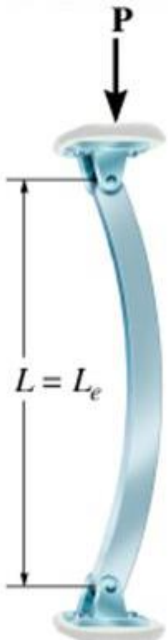
Lecture 37 Stability of Elastic Columns (II)
Columns with other boundary conditions

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$



Recap

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{P}{EI}y$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

(1) When $P < P_{cr}$ or $\sigma < \sigma_{cr}$, the column is straight.

(2) Pertubed configuration is a neutral equilibrium.

(3) When $P > P_{cr} \quad \rightarrow \quad y_{max} \rightarrow \infty$.

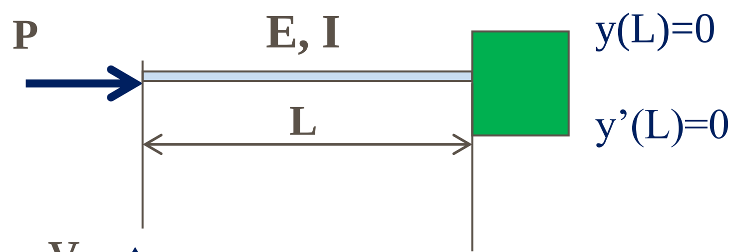
$y(x) = A \sin \lambda x$, *where A is arbitrary.*

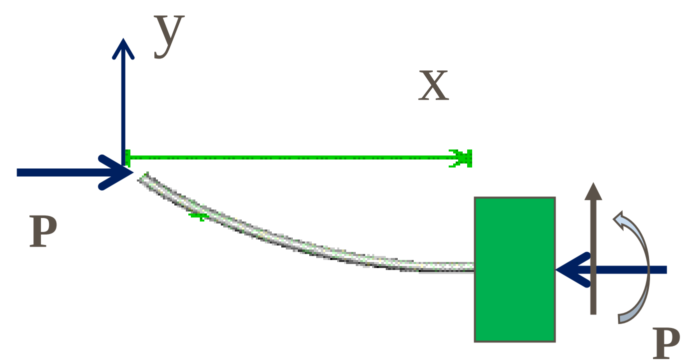
- Solution with assumed configuration can only be obtained if

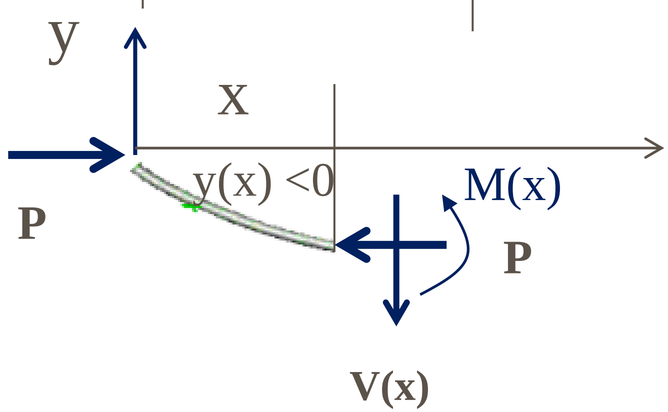
$$P > P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$\sigma = \frac{P}{A} > \sigma_{cr} = \frac{\pi^2 E\left(Ar^2\right)}{L^2 A} = \frac{\pi^2 E}{(L/r)^2}$$

(2) Cantilever Column







$y(L) \neq 0$ in floating coordinate;
but $y'(L) = 0$.

$$\sum F_y = 0 \quad \rightarrow \quad V(x) = 0;$$

because $y(x) < 0$

$$\sum M_x \Big|_x = 0, \quad \rightarrow \quad M(x) - P|y(x)| = 0 \quad \rightarrow \quad M(x) = P|y(x)| = -Py(x)$$

$$\rightarrow \quad M(x) = EI\frac{d^2y}{dx^2} = -Py(x) \quad \rightarrow \quad \frac{d^2y}{dx^2} + \lambda^2 y = 0, \quad \lambda^2 = \frac{P}{EI}$$

$$y(x) = A\sin\lambda x + B\cos\lambda x; \quad \text{with} \quad y(0) = 0 \;\; \& \;\; y'(L) = 0$$

$$y(0) = 0 \;\rightarrow\; B = 0; \quad \rightarrow \quad y(x) = A\sin\lambda x, \quad y'(x) = \lambda A\cos\lambda x$$

$$y'(L) = 0 \;\rightarrow\; A\lambda\cos\lambda L = 0 \;\rightarrow\; \lambda_n L = \frac{(2n+1)\pi}{2}, \quad \lambda_n = \frac{\pi}{2L}, \; \frac{3\pi}{2L}, \; \frac{5\pi}{2L}, \; \cdots$$

$$n = 0, \quad 1, \quad 2$$

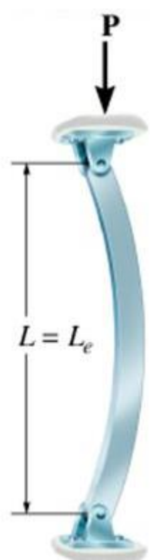
$$\lambda_{min} = \lambda_0 \;\rightarrow\; \lambda_0^2 = \frac{P_{cr}}{EI} = \left(\frac{\pi}{2L}\right)^2 \;\rightarrow\; P_{cr} = \frac{\pi^2 EI}{4L^2},$$

Comparing with $P_{cr}^{Euler} = \dfrac{\pi^2 EI}{L^2}$ for cantilever column, we can write :

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}, \quad L_e = 2L \;;$$

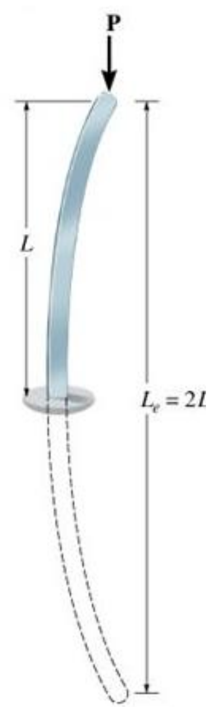
What is L_e ?

