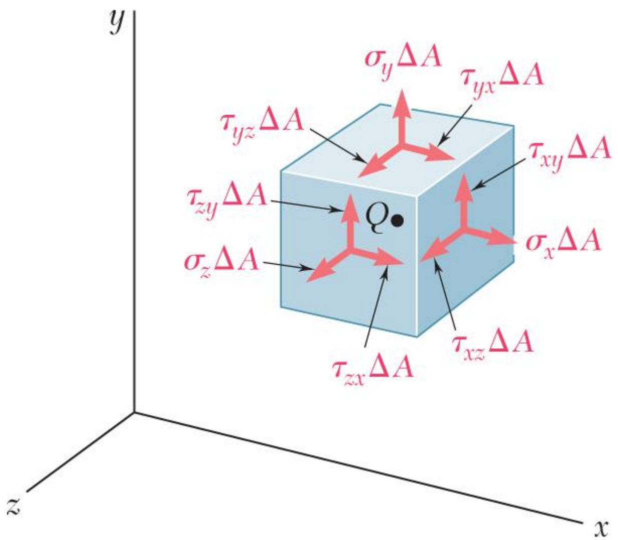
Lecture 33 Transformations of Stress (I)



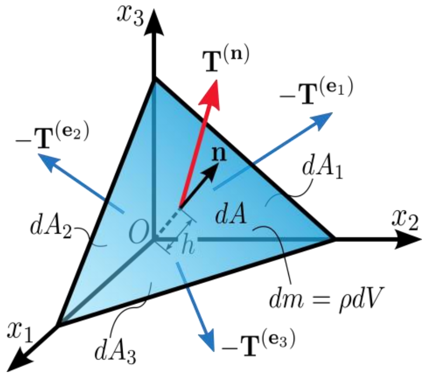
Augstin Louis Cauchy (1787-1857)

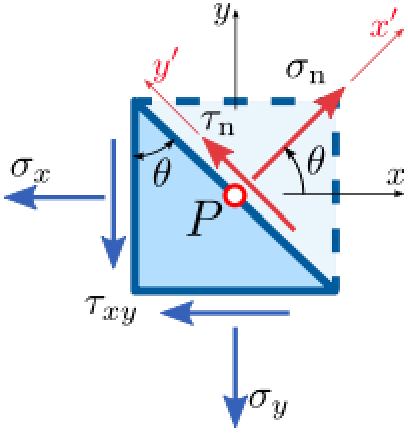
Cauchy Stress Tensor,

$$(3D) \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Cauchy Theorem $\longrightarrow$ Cauchy Stress

$$(2D) \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix}$$



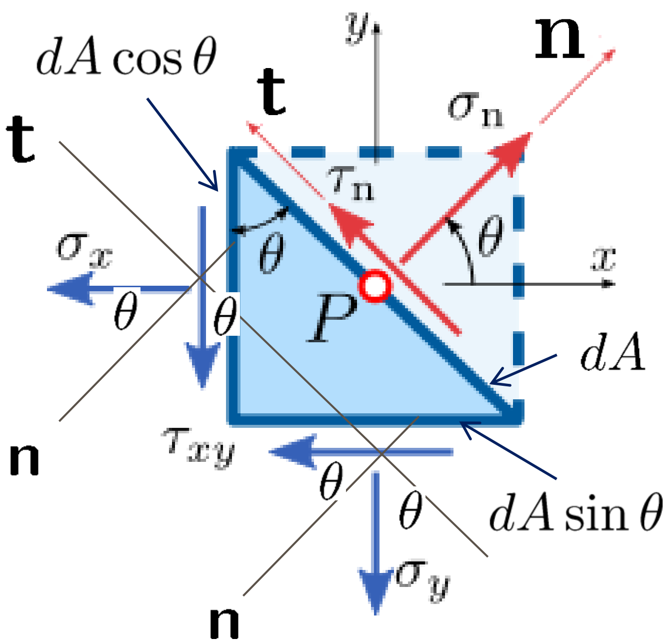


In Lecture 18:

We shall prove this !

