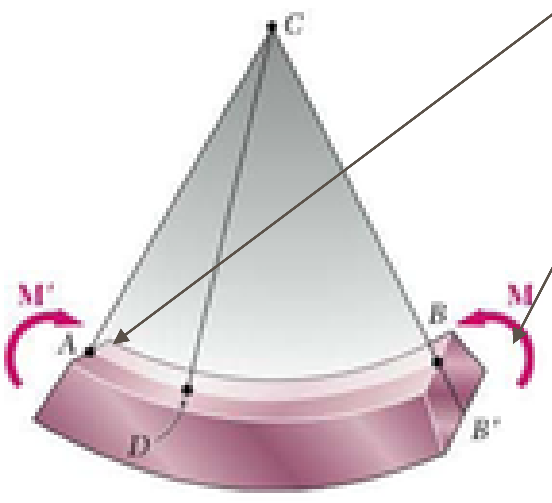
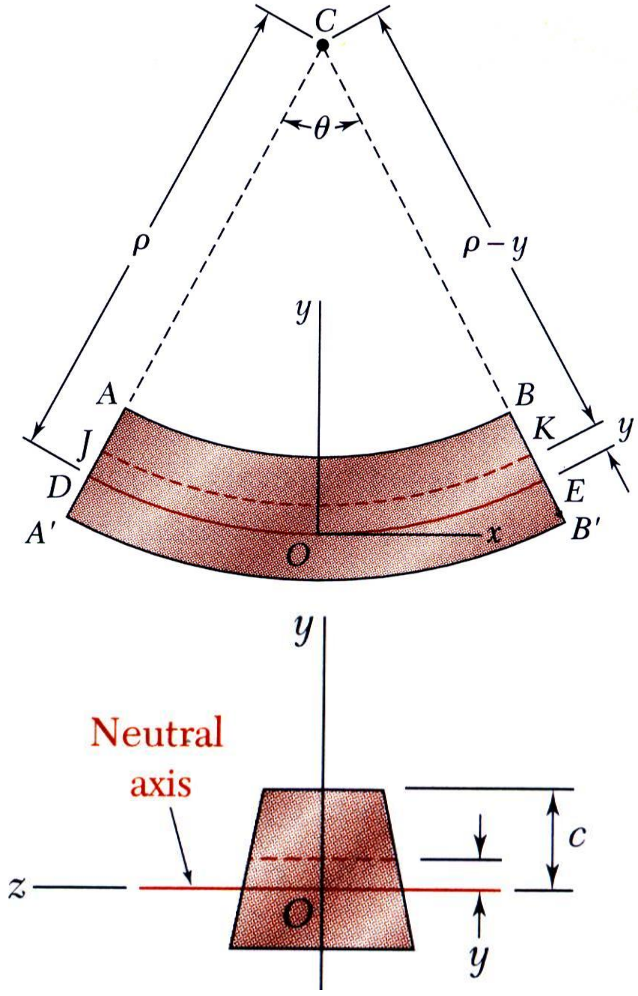
Lecture 27 Beam Bending (II)

Structural member	stress resultant	Corresponding stress
Bar	P	$\sigma = \dfrac{P}{A}$
Shaft	T	$\tau = \dfrac{T\rho}{J}$
Beam	M, V	?, ?



Pure Bending

Normal Strain Due to Pure Bending



Consider a beam segment of length *L*.

After deformation, the length of the neutral surface remains *L*. At other sections,

$$L' = \overline{JK}$$

$$\begin{aligned} L' &= (\rho - y)\theta \\ \Delta &= L' - L = (\rho - y)\theta - \rho\theta = -y\theta \\ \epsilon_x &= \frac{\Delta}{L} = -\frac{y\theta}{\rho\theta} = -\frac{y}{\rho} = -\left(\frac{c}{\rho}\right)\frac{y}{c} \end{aligned}$$

Strain varies linearly

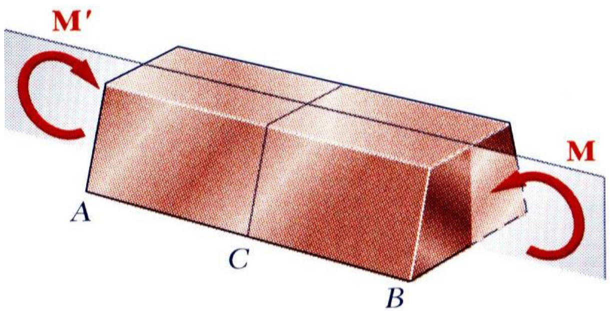
$$\epsilon_m = \left|\frac{c}{\rho}\right|, \quad \text{or} \quad \boxed{\rho = \frac{1}{\kappa} = \frac{c}{\epsilon_m}}$$

