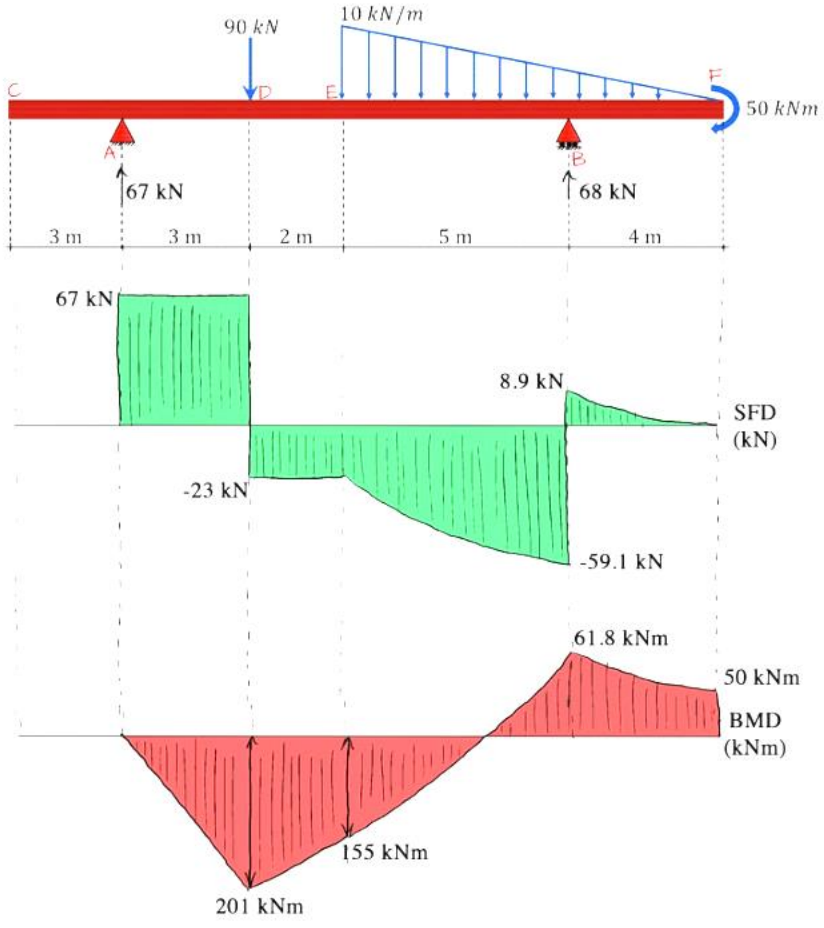
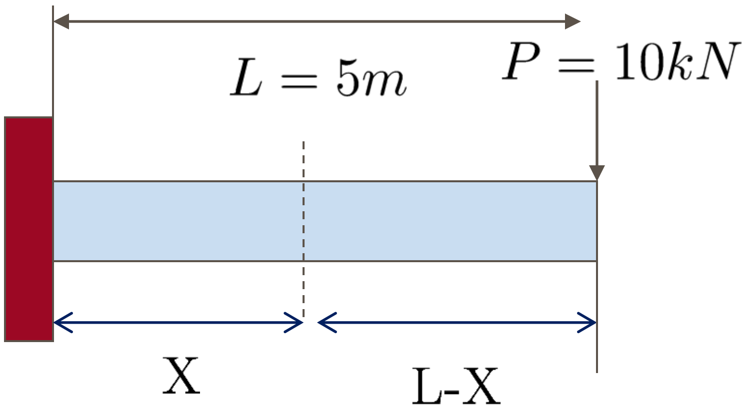
# Lecture 25 Bending Moment Diagram



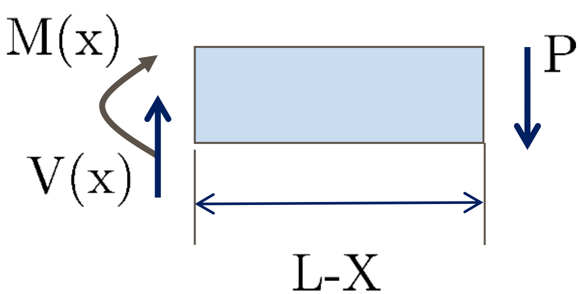
# Example II. Galileo's Problem




$$\sum F_y = 0 \ \rightarrow$$

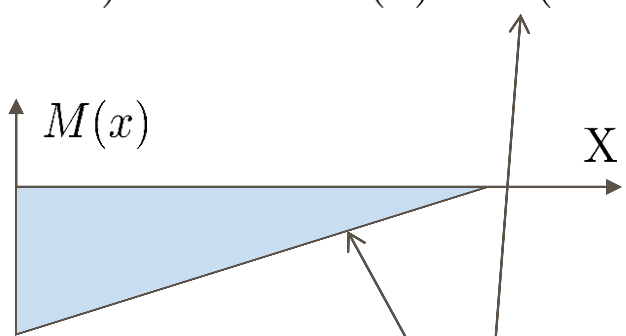
$$V(x) - P = 0 \ \rightarrow \ V(x) = P;$$


## Free-body diagram




$$\sum M_x = 0 \ \rightarrow$$

$$-M(x) - P(L - x) = 0 \ \rightarrow \ M(x) = P(x - L);$$



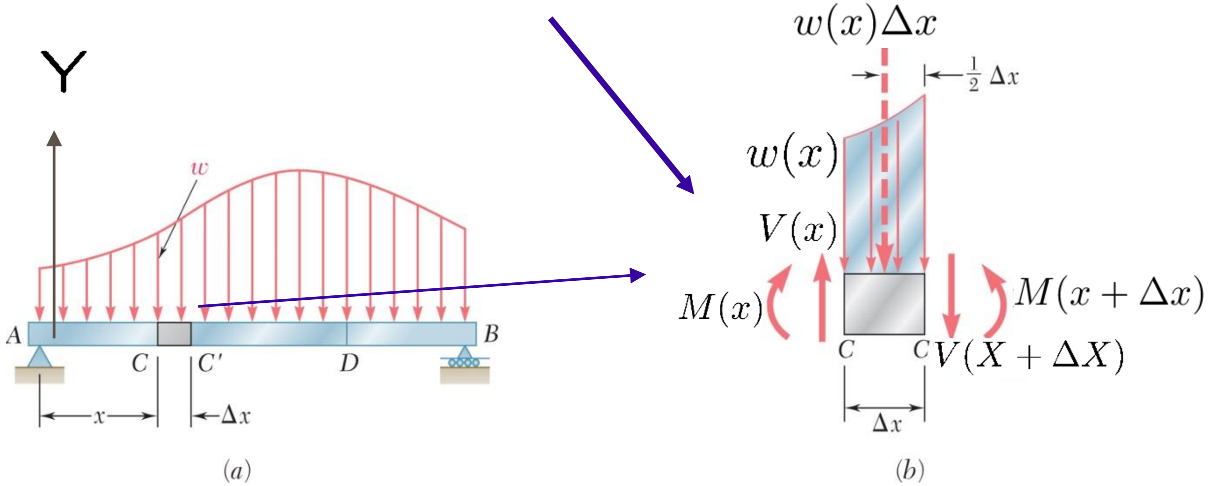

$$M_{max} = -PL$$

$$M_{\max} = M(0) = -PL = -5 \times 10^4 \, N \cdot m$$

**What is the slop of this line ?**

# Differential Equation Approach to Moment and Shear

Representative element approach




We study the equilibrium condition:

