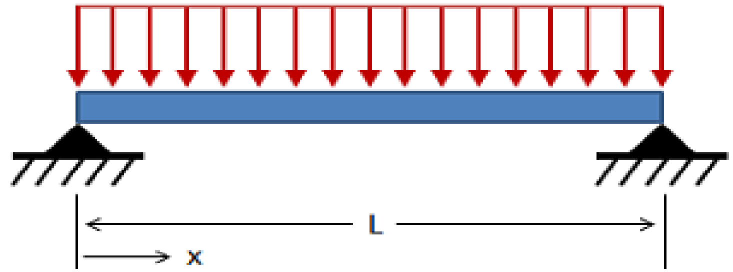
$$\rightarrow \quad -w_0 L^2/2 + C_1 L = 0, \quad \rightarrow \quad C_1 = \frac{1}{2} w_0 L = V(0)$$

$$EIy' = -\frac{w_0 x^3}{6} + \frac{w_0 L x^2}{4} + C_3$$

$$EIy(x) = -\frac{w_0 x^4}{24} + \frac{w_0 L x^3}{12} + C_3 x + C_4 \qquad y(0) = 0, \quad \rightarrow \quad C_4 = 0$$

$$EIy(L) = -\frac{w_0 L^4}{24} + \frac{w_0 L^4}{12} + C_3 L = 0 \quad \rightarrow \quad C_3 = -\frac{w_0 L^3}{24} = \theta(0)$$

$$EIy^{(iv)}(x) = -w_0$$



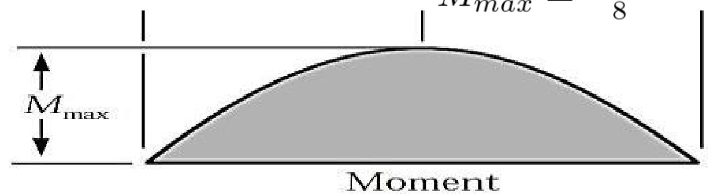
$$EIy'''(x) = V(x) = -w_0 x + \frac{w_0 L}{2}$$

$$-\frac{w_0 l}{2}$$

$$V(0) = \frac{w_0 l}{2}$$

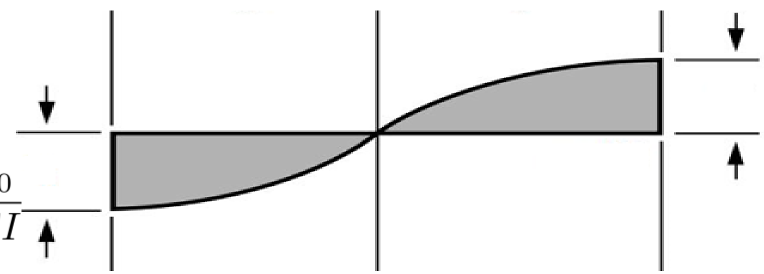
$$M_{max} = \frac{w_0 l^2}{8}$$

$$EIy''(x) = M(x) = -\frac{w_0 x^2}{2} + \frac{w_0 L x}{2}$$



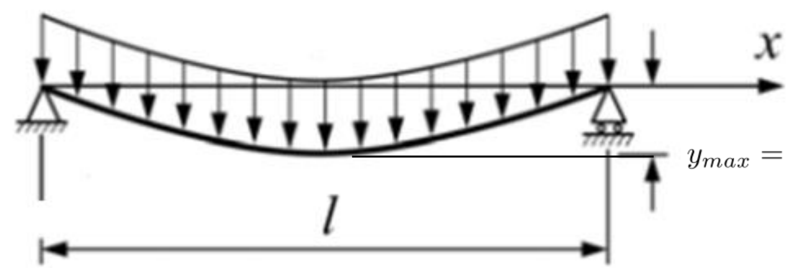
$$y'(x) = \theta(x) = -\frac{w_0}{24EI}\left(4x^3 - 6Lx^2 + L^3\right)$$

