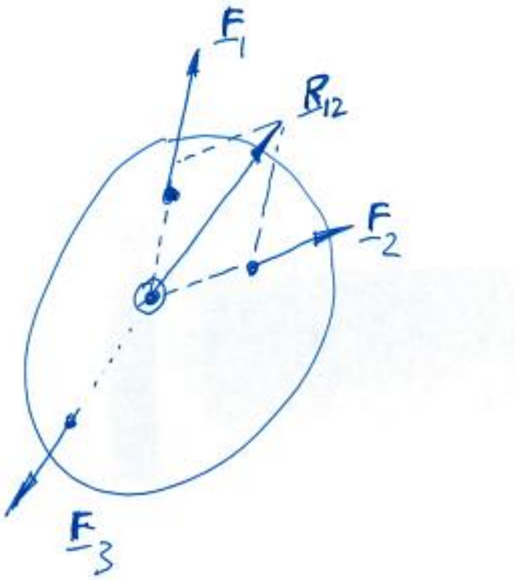
Lecture 7
Equilibrium of Rigid Bodies (3D)

Three-force member

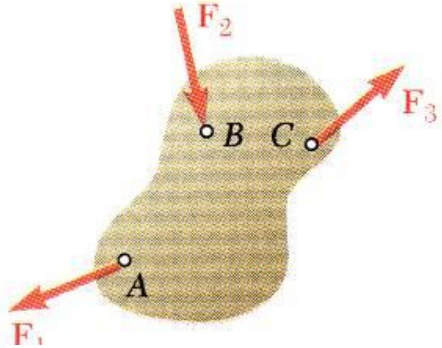
For three non-parallel forces acting on a rigid body, the line of action of the three forces must interect at on common point.

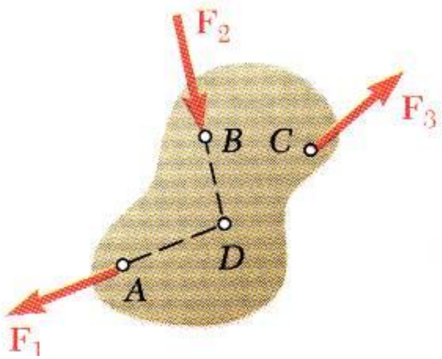


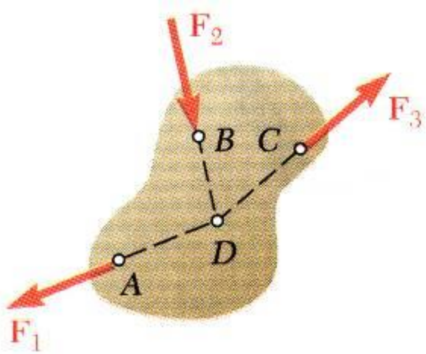
Why ?

because the resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$, $\mathbf{R}_{12}$, will form a two-force system with the remaining force $\mathbf{F}_3$.

Equilibrium of a Three-Force Body







- Consider a rigid body subjected to forces acting at only 3 points.

- Assuming that their lines of action intersect, the moment of $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ about the point of intersection represented by *D* is zero.

- Since the rigid body is in equilibrium, the sum of the moments of $\boldsymbol{F_1}$, $\boldsymbol{F_2}$, and $\boldsymbol{F_3}$ about any axis must be zero. It follows that the moment of $\boldsymbol{F_3}$ about *D* must be zero as well and that the line of action of $\boldsymbol{F_3}$ must pass through *D*.

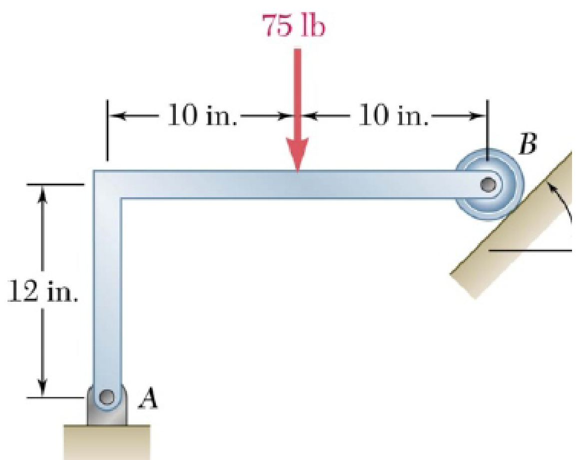
- The lines of action of the three forces must be **concurrent** or parallel.

Definition:

If three **non-parallel** forces act on a rigid body in equilibrium, it is known as a three-force member.

Three-force member principle

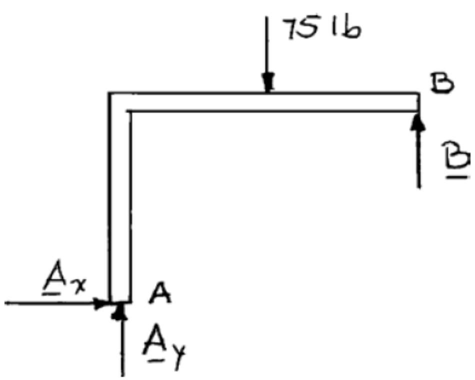
If a three-force member is in equilibrium, the line of action of all three forces must intersect at a common point; and the total resultant is zero. In other words, any single force is the equilibrant of the two other forces.



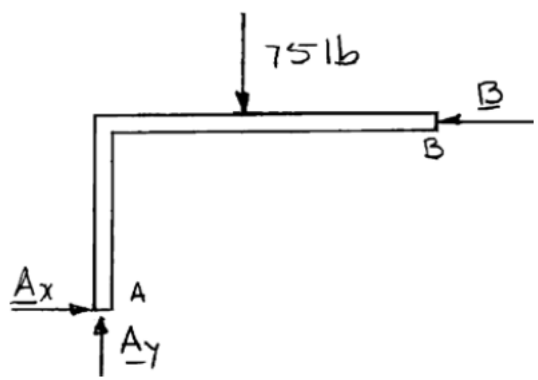
PROBLEM 4.13

Determine the reactions at A and B when (*a*) $\alpha = 0$, (*b*) $\alpha = 90°$, (*c*) $\alpha = 30°$.