$M(0) = 0$



$M(L_e) = 0$

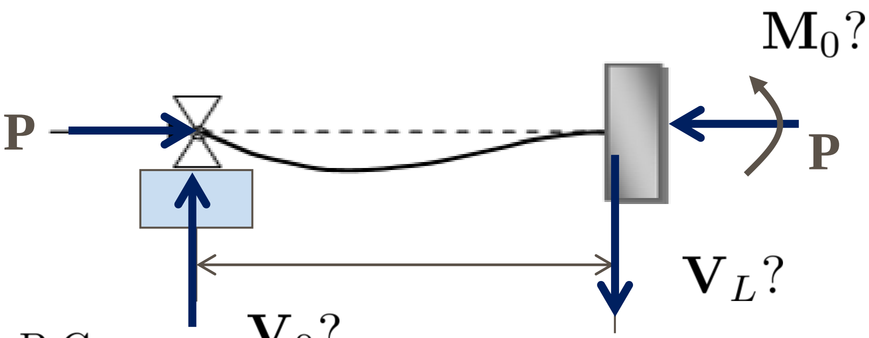
$M(0) = 0$



L_e is the distance between two points that have zero moment.

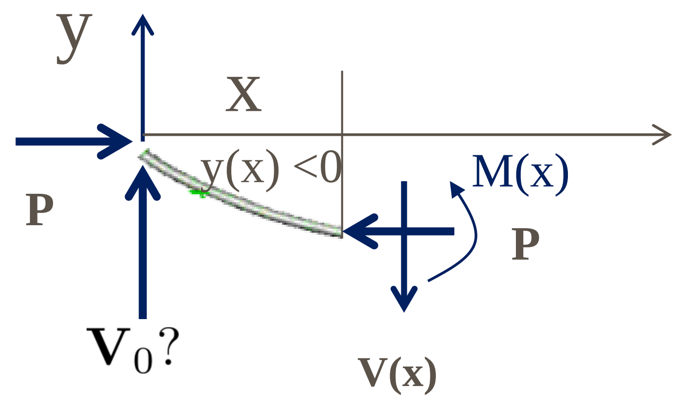
$M(L_e) = 0$

(3) Column with roller and built-in support



B.C.:
$y(0) = 0;$
$M(0) = 0;$

$y(L) = 0;$
$y'(L) = 0;$



$$\sum M_z \Big|_{@x} = 0$$

Remark:
This is a statically indeterminant system of degree one.

$$M(x) - P|y(x)| - V_0 x = 0 \;\rightarrow\; M(x) + Py(x) = V_0 x$$

$$EI\frac{d^2y}{dx^2} + Py(x) = V_0 x \quad\rightarrow\quad \frac{d^2y}{dx^2} + \lambda^2 y(x) = \frac{V_0}{EI}x, \quad \lambda^2 = \frac{P}{EI}$$

General solution = Homogenous solution + particular solution

$$y(x) = y_h(x) + y_p(x)$$

Homogeneous solution $y_h'' + \lambda^2 y_h = 0 \;\rightarrow\;:\qquad y_h(x) = A\cos\lambda x + B\sin\lambda x$

How to find the particular solution ? $\quad\rightarrow\quad$ by inspection

Let $y_p(x) = Cx$, where C is unknown constant, so that $y_p'' = 0$.

$$\lambda^2(Cx) = \frac{V_0}{EI}x \quad\rightarrow\quad C = \frac{V_0}{\lambda^2 EI} = \frac{V_0}{P}$$

$$\rightarrow\quad y(x) = A\cos\lambda x + B\sin\lambda x + \frac{V_0}{P}x$$

Boundary conditions: $y(0) = 0$, $y(L) = 0$ and $y'(L) = 0$.

(1) $y(0) = 0 \quad\rightarrow\quad A = 0;$

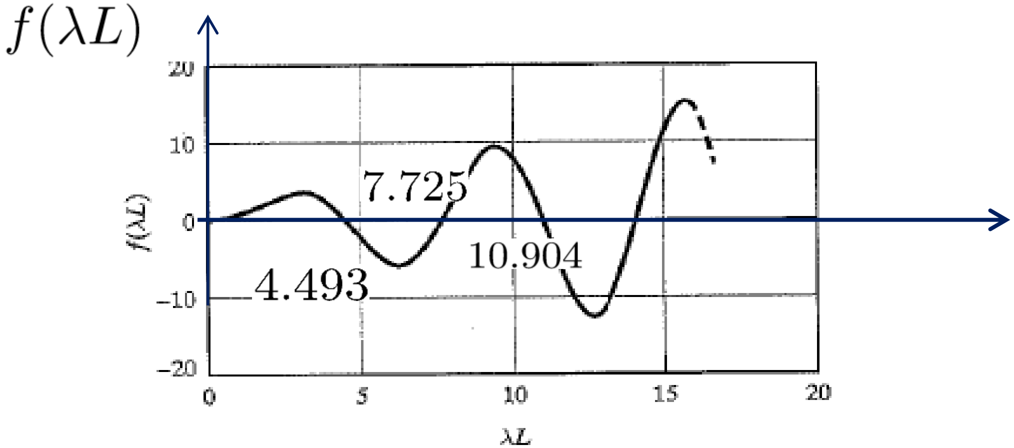
(2) $y(L) = 0 \quad\rightarrow\quad B\sin\lambda L + \frac{V_0}{P}L = 0;$