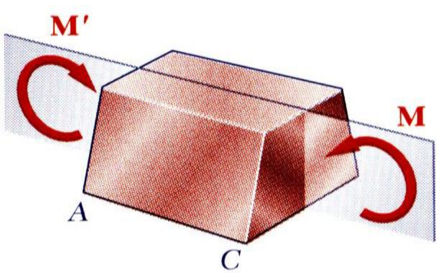
$$\epsilon_x = -\epsilon_m \frac{y}{c} \quad \Rightarrow \quad \boxed{\sigma_x = -\sigma_m\left(\frac{y}{c}\right)}$$

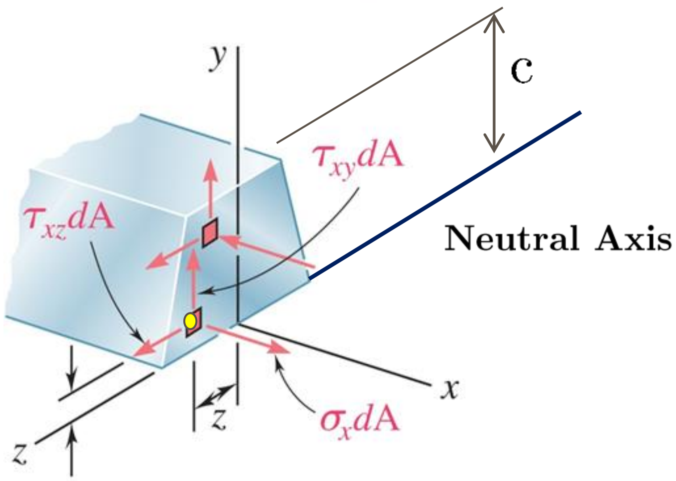
$$\sigma_x = E\varepsilon_x = -\frac{y}{c}E\varepsilon_m = -\frac{y}{c}\sigma_m \quad \text{(stress varies linearly)}$$

c is the upper depth of the beam.

Internal Forces in Pure Bending







The upper part of the beam section

- Internal forces in any cross section are equivalent to a couple. The moment of the couple is the section *bending moment*.

This is the statics part.

$$F_x = \int \sigma_x \, dA = 0$$

$$M_z = \int -y\sigma_x \, dA = M \qquad \sigma_x = -\sigma_m\left(\frac{y}{c}\right)$$

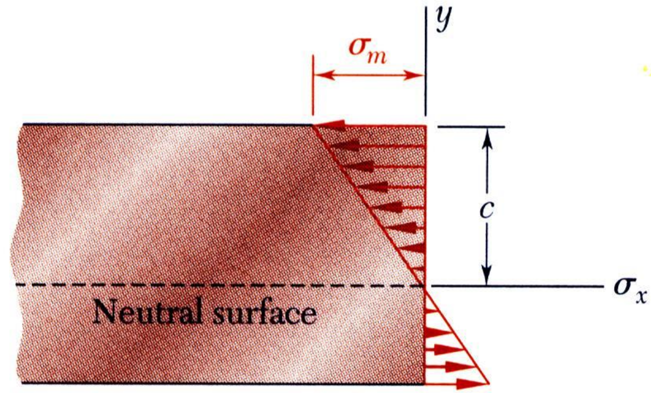
- For static equilibrium,

$$F_x = 0 = \int \sigma_x \, dA = \int -\frac{y}{c}\sigma_m \, dA$$

$$0 = -\frac{\sigma_m}{c}\int y \, dA$$ What does this mean ?

The neutral axis is the centroidal axis.