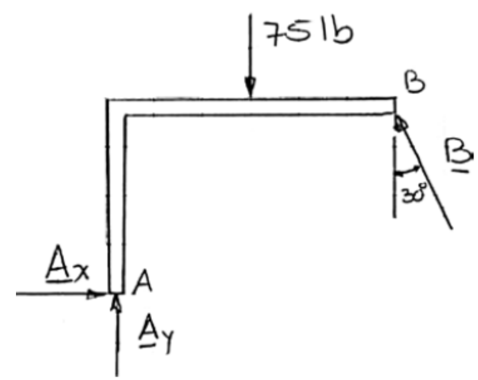
$\alpha = 0$

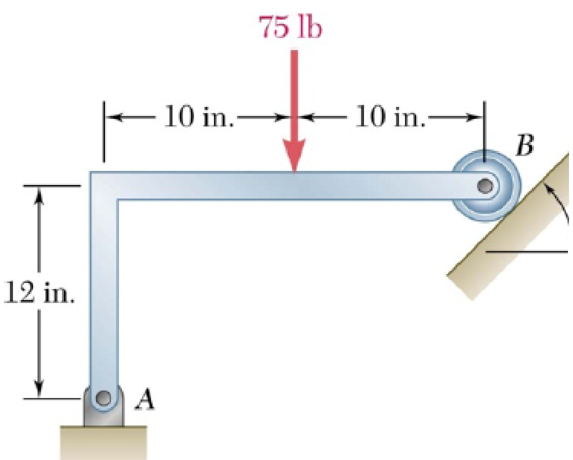


$\alpha = 90^o$



$\alpha = 30^o$

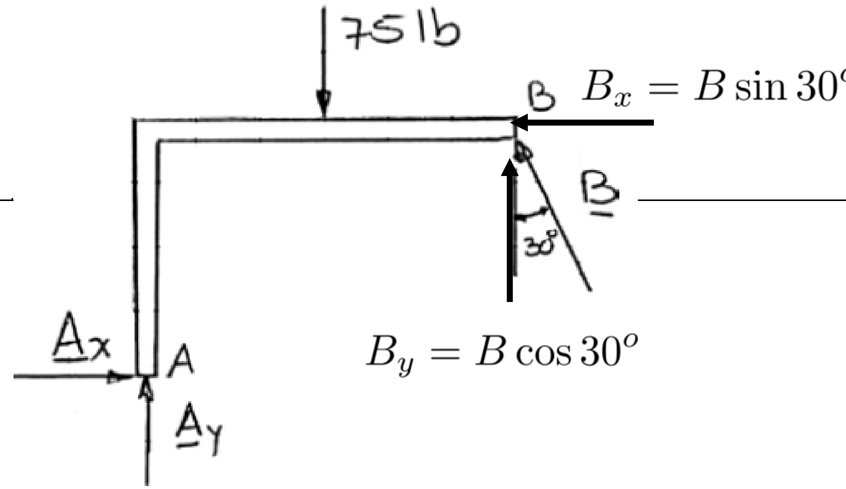




PROBLEM 4.13

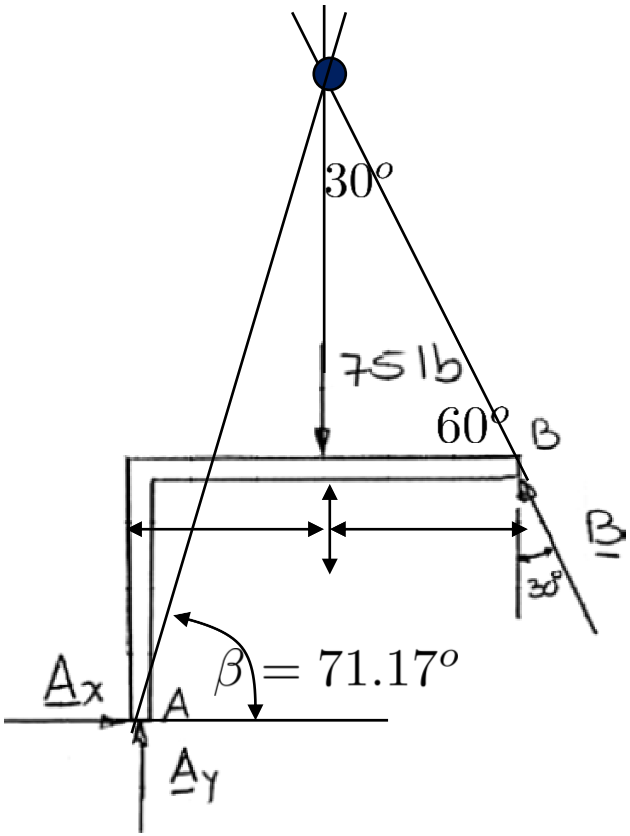
Determine the reactions at A and B when (a) $\alpha = 0$, (b) $\alpha = 90°$, (c) $\alpha = 30°$.

$$\alpha = 30^o$$

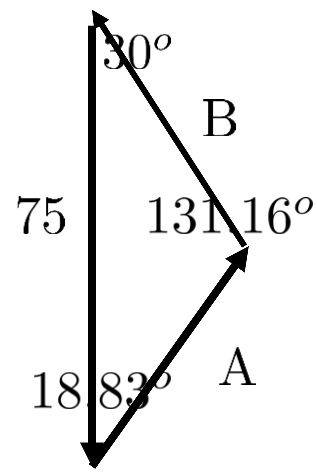


$$\sum M_A = 0 \;\rightarrow\; 12B_x + 20B_y - 75\times 10 = 0$$

$$B = \frac{750}{12\sin 30^o + 20\cos 30^o}$$



$$\tan\beta = \frac{10\tan 60^o + 12}{10} = 2.932$$



$$\frac{75}{\sin 90.57^o} = \frac{B}{\sin 59.43^o} = \frac{A}{\sin 30^o}$$

Equilibrium of a Rigid Body in Three Dimensions

$$\sum \vec{F} = 0 \quad \sum \vec{M}_O = \sum \left(\vec{r} \times \vec{F} \right) = 0$$

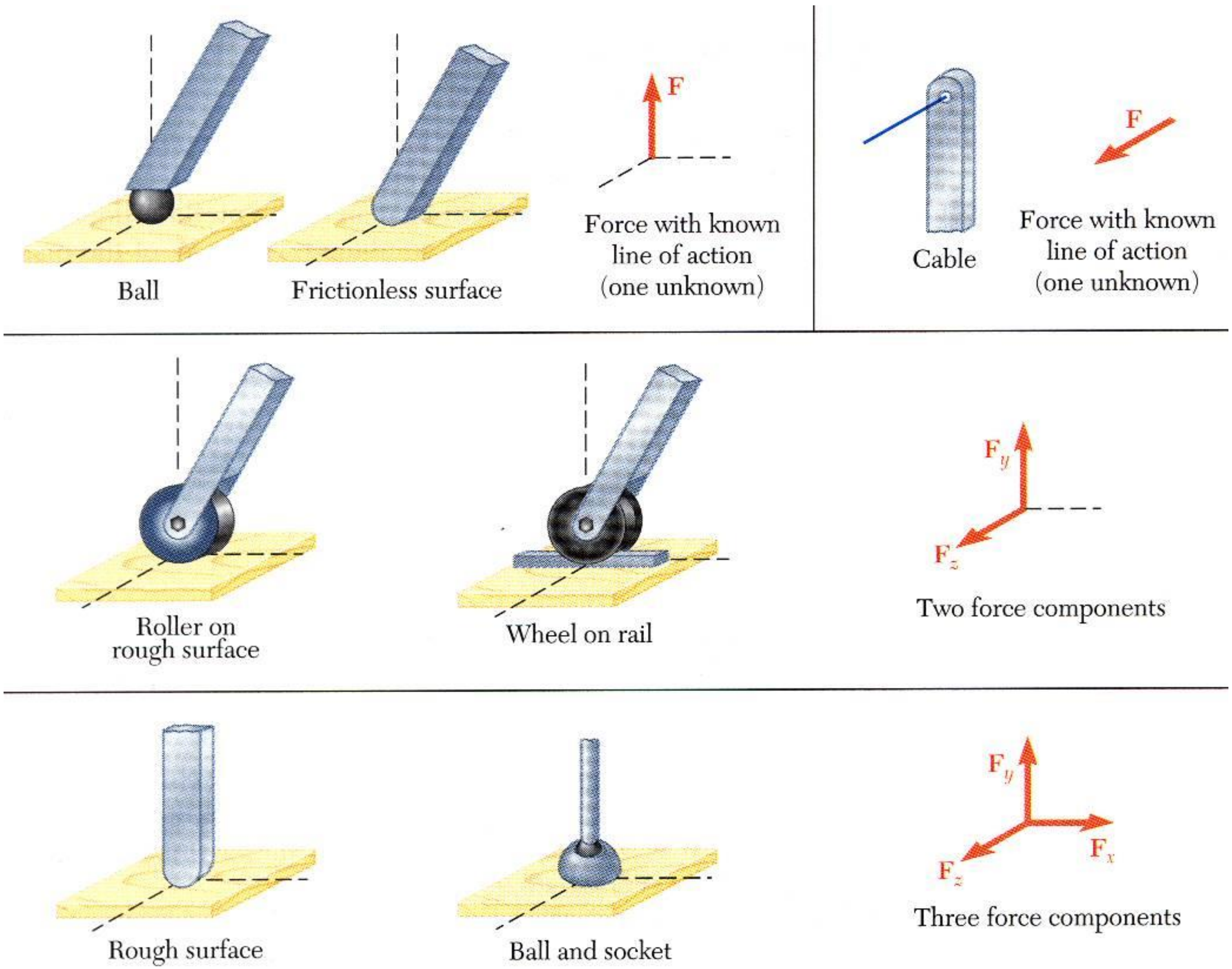
- Six scalar equations are required to express the conditions for the equilibrium of a rigid body in the general three dimensional case.