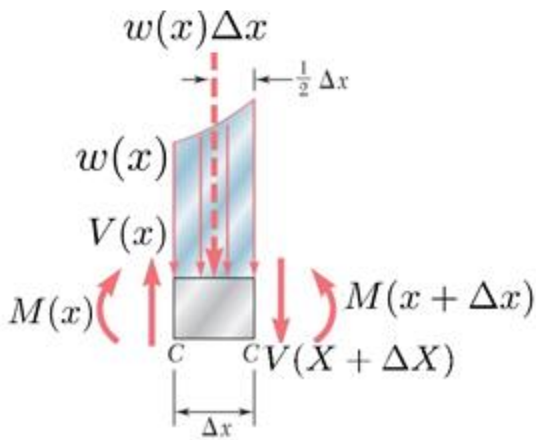
$$(1) \quad \sum F_y = 0 \qquad\qquad V(x) - V(x+\Delta x) - w(x)\Delta x = 0$$

$$\frac{V(x+\Delta x) - V(x)}{\Delta x} = -w(x) \quad \Rightarrow \quad \boxed{\frac{dV}{dx} = -w(x)}$$

$$\boxed{V(x) - V(0) = -\int_0^x w(x)dx}$$

w(x) is downward.

$$(\mathbf{2})\ \sum M_{x+\Delta x} = 0$$




$$(M(x+\Delta x) - M(x) - V(x)\Delta x + (w\Delta x)\frac{\Delta x}{2} = 0;$$

$$\frac{M(x+\Delta x) - M(x)}{\Delta x} = V(x) - \frac{1}{2}w(x)\Delta x \quad \rightarrow \quad \boxed{\frac{dM}{dx} = V(x)}$$

Therefore,

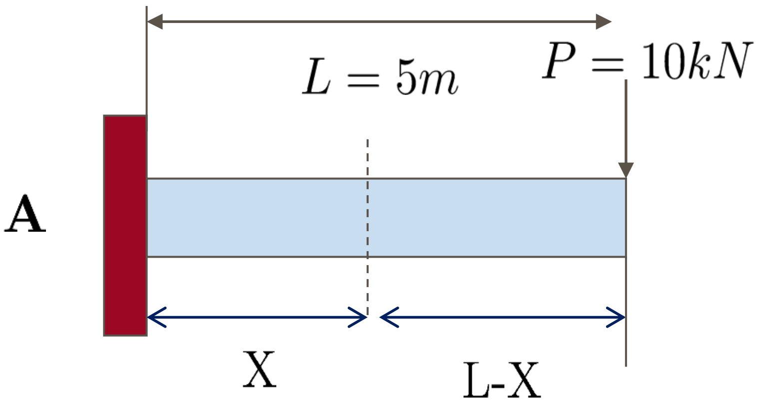
$$\boxed{M(x) - M(0) = \int_0^x V(x')dx'}$$

# Summary:

$$\boxed{\frac{dV}{dx} = -w(x)} \rightarrow \boxed{V(x) - V(0) = -\int_0^x w(x)dx}$$

$$\boxed{\frac{dM}{dx} = V(x)} \rightarrow \boxed{M(x) - M(0) = \int_0^x V(x')dx'}$$

**Example III**



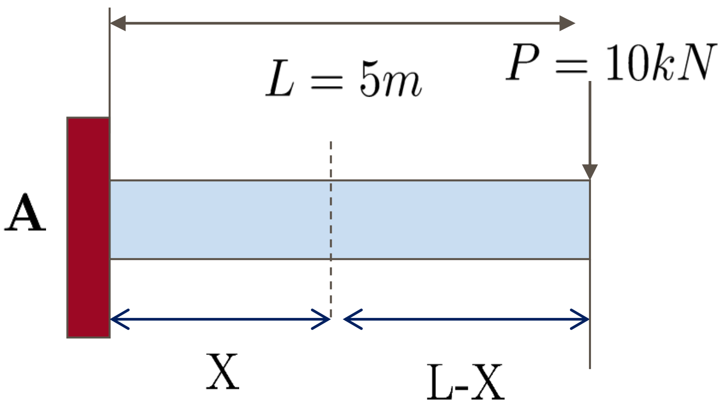

[Solution]

$$V(0) = R_A = P \text{ and } M(0) = M_A = -PL,$$

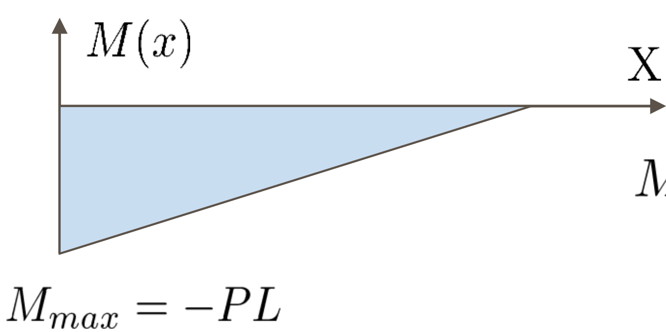
$$w(x) = 0$$

$$V(x) - V(0) = 0 \rightarrow V(x) = R_A = P$$

**Example III**








[Solution]

$V(0) = R_A$ and $M(0) = M_A$, **Why?**

$$w(x) = 0$$

$$V(x) - V(0) = 0 \rightarrow V(x) = P$$

**What is $V(L)$?**

$$\frac{dM}{dx} = V(x) \qquad M(0) = M_A = -PL$$

$$M(x) - M(0) = \int_0^x P dx = Px$$

$$M(x) = M(0) + Px = -PL + Px = P(x - L)$$

**What is $M(L)$?**

Technically speaking, there are two types of structural engineers …..

Blue collar type

White collar type

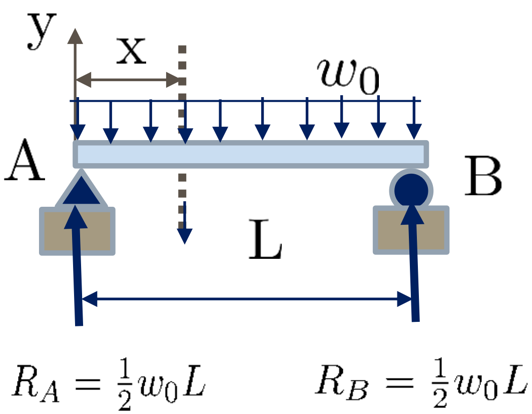
Can fight

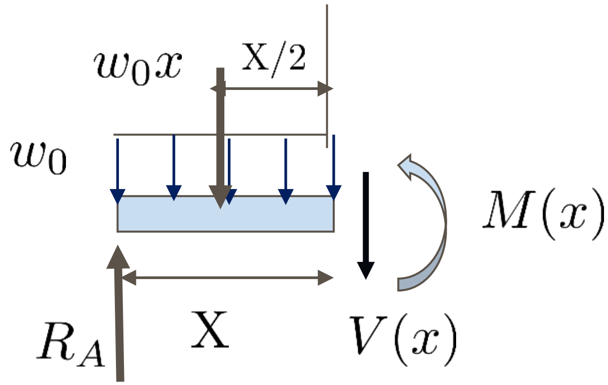
Can write

Which type do you want to be …?

**Example IV** [Solution] By symmetry $R_A = R_B$.

$$R_A = R_B = \frac{1}{2} w_0 L$$




$$\sum F_y = 0 \;\rightarrow\; R_A - w_0 x - V(x) = 0$$

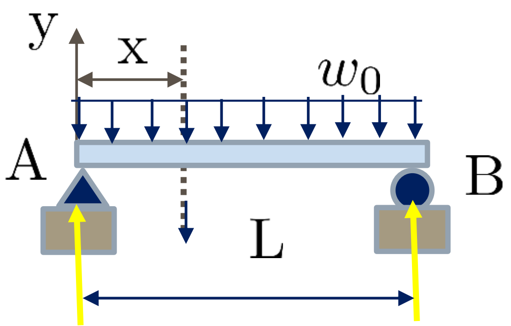
$$V(x) = R_A - w_0 x = w_0 (L/2 - x)$$

$$\sum M_x = 0 \;\rightarrow\; -R_A x + w_0 x (x/2) + M(x) = 0$$

$$M_x = R_A x - w_0 x (x/2) = \frac{w_0}{2} (Lx - x^2)$$

Blue collar type

## Example IV




$R_A = \frac{1}{2}w_0 L$ $R_B = \frac{1}{2}w_0 L$

White collar type

[Solution] $R_A = R_B = \dfrac{1}{2}w_0 L$

$V(0) = R_A$ and $M(0) = 0$. **Why?**

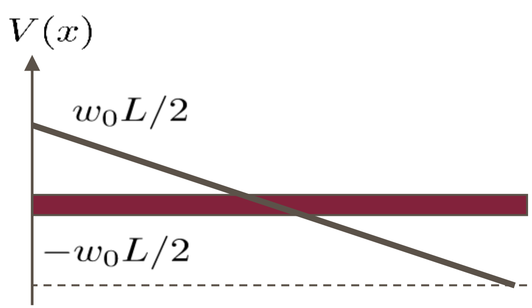
What is $V(L)$? $V(L) = -R_B = -\frac{1}{2}w_0 L$