(3) $y'(L) = 0 \quad\rightarrow\quad B\lambda\cos\lambda L + \frac{V_0}{P} = 0;$

(2) $- L \times$ (3) $\quad\rightarrow\quad B\sin\lambda L - B\lambda\cos(\lambda L)L = 0 \quad\rightarrow\quad \tan\lambda L - \lambda L = 0;$

This is a transcendental equation, whose roots are irrational numbers.

Let $f(\lambda L) = \tan(\lambda L) - \lambda L.$



$$(\lambda L)_1 = 4.493\cdots\cdots$$

$$(\lambda L)_2 = 7.725\cdots\cdots$$

$$(\lambda L)_1 = 10.904\cdots\cdots$$

$$\lambda_{min} = \lambda_1 \;\rightarrow\; \lambda_1^2 = \frac{P_{cr}}{EI} \;\rightarrow\; P_{cr} = \lambda_1^2 EI = \left(\frac{4.493}{L}\right)^2 EI$$

$$P_{cr} = \frac{\pi^2 EI}{\left(L\pi/4.493\right)^2} = \frac{\pi^2 EI}{L_e^2} \;\rightarrow\; L_e^2 \qquad L_e = \frac{\pi L}{4.493} \approx 0.7L < L$$

Question:
Why is this column harder to buckle than the Euler column ?

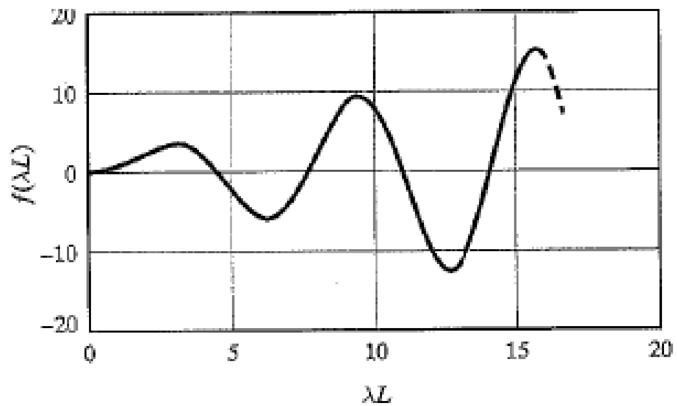
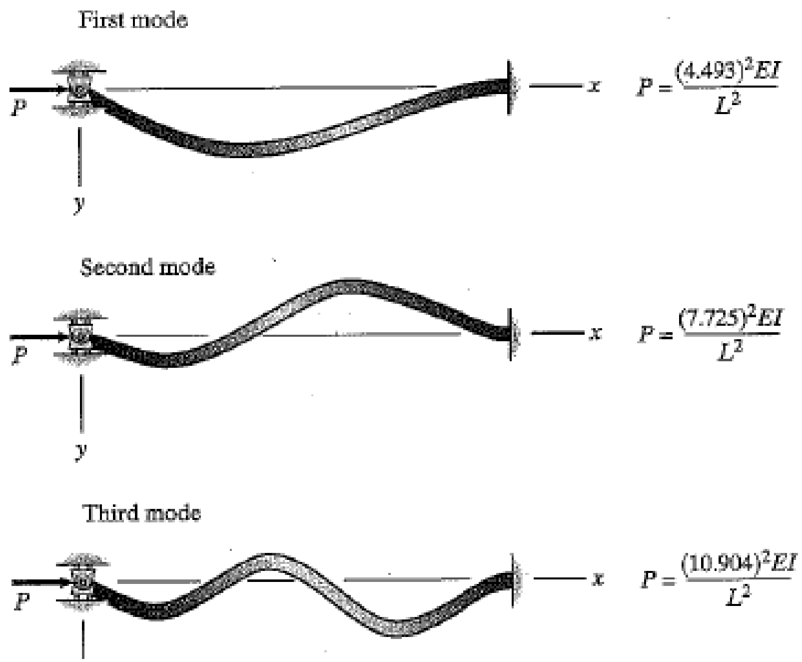
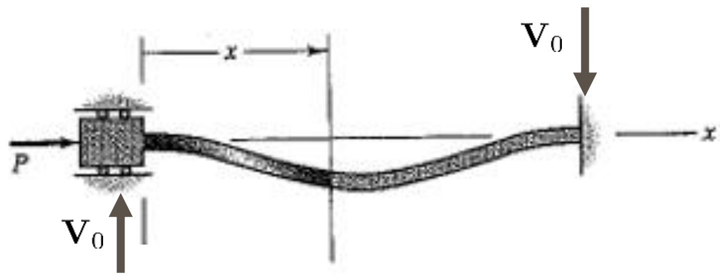


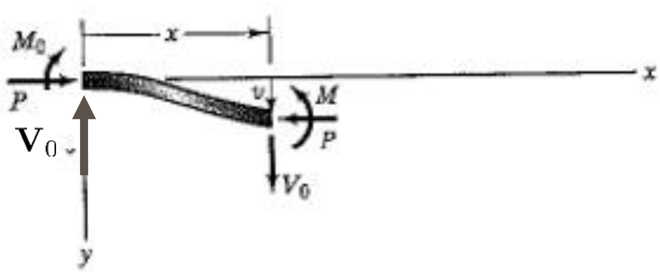
FIGURE 10-14 Graph of the characteristic function.