$$-\frac{l^3 w_0}{24EI}$$



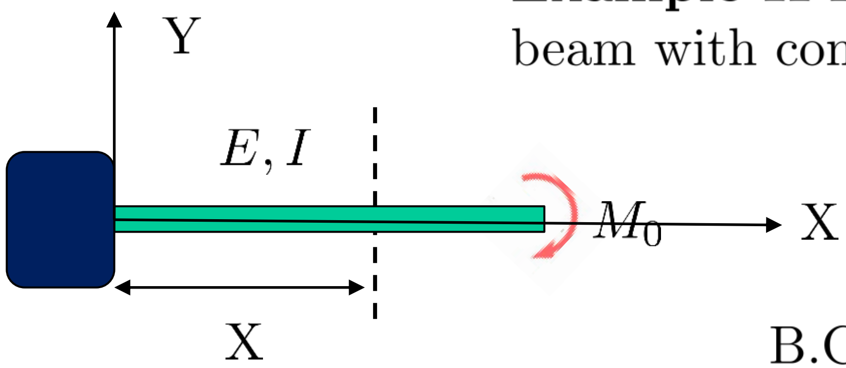
$$\frac{l^3 w_0}{24EI}$$

$$\boxed{y(x) = -\frac{w_0}{24EI}\left[x^4 - 2Lx^3 + L^3 x\right]}$$



$$y_{max} = -\frac{5w_0 l^4}{384EI}$$

Example II Find the deflection of a cantilever beam with constant bending moment.



B.C.: $y(0) = 0$ and $y'(0) = 0$

$$M(x) = -M_0$$

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} = -\frac{M_0}{EI};$$

$$\frac{dy}{dx} = -\frac{M_0 x}{EI} + C_1, \quad C_1 = 0.$$

$$y(x) = -\frac{M_0 x^2}{2EI} + C_2, \quad C_2 = 0.$$

This is not a circle !

Remark

(1) $y(x)$ — the deflection of the beam at given location of a cross-section.

We made two approximations:

$d\ell \approx dx$

$$y' = \tan\theta \approx \theta, \quad \text{and} \quad \frac{d\theta}{d\ell} \approx \frac{d\theta}{dx} = y''$$

(2) $y'(x) \approx \theta(x)$ — the rotation at the specific cross section;

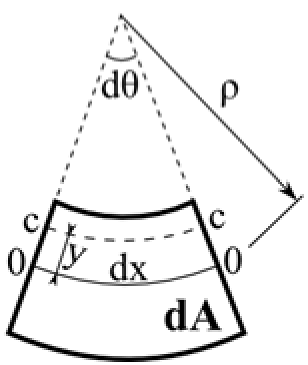
(3) $y''(x) \approx \kappa = 1/\rho$ —- the curvature at location x, which is related with $EIy''(x) \approx M(x)$ — the internal moment;

(4) $V(x) = \dfrac{dM(x)}{dx} = EIy'''(x)$ — the internal shear force.

Recall

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

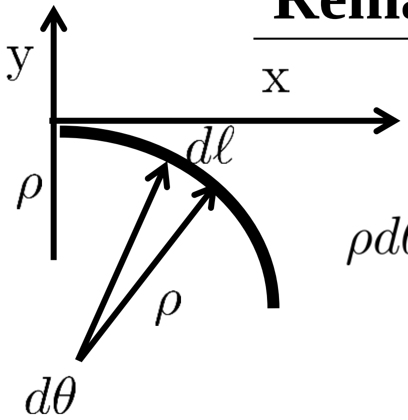
In general

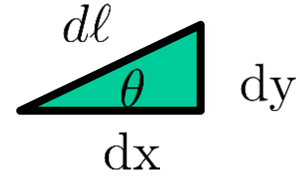


$$\theta = \frac{dy}{d\ell} \quad \text{and} \quad \tan\theta = \frac{dy}{dx} \quad \to \quad \theta = \tan^{-1} y'$$

$$d\ell = \sqrt{dx^2 + dy^2} = \sqrt{1 + (y')^2}dx$$

Remark: Example II





$$\rho d\theta = d\ell$$

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{d\ell} = \frac{d\theta}{dx}\frac{dx}{d\ell} = \cos\theta \frac{d\theta}{dx}$$