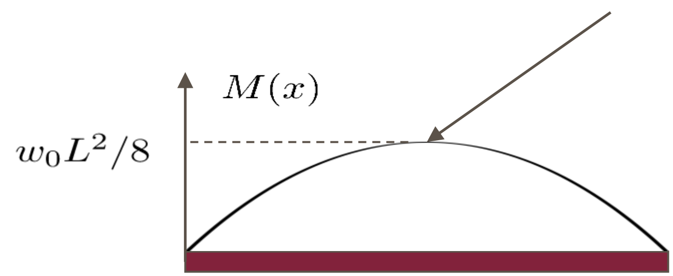
$$V(x) - V(0) = -\int_0^x w_0 dx = -w_0 x \quad \rightarrow$$

$$\rightarrow \quad V(x) = -w_0 x + \frac{w_0 L}{2}$$

$$M(x) - M(0) = \int_0^x V(x)dx = -\frac{w_0 x^2}{2} + \frac{w_0 L}{2}x$$

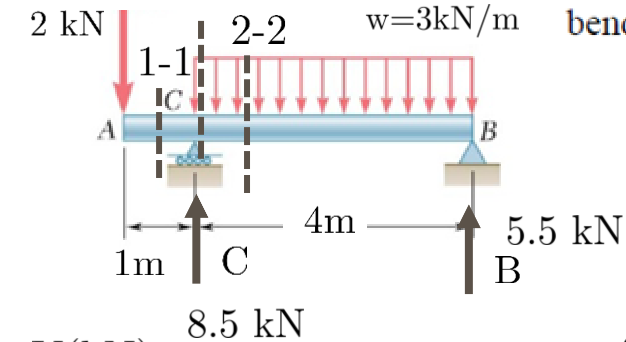
$$M(x) = \frac{w_0}{2}(Lx - x^2) \quad M_{max} = M((L/2) = w_0 L^2/8$$



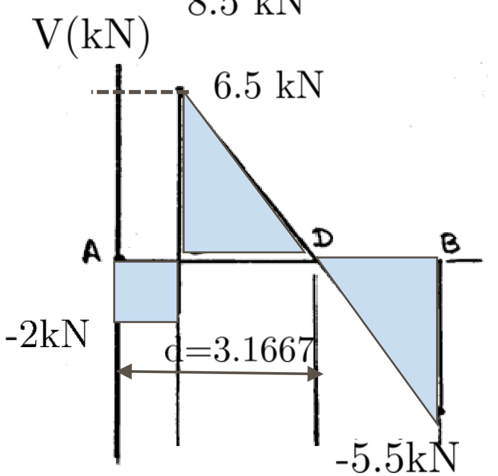
## PROBLEM 12.53

Draw the shear and bending-moment diagrams for the beam and loading shown and determine the maximum normal stress due to bending.

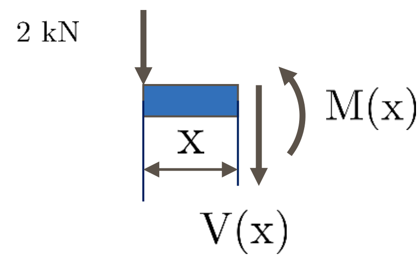


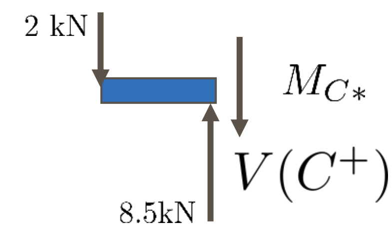

$$+\circlearrowleft \Sigma M_C = 0:\ (2)(1) - (3)(4)(2) + 4B = 0 \qquad B = 5.5\ \text{kN}$$

$$+\circlearrowleft \Sigma M_B = 0:\ (5)(2) + (3)(4)(2) - 4C = 0 \qquad C = 8.5\ \text{kN}$$




Cut 1-1:



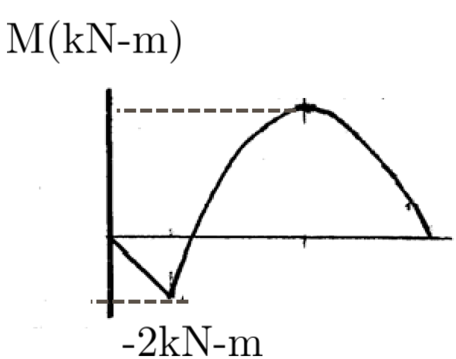




$$\sum F_y = 0 \ \rightarrow\ -V(x) - 2 = 0 \ \rightarrow\ V(x) = -2kN \qquad 0 < x < C^-$$

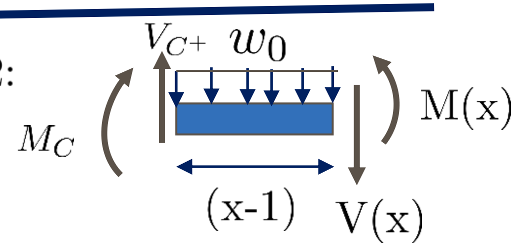
$$\text{At } C^+ : \sum F_y = 0 :\ \ -2 + 8.5 - V_{C^+} = 0 \rightarrow V_{C^+} = -2 + 8.5 = 6.5\ \text{kN}$$

$$\sum M_x = 0 \ \rightarrow\ 2x + M(x) = 0 \ \rightarrow\ M(x) = -2x\ kN{-}m$$

At C: $M_C = -2$ kN-m




Cut 2-2:

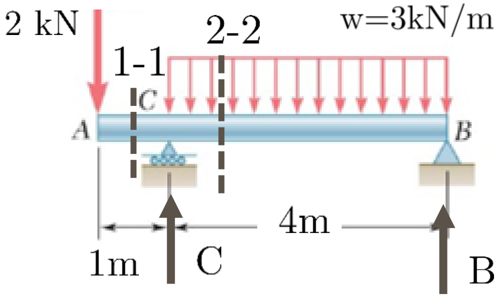



$C^+ \le x \le 5$

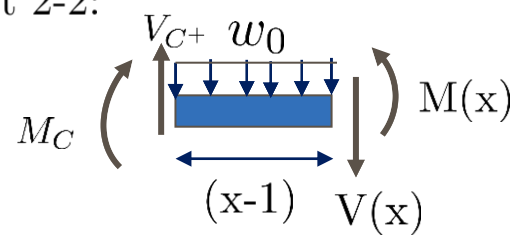
$$\sum F_y = 0 \ \rightarrow\ -V(x) + V_{C^+} - w_0(x-1) = 0 \ \rightarrow\ V(x) = 9.5 - 3x\ \ kN,\ \ 1^+ < x < 5$$