Elastic Flexure Formula



- For a linearly elastic material,

$$\sigma_x = -\sigma_m\left(\frac{y}{c}\right)$$

$$F_x = \int_A \sigma_x dA = 0$$

$$M_z = \int_A (-y)\sigma_x dA = M$$

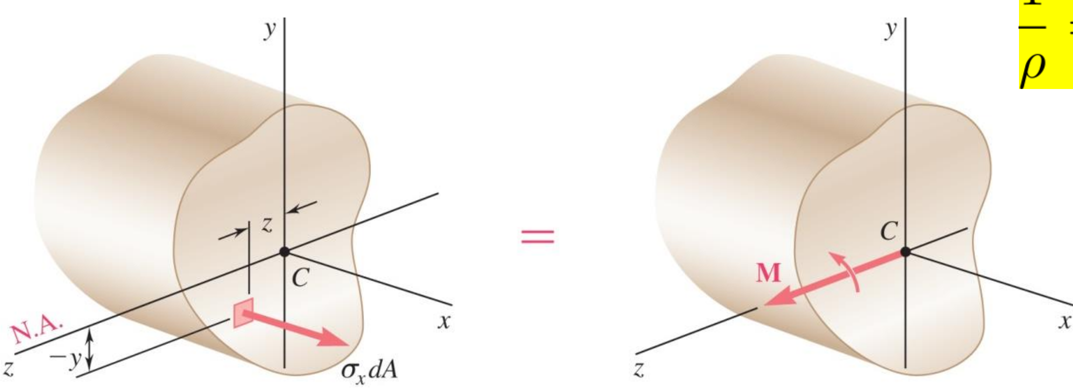
$$M_z = \int_A (-y\sigma_x) dA = \int_A (-y)\left(-\sigma_m \frac{y}{c}\right) dA$$

$$M_z = \frac{\sigma_m}{c}\int_A y^2 dA = \frac{\sigma_m I_z}{c} \, .$$

Substituting $\dfrac{\sigma_m}{c} = \dfrac{M_z(x)}{I_z} \rightarrow \sigma_x = -\sigma_m \dfrac{y}{c}$

$$\frac{1}{\rho} = \frac{\epsilon_m}{c} = \frac{M_z}{EI_z}$$

$$\boxed{\sigma_x = -\frac{M_z(x)y}{I_z}}$$



Remark: $\epsilon_x = -\dfrac{y}{\rho} = -\kappa y;$

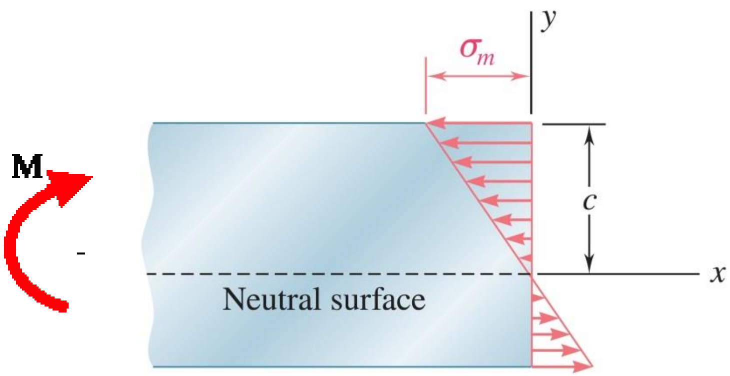
The minus sign means that under positive bending moment the upper part of the beam ($y > 0$) is in compression, wherea the lower part of the beam ($y < 0$) is in tension.

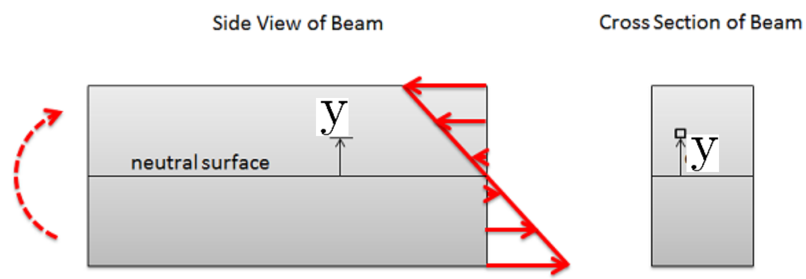
Step 3 Stress

$$\begin{aligned} \epsilon_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}; \\ \rightarrow \quad & \sigma_x = E\epsilon_x = -E\left(\frac{y}{\rho}\right) \end{aligned}$$

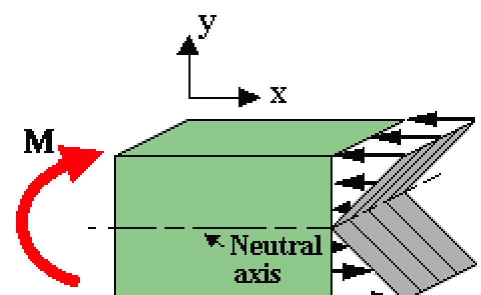
Assumption

$$\begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y = 0 & 0 \\ 0 & 0 & \sigma_z = 0 \end{pmatrix}$$





When $y = 0, \sigma_x = 0$, therefore we call z-axis as **neutral axis**.