$$\rightarrow \quad \cos\theta d\theta = \kappa dx \quad \sin\theta = \kappa x + C_1$$

$$\text{Consider } x = 0, y'(0) = 0 \quad \rightarrow \quad C_1 = 0.$$

$$\text{Since } \frac{dy}{dx} = \tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{\kappa x}{\sqrt{1-(\kappa x)^2}}$$

which leads to

$$y(x) = \frac{1}{\kappa}\sqrt{1-(\kappa x)^2} + C_2$$

Consider $x = 0, y(0) = 0 \quad \rightarrow \quad C_2 = -\rho.$ This leads to

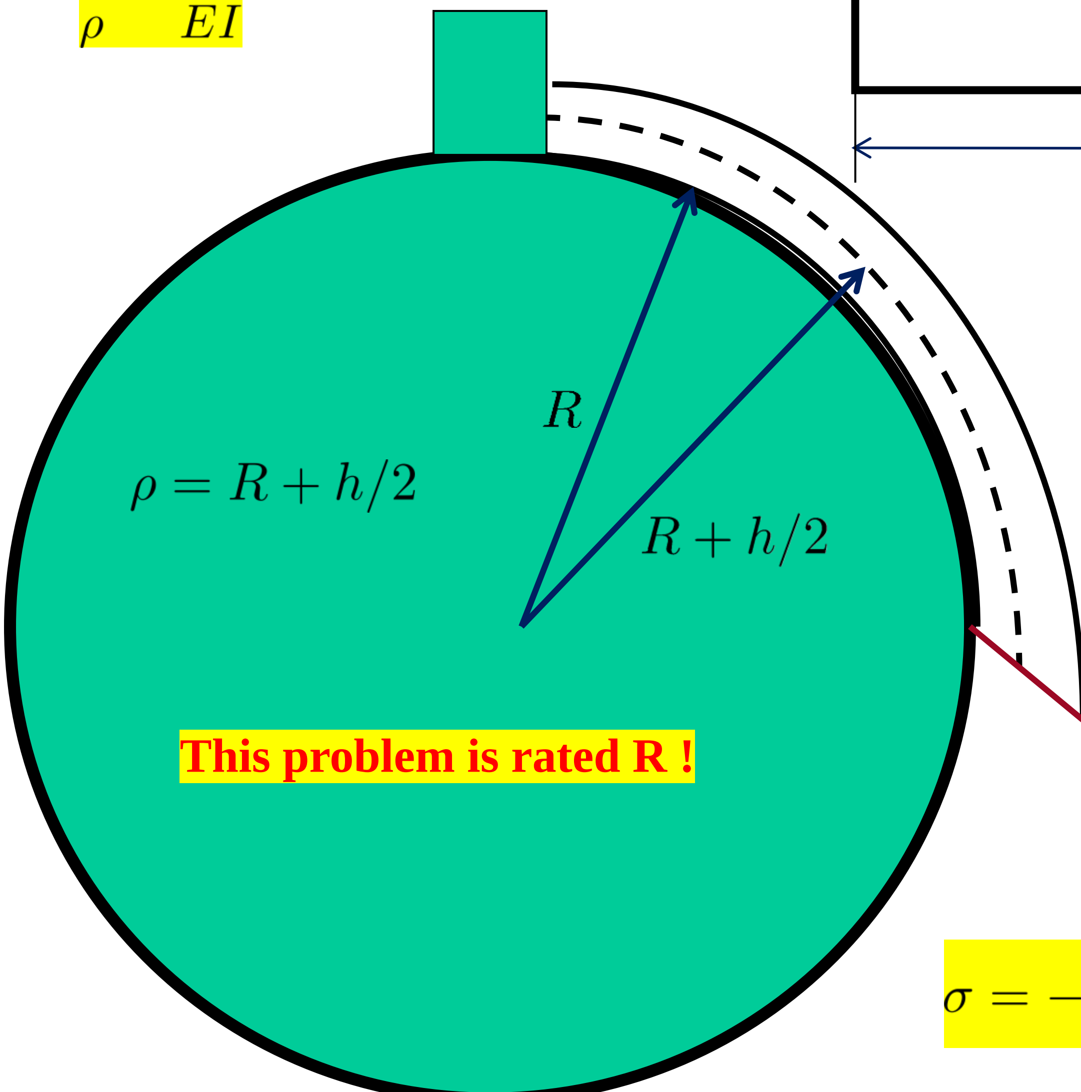
$$x^2 + (y+\rho)^2 = \frac{1}{\kappa^2} = \rho^2$$