(4) A beam with a support that prevents lateral deflection and rotation at the left and built-in support at the right end.

Boundary Conditions:
$y(0) = 0,\ y'(0) = 0;$
$y(L) = 0,\ y'(L) = 0.$





$$\sum M_z \Big|_{@x} = 0$$

$$M(x) - V_0 x - M_0 - P|y(x)| = 0$$

$$M(x) = V_0 x + M_0 - P y(x), \quad (y(x) < 0)$$

Remark:
This is a statically indeterminant system of degree two.

$$
\begin{aligned}
EI\frac{d^2y}{dx^2} &= M(x) = V_0x + M_0 + P|y(x)| \\
&= -Py(x) + M_0 + V_0x, \qquad (y(x) < 0) \qquad\qquad (1)
\end{aligned}
$$

$$
\text{Let } \lambda^2 = \frac{P}{EI}, \quad \rightarrow \quad \frac{d^2y}{dx^2} + \lambda^2 y = \frac{1}{EI}(V_0x + M_0);
$$

$$
y(x) = y_h(x) + y_p(x) \text{ with } y_h(x) = A\cos\lambda x + B\sin\lambda x;
$$

$$
\text{Let } y_p(x) = Cx + D, \quad \rightarrow \quad \lambda^2(Cx + D) = \tfrac{1}{EI}(V_0x + M_0)
$$

$$
\text{We then have} : C = \frac{V_0}{\lambda^2 EI} = \frac{V_0}{P}, \;\; D = \frac{M_0}{\lambda^2 EI} = \frac{M_0}{P}\;.
$$

$$
\text{Hence, } y(x) = y_h(x) + y_p(x) = A\cos\lambda x + B\sin\lambda x + \tfrac{1}{P}(V_0x + M_0);
$$

Consider B.C.

$$
(1) \quad y(0) = 0, \quad \rightarrow \quad A + \frac{M_0}{P} = 0, \text{ or } \frac{M_0}{P} = -A
$$

Consider $y'(x) = -\lambda A \sin \lambda x + \lambda B \cos \lambda x + \frac{V_0}{P}$ and

$$y'(x)\Big|_{x=0} = \lambda B + \frac{V_0}{P} = 0, \quad \rightarrow \quad \frac{V_0}{P} = -\lambda B \qquad \rightarrow B = 0$$

When $y(x)$ is symmetric.

Hence,

$$\begin{aligned} y(x) &= A \cos \lambda x - A; \quad \text{and} \\ y'(x) &= -\lambda A \sin \lambda x \end{aligned}$$

Then BCs: $y(L) = y'(L) = 0$ lead to

$$A(\cos \lambda L - 1) = 0$$

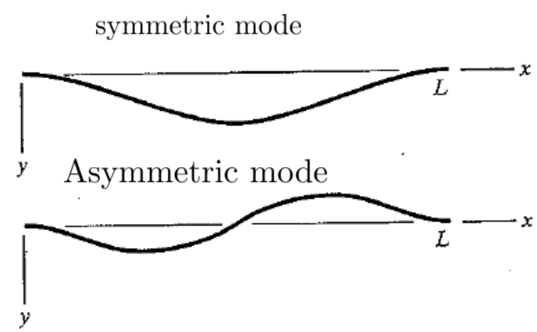
$$-\lambda A \sin \lambda L = 0$$

$$\lambda_n L = 2n\pi$$

$$P_{cr} = \frac{4\pi^2 EI}{L^2} = \frac{\pi^2 EI}{L_e^2}$$

where $L_e = L/2$ and

$$P_{cr} = 4P_{cr}^{Euler}$$



In general, we consider asymmetric modes,

$$\begin{aligned} y(x) &= A\cos\lambda x + B\sin\lambda x - \lambda B x - A; \quad \text{and} \\ y'(x) &= -\lambda A\sin\lambda x + B\lambda\cos\lambda x - \lambda B; \end{aligned}$$

Then BCs: $y(L) = y'(L) = 0$ lead to

$$\begin{cases} A\cos\lambda L + B\sin\lambda L - \lambda B L - A &= 0 \\ -\lambda A\sin\lambda L + B\lambda\cos\lambda L - \lambda B &= 0 \end{cases}$$

