Quantum Computing and Machine Learning

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Implement the HHL Algorithm for a 2x2 and 4x4 System

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Agenda

Project Goals:

- Review the HHL algorithm,
- Implement the algorithm for a 2x2 system,
- Compare with the classical solution,
- Implement HHL for a 4x4 system.

Systems of equations - reminder

$$\begin{cases} 2x_1 + 3x_2 = 8 \\ 4x_1 - x_2 = 3 \end{cases} \implies \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix} \overset{R_2 - 2R_1}{\implies} \begin{bmatrix} 2 & 3 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ -13 \end{bmatrix}$$

$$x_2 = \frac{13}{7} \implies x_1 = \frac{17}{14}$$

HHL - Harrow–Hassidim–Lloyd algorithm

It's a quantum algorithm that obtains limited information about the solution to a system of linear equations.

It was first proposed in the 2008 white paper *"Quantum algorithm for solving linear systems of equations"*.

[1] HHL algorithm

HHL - Authors

Figure 1: Aram Harrow

Figure 2: Avinatan Hassidim

Figure 3: Seth Lloyd

[2] Aram Harrow
[3] Avinatan Hassidim
[4] Seth Lloyd

The problem:
given a matrix A and a vector $\vec{b}$, find a vector $\vec{x}$ such that $A\vec{x} = \vec{b}$.

Proposed solution:
We consider the case where one doesn't need to know the solution $\vec{x}$ itself, but rather an approximation of the expectation value of some operator associated with $\vec{x}$, e.g., $\vec{x}^\dagger M\vec{x}$ for some matrix M.

[5] Quantum algorithm for solving linear systems of equations

Conjugate gradient method time complexity:

$$\tilde{O}\left(N\sqrt{\kappa}\right) \tag{1}$$

HHL time complexity:

In case when A is Hermitanian, s-sparse and efficiently row computable, then:

$$\tilde{O}\left(log(N)s^2\kappa^2/\epsilon\right) \tag{2}$$

s, κ - derived from A.

ϵ - is the additive error achieved in the output state.

2008

- Original paper is published

2013

- First implementation of HHL by 3 independed papers.

2018

- general-purpose version of the algorithm published.

2024

- Surve on Quantum Linear System Solvers, compares different post-HHL algorithms.

1. Represent $\vec{b}$ as a quntum state $|b\rangle = \sum_{i=1}^{N} b_i |i\rangle$,
2. Perform QPE (Quantum Phase Estimation)
3. Apply controlled rotation and measure the ancilla qubit
4. Perform IQPE (Inverse Quantum Phase Estimation)
5. Measure $|b\rangle$

[6] HHL Algorithm Walkthrough

Input data:

$$A = \begin{pmatrix} 1 & -\frac{1}{3} \\ -\frac{1}{3} & 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} \frac{3}{8} \\ \frac{9}{8} \end{pmatrix}$$

Eigenvectors and eigenvalues of A:

$$\lambda_1 = \frac{2}{3}, \qquad \vec{u_1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad \lambda_2 = \frac{4}{3}, \quad \vec{u_2} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Preparing the circuit

Matrix A can be written as a **linear combination** of the outer products of its eigenvectors

$$A = \sum_{i=0}^{2^{n_b}-1} \lambda_i |u_i\rangle\langle u_i|$$

Vector $\vec{b}$ can be also expressed in the **basis formed by the eigenvectors** of A

$$\vec{b} = \sum_{j=0}^{2^{n_b}-1} b_j |u_i\rangle$$

Therefore our **solution** can be encoded as

$$\vec{x} = A^{-1}|b\rangle = \sum_{i=0}^{2^{n_b}-1} \lambda_i^{-1} b_i |u_i\rangle$$

Preparing the circuit

Algorithm uses three main quantum registers:

- **Ancilla register**

$$|0\rangle_a$$

  is used for the controlled rotation implementing $1/\lambda_i$.
- **c-register**

$$|c\rangle$$

  store the values of the phase of the eigenvalues λ_i of matrix A.
- **b-register**

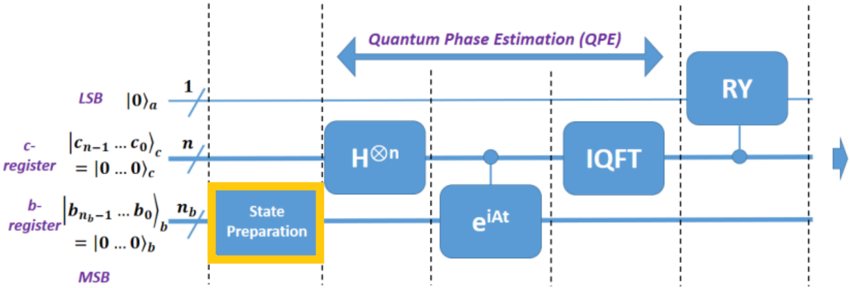
$$|b\rangle$$

  stores the solution vector $|x\rangle$.