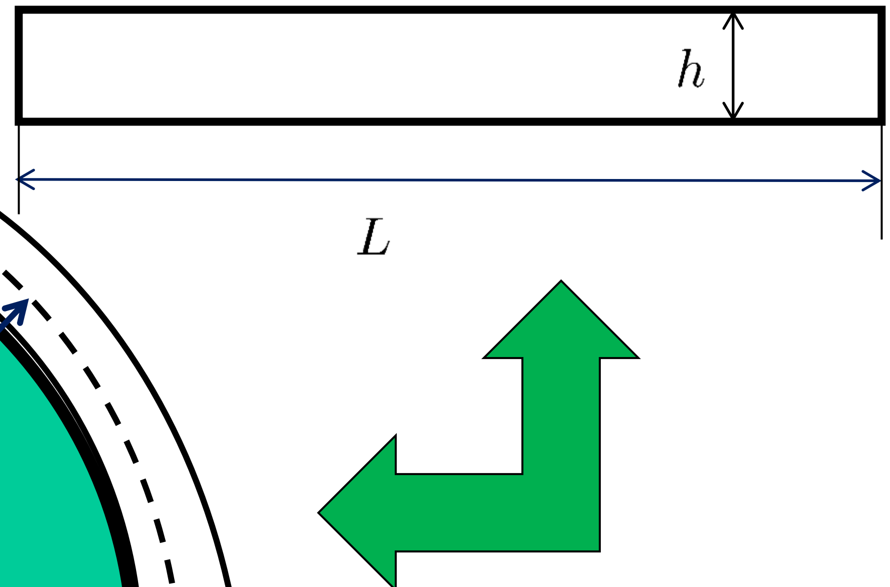
A Circle!