Proof of Cauchy Theorem

θ is an angle between n-x or t-y;



Step 1: $\sum F_n = 0;$

$$\sigma_n dA - \sigma_x \cos\theta(dA\cos\theta)$$
$$-\tau_{xy}\sin\theta(dA\cos\theta)$$
$$-\tau_{yx}\cos\theta(dA\sin\theta)$$
$$-\sigma_y\sin\theta(dA\sin\theta) = 0$$

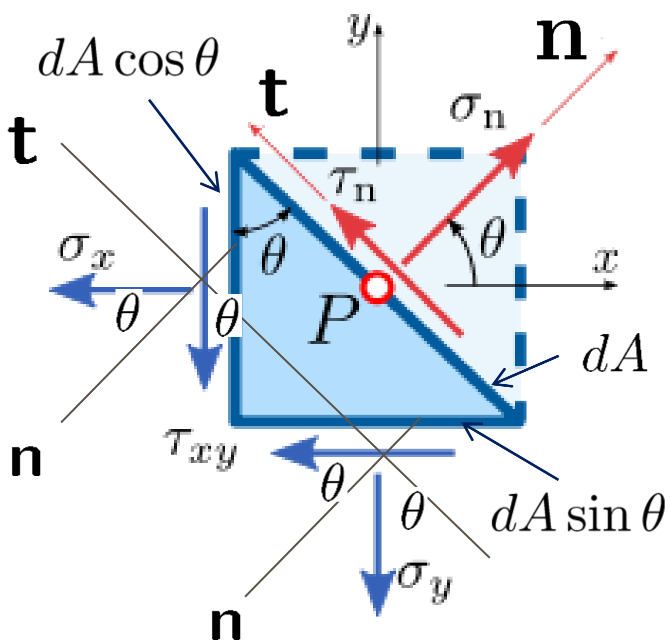
$$\sigma_n = \sigma_x\cos^2\theta + 2\tau_{xy}\cos\theta\sin\theta + \sigma_y\sin^2\theta$$

Consider

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta) \quad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \quad 2\cos\theta\sin\theta = \sin 2\theta$$

We have

$$\sigma_n(\theta) = \frac{1}{2}(\sigma_x+\sigma_y) + \frac{1}{2}(\sigma_x-\sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta \quad (1)$$



Step 2: $\sum F_t = 0$

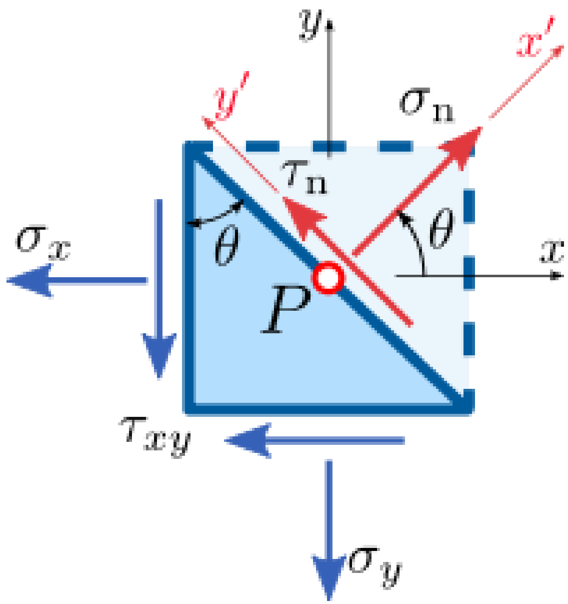
$$\tau_n dA + \sigma_x \sin\theta(dA\cos\theta) - \tau_{xy}\cos\theta(dA\cos\theta)$$
$$+\tau_{xy}\sin\theta(dA\sin\theta) - \sigma_y\cos\theta(dAsin\theta) = 0$$

$$\tau_n = -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\tau_n(\theta) = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (2)$$

$$\sigma_n(\theta) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (1)$$

Summary



$$\sigma_n(\theta) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (1)$$

$$\tau_n(\theta) = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (2)$$

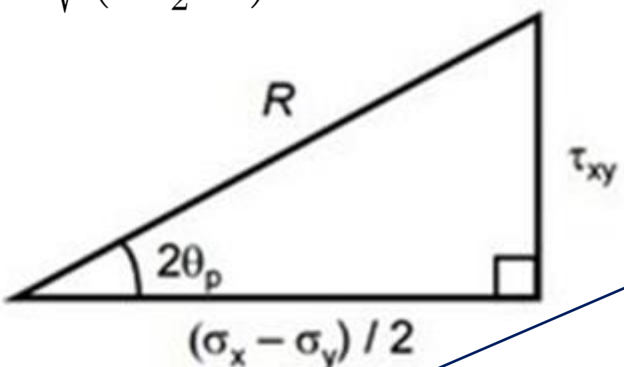
At which θ, $\sigma_n \rightarrow \sigma_{max}$? $\tau_n \rightarrow \tau_{max}$?

