

Hamilton Paths and Hamilton Cycles

Overview

Visiting All the Vertices

Subgraph Test for Hamilton Paths and Cycles

The Path/Cycle Principle

Outline

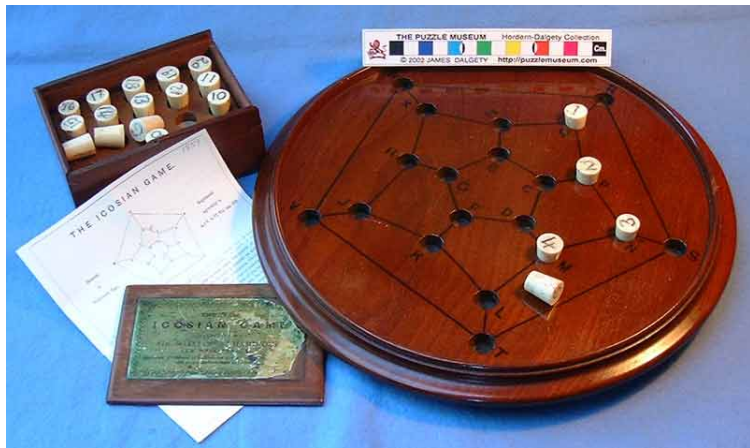
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Hamilton's Game

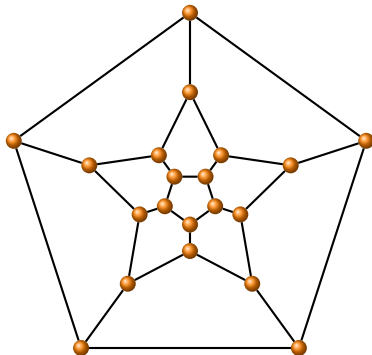
William Rowan Hamilton invented the Icosian Game in 1857.



Hamilton's Problem

Is it possible to start at one of the 20 vertices, and, by following edges, visit every other vertex exactly once before returning to the starting point?

Equivalently: Is there a cycle through all the vertices?



Hamiltonian Graphs

Definition

A **Hamilton path** in a graph G is a path that contains each vertex of G exactly once.

Definition

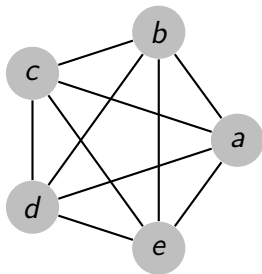
A **Hamilton cycle** in a graph G is a closed path that passes through each vertex exactly once and in which all the edges are distinct.

Definition

A **Hamiltonian graph** is a graph containing a Hamilton cycle.

Example

Does this graph have a Hamilton cycle?



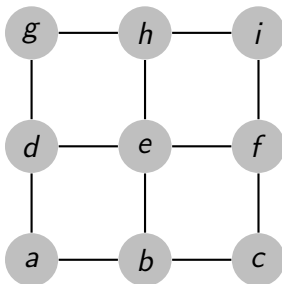
Hamiltonian Complete Graphs

Theorem

K_n has a Hamilton cycle for $n \geq 3$.

Example

Does this graph have a Hamilton cycle?



Hamilton Cycles in Bipartite Graphs

Theorem

If a bipartite graph has a Hamilton cycle, then it must have an even number vertices.

Hamilton Cycles in Bipartite Graphs

Theorem

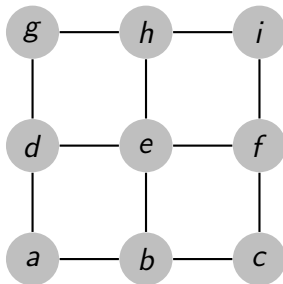
If a bipartite graph has a Hamilton cycle, then it must have an even number vertices.

Theorem

$K_{m,n}$ has a Hamilton cycle if and only if $m = n \geq 2$.