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0$$
$$\sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0$$

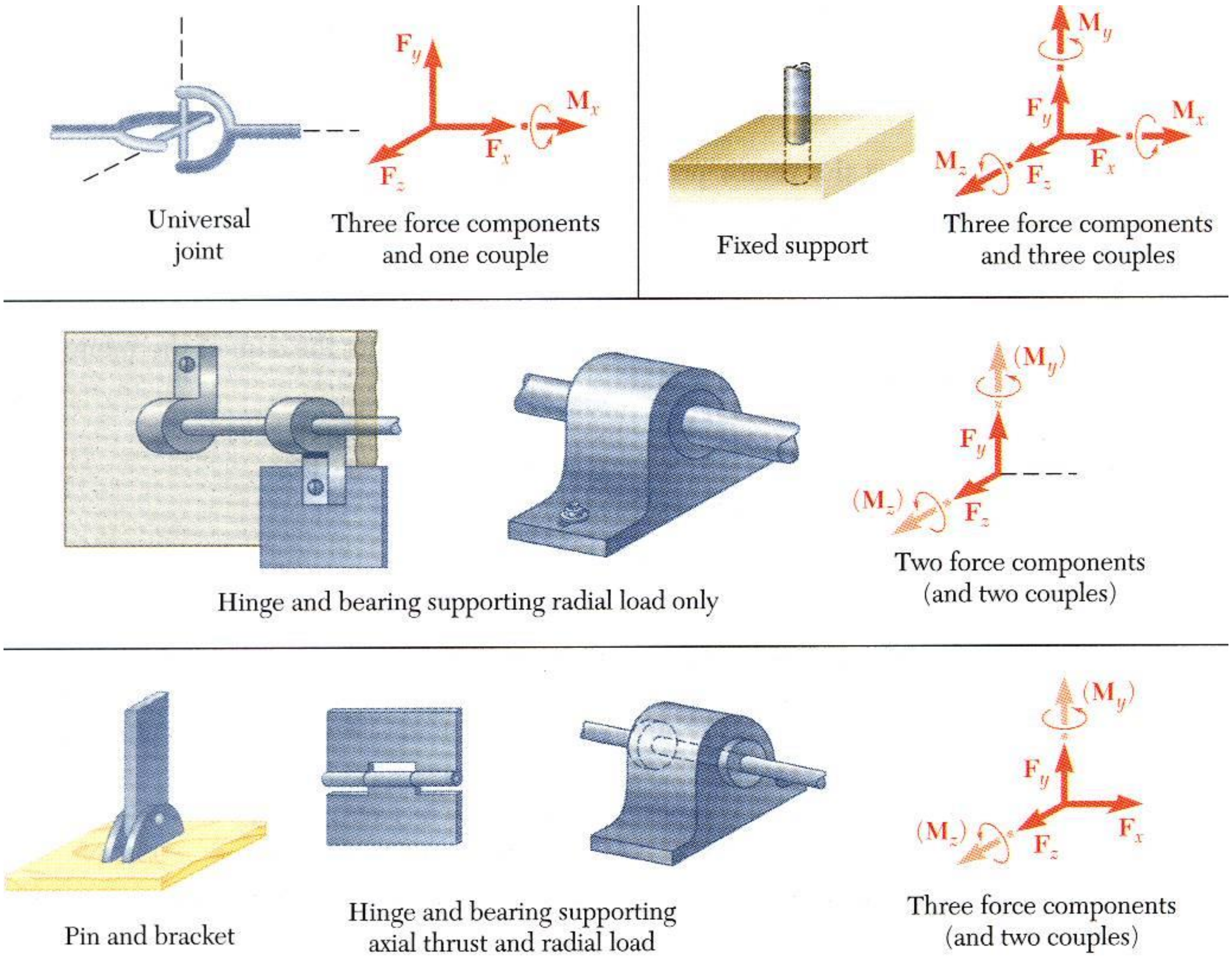
- These equations can be solved for no more than 6 unknowns which generally represent reactions at supports or connections.