$$\text{Let } V(d) = 9.5 - w_0 d = 0 \ \rightarrow\ d = 9.5/3 = 3.16667m$$

$$V_B = V(5) = 9.5 - 3 \times 5 = -5.5kN$$



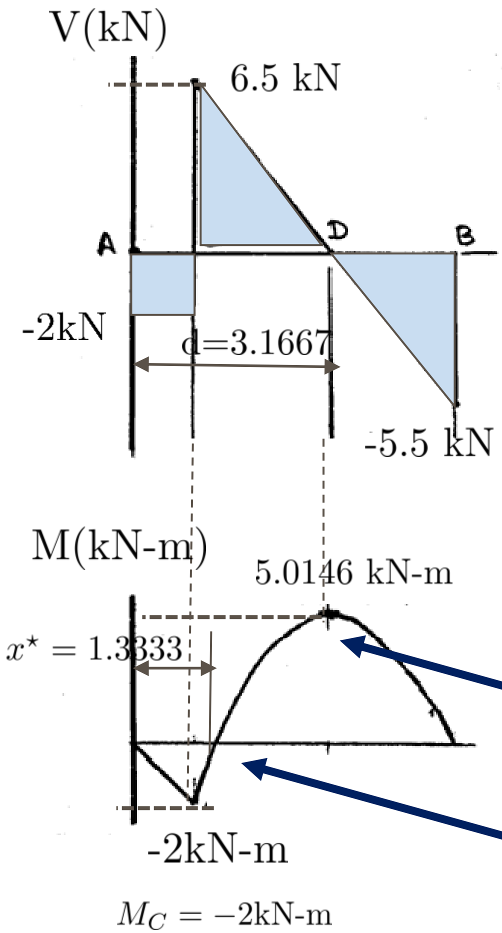

Cut 2-2:




$C^+ \le x \le 5$

$$\sum F_y = 0 \rightarrow \ -V(x) + V_{C^+} - w_0(x-1) = 0 \ \rightarrow$$

$$V(x) = 9.5 - 3x kN, \ 1^+ < x < 5$$




Let $V(d) = 9.5 - w_0 d = 0 \ \rightarrow$ $d = 3.1667m$

$d = 9.5/3 = 3.16667m$

$V_B = V(5) = 9.5 - 3 \times 5 = -5.5kN \quad V_B \neq R_B$

$$\sum M_x = 0 \ \rightarrow \ -M_c - (x-1)V_{C^+} + w_0 \frac{(x-1)^2}{2} + M(x) = 0 \ \rightarrow$$

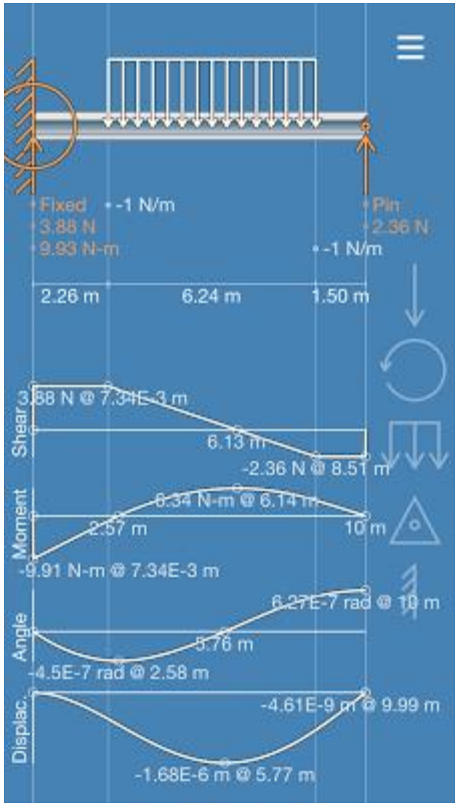
$$M(x) = -\frac{3}{2}(x-1)^2 + 6.5(x-1) - 2$$ kN-m

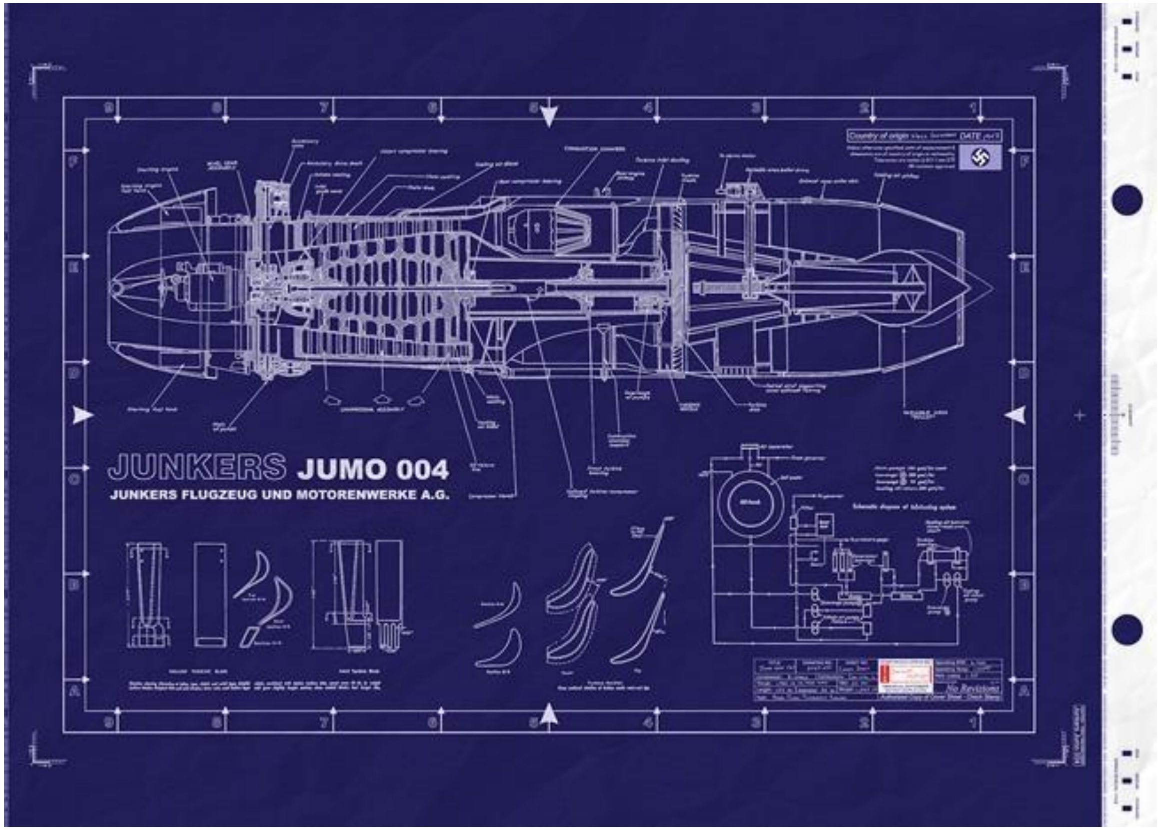
**At which point $M(x) = M_{max}$?**

$M(d) = -\frac{3}{2}(3.1667-1)^2 + 6.5(3.1667-1) - 2 = 5.04167$kN-m

Let $M(x^*) = -\frac{3}{2}(x^*-1)^2 + 6.5(x^*-1) - 2 = 0 \ \rightarrow \ x^* = 1.333m$