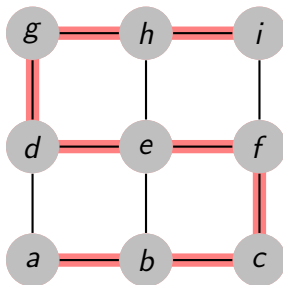
Example

Does this graph have a Hamilton **path**?



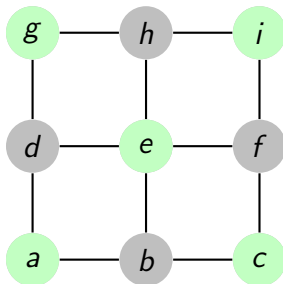
Example

Does this graph have a Hamilton **path**?



Example

Note that this graph is bipartite with $m = 4$ black and $n = 5$ green vertices.



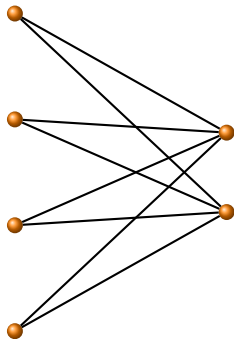
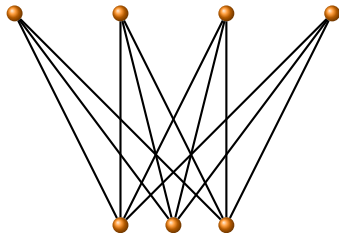
Hamilton Paths in Bipartite Graphs

Theorem

Let G be a bipartite graph with m vertices of one color and n vertices of the other color. If $|m - n| \geq 2$, then G contains no Hamilton path.

Example

Which graph has a Hamilton path?



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Number of Components After Removing Vertices

- ▶ If k vertices are removed from a graph that contains a Hamilton path, then the number of components in the resulting subgraph is at most $k + 1$.
- ▶ If k vertices are removed from a graph that contains a Hamilton cycle, then the number of components in the resulting subgraph is at most k .

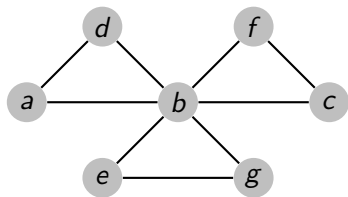
Subgraph Test for Hamilton Paths and Cycles

Let k be a positive integer and G a graph.

1. Suppose that G contains a set of k vertices such that when these vertices are removed, the resulting subgraph contains at least $k + 2$ components. Then G contains no Hamilton path.
2. Suppose that G contains a set of k vertices such that when these vertices are removed, the resulting subgraph contains at least $k + 1$ components. Then G contains no Hamilton cycle.

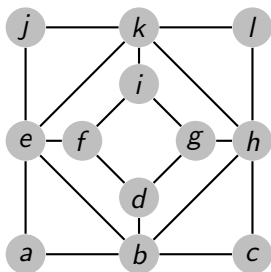
Example

Apply the Subgraph Test to the graph.



Example

Apply the Subgraph Test to the graph.



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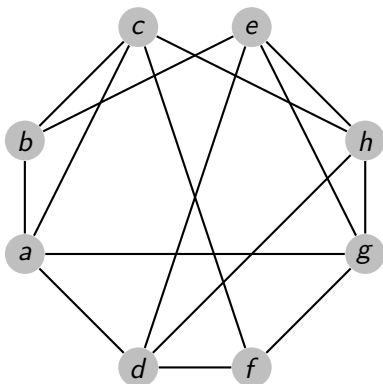
The Path/Cycle Principle

Theorem

Let G be a simple graph with $p \geq 3$ vertices. If G contains a Hamilton path whose endpoints (call them A and B) satisfy the condition $\deg(A) + \deg(B) \geq p$, then G contains a Hamilton cycle.

An Obvious Hamilton Path

This graph has an obvious Hamilton path. Does it have a Hamilton cycle?



Acknowledgements

Statements of results follow the notation and wording of Anderson's *First Course in Discrete Mathematics*.