Today's Lecture Attendance Password is: 3D Problem

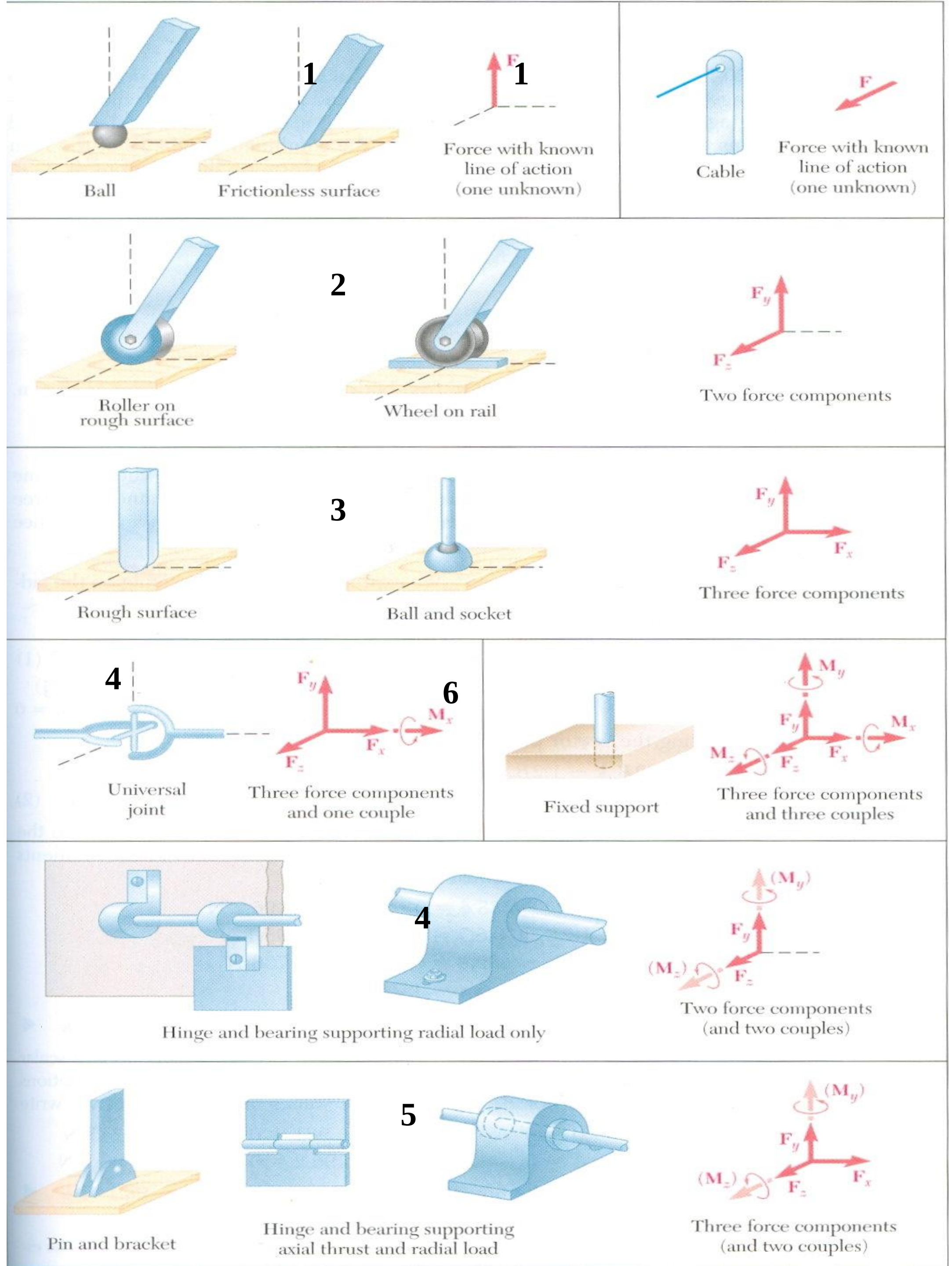
Reactions at Supports and Connections for a 3D Structure



Reactions at Supports and Connections for a 3D Structure

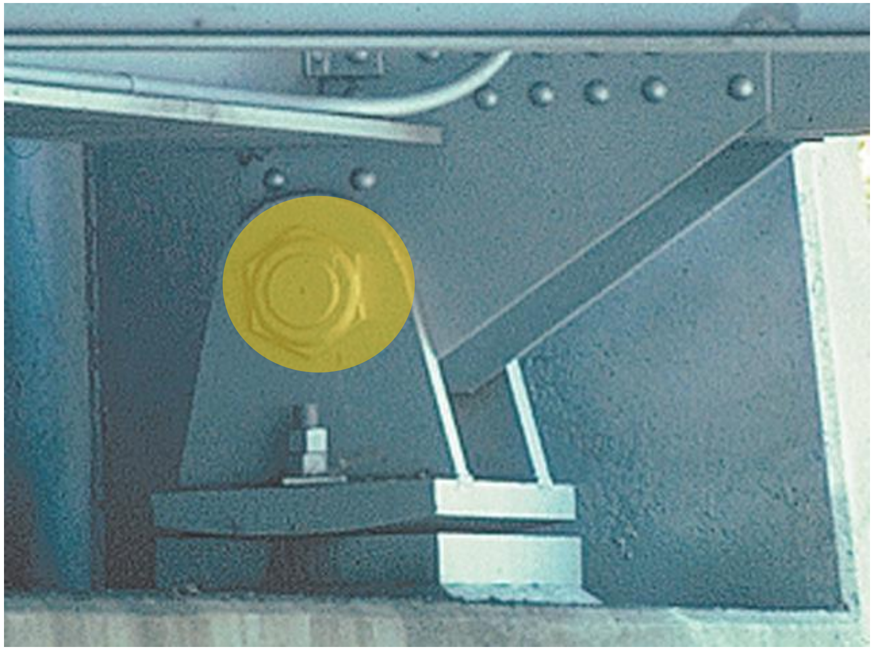


Boundary Support Summary



Rocker Bearing used to Support the Roadway of a Bridge

Pin connections allow rotation. Reactions at pins are forces and NOT MOMENTS.



4.53 A 4 × 8-ft sheet of plywood weighing 40 lb has been temporarily propped against column *CD*. It rests at *A* and *B* on small wooden blocks and against protruding nails. Neglecting friction at all surfaces of contact, determine the reactions at *A*, *B*, and *C*.

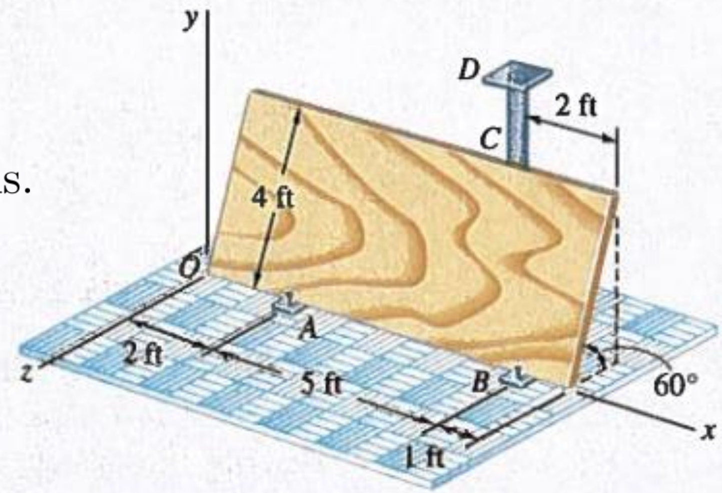
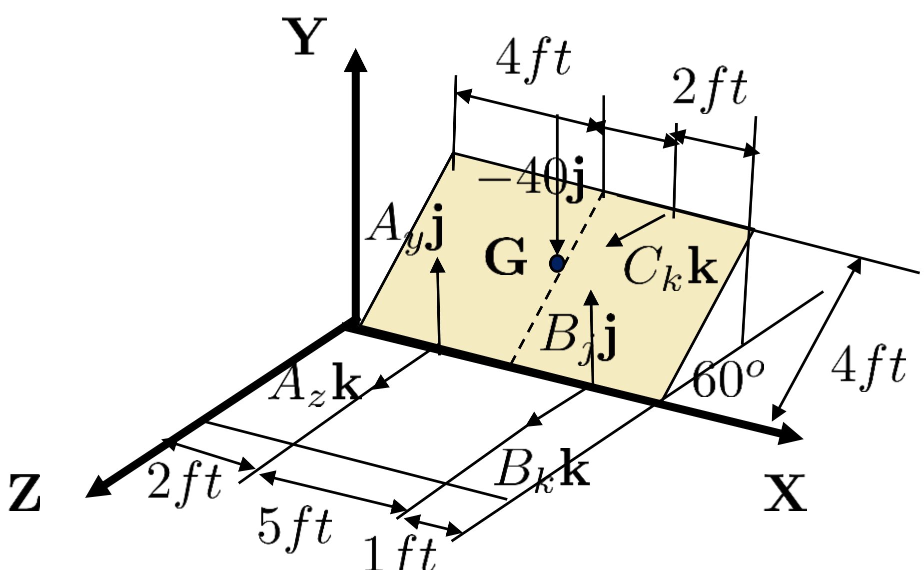


Fig. P4.53

Freebody-Diagram

We have five unknows and six equations.