Beam Section Property: Section Modulus

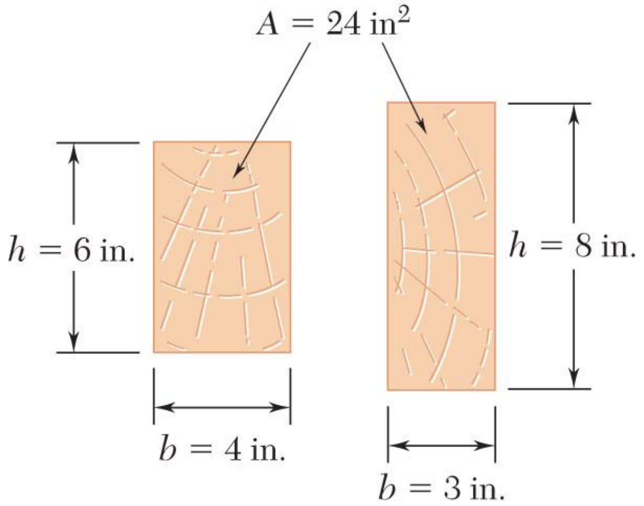


Fig. 11.12 Wood beam cross sections.

- The maximum normal stress due to bending,

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

$$I = \text{section moment of inertia}$$

$$S = \frac{I}{c} = \text{section modulus}$$

A beam section with a larger section modulus will have a lower maximum stress

- Consider a rectangular beam cross section,

$$S = \frac{I}{c} = \frac{\frac{1}{12} bh^3}{h/2} = \frac{1}{6} bh^3 = \frac{1}{6} Ah$$

Between two beams with the same cross sectional area, the beam with the greater depth will be more effective in resisting bending.

- Structural steel beams are designed to have a large section modulus.

Summary: Elastic Flexure Formula

1. Kinematic Assumption : $\sigma_x = -\sigma_m \left(\frac{y}{c} \right)$ $\qquad \epsilon_m = \frac{c}{\rho}$

2 From equilibrium $\rightarrow \frac{\sigma_m}{c} = \frac{M_z}{I_z}$ $\qquad \frac{1}{\rho} = \frac{M}{EI}$

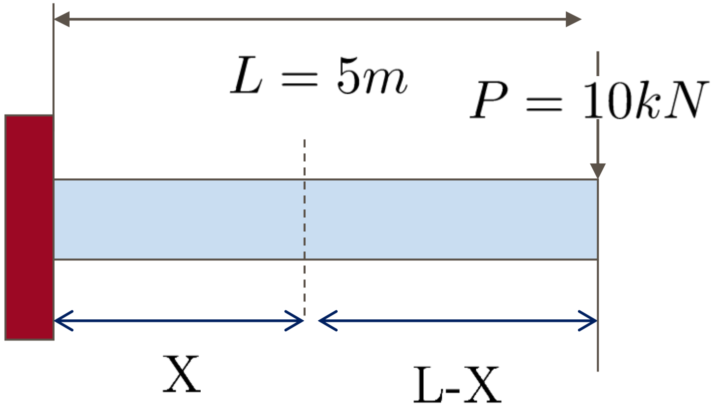
3. Elastic Flexure Formula : $\boldsymbol{\sigma}_x = -\frac{M_z y}{I_z}$

4. We define $S := I_z / c$ as the section modulus.

5. Parallel Axis Theorem : $I_Z = I_{Z_c} + d^2 A.$

Today's Lecture Password is: **Pure Bending**

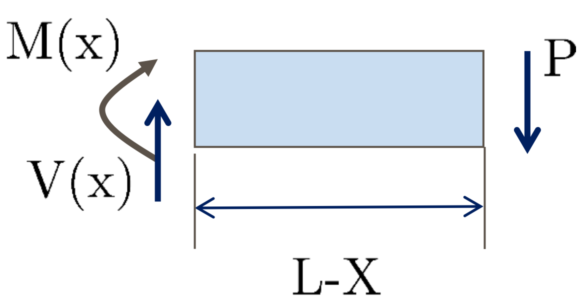
II. Example: Galileo's Problem



$$\sum F_y = 0 \;\rightarrow$$

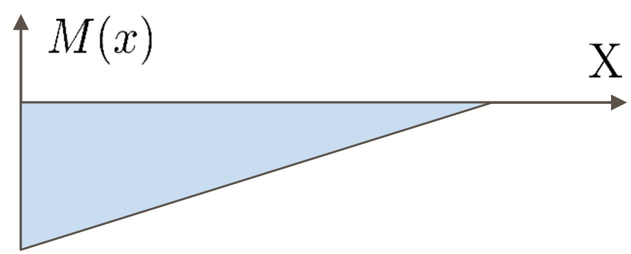
$$V(x) - P = 0 \;\rightarrow\; V(x) = P;$$