Matrix form,

$$\begin{bmatrix} \cos\lambda L - 1 & \sin\lambda L - \lambda L \\ -\lambda\sin\lambda L & \lambda\cos\lambda L - \lambda \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0 \; .$$

$$f(\lambda L) = 2 - 2\cos\lambda L - (\lambda L)\sin\lambda L$$

$$f(\lambda L) = 2 - 2\cos\lambda L - (\lambda L)\sin\lambda L$$

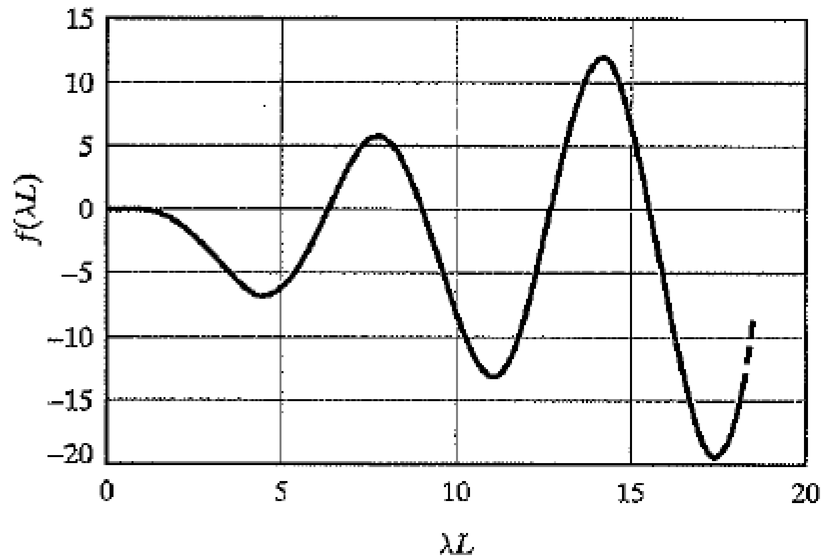


FIGURE 10-19 Graph of the characteristic function.

$$\begin{aligned}(\lambda L)_1 &= 2\pi\\ (\lambda L)_2 &= 8.999\\ (\lambda L)_3 &= 12.566\\ \cdots & \quad \cdots\end{aligned}$$

Hence,

$$\lambda_{min} = \lambda_1 = \frac{2\pi}{L} \rightarrow \frac{P_{cr}}{EI} = \lambda_1^2;$$

$$P_{cr} = \frac{4\pi^2 EI}{L^2} = \frac{\pi^2 EI}{L_e^2}$$

where $L_e = L/2$ and

$$P_{cr} = 4P_{cr}^{Euler}$$

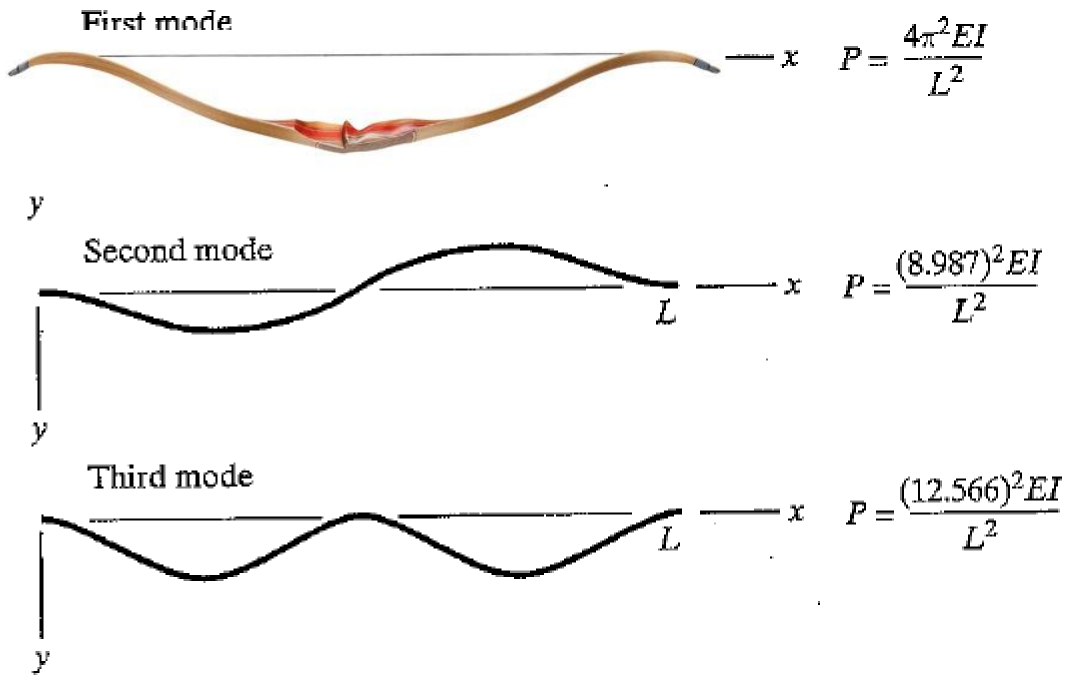
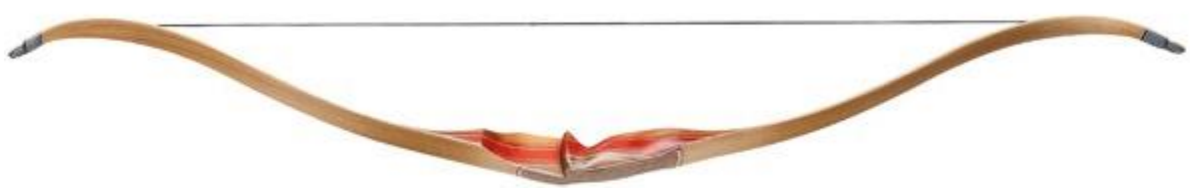


FIGURE 10-20 First three buckling modes.