The plywood sheet is free to move in x-direction, but ($\sum F_x = 0$).

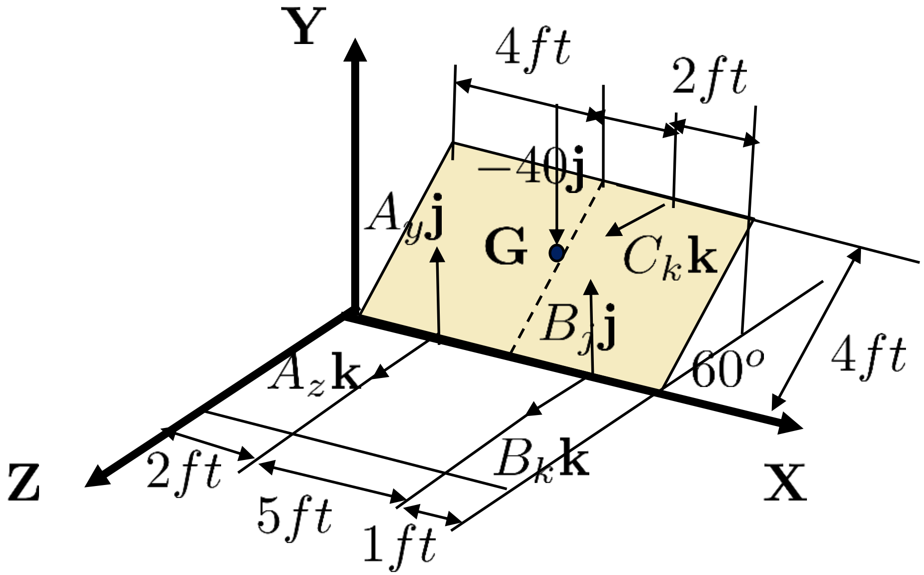


A:(2,0,0)

B:(7,0,0)

C : $(6, 2\sqrt{3}, -2)$

G : $(4, \sqrt{3}, -1)$



A:(2,0,0)

B:(7,0,0)

$\mathbf{C}:(6,2\sqrt{3},-2)$

$\mathbf{G}:(4,\sqrt{3},-1)$

$$\mathbf{r}_{AB}=5\mathbf{i}$$

$$\mathbf{r}_{AC}=4\mathbf{i}+4\sin 60^o\mathbf{j}-4\cos 60^o\mathbf{k}$$

$$\mathbf{r}_{AG}=2\mathbf{i}+2\sin 60^o\mathbf{j}-2\cos 60^o\mathbf{k}$$

$$\Sigma M_A=0:\quad \mathbf{r}_{B/A}\times(B_y\,\mathbf{j}+B_z\mathbf{k})+\mathbf{r}_{C/A}\times C\,\mathbf{k}+\mathbf{r}_{G/A}\times(-40\text{ lb})\mathbf{j}=0$$

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5 & 0 & 0\\ 0 & B_y & B_z\end{vmatrix}+\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 4\sin 60° & -4\cos 60°\\ 0 & 0 & C\end{vmatrix}+\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 2\sin 60° & -2\cos 60°\\ 0 & -40 & 0\end{vmatrix}=0$$

$$(4C\sin 60°-80\cos 60°)\mathbf{i}+(-5B_z-4C)\mathbf{j}+(5B_y-80)\mathbf{k}=0$$

$\mathbf{i}$: $\quad 4C\sin 60°-80\cos 60°=0 \qquad C=11.5470\text{ lb}$

$\mathbf{j}$: $\quad -5B_z-4C=0 \qquad B_z=9.2376\text{ lb}$

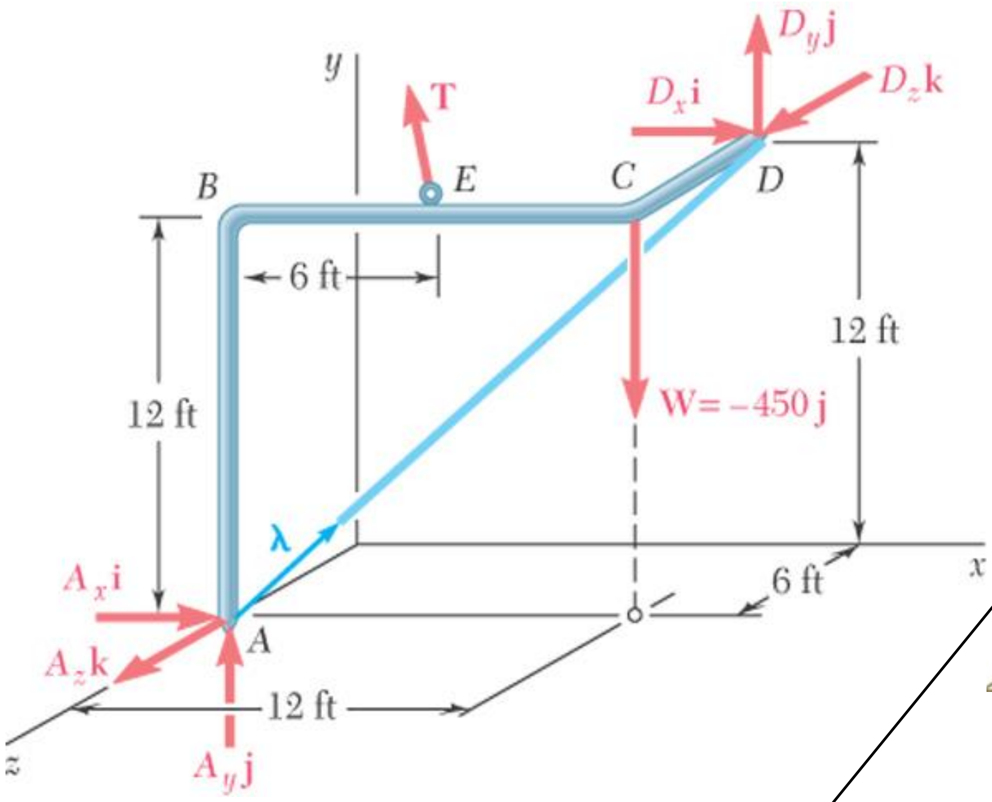
$\mathbf{k}$: $\quad 5B_y-80=0 \qquad B_y=16.0000\text{ lb}$