Free-body diagram



$$\sum M_z = 0 \;\rightarrow$$

$$-M(x) - P(L-x) = 0 \;\rightarrow\; M(x) = P(x-L);$$



$$M_{max} = -PL$$

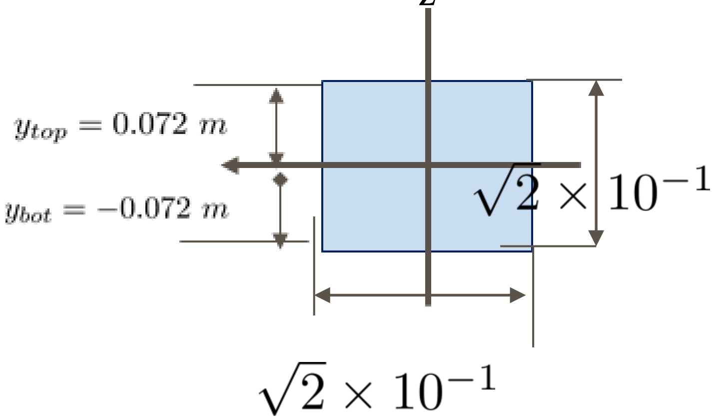
$$M_{\max} = M(0) = -PL = -5\times10^4\, N\cdot m$$

Design I: Square section

$$M_{max} = -5 \times 10^4 N - m$$

$$\sigma_x = -\frac{M_z y}{I_z}$$

$$A = (\sqrt{2} \times 10^{-1})^2 = 0.02m^2$$



$$I_z = \frac{bh^3}{12} = \frac{(\sqrt{2} \times 10^{-1})(\sqrt{2} \times 10^{-1})^3}{12} \approx 3.33 \times 10^{-5} m^4$$

$$y_{top} = \frac{\sqrt{2}}{2} \times 10^{-1} \approx 0.072\ m$$

$$y_{bot} = -\frac{\sqrt{2}}{2} \times 10^{-1} \approx -0.072\ m$$

At the top surface

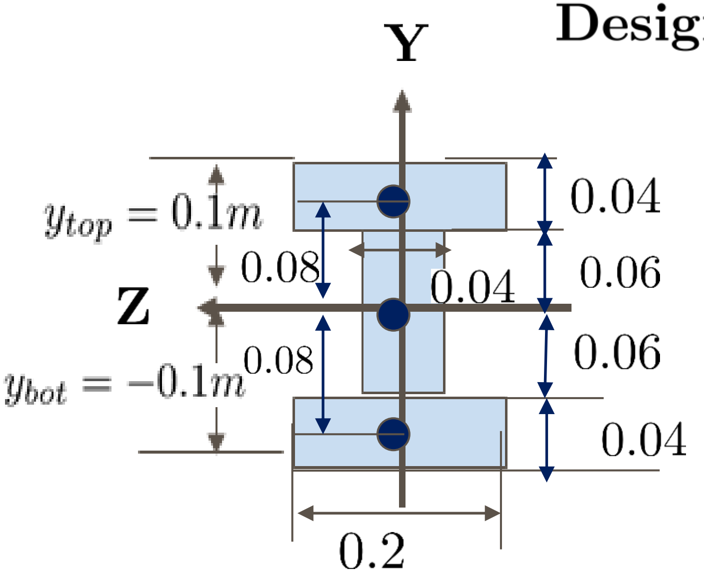
$$\sigma_{max} = -\frac{(-5 \times 10^4) \times (0.072)}{3.33 \times 10^{-5}} = 100.6MP_a, \quad \text{in tension}$$

$$S = I/c = 0.4625 \times 10^{-3} m^3$$

At the lower surface

$$\sigma_{min} = -\frac{(-5 \times 10^4) \times (-0.072)}{3.33 \times 10^{-5}} = -100.6MP_a, \quad \text{in compression}$$

Design II: I-beam (two flange and a web)



$$A = 2 \times (0.2 \times 0.04) + 0.04 \times 0.12$$
$$= 0.0208 \approx 0.02m^2$$

$$I_z = 2I_f + I_w$$

$$I_w = \frac{1}{12}(0.04) \times (0.120)^3 = 0.576 \times 10^{-5} m^4$$

$$I_f = I_{fc} + d^2 A = \frac{0.2 \times (0.04)^3}{12} + (0.08)^2 \times (0.04) \cdot (0.2)$$
$$= 5.227 \times 10^{-5} m^4$$

$$I_z = 2I_f + I_w = 2 \times 5.227 \times 10^{-5} + 0.576 \times 10^{-5} = 11.03 \times 10^{-5} m^4$$

At the top surface

y_{top}

$$\sigma_{max} = -\frac{(-5 \times 10^4) \times (0.1)}{11.03 \times 10^{-5}} = 45.33 MP_a, \quad \text{in tension}$$