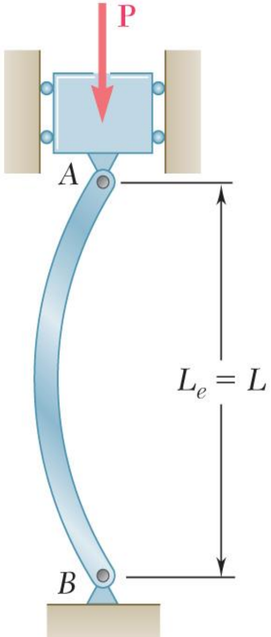
Euler’s Formula with different boundary conditions

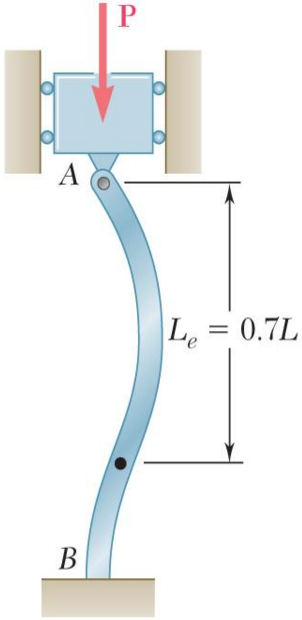
(*a*) One fixed end, one free end

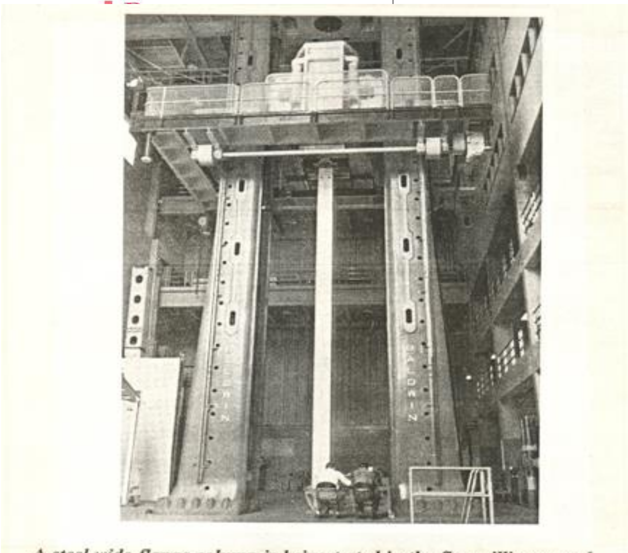
(*b*) Both ends pinned

(*c*) One fixed end, one pinned end

(*d*) Both ends fixed





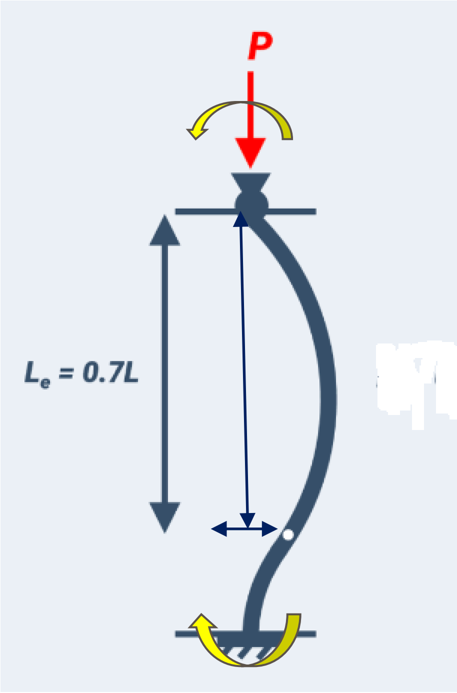


A steel wide-flange column is being tested in the five-million-pound universal testing machine at Lehigh University, Bethlehem,

$$P_{cr} = P_e/4 \qquad P_{cr} = \frac{EI\pi^2}{L^2} \qquad P_{cr} = 2.047P_e \qquad P_{cr} = 4P_e$$

Why does the critical load increase in this direction ?

How to calculate Le?



$$\text{Why } L_e = 0.7L \quad (L_e = (\pi/4.493)L) \text{ ?}$$

$$M(x) = EI\frac{d^2y}{dx^2} = -\lambda^2 B\sin\lambda x$$

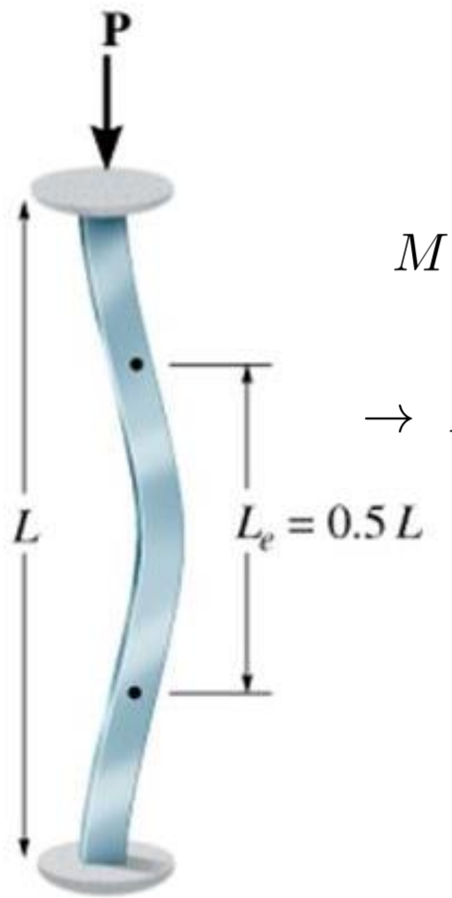
$$\text{Then} \quad M(L_e) = -\lambda^2 B\sin\lambda_1 L_e$$

$$= -\lambda^2 B\sin\left(\frac{4.493}{L}\right)\left(\frac{\pi}{4.493}\right)L = -\lambda^2 B\sin\pi = 0$$