1. What is the bending moment in this beam ?

$$M = \frac{EI}{R + h/2}$$

$$\frac{1}{\rho} = \frac{M}{EI}$$



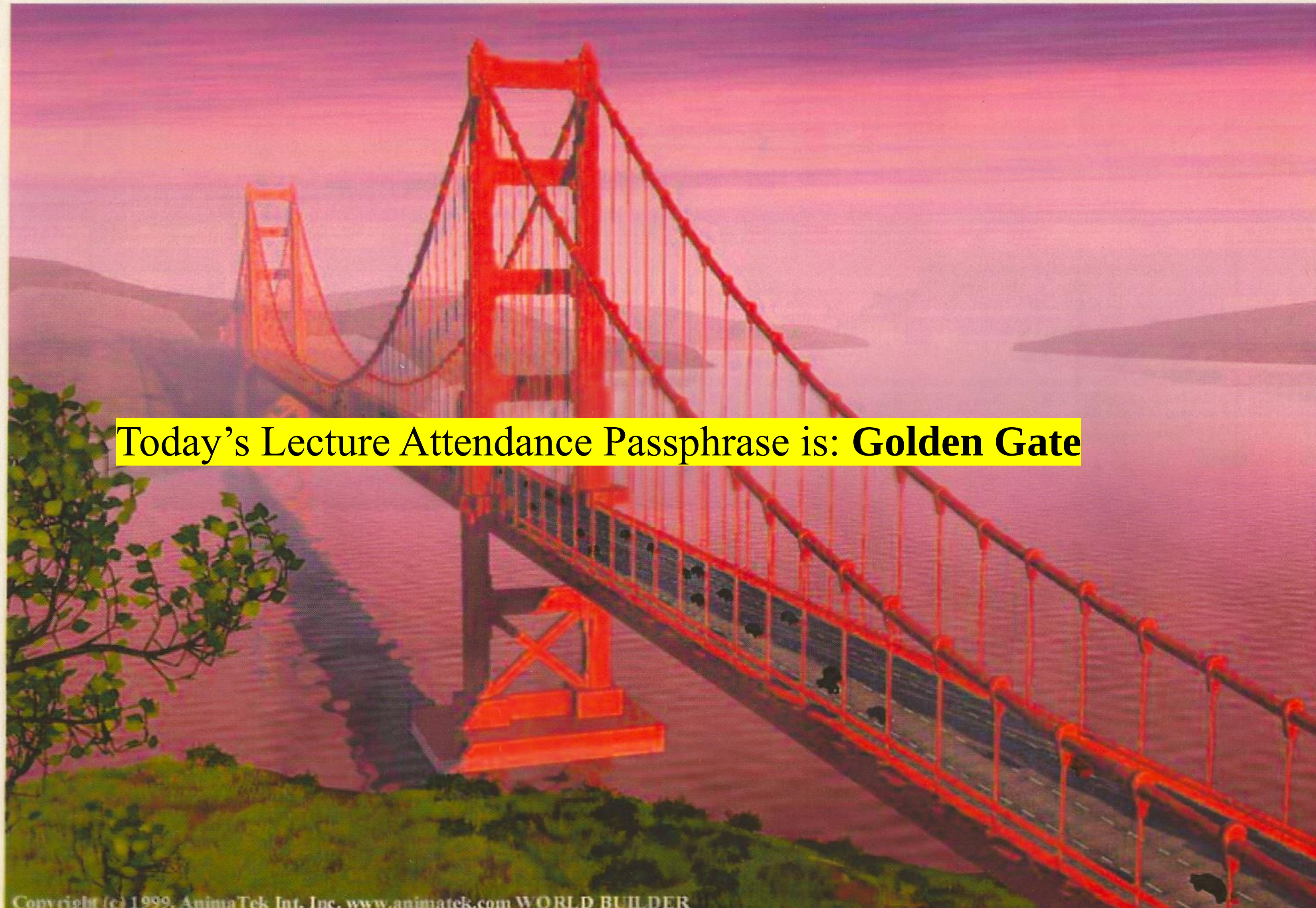


2. What is the maximum stress inside the beam?

$$\sigma_m = E\frac{h/2}{R + h/2}$$

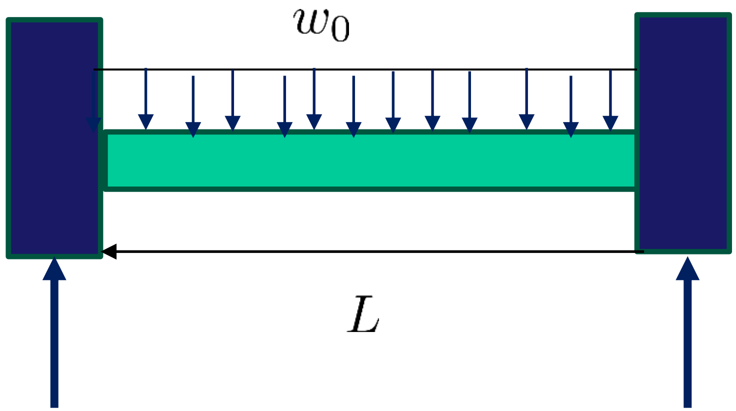
$$\sigma = -\frac{My}{I}$$

Golden Gate Bridge



Today’s Lecture Attendance Passphrase is: **Golden Gate**

$$y(0) = 0, y'(0) = 0 \qquad y(L) = 0, y'(L) = 0$$



$$EIy^{(iv)} = -w_0$$

$$EIy''' = -w_0 x + C_1$$

$V(0) = \frac{w_0 L}{2}$

$$C_1 = \frac{w_0 L}{2}$$

$$EIy'' = -\frac{w_0 x^2}{2} + \frac{w_0 L x}{2} + C_2,$$

$$EIy' = -\frac{w_0 x^3}{6} + \frac{w_0 L x^2}{4} + C_2 x + C_3 \qquad y'(0) = 0 \rightarrow C_3 = 0$$