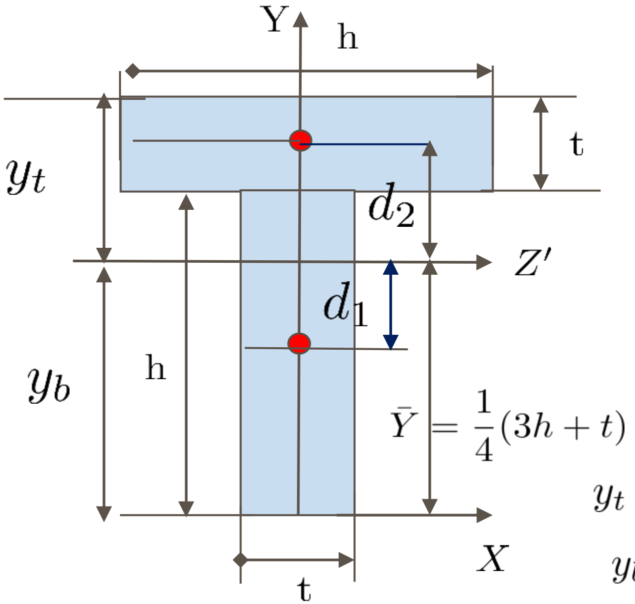
At the bottom surface

y_{bot}

54.94 % reduction

$$\sigma_{max} = -\frac{(-5 \times 10^4) \times (-0.1)}{11.03 \times 10^{-5}} = -45.33 MP_a, \quad \text{in compression}$$

Design III: T-beam (a flange and a web)



$$A = 0.2m^2 \; h = 0.2m, \; t = 0.05m$$

$$d_1 = \bar{Y} - \bar{y}_1$$
$$= 0.1625 - 0.1 = 0.0625$$

$$d_2 = \bar{y}_2 - \bar{Y}$$
$$= 0.225 - 0.1625 = 0.0625$$

$$y_t = 0.25 - 0.1625 = 0.0875$$

$$y_b = \bar{Y} = 0.1625$$

	$\bar{A}_i$	$\bar{y}_i$	$A_i\bar{y}_i$
1	0.01	0.1	0.001
2	0.01	0.225	0.00225
$\sum$	0.02		0.00325

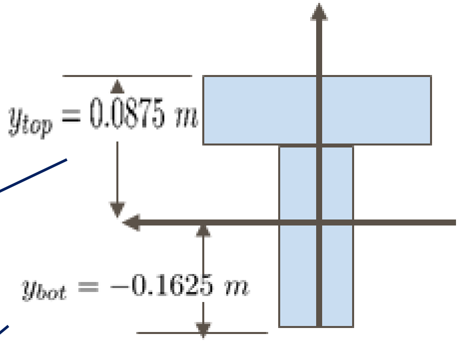
$$\bar{Y} = \frac{0.00325}{0.02} = 0.1625$$

$$I_Z^{(1)} = \frac{b_1 h_1^3}{12} + d_1^2 A_1 = \frac{(0.05)(0.2)^3}{12} + (0.0625)^2 \times 0.01 = 0.7236 \times 10^{-5} m^4$$

$$I_Z^{(2)} = \frac{b_2 h_2^3}{12} + d_2^2 A_2 = \frac{(0.2)(0.05)^3}{12} + (0.0625)^2 \times 0.01 = 4.115 \times 10^{-5} m^4$$

$$I_Z = I_Z^{(1)} + I_Z^{(2)} = 11.35 \times 10^{-5} m^4$$

T-beam design (Cont'd)



$$I_z = 11.35 \times 10^{-5} m^4$$

At the top surface

$$\sigma_{max} = -\frac{(-5 \times 10^4) \times (0.0875)}{11.35 \times 10^{-5}} = 38.54 MP_a, \quad \text{in tension}$$

61.19 % reduction

At the lower surface

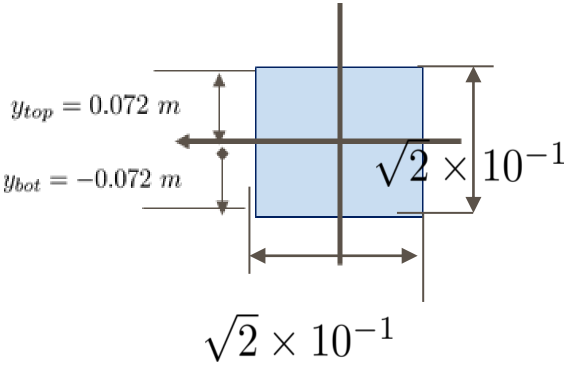
$$\sigma_{max} = -\frac{(-5 \times 10^4) \times (0.1625)}{11.35 \times 10^{-5}} = -71.58 MP_a, \quad \text{in compression}$$