To find σ_{max} or σ_{min}, we take the derivative of σ_n with θ,

$$\sigma_n(\theta) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (1)$$

$$\left.\frac{d\sigma_n}{d\theta}\right|_{\theta_p} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0 \Rightarrow \tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}, \qquad (*1)$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$



$$\sin 2\theta_p = \frac{\tau_{xy}}{R}$$

$$\cos 2\theta_p = \frac{\sigma_x - \sigma_y}{2R}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + R$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\left(\frac{\sigma_x - \sigma_y}{2R}\right) + \tau_{xy}\frac{\tau_{xy}}{R}$$

$$\sigma_n(\theta) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (1)$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + R, \qquad \tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}, \qquad (*1)$$

Remarks:

The range of θ_p is: $0 \le 2\theta_p \le 2\pi$; Eq. (*1) has two solutions:

$$\tan 2\theta_p = \tan(2\theta_p + \pi) = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$

We know that

$$\sin(2\theta_p + \pi) = -\sin 2\theta_p = -\frac{\tau_{xy}}{R}, \quad \cos(2\theta_p + \pi) = -\cos 2\theta_p = -\frac{(\sigma_x - \sigma_y)/2}{R},$$

Substituting them to (1), we have

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \frac{(\sigma_x - \sigma_y)}{2}\frac{(\sigma_x - \sigma_y)/2}{R} - \tau_{xy}\frac{\tau_{xy}}{R}$$

$\theta_p + \pi/2$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - R$$

Substituting

$$\sin(2\theta_p) = \frac{\tau_{xy}}{R}, \quad \cos(2\theta_p) = \frac{(\sigma_x - \sigma_y)/2}{R},$$

into $$\tau_n(\theta) = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \qquad (2)$$

$$\Downarrow$$

$$\tau_n(\theta_s) = -\frac{\sigma_x - \sigma_y}{2} \frac{\tau_{xy}}{R} + \tau_{xy} \frac{\sigma_x - \sigma_y}{2R} = 0!$$

Therefore, the definition of the principal planes is:
The principal planes are the planes on which shear stresses are zero.

Note that when you first get the value from the following equation,

$$\tan 2\theta_p = \tan(2\theta_p + \pi) = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}, \qquad 0 \leq \theta_p \leq \pi$$

We do not know the angle that we obtained is θ_p or $\theta_p + \pi/2$?

Now consider : $\tau_n(\theta) = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (2)$

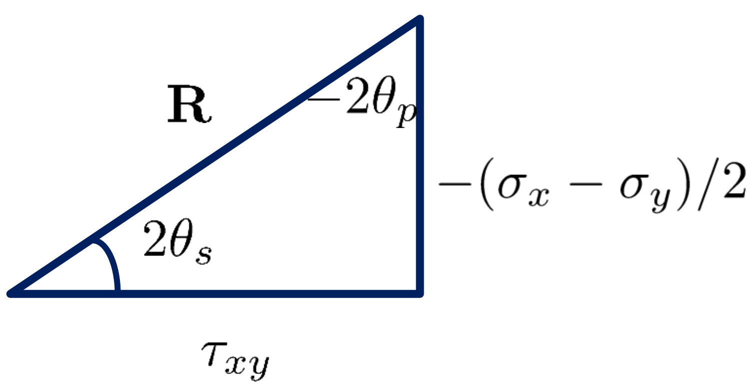
To find τ_{max} or τ_{min}, we take the derivative of τ_n with θ,

$$\frac{d\tau_n}{d\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}}, \qquad (*2)$$

Compare

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$



$$\sin 2\theta_s = -\frac{(\sigma_x - \sigma_y)/2}{R}$$

$$\cos 2\theta_s = \frac{\tau_{xy}}{R}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = R$$

Remarks:

The range of θ_s is: $0 \le \theta_s \le \pi$; Eq. (*2) has two solutions:

$$\tan 2\theta_s = \tan(2\theta_s + \pi)\ .$$

$$\cos(2\theta_s + \pi) = -\cos 2\theta_s = -\frac{\tau_{xy}}{R}$$

$$\sin(2\theta_s + \pi) = -\sin 2\theta_s = \frac{(\sigma_x - \sigma_y)/2}{R}$$

$$\tau_n(\theta) = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (2)$$

$$\tau_n(\theta_s + \pi/2) = \tau_{min} = -\frac{\sigma_x - \sigma_y}{2R}\frac{\sigma_x - \sigma_y}{2} - \tau_{xy}\frac{\tau_{xy}}{R} = -R$$

$$\tau_{min} = -R$$

Substituting

$$\sin(2\theta_s) = -\frac{(\sigma_x - \sigma_y)/2}{R}, \quad \cos(2\theta_s) = \frac{\tau_{xy}}{R},$$

into $$\sigma_n(\theta) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (1)$$