$$-\frac{w_0 L^3}{6} + \frac{w_0 L^3}{4} + C_2 L = 0 \;\rightarrow\; C_3 = -\frac{w_0 L^2}{12}$$

$$EIy(x) = -\frac{w_0 x^4}{24} + \frac{w_0 L x^3}{12} - \frac{w_0 L^2 x}{12} + C_4 \qquad y(0) = 0 \rightarrow C_4 = 0$$

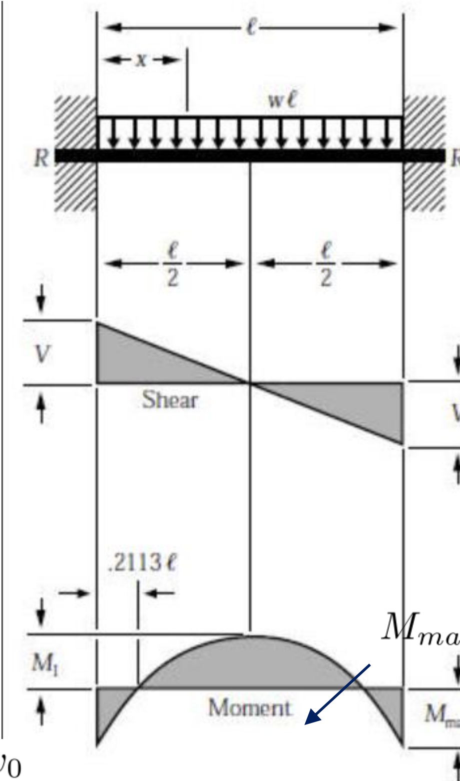
$$EIy^{(iv)}(x) = -w_0$$

$$EIy'''(x) = V(x) = -w_0 x + \frac{w_0 L}{2}$$

$$EIy''(x) = M(x) = -\frac{w_0 x^2}{2} + \frac{w_0 L x}{2} - \frac{w_0 L^2}{12}$$