28.85 % reduction

$$S = \frac{I}{c} = \frac{11.35 \times 10^{-5}}{0.0875} = 1.297 \times 10^{-3} m^3$$

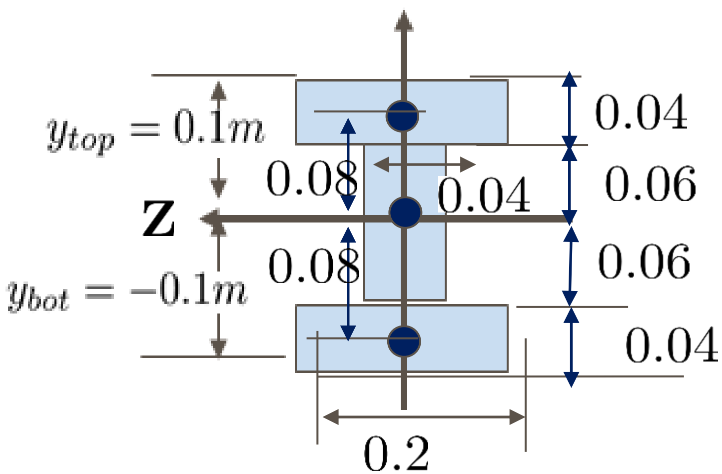
Comparison

$$M_{max} = -5 \times 10^4 N - m$$



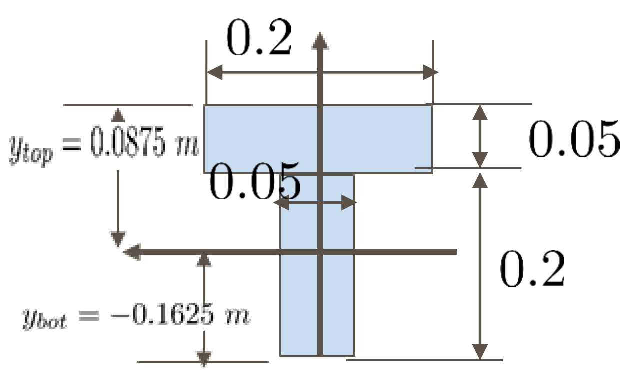
$$S = I/c = 0.4625 \times 10^{-3} m^3$$

$$\sigma_{max} = -\frac{(-5 \times 10^4)}{0.4625 \times 10^{-3}} = 100.6 MP_a\ .$$



$$S = I/c^+ = 1.103 \times 10^{-3} m^3$$

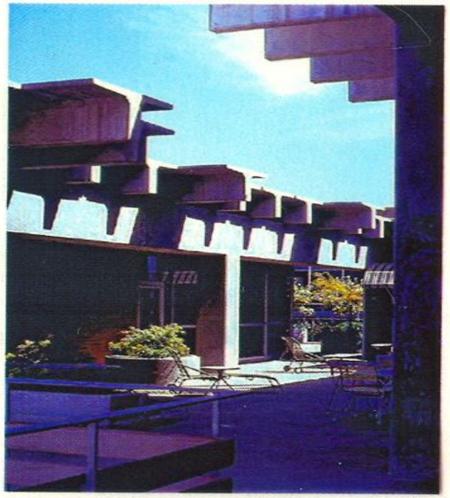
$$\sigma_{max} = -\frac{(-5 \times 10^4)}{1.103 \times 10^{-3}} = 45.33 MP_a \quad 54.94\%$$