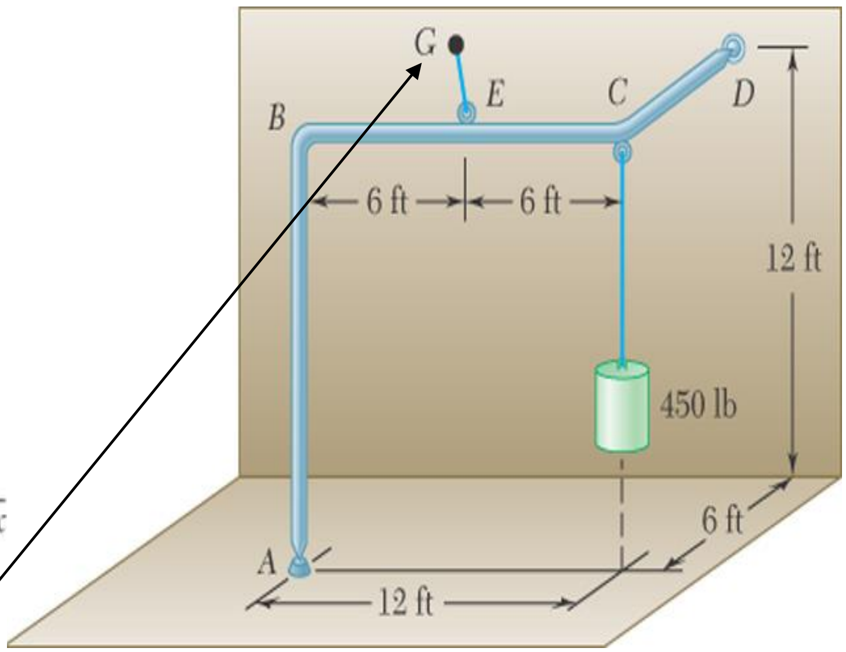
$\Sigma F_y=0$: $\quad A_y+B_y-40=0 \qquad A_y=40-16.0000=24.000\text{ lb}$

$\Sigma F_z=0$: $\quad A_z+B_z+C=0 \qquad A_z=9.2376-11.5470=-2.3094\text{ lb}$

$$\mathbf{A}=(24.0\text{ lb})\mathbf{j}-(2.31\text{ lb})\mathbf{k};\quad \mathbf{B}=(16.00\text{ lb})\mathbf{j}-(9.24\text{ lb})\mathbf{k};\quad \mathbf{C}=(11.55\text{ lb})\mathbf{k}$$ ◀

Sample Problem 4.10

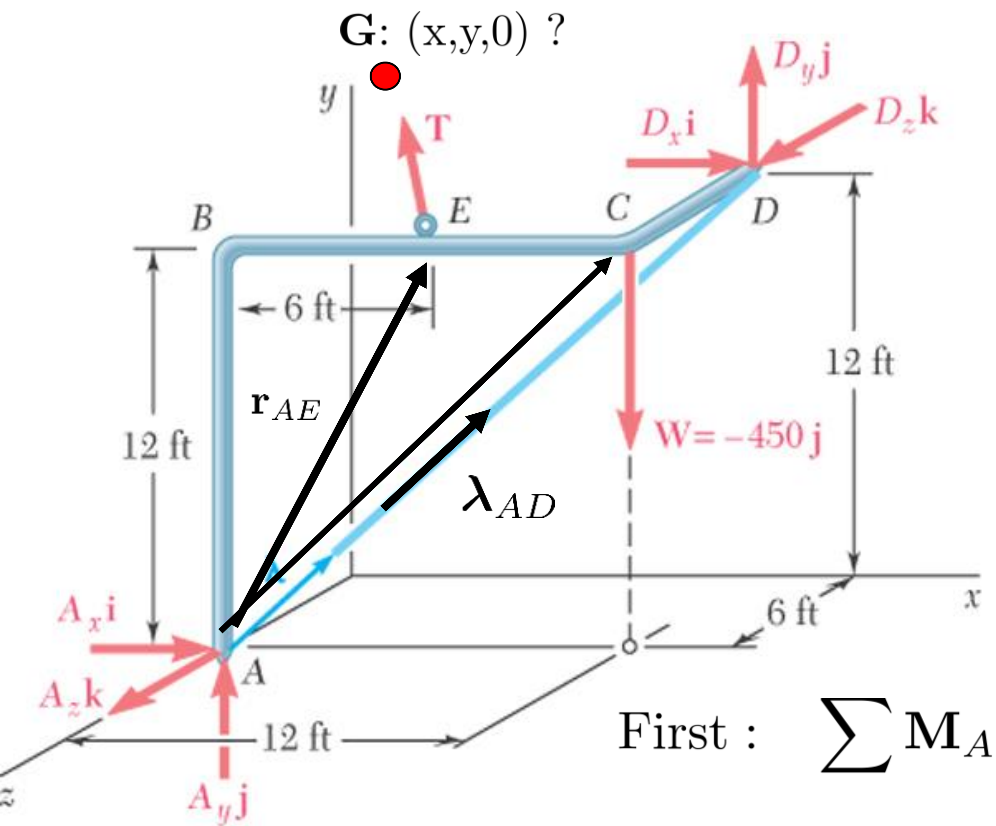




Determine (a) where G should be located if the tension in the cable is to be minimum,

(b) the corresponding minimum value of the tension.

$$A:(0,0,6),\ \ C:(12,12,6),\ \ D:(12,12,0)\ \ E:(6,12,6)$$



$$\begin{aligned}\mathbf{r}_{AE} &= (6-0)\mathbf{i}+(12-0)\mathbf{j}+(6-6)\mathbf{k}\\ &= 6\mathbf{i}+12\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{r}_{AC} &= (12-0)\mathbf{i}+(12-0)\mathbf{j}+(6-6)\mathbf{k}\\ &= 12\mathbf{i}+12\mathbf{j}\end{aligned}$$

$$\begin{aligned}\mathbf{r}_{AD} &= (12-0)\mathbf{i}+(12-0)\mathbf{j}+(0-6)\mathbf{k}\\ &= 12\mathbf{i}+12\mathbf{j}-6\mathbf{k}\end{aligned}$$

$$|\mathbf{r}_{AD}| = 18$$

$$\boldsymbol{\lambda}_{AD} = \frac{\mathbf{r}_{AD}}{|\mathbf{r}_{AD}|} = \frac{2}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}-\frac{1}{3}\mathbf{k}$$

$$\text{First}:\quad \sum \mathbf{M}_A = 0 \ \rightarrow\ \sum M_{AD} = 0 \ \rightarrow$$

Why ?

$$\boldsymbol{\lambda}_{AD}\cdot\{\mathbf{r}_{AE}\times\mathbf{T}_{EG}+\mathbf{r}_{AC}\times\mathbf{W}+\mathbf{r}_{AD}\times\mathbf{D}\}=0\ .$$

$$\mathbf{W} = -450\mathbf{j}$$

$$\text{where}\quad \boldsymbol{\lambda}_{AD} = \frac{\mathbf{r}_{AD}}{|\mathbf{r}_{AD}|} = \frac{1}{18}\Big(12\mathbf{i}+12\mathbf{j}-6\mathbf{k}\Big)\ \text{ and}$$

$$\mathbf{r}_{AC}\times\mathbf{W} = (12\mathbf{i}+12\mathbf{j})\times(-450\mathbf{j}) = -5400\mathbf{k}$$

$$\boldsymbol{\lambda}_{AD} \cdot (\mathbf{r}_{AE} \times \mathbf{T}_{EG}) + \boldsymbol{\lambda}_{AD} \cdot (\mathbf{r}_{AC} \times \mathbf{W}) = 0, \quad \Rightarrow$$

$$\boldsymbol{\lambda}_{AD} \cdot (\mathbf{r}_{AE} \times \mathbf{T}_{EG}) = -\left(\frac{2\mathbf{i} + 2\mathbf{j} - 1\mathbf{k}}{3}\right) \cdot (-5400\mathbf{k}) = -1800 \ .$$

$$\boldsymbol{\lambda}_{AD} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$$

$$\boldsymbol{\lambda}_{AD} \cdot (\mathbf{r}_{AE} \times \mathbf{T}_{EG}) = \mathbf{T}_{EG} \cdot (\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE}) = 1800$$

$$\mathbf{r}_{AE} = 6\mathbf{i} + 12\mathbf{j}$$

$$\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}, \quad \text{and} \quad |\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE}| = 6 \ .$$