Recall that L_e is the distance between $M(L_e) = 0$ and $M(0) = 0$.

$$(\lambda L)_1 = 4.493\cdots\cdots$$

$$M(L_e) = 0.$$

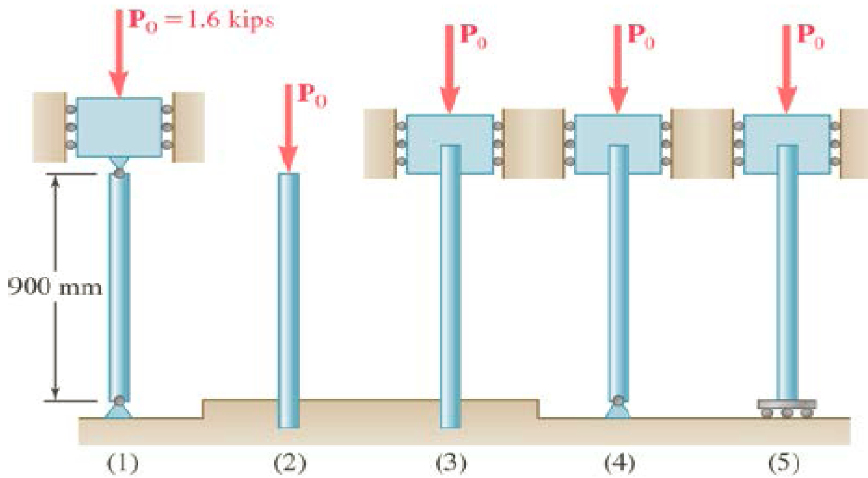


$$y''(x) = -A\lambda^2 \cos \lambda x; \quad \lambda = \frac{2\pi}{L}$$

$$M(x) = -EIA\lambda^2 \cos \lambda x \;\rightarrow\; M(L/4) = -AP\cos\frac{\pi}{2} = 0$$

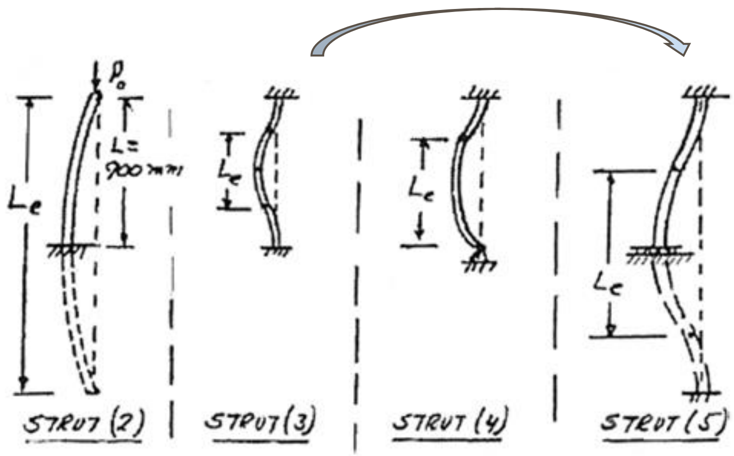
$$\rightarrow\; M(3L/4) = -A\lambda^2 \cos\frac{3\pi}{2} = 0 \;\rightarrow\; L_e = \frac{3L}{4} - \frac{L}{4} = \frac{L}{2}$$

For symmetric mode



PROBLEM 16.24

Each of the five struts shown consists of a solid steel rod. (*a*) Knowing that the strut of Fig. (1) is of a 0.8-in. diameter, determine the factor of safety with respect to buckling for the loading shown. (*b*) Determine the diameter of each of the other struts for which the factor of safety is the same as the factor of safety obtained in part *a*. Use $E = 29 \times 10^6$ psi .



$L_e = L$ $\quad L_e = 2L$ $\quad L_e = L/2$ $\quad L_e = 0.7L$ $\quad L_e = L$

(1) (2) (3) (4) (5)

Solid circular cross section: $\quad c = \frac{1}{2}d = 0.40$ in.

$$I = \frac{\pi}{4}c^4 = \frac{\pi}{4}(0.40)^4 = 20.106 \times 10^{-3} \text{ in}^4$$

$$E = 29 \times 10^6 \text{ psi} = 29 \times 10^3 \text{ ksi}$$

$$EI = (29 \times 10^3)(20.106 \times 10^{-3}) = 583.07 \text{ kip} \cdot \text{in}^2$$

For strut (1), $\quad L_e = L = 36$ in.

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(583.07)}{(36)^2} = 4.4403 \text{ kips}$$

(*a*) $\quad F.S. = \frac{P_{cr}}{P_0} = \frac{4.4403 \text{ kips}}{1.6 \text{ kips}} \qquad$ F.S. $=2.78$

(*b*) For the same factor of safety, the struts must have the same critical load.