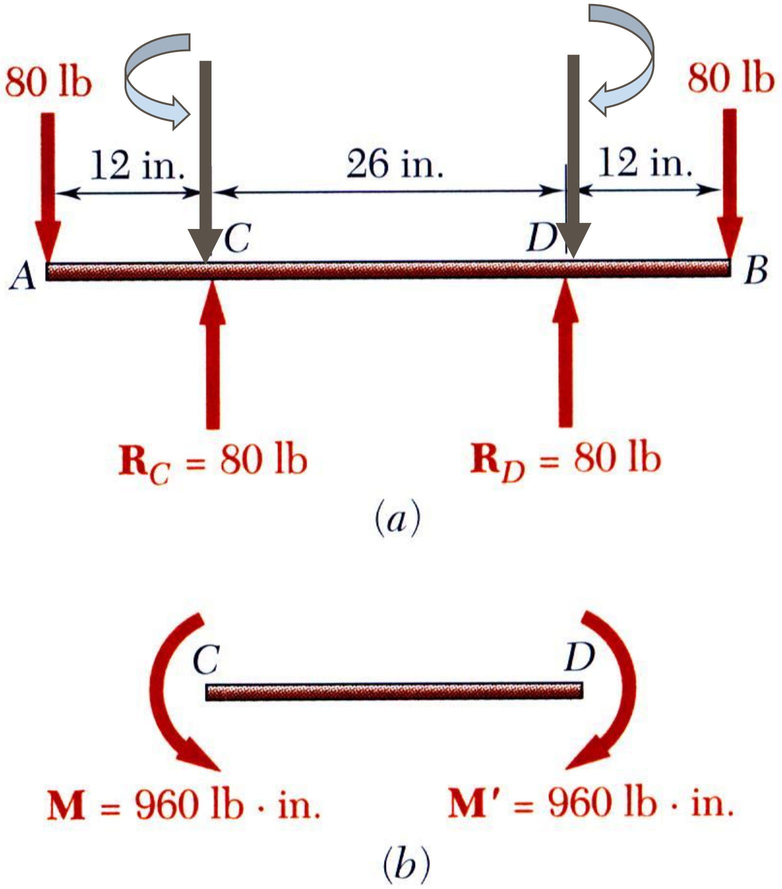
# Take-Home Message: Attention to Details
# Today's Lecture Passphrase is: Perfection



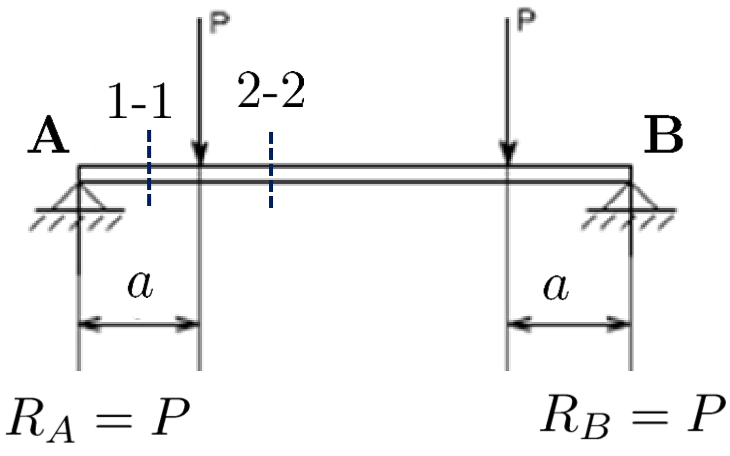

**Be a meticulous engineer who is in pursuit of perfection!**

# What is Pure Bending ?


*Pure Bending*: Prismatic members subjected to equal and opposite couples acting in the same longitudinal plane

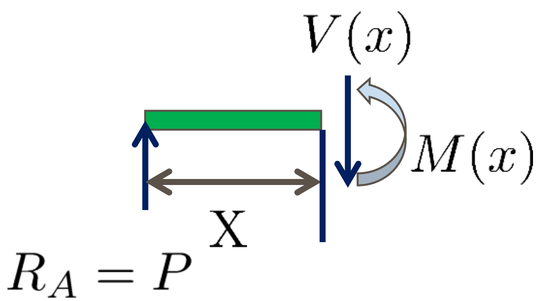
# Draw shear and moment diagrams




**Solution:**

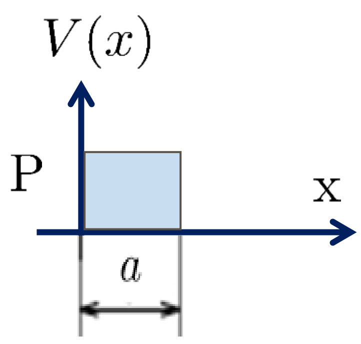
**Cut 1-1:**

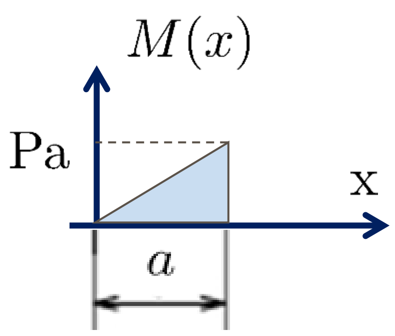
**Step 1**



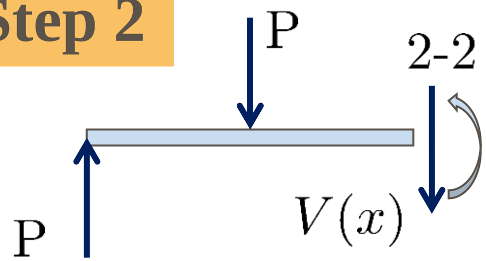

$$\sum F_y = 0, \;\rightarrow\; P - V(x) = 0 \;\rightarrow\; V(x) = P, \quad 0 < x < a$$

$$\sum M_x = 0, \;\rightarrow\; M(x) - Px = 0 \;\rightarrow\; M(x) = Px, \quad 0 < x < a$$



**Step 2**

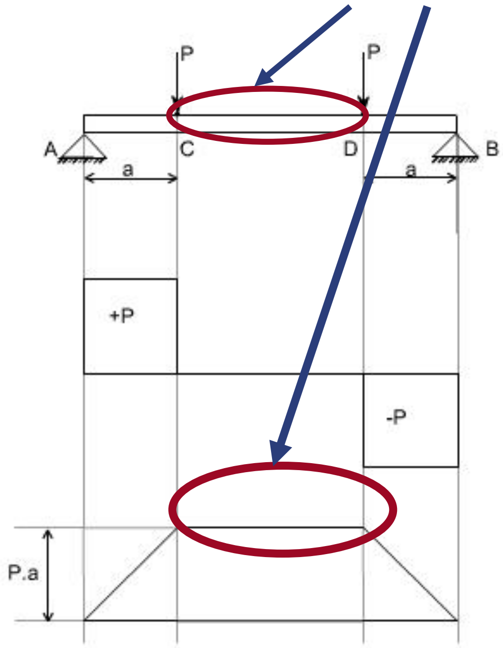



**Cut 2-2**

For $a < x < L - a$,

$$\sum F_y = 0, \ \rightarrow \ P - P - V(x) = 0 \ \rightarrow \ V(x) = 0;$$

**Pure Bending !**




$$\sum M_x = 0, \ \rightarrow \ M(x) - Px + P(x-a) = 0 \ \rightarrow \ M(x) = Pa;$$