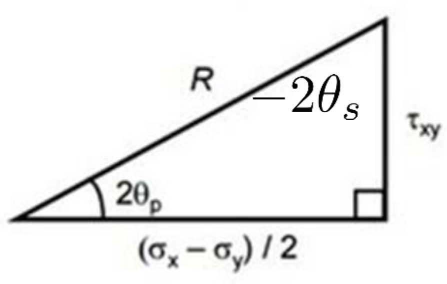
$$\sigma_n(\theta_s) = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\frac{(\sigma_{xy})}{R} - \tau_{xy}\frac{(\sigma_x - \sigma_y)}{2R} = \frac{\sigma_x + \sigma_y}{2} \neq 0 \ .$$

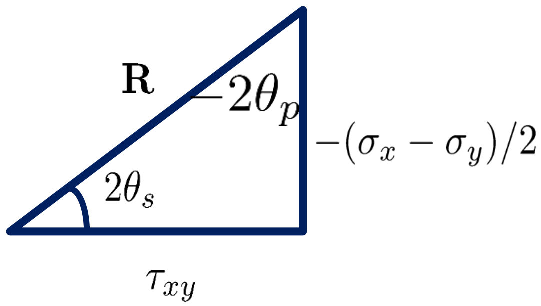
Therefore, the normal stresses are NOT zero on the planes on which shear stresses are zero. On the maximum shear stress plane, the normal stress is the average stress, i.e.

$$\boxed{\sigma_n(\theta_s) = \frac{\sigma_x + \sigma_y}{2} = \sigma_{ave}} \ .$$

Relation between principal plane and maximum shear stress plane:

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} \;,\quad \tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}}$$



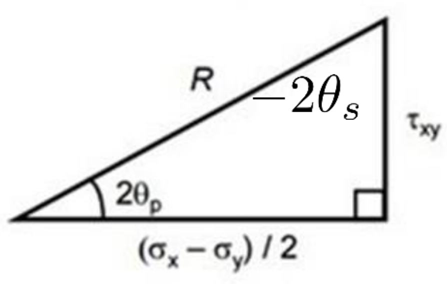


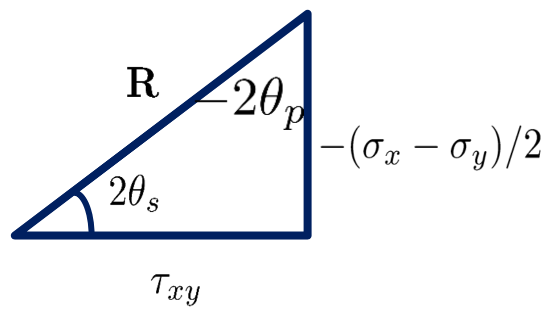
$$\rightarrow \quad 2(\theta_s - \theta_p) = \pm\frac{\pi}{2} \quad \rightarrow \qquad \theta_s = \theta_p \pm \frac{\pi}{4}$$

The maximum shear stress occurs on the plane that forms a 45^o angle with the principal planes.

Relation between principal plane and maximum shear stress plane:

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} \;, \quad \tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}}$$





$$\tan 2\theta_p = -\frac{1}{\tan 2\theta_s} \quad \rightarrow \quad \frac{\sin 2\theta_p}{\cos 2\theta_p} + \frac{\cos 2\theta_s}{\sin 2\theta_s} = 0 \quad \times \cos 2\theta_p \sin 2\theta_s \;\rightarrow$$

$$\sin 2\theta_s \sin 2\theta_p + \cos 2\theta_s \cos 2\theta_p = 0 \;\rightarrow\; \cos 2(\theta_s - \theta_p) = 0 \;.$$

$$\cos \pm\frac{\pi}{2} = 0 \quad \rightarrow \quad 2(\theta_s - \theta_p) = \pm\frac{\pi}{2} \;\rightarrow \qquad \theta_s = \theta_p \pm \frac{\pi}{4}$$