$$y'(x) = \theta(x) = -\frac{w_0}{24EI}\left(4x^3 - 6Lx^2 + 2L^2 x\right)$$

$$y(x) = -\frac{w_0}{24EI}\left(x^4 - 2Lx^3 + L^2 x^2\right)$$

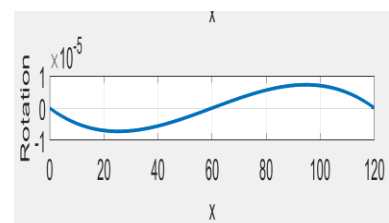


$$V(0) = \frac{w_0 l}{2}$$

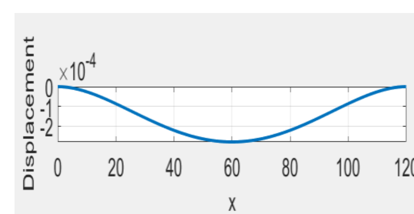
$$-\frac{w_0 l}{2}$$

$$M_{max} = \frac{w_0 L^4}{24EI}$$

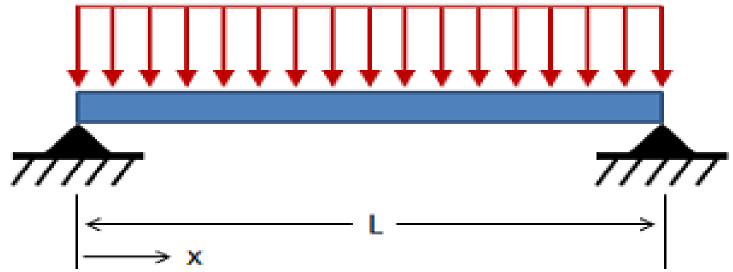
$$M(0) = -\frac{L^3 w_0}{12EI}$$



$$y_{max} = -\frac{w_0 l^4}{384EI}$$



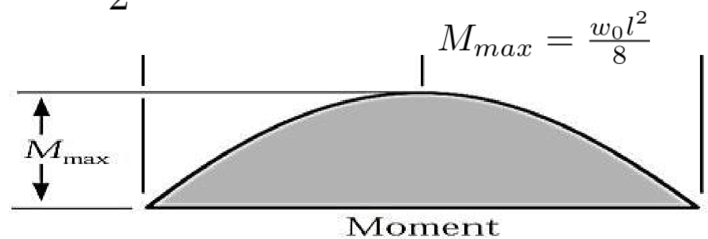
Compare



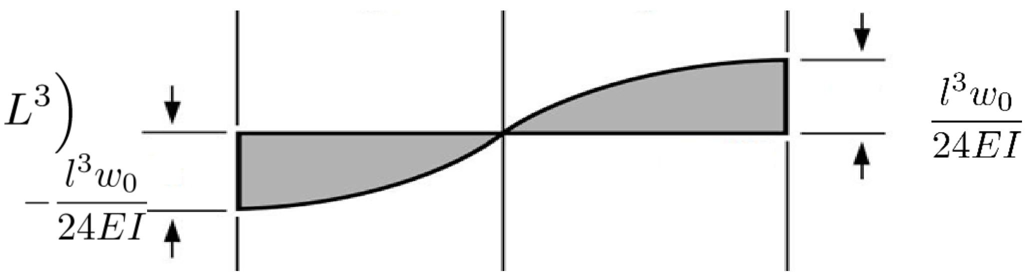
$$EIy^{(iv)}(x) = -w_0$$

$$EIy'''(x) = V(x) = -w_0 x + \frac{w_0 L}{2}$$

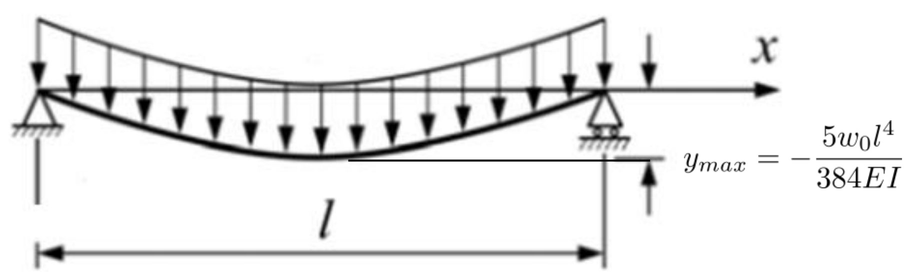
$$EIy''(x) = M(x) = -\frac{w_0 x^2}{2} + \frac{w_0 L x}{2}$$

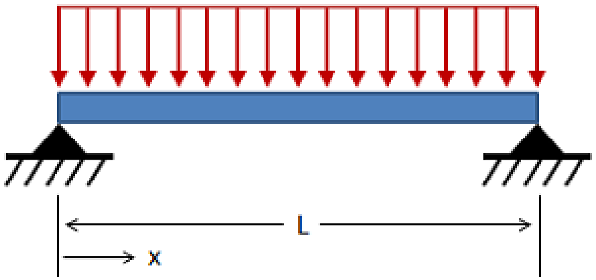


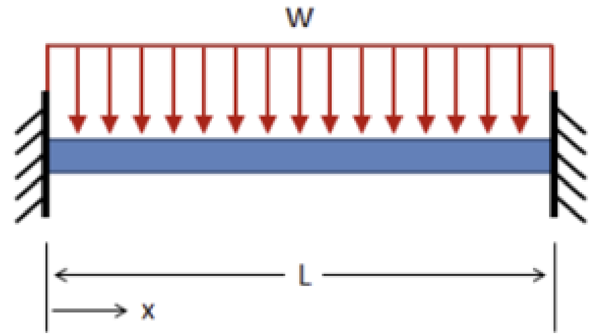
$$y'(x) = \theta(x) = -\frac{w_0}{24EI}\left(4x^3 - 6Lx^2 + L^3\right)$$



