Strategic Ratification: Supplementary Proofs

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We prove the following claim.

Claim 1 *Let $\mathcal{C}$ be representable as a compact convex subset of $\mathbb{R}^1$ and let preferences over elements of $\mathcal{C}$ be represented with a concave utility function for each of i and j with $u_i' > 0$, $u_j' < 0$ and $u_k(x) > 0$ for $x \in \mathcal{C}$, $k \in \{i, j\}$. Then, for points a, b, c, d on $\mathcal{C}$, with $c > d$, $\delta \in (0, 1)$, $u_i(d) = \delta u_i(b)$ and $u_j(b) = \delta u_j(d)$:*
(I) $u_j(a) = \delta u_j(c)$ implies $\delta u_i(a) < u_i(c)$
(II) $u_i(c) = \delta u_i(a)$ implies $u_j(a) < \delta u_j(c)$.

According to Claim 1 Condition $(*)$ from the main text holds for any concave utility functions when contract curves are linear. The proof of the claim makes use of the following Lemma:

Lemma 2 *Let w, x, y, z denote four distinct points on $\mathbb{R}^1$ with $w < x < z$ and $w < y < z$ and let $f : \mathbb{R}^1 \to \mathbb{R}^1$ denote a concave function. Then: $\frac{f(x)-f(w)}{x-w} \geq \frac{f(z)-f(y)}{z-y}$.*

Proof. Without loss of generality assume that $w = f(w) = 0$. The chord joining w and z has slope $\frac{f(z)-f(w)}{z-w} = \frac{f(z)}{z}$ and intercept $f(w) = 0$; values on the chord corresponding to any point $q \in \mathbb{R}^1$ are described by the function $g : \mathbb{R}^1 \to \mathbb{R}^1$ where $g(q) = \frac{f(z)}{z}q$. Note that with x and y both lying between w and z we have from the concavity of f that $f(x) \geq g(x)$ and $f(y) \geq g(y)$. Now, $f(y) \geq g(y) = \frac{f(z)}{z}y$ implies that $zf(y) \geq yf(z)$ and hence, subtracting $z(f(z))$ from both sides, $z(f(y) - f(z)) \geq f(z)(y - z)$ which, with $z > y$, implies that $\frac{f(z)}{z} \geq \frac{f(z)-f(y)}{z-y}$. Also, $f(x) \geq g(x) = \frac{f(z)}{z}x$ implies that $\frac{f(x)}{x} \geq \frac{f(z)}{z}$. Together these imply that $\frac{f(x)-f(w)}{x-w} = \frac{f(x)}{x} \geq \frac{f(z)-f(y)}{z-y}$. ■

Proof of claim. Let the negotiation set and utilities be as in the statement of the claim and note that $c > d$ implies $u_j(d) > u_j(c)$ and $u_i(d) < u_i(c)$.

Assume first that $u_j(a) = \delta u_j(c)$. In this case $u_j(b) = \delta u_j(d)$ implies $b > d$ and $u_j(a) = \delta u_j(c)$ implies $a > c$. Furthermore, $c > d$ implies $a > b$ (since $c > d \Rightarrow u_j(d) > u_j(c) \Rightarrow u_j(b) = \delta u_j(d) > \delta u_j(c) = u_j(a) \Rightarrow a > b$). The slope of the chord between $(d, u_j(d))$ and $(b, u_j(b))$ is $\frac{u_j(d)(\delta-1)}{b-d}$; the slope between $(c, u_j(c))$ and $(a, u_j(a))$ is $\frac{u_j(c)(\delta-1)}{a-c}$. With u_j concave, $d < c$ and $b < a$ we have from Lemma 2 that $\frac{u_j(d)(\delta-1)}{b-d} \geq \frac{u_j(c)(\delta-1)}{a-c}$. Hence, with $\delta < 1$, $\frac{u_j(d)}{b-d} \leq \frac{u_j(c)}{a-c}$. But, then $u_j(d) > u_j(c)$ implies $a - c < b - d$. The slope of the chord between $(d, u_i(d))$ and $(b, u_i(b))$ is $\frac{u_i(d)\frac{1-\delta}{\delta}}{b-d}$; the slope between $(c, u_i(c))$ and $(a, u_i(a))$ is $\frac{u_i(a)-u_i(c)}{a-c}$. If, contrary to the proposition, $\delta u_i(a) \geq u_i(c)$, then $u_i(a) - u_i(c) \geq \frac{1}{\delta}u_i(c) - u_i(c) = u_i(c)\frac{(1-\delta)}{\delta}$, and again using the concavity of u_i we then have that $d < c$ and $b < a$ imply $\frac{u_i(d)\frac{1-\delta}{\delta}}{b-d} \geq \frac{u_i(a)-u_i(c)}{a-c} = \frac{u_i(c)\frac{1-\delta}{\delta}}{a-c}$ and so $\frac{u_i(d)}{b-d} \geq \frac{u_i(c)}{a-c}$. As $u_i(d) < u_i(c)$ we have then that $b-d < a-c$, a contradiction that proves Part I of the claim.

The proof for Part II is essentially identical: $\delta u_i(a) = u_i(c)$ implies $a > c$ and $c > d$ implies $b < a$ (since $c > d \Rightarrow u_i(a) = \frac{1}{\delta}u_i(c) > \frac{1}{\delta}u_i(d) = u_i(b) \Rightarrow a > b$). With u_i concave, $d < c$ and $b < a$ we have $\frac{u_i(d)\frac{(1-\delta)}{\delta}}{b-d} \geq \frac{u_i(c)\frac{(1-\delta)}{\delta}}{a-c}$ and hence $\frac{u_i(d)}{b-d} \geq \frac{u_i(c)}{a-c}$ and so, using, $u_i(d) < u_i(c)$ we have $a - c > b - d$. If, contrary to the proposition $u_j(a) \geq \delta u_j(c)$, then $u_j(a) - u_j(c) \geq u_j(c)(\delta - 1)$ and, again using the concavity of u_j we then have $\frac{u_j(d)(\delta-1)}{b-d} \geq \frac{u_j(a)-u_j(c)}{a-c} \geq \frac{u_j(c)(\delta-1)}{a-c}$, and hence $u_j(d) > u_j(c)$ implies $a - c < b - d$—a contradiction, proving Part II of the claim. ■