Summary

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + R, \qquad \tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}, \tag{*1}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - R, \quad \theta_{p2} = \theta_p + \frac{\pi}{2} \tag{*2}$$

$$\tau_n(\theta_p) = 0. \tag{*3}$$

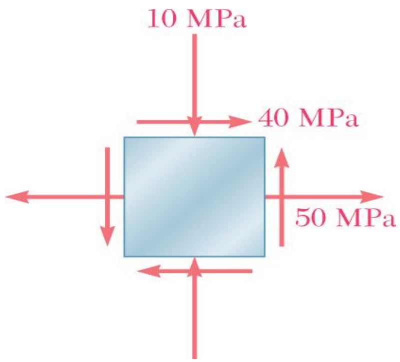
$$\tau_{max} = R, \qquad \tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}}, \tag{*4}$$

$$\tau_{min} = -R, \quad \theta_{s2} = \theta_s + \frac{\pi}{2}, \tag{*5}$$

$$\sigma_n(\theta_s) = \frac{\sigma_x + \sigma_y}{2} = \sigma_{ave} \ . \tag{*6}$$

$$\theta_s = \theta_p \pm \frac{\pi}{4} \tag{*6}$$

Example I



For the state of plane stress shown, determine (a) the principal planes, (b) the principal stresses, (c) the maximum shearing stress and the corresponding normal stress.

SOLUTION:

- Find the element orientation for the principal stresses from

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

- Determine the principal stresses from

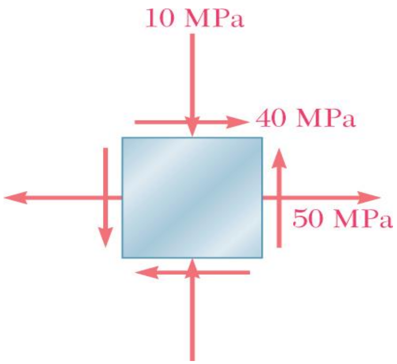
$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- Calculate the maximum shearing stress with

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

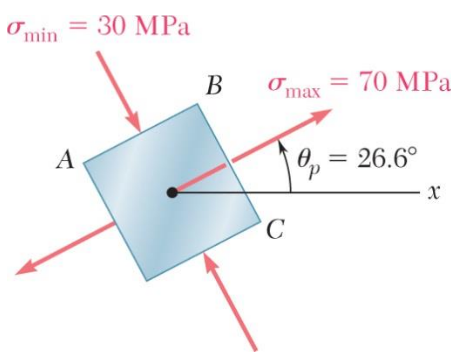
$$\sigma' = \frac{\sigma_x + \sigma_y}{2}$$

$$\sigma_n(\theta) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (1)$$



$$\sigma_x = +50\,\text{MPa} \qquad \tau_{xy} = +40\,\text{MPa}$$

$$\sigma_y = -10\,\text{MPa}$$



SOLUTION:

- Find the element orientation for the principal stresses from

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(+40)}{50-(-10)} = 1.333$$

$$2\theta_p = 53.1°,\ 233.1°$$

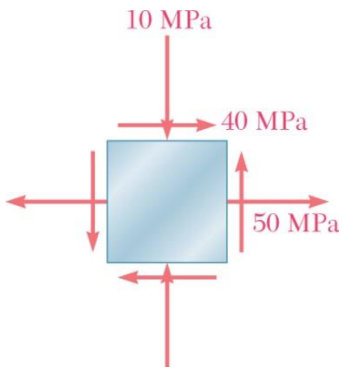
$$\boxed{\theta_p = 26.6°,\ 116.6°}$$

- Determine the principal stresses from

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 20 \pm \sqrt{(30)^2 + (40)^2}$$

$$\boxed{\begin{aligned}\sigma_{\max} &= 70\,\text{MPa}\\ \sigma_{\min} &= -30\,\text{MPa}\end{aligned}}$$



- Calculate the maximum shearing stress with

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$
$$= \sqrt{(30)^2 + (40)^2}$$

$$\boxed{\tau_{\max} = 50\,\text{MPa}}$$

$$\theta_s = \theta_p \pm 45^o$$

$$\theta_{s1} = 26.6 - 45 = -18.4^o$$

$$\theta_{s2} = 26.6 + 45 = 71.6^o$$

$$\boxed{\theta_s = -18.4°,\ 71.6°}$$

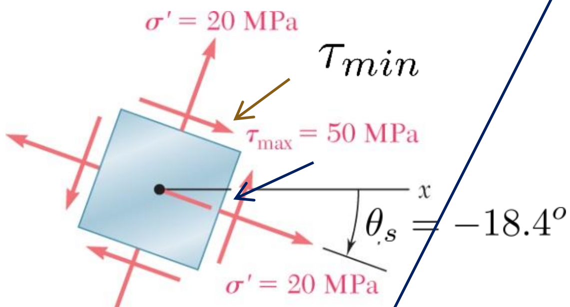
$$\sigma_x = +50\,\text{MPa} \qquad \tau_{xy} = +40\,\text{MPa}$$