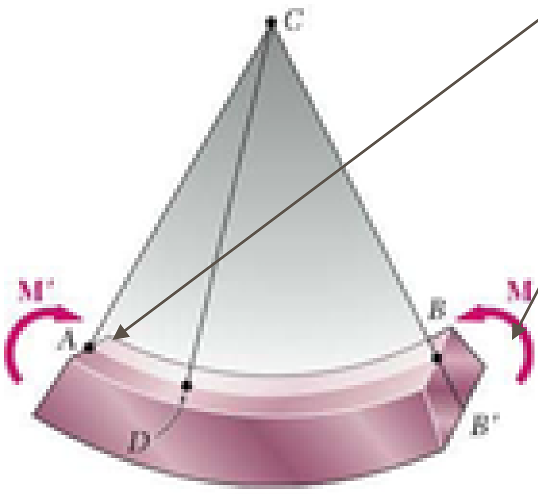
**For** $a < x < L - a$

$V(x) = 0$ **and** $M(x) = Pa$

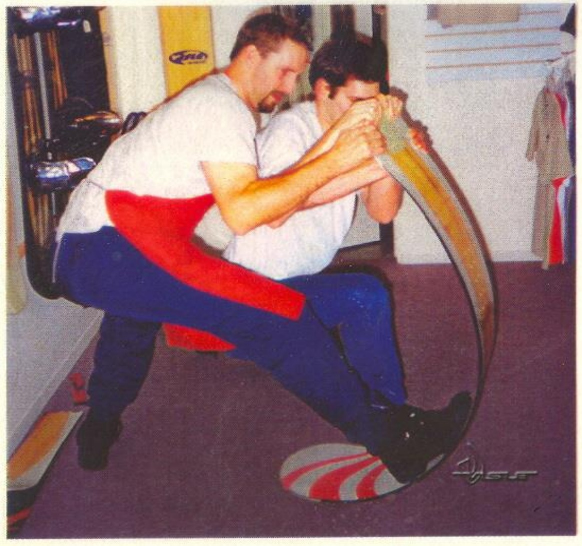
## Pure Bending

# Lecture 26 Beam Bending

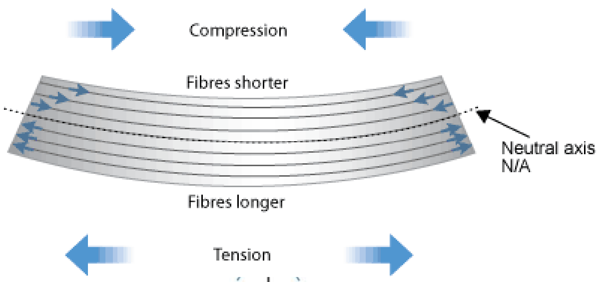
| Structural member | stress resultant | Corresponding stress |
|---|---|---|
| Bar | $P$ | $\sigma = \dfrac{P}{A}$ |
| Shaft | $T$ | $\tau = \dfrac{T\rho}{J}$ |
| Beam | $M, V$ | ?, ? |

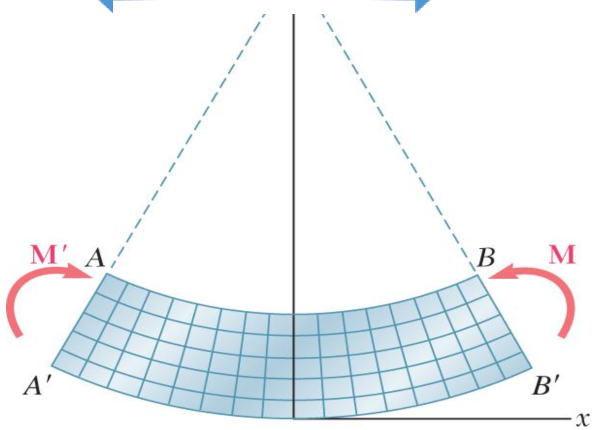



**Pure Bending**

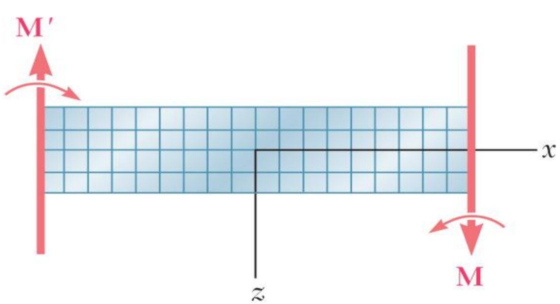


# How does a beam bend in pure bending ?




($a$) Longitudinal, vertical section (plane of symmetry)

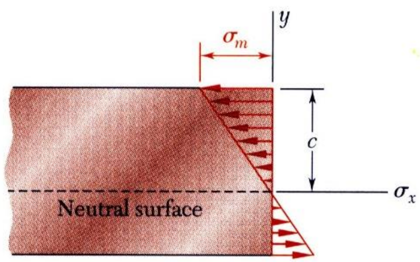


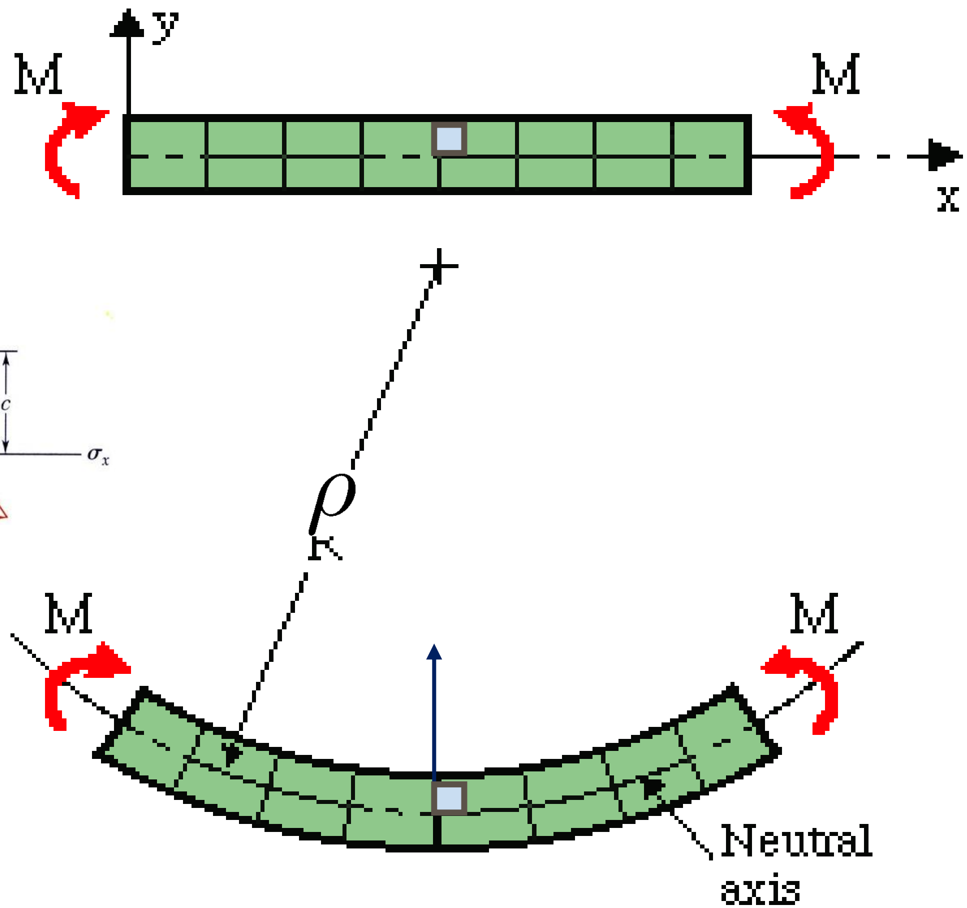

($b$) Longitudinal, horizontal section

**Beam with a plane of symmetry in pure bending:**

- member remains symmetric
- bends uniformly to form a circular arc
- cross-sectional plane passes through arc center and remains planar
- length of top decreases and length of bottom increases
- a *neutral surface* must exist that is parallel to the upper and lower surfaces and for which the length does not change
- stresses and strains are negative (compressive) above the neutral plane and positive (tension) below it

# Kinematic Assumption of Bernoulli-Euler Beam




Pure bending

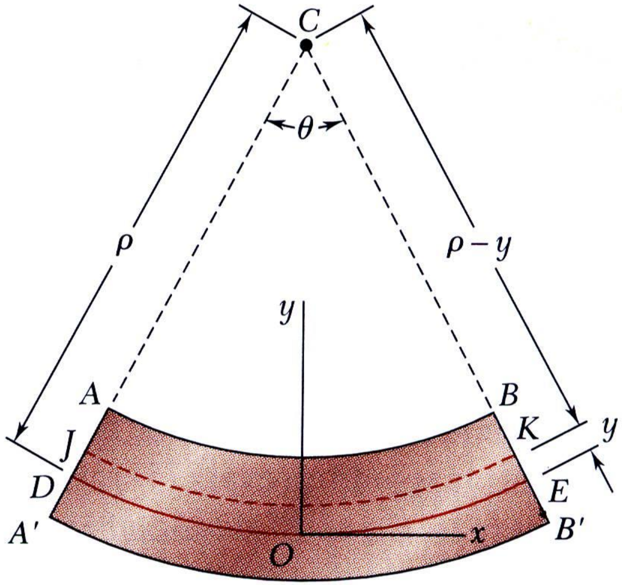
$$\begin{cases} M = const. \\ V = 0\ . \end{cases}$$

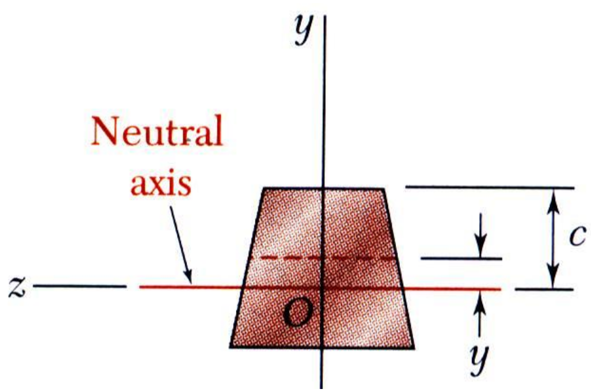
$$\text{Curvature :} \quad \kappa = \frac{1}{\rho}$$

**No shear strain !**

**Why Neutral Axis ?**

# Normal Strain Due to Pure Bending




Consider a beam segment of length $L$.