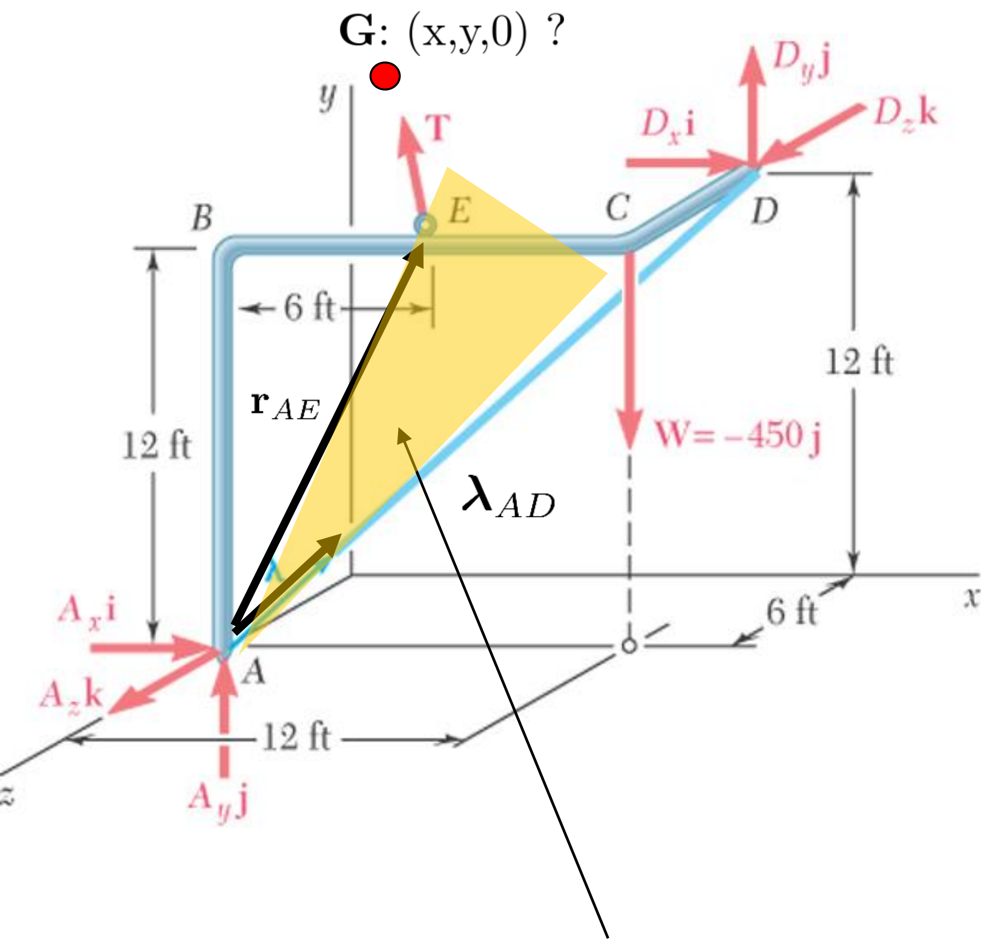
$$\mathbf{T}_{EG} \cdot (\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE}) = T_{EG}|\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE}| \cos\theta = 6T_{EG}\cos\theta = 1800$$

$$\mathbf{T}_{EG} = T_{EG}\boldsymbol{\lambda}_{EG} \qquad \boldsymbol{\lambda}_{EG} \cdot \frac{\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE}}{|\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE}|} = \cos\theta$$

$$\boxed{\mathbf{T}_{EG} = T_{min}\frac{(\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE})}{|\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE}|} = T_{min}\frac{(\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE})}{6}}$$

Why?

$$\cos\theta = 1 \ \rightarrow \ \boldsymbol{\lambda}_{EG} // \frac{\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE}}{|\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE}|} \ \rightarrow \ \theta = 0 \quad T_{EG} = T_{min}$$



If T_{EG} is minimum, $\mathbf{T}_{EG}$ should be perpendicular to the plane spaned by $\boldsymbol{\lambda}_{AD}$ and $\mathbf{r}_{AE}$.

$$\text{Hence : } \quad \mathbf{T}_{EG} = T_{min}\frac{(\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE})}{|\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE}|} = T_{min}\frac{(\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE})}{6}$$

$$\mathbf{T}_{EG} \cdot (\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE}) = T_{min}|\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE}| = 6T_{min}$$

$$\text{Recall : } \quad \boldsymbol{\lambda}_{AD} \cdot (\mathbf{r}_{AE} \times \mathbf{T}_{EG}) = \mathbf{T}_{EG} \cdot (\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE}) = 1800$$

$$6T_{min} = 1800 \quad \rightarrow \quad T_{min} = 300lb;$$

$$\boxed{\mathbf{T}_{EG} = T_{min}\frac{(\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE})}{|\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE}|} = T_{min}\frac{(\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE})}{6}}$$

$$\text{Recall } \quad \boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

$$\mathbf{T}_{EG} = T_{min}\frac{(\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE})}{6} = (50)\Big(4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}\Big)$$

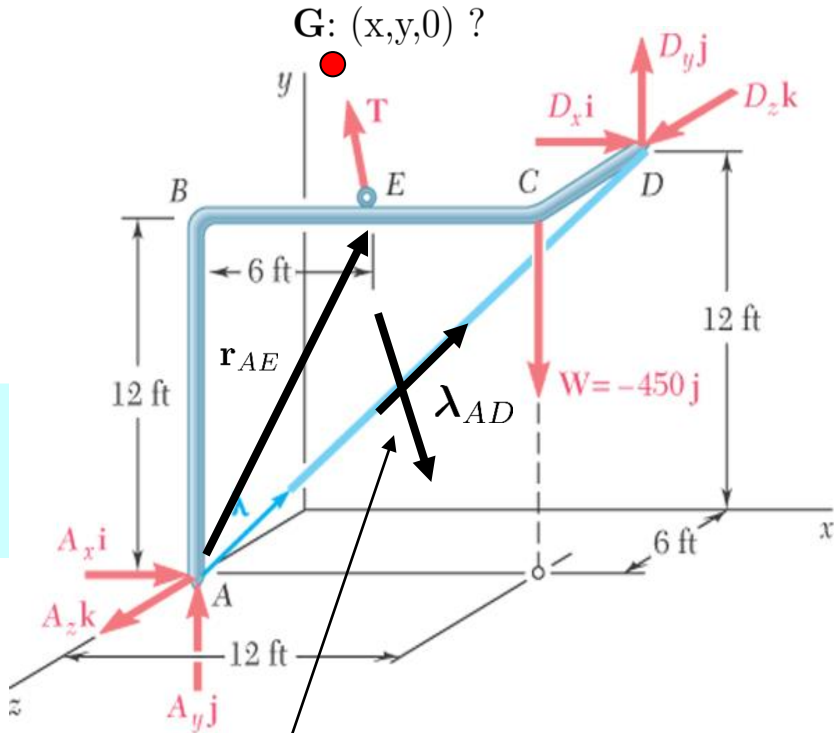
G: (x,y,0)

$$\mathbf{T}_{EG} = T_{min}\frac{(\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE})}{6} = (50)\Big(4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}\Big)$$

$$E : (6, 12, 6),\ \ G(x, y, 0)$$

$$\mathbf{r}_{EG} = (x - 6)\mathbf{i} + (y - 12)\mathbf{j} + (0 - 6)\mathbf{k}$$

$$\mathbf{r}_{EG} // (4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$$



$\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and
$|\boldsymbol{\lambda}_{AD} \times \mathbf{r}_{AE}| = 6.$