$$\sigma_x = -10\,\text{MPa}$$

- The corresponding normal stress is

$$\sigma' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{50 - 10}{2}$$

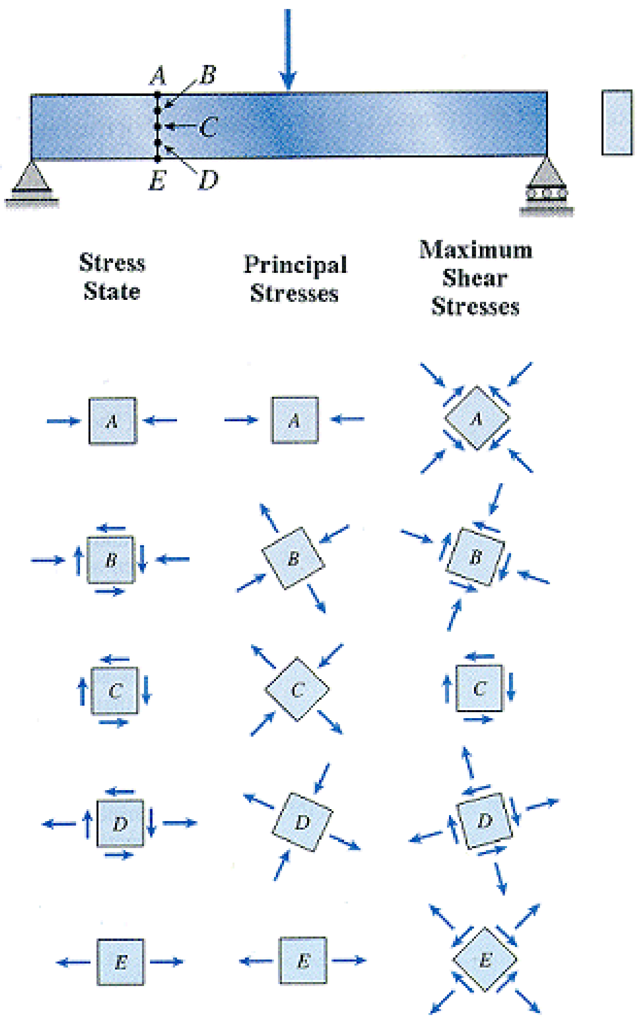
$$\boxed{\sigma' = 20\,\text{MPa}}$$

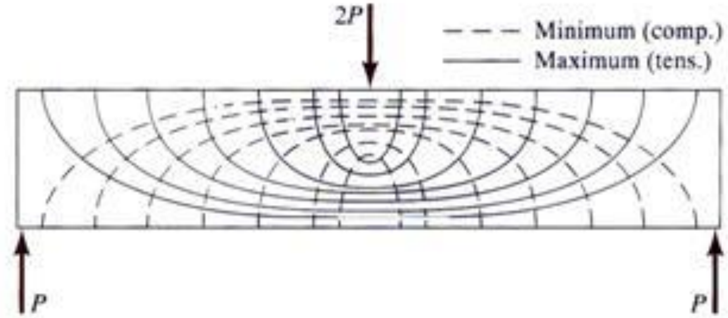


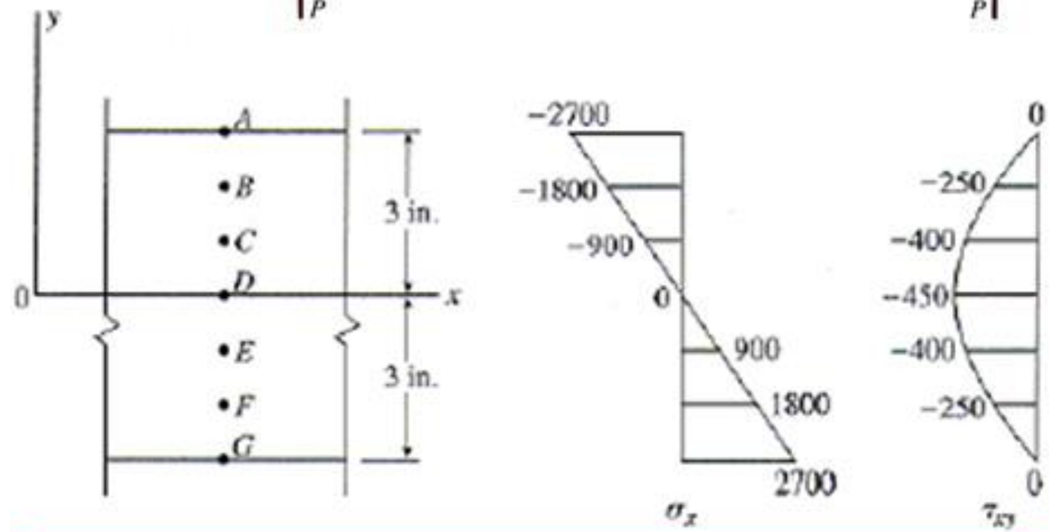
$$\text{Verify :} \quad \tau_n(\theta) = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (2)$$