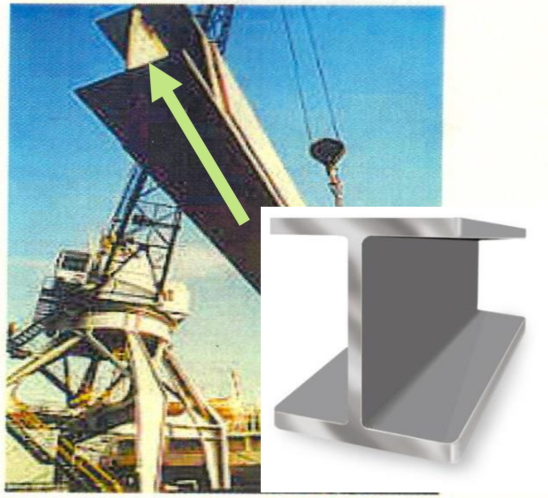


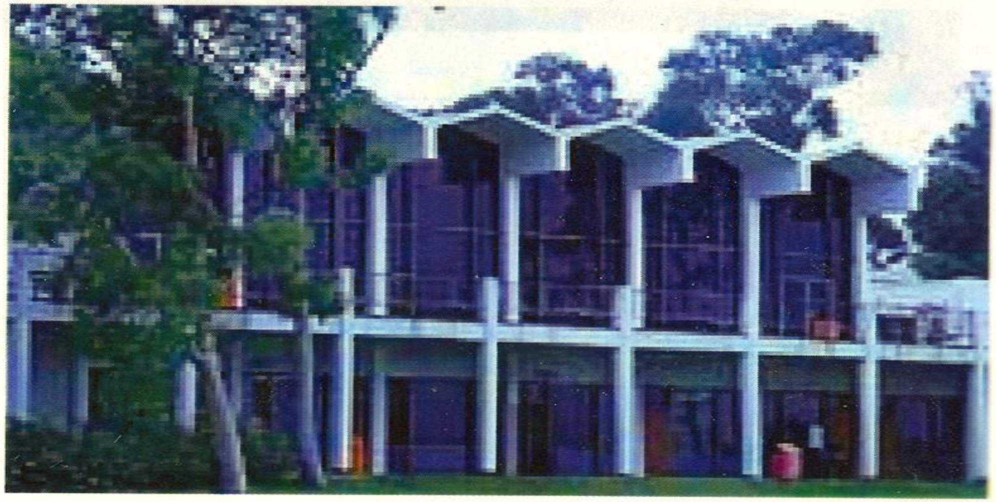
$$\begin{aligned} S &= \frac{I}{c^+} = \frac{11.35 \times 10^{-5}}{0.0875} \\ &= 1.297 \times 10^{-3} m^3 \end{aligned}$$

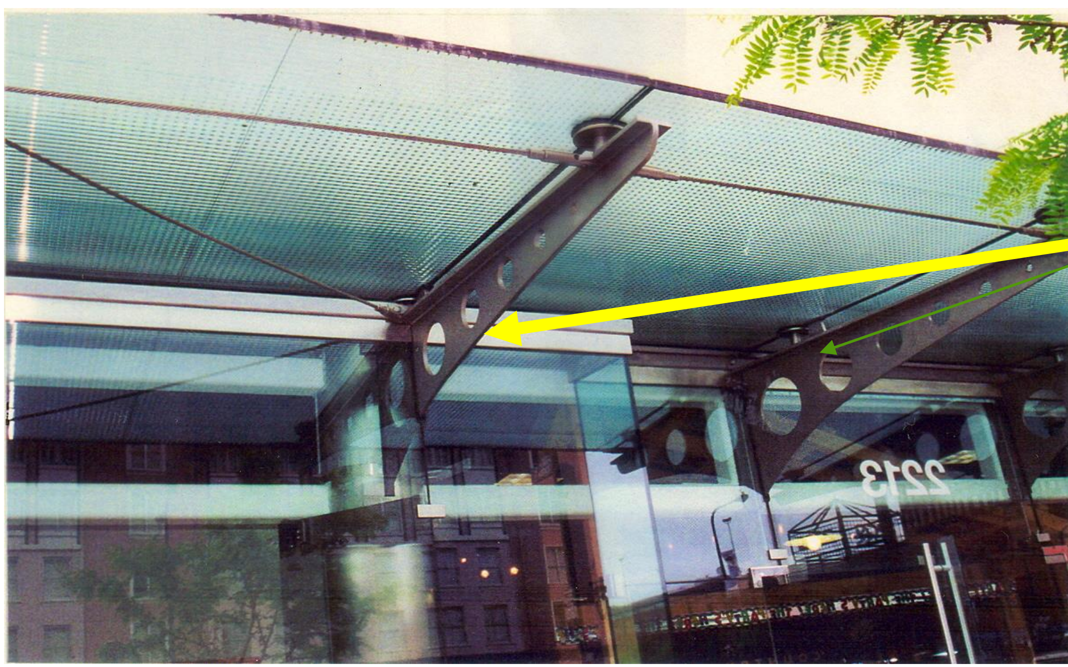
$$\sigma_{max} = -\frac{(-5 \times 10^4)}{1.297 \times 10^{-3}} = 38.54 MP_a, \quad 61.19\%$$

Examples of I-beam, T-beam, and Y-beam

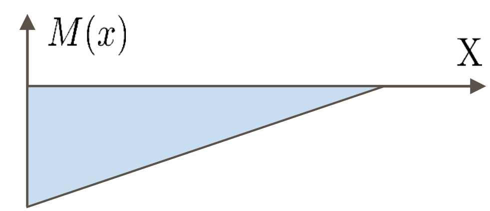








Why do they drill holes inside the T-beam ?



$$M_{max} = -PL$$