After deformation, the length of the neutral surface remains $L$. At other sections,

$$L' = \overline{JK}$$

$$\begin{aligned} L' &= (\rho - y)\theta \\ \Delta &= L' - L = (\rho - y)\theta - \rho\theta = -y\theta \\ \epsilon_x &= \frac{\Delta}{L} = -\frac{y\theta}{\rho\theta} = -\frac{y}{\rho} = -\left(\frac{c}{\rho}\right)\frac{y}{c} \end{aligned}$$

Strain varies linearly

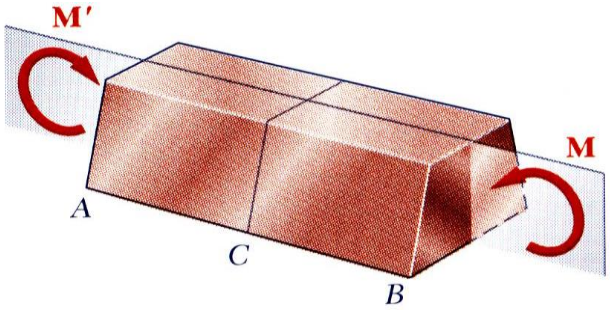
$$\epsilon_m = \left|\frac{c}{\rho}\right|, \quad \text{or} \quad \boxed{\rho = \frac{1}{\kappa} = \frac{c}{\epsilon_m}}$$

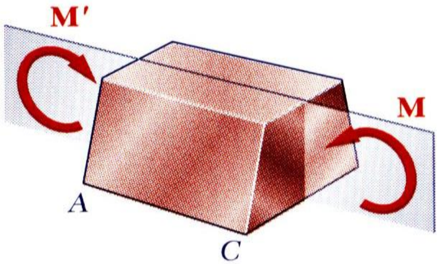
$$\epsilon_x = -\epsilon_m \frac{y}{c} \quad \Rightarrow \quad \boxed{\sigma_x = -\sigma_m\left(\frac{y}{c}\right)}$$

$$\sigma_x = E\varepsilon_x = -\frac{y}{c}E\varepsilon_m = -\frac{y}{c}\sigma_m \quad \text{(stress varies linearly)}$$

c is the upper depth of the beam.

# Internal Forces in Pure Bending



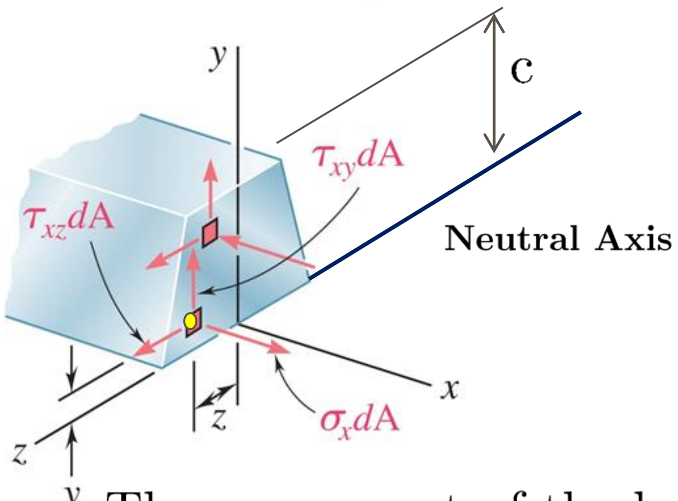




- Internal forces in any cross section are equivalent to a couple. The moment of the couple is the section *bending moment*.

This is the statics part.

$$F_x = \int \sigma_x \, dA = 0$$

$$M_z = \int -y\sigma_x \, dA = M \qquad \sigma_x = -\sigma_m \left(\frac{y}{c}\right)$$




The upper part of the beam section

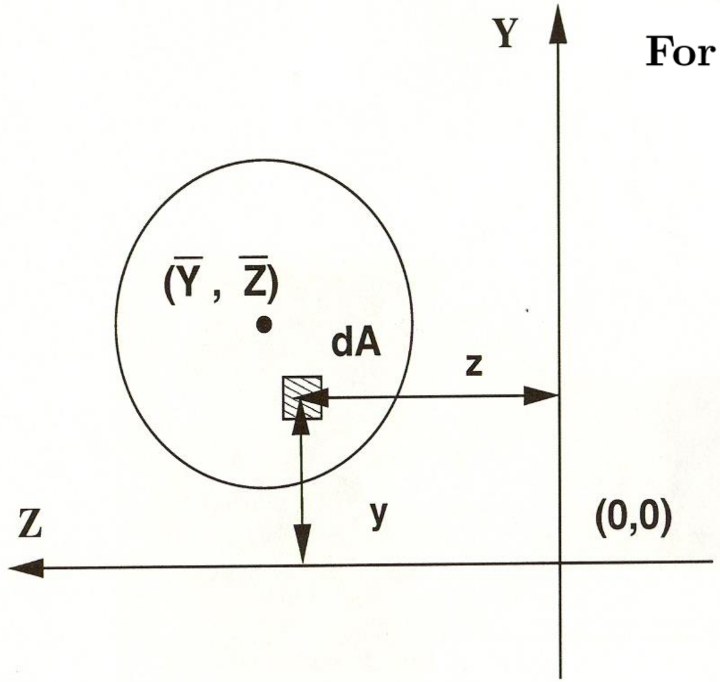
- For static equilibrium,

$$F_x = 0 = \int \sigma_x \, dA = \int -\frac{y}{c}\sigma_m \, dA$$

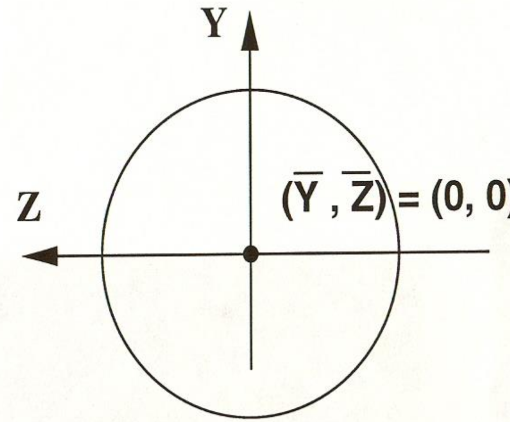
$$0 = -\frac{\sigma_m}{c}\int y \, dA \qquad \text{What does this mean ?}$$