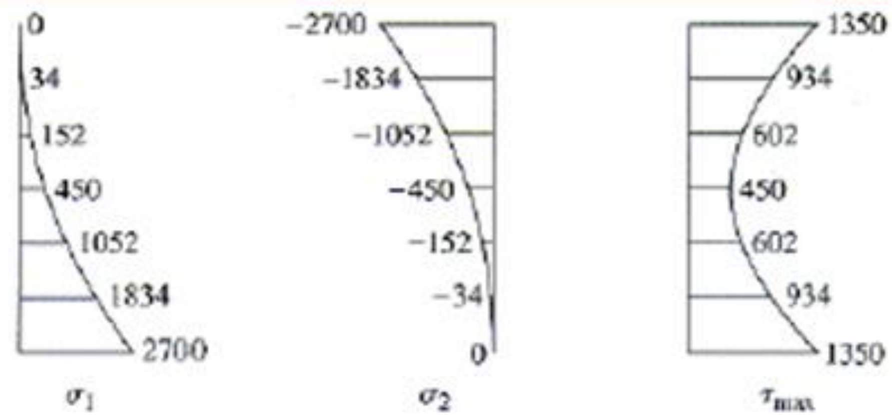
Applications

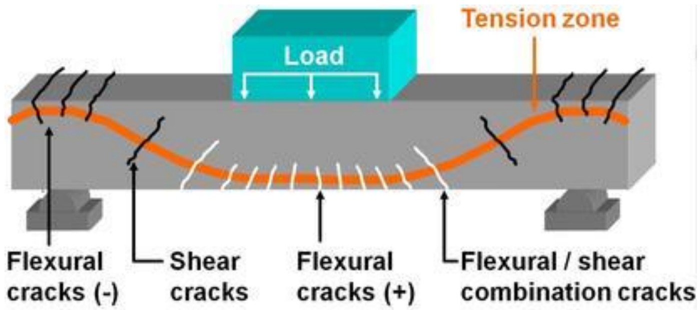
$$\sigma_x = -\frac{M_z y}{I_z} \qquad \tau_{xy} = -\frac{V(x)Q(y)}{I_z b}$$