$$y(x) = -\frac{w_0}{24EI}\left[x^4 - 2Lx^3 + L^3 x\right]$$



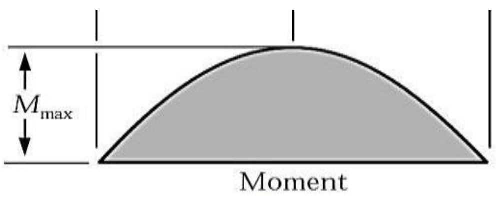




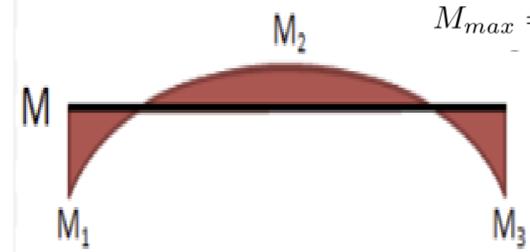
x=0, y(0) =0;
x=0, M(0) =0

x=0, y(0) =0;
x=0, θ(0) =0

$$M_{max} = \frac{w_0 l^2}{8}$$



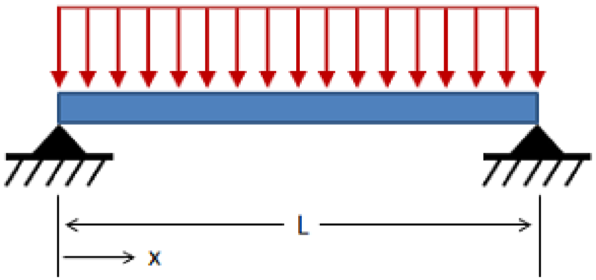
$$M_{max} = \frac{w_0 L^2}{24}$$

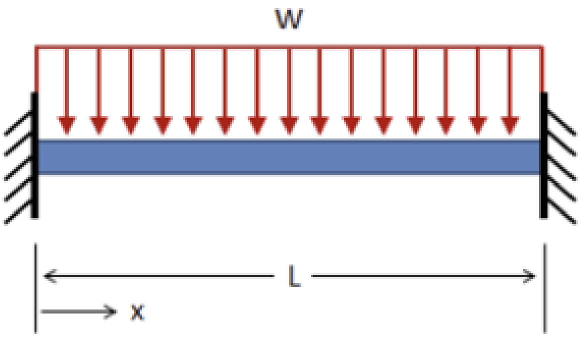


$$M(x) = -\frac{w_0 x^2}{2} + \frac{w_0 L x}{2}$$

$$M(x) = -\frac{w_0 x^2}{2} + \frac{w_0 L x}{2} - \frac{w_0 L^2}{12}$$

For the same external load, the statistically indeterminant system has lower value of the maximum internal force. However, …..





x=0, y(0) =0;
x=0, M(0) =0

x=0, y(0) =0;
x=0, θ(0) =0

$$y(x) = -\frac{w_0}{24EI}[x^4 - 2Lx^3 + L^3x]$$

$$y(x) = -\frac{w_0}{24EI}\left(x^4 - 2Lx^3 + L^2x^2\right)$$

$$y_{max} = -\frac{5w_0L^4}{(16\times24)EI}$$

$$= -\frac{5M_{max}L^2}{48EI}$$

$$M_{max} = w_0L^2/8$$

$$y_{max} = -\frac{w_0L^4}{(16\times24)EI}$$

$$= -\frac{M_{max}L^2}{EI}$$

$$M_{max} = w_0L^2/24$$

$$M_{max} = 9.6|y_{max}|EI/L^2$$

$$M_{max} = 16|y_{max}|EI/L^2$$

For the same maximum deflection, the statistically determinant system has lower value of the maximum internal force.

San Francisco 1989 Earthquake

Golden Gate Bridge

Oakland Bay Bridge

……whereas for the golden gate bridge, its flexible suspension structure might have been its salvation, …..