**The neutral axis is the centroidal axis.**

# Definition of Centroid of an Area




**For centroidal axis :** $Q_z = \int_A y dA = 0$




$Q_y = 0$ and $Q_z = 0$

**Centroidal axis**

The first moment of an area A with respect to Z-axis and with respect to Y-axis are:

$$Q_z = \int_A y dA, \qquad Q_y = \int_A z dA$$

The centroid of the area A is defined as

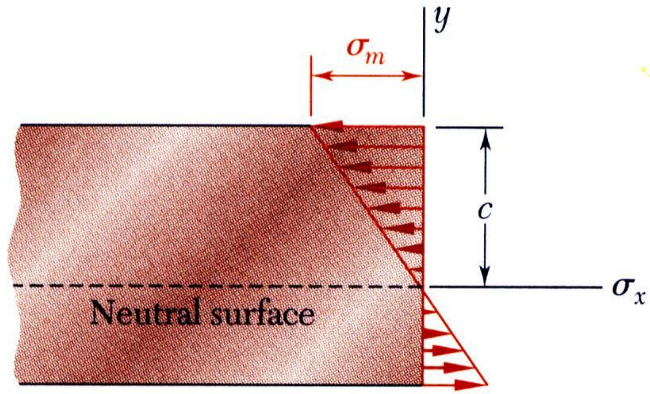
$$\bar{Z} := \frac{\int_A z dA}{A} \text{ and } \bar{Y} := \frac{\int_A y dA}{A}$$

# Elastic Flexure Formula

- For a linearly elastic material,

$$\sigma_x = E\varepsilon_x = -\frac{y}{c}E\varepsilon_m$$

$$= -\frac{y}{c}\sigma_m \quad \text{(stress varies linearly)}$$



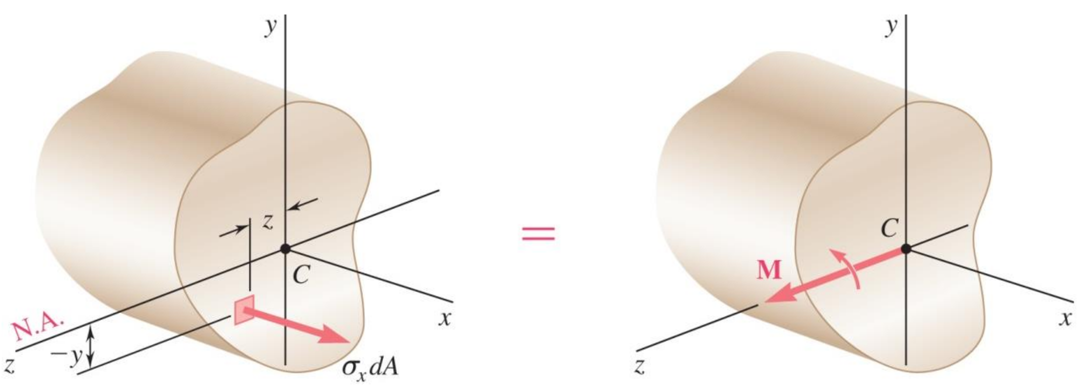

- For static equilibrium,

$$F_x = 0 = \int \sigma_x \, dA = \int -\frac{y}{c}\sigma_m \, dA$$

$$0 = -\frac{\sigma_m}{c}\int y \, dA$$

What does this mean ?

**The neutral axis is the centroidal axis.**




- For static equilibrium,

$$M_z = \int -y\sigma_x \, dA = \int -y\left(-\frac{y}{c}\sigma_m\right) dA$$

$$M = \frac{\sigma_m}{c}\int y^2 \, dA = \frac{\sigma_m I}{c}$$

$$\sigma_m = \frac{Mc}{I}$$

$$\frac{\sigma_m}{c} = \frac{M}{I}$$

Substituting $\sigma_x = -\frac{y}{c}\sigma_m$

$$\sigma_x = -\frac{My}{I}$$

$$\boxed{\sigma_x = -\frac{M_z(x)y}{I_z}}$$

# Elastic Flexure Formula

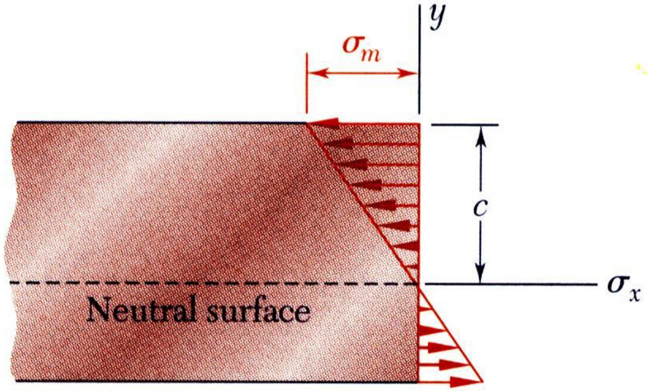
- For a linearly elastic material,

$$\sigma_x = -\sigma_m \left(\frac{y}{c}\right)$$

$$F_x = \int \sigma_x \, dA = 0$$

$$M_y = \int z\sigma_x \, dA = 0$$

$$M_z = \int -y\sigma_x \, dA = M$$




$$M_z = \int_A (-y\sigma_x)dA = \int_A (-y)\left(-\sigma_m \frac{y}{c}\right)dA$$

$$\begin{aligned} M_y &= \int_A z\sigma_x dA = -\frac{\sigma_m}{c}\int_A zy dA \\ &= 0, \end{aligned}$$

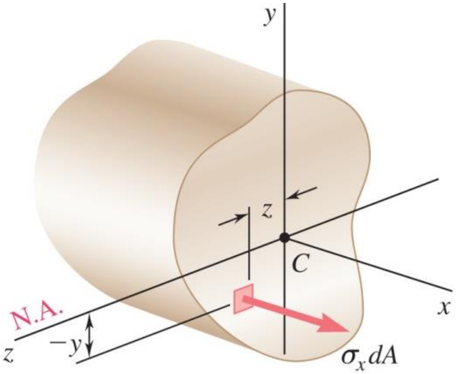
if y−z are centridoal axes.

$$M_z = \frac{\sigma_m}{c}\int_A y^2 dA = \frac{\sigma_m I_z}{c}\ .$$

Substituting $\frac{\sigma_m}{c} = \frac{M_z(x)}{I_z} \rightarrow \sigma_x = -\sigma_m \frac{y}{c}$

$$\frac{1}{\rho} = \frac{\epsilon_m}{c} = \frac{M_z}{EI_z}$$

$$\boxed{\sigma_x = -\frac{M_z(x)y}{I_z}}$$

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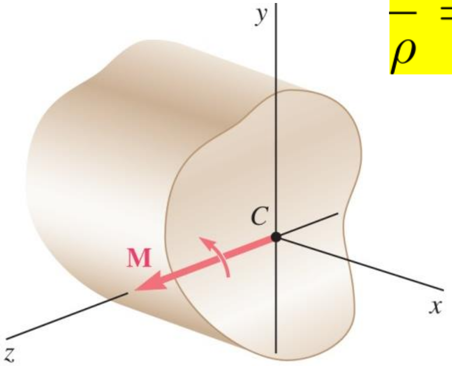

# Summary: Elastic Flexure Formula

**1.** Kinematic Assumption : $\sigma_x = -\sigma_m \left(\frac{y}{c}\right)$ $\qquad \epsilon_m = \frac{c}{\rho}$

**2** From equilibrium $\rightarrow$ $\frac{\sigma_m}{c} = \frac{M_z}{I_z}$ $\qquad \frac{1}{\rho} = \frac{M}{EI}$

**3. Elastic Flexure Formula :** $\boldsymbol{\sigma}_x = -\frac{M_z y}{I_z}$

| | Pure Bending | Torsion |
|---|---|---|
| Member | Bar (rod) | Shaft |
| Internal force | Bending Moment | Torque T |
| Constitutive law | $\sigma = E\epsilon$ | $\tau = G\gamma$ |
| Kinematic Assumption | Only allow axial deformation | The cross section of the shaft remains a plane after the twist. |
| Relation between internal force and stress | $\sigma = -\frac{M(x)y}{I_z}$ | $\tau = \frac{T\rho}{J}$ |
| Deformation | | $\Delta\phi = \frac{TL}{GJ}$ |
| Flexibility & Stiffness | | $f = \frac{L}{GJ}$ & $k = \frac{JG}{L}$ |