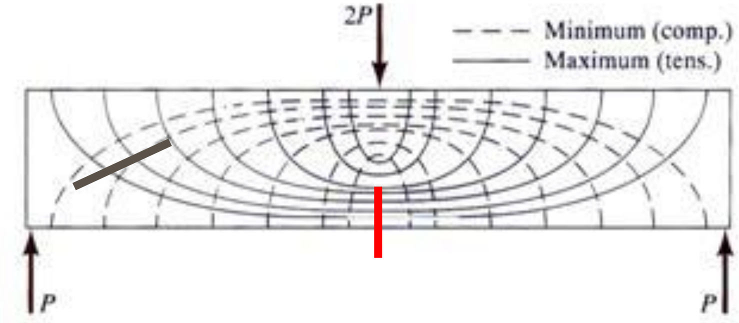










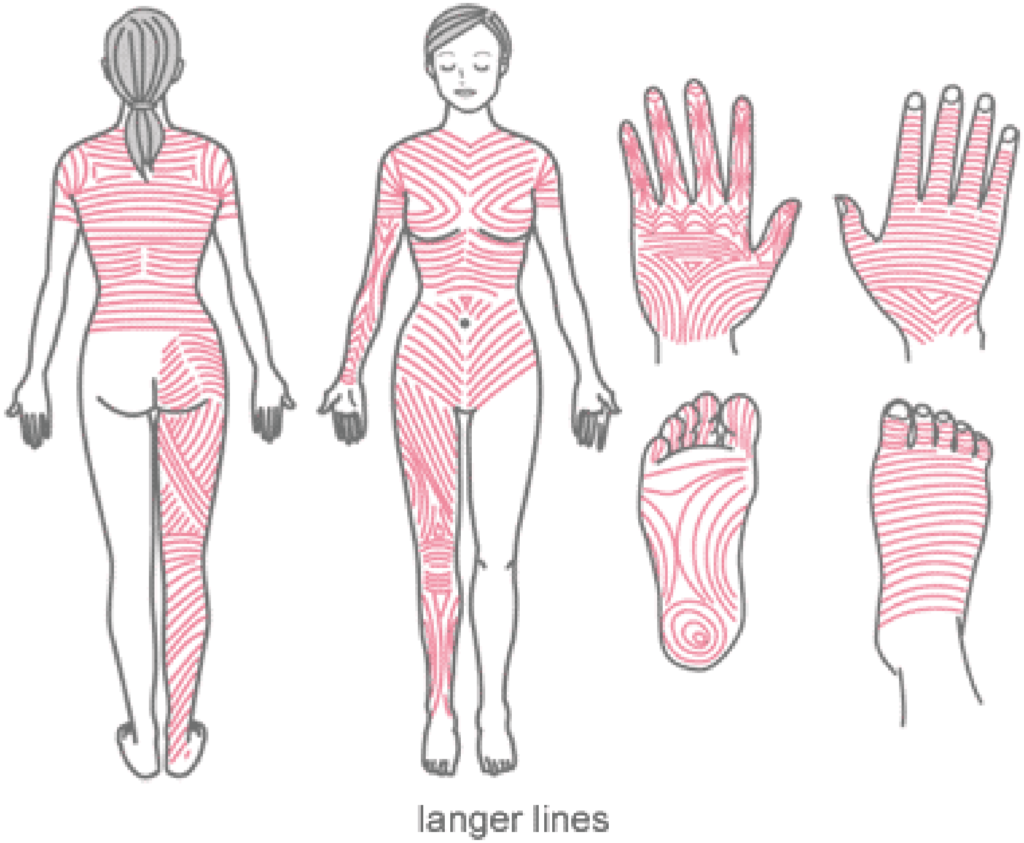


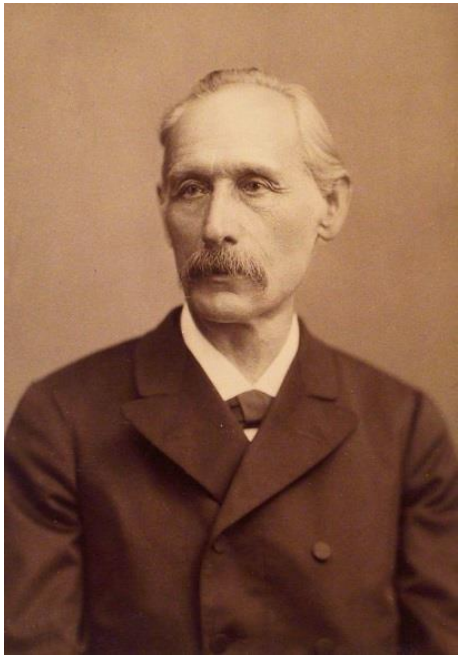
Opening crack is perpendicular to the maximum stress contour.

Flexural crack

Shear crack

Langer's Lines are the principal stress Trajectories on human body (skin).

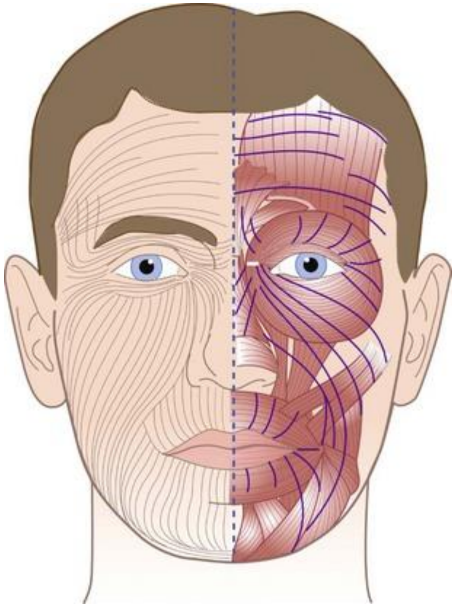




Karl Langer (1819-1887)

Langer's lines, Langer lines of skin tension,
are topological lines drawn on a map of the human body.
They are parallel to the natural orientation of collagen
fibers in the dermis, as well as the underlying muscle fibers.

What is the takeaway ?



When you can apply the mechanic’s principle to Solve practical problems, it becomes beautiful !