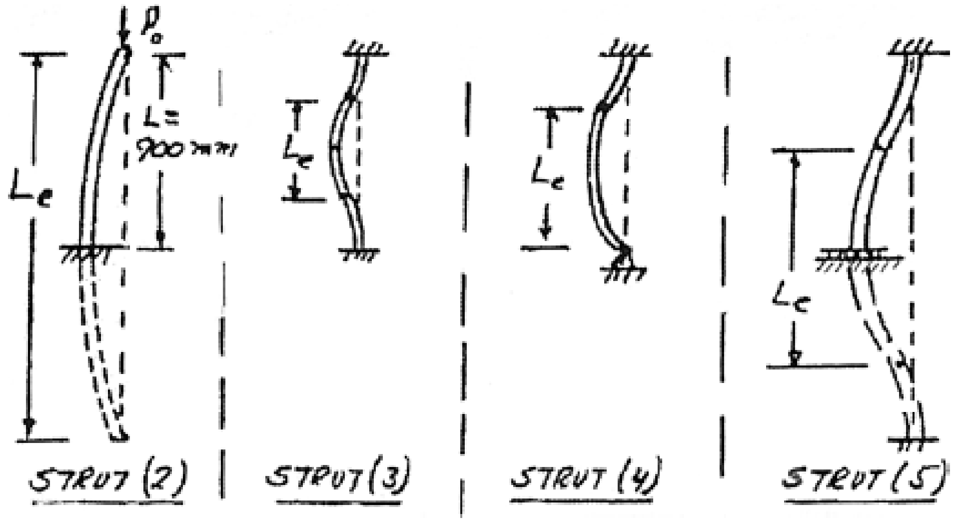
$$P_{cr} = \frac{\pi^2 EI_i}{L_i^2} \text{ where } i = 1, 2, 3, 4, \text{ and } 5$$

For $i = 2, 3, 4,$ and 5, $\quad \frac{I_i}{L_i^2} = \frac{I_1}{L_1^2} \quad$ or $\quad \frac{I_i}{I_1} = \frac{L_i^2}{L_1^2}$

Since I is proportional to d^4, $\quad \frac{d_i^4}{d_1^4} = \frac{L_i^2}{L_1^2}$

or $\quad \frac{d_i}{d_1} = \sqrt{\frac{L_i}{L_1}}$, where L_i is the effective length.

$L_e = L$ $L_e = 2L$ $L_e = L/2$ $L_e = 0.7L$ $L_e = L$



<u>Strut (1)</u>: $L_1 = 36$ in. $d_1 = 0.800$ in. $900\ mm \rightarrow 35.4in$

<u>Strut (2)</u>: $L_2 = 2L_1 = 72$ in. $(L_e = 2L)$

$$\frac{d_2}{0.80} = \sqrt{\frac{72}{36}} \qquad \frac{d_i}{d_1} = \sqrt{\frac{L_i}{L_1}}. \qquad d_2 = 1.131 \text{ in.} \blacktriangleleft$$

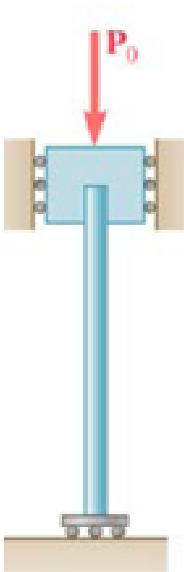
<u>Strut (3)</u>: $L_3 = \frac{1}{2}L_1 = 18$ in. $(L_e = L/2)$

$$\frac{d_3}{0.80} = \sqrt{\frac{18}{36}} \qquad d_3 = 0.566 \text{ in.} \blacktriangleleft$$

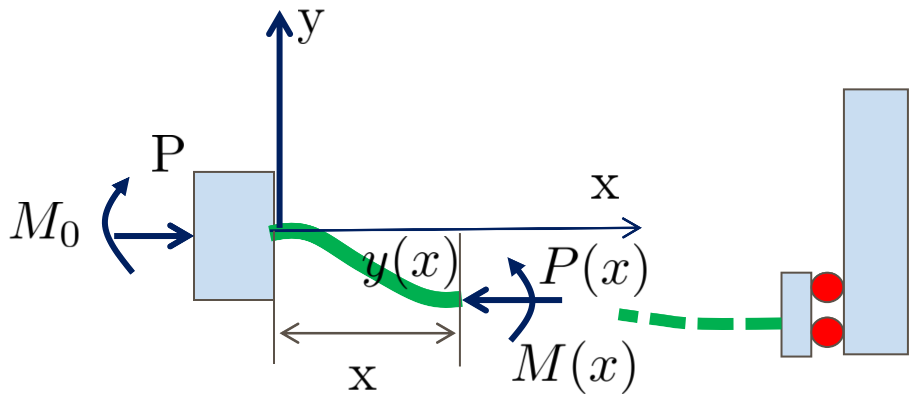
<u>Strut (4)</u>: $L_4 = 0.669 \quad L_1 = 629.1$ mm $(L_e = 0.7L)$

$$\frac{d_4}{0.80} = \sqrt{\frac{25.2}{36}} \qquad d_4 = 0.669 \text{ in.} \blacktriangleleft$$

<u>Strut (5)</u>: $L_5 = L_1 = 36$ in. $(L_e = L)$ $d_5 = 0.800$ in. $\blacktriangleleft$



(5)



$y(0) = 0;$

$y'(0) = 0;$

$y'(L) = 0;$

$$\sum M_z \Big|_{@x} = 0 \quad \rightarrow \quad M(x) - M_0 - P|y(x)| = 0$$

This problem is rated R !