They must be in a parallel direction; that is

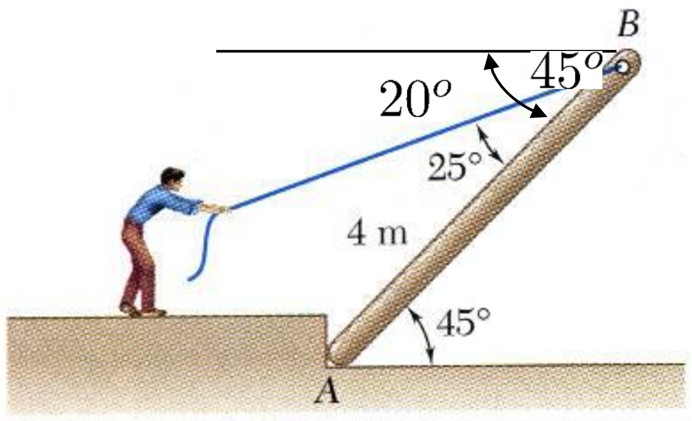
$$\frac{x-6}{4} = \frac{y-12}{-2} = \frac{0-6}{4}$$

$$E : (6, 12, 6)$$

$$G : (x, y, 0)$$

Hence x=0, y=15 ft

Sample Problem 4.6



A man raises a 10 kg joist, of length 4 m, by pulling on a rope.

Find the tension in the rope and the reaction at *A*.

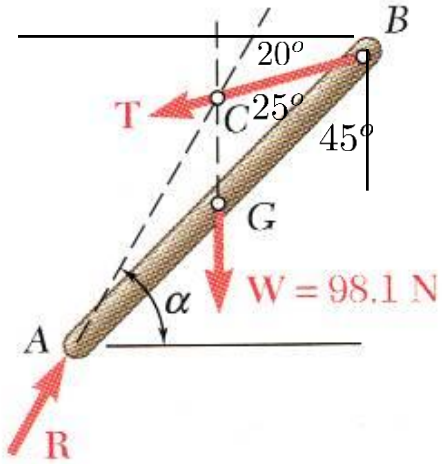
SOLUTION:

- Create a free-body diagram of the joist. Note that the joist is a 3 force body.

- The three forces must be concurrent for static equilibrium. Therefore, the reaction ***R*** must pass through the intersection of the lines of action of the weight and rope forces.

- Utilize a force triangle to determine the magnitude of the reaction force ***R***.

- **Create a free-body diagram of the joist.**

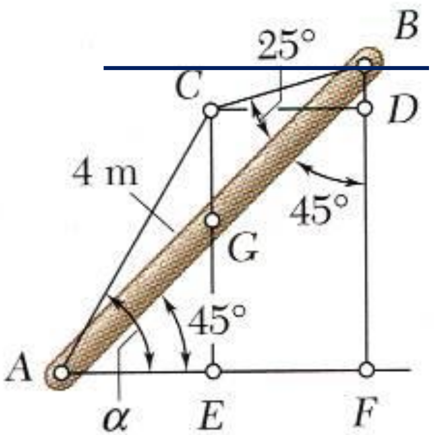


- Determine the direction of the reaction force ***R***.

$$AF = AB\cos 45 = (4\,\text{m})\cos 45 = 2.828\text{m}$$

$$CD = AE = \frac{1}{2}AF = 1.414m$$

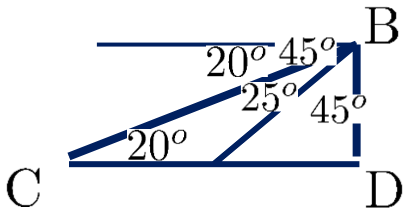
$$BD = CD\cot(45+25) = CD\tan 20 = 1.414\tan 20 = 0.515m$$

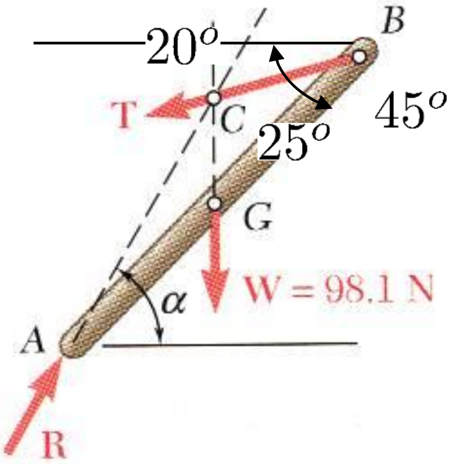


$$CE = BF - BD = (2.828 - 0.515)\text{m} = 2.313\text{m}$$

$$\tan\alpha = \frac{CE}{AE} = \frac{2.313}{1.414} = 1.636$$

$$\boxed{\alpha = 58.6^\circ}$$





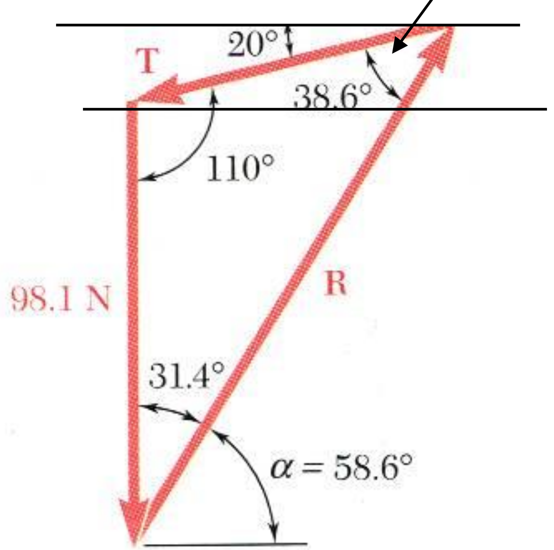
This is a three-force system.

$$\alpha - 20^o = 38.6^o$$

- Determine the magnitude of the reaction force ***R***.

$$\frac{T}{\sin 31.4^\circ} = \frac{R}{\sin 110^\circ} = \frac{98.1\,\text{N}}{\sin 38.6^\circ}$$

$$38.6^o = 58.6^o - 20^o$$



$$31.4^o = 90^o - 58.6^o$$

$$\boxed{\begin{aligned} T &= 81.9\,\text{N} \\ R &= 147.8\,\text{N} \end{aligned}}$$