Creating the circuit - state preparation

The full quantum state for 2x2 matrix is:

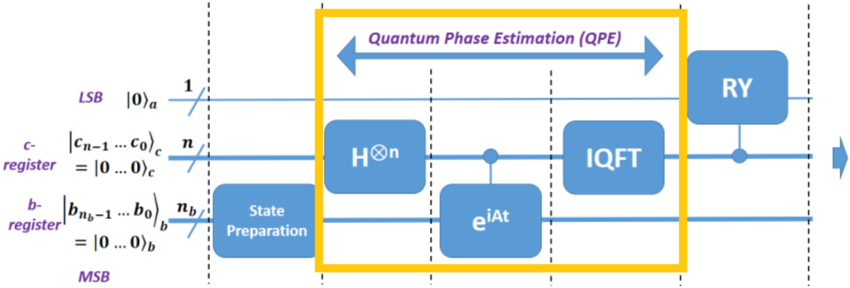
$$|\psi\rangle = |b\rangle_b |00\rangle_c |0\rangle_a$$



QPE has three components:

1. Superposition of the clock qubits through Hadamard gates,
2. Controlled rotation,
3. Inverse Quantum Fourier Transform (IQFT).

Creating the circuit - QPE

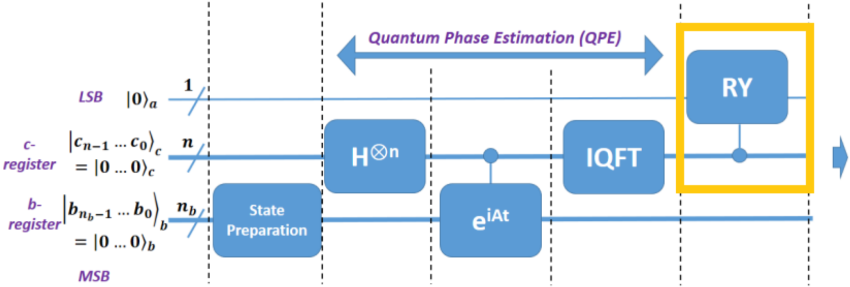


Main goal is to estimate the phase of the eigenvalues of the unitary rotation matrix $U = e^{iAt}$

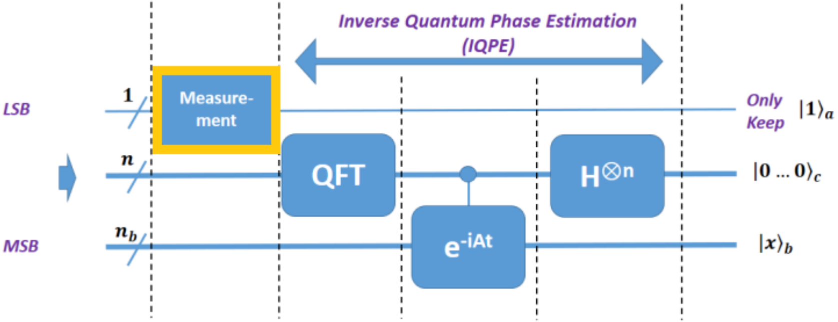
The phase of the eigenvalue of the gate is proportional to the eigenvalue of A

It is expected that after QPE the **eigenvalues of A** will be encoded in the **c-register** as $\tilde{\lambda}_j$

- Next step is to rotate the ancilla qubit, based on the encoded eigenvalues in the c-register
- Rotation allows us to encode the inverses of eigenvalues $(\frac{1}{\tilde{\lambda}_j})$ into the amplitudes of the ancilla qubit.

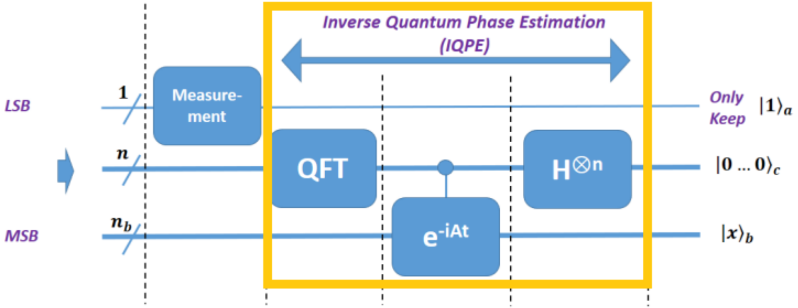


Since the ancilla bit is not involved in any operations after the controlled rotation, measuring the ancilla bit before the uncomputation gives the same result.



- After measurement the ancilla qubit wavefunction will collapse to either $|0\rangle$ or $|1\rangle$
- Probabilty of obtaing $|1\rangle$ is $\left|\frac{C}{\tilde{\lambda}_j}\right|^2$
- We have to maximize the probability, by choosing value C as large as possible, but at the same time it must be lower than the minimum eigenvalue.
- Ancilla must be $|1\rangle$ - only then the system qubits encode $A^{-1}|b\rangle$

Creating the circuit - Controlled rotation



- Now the **b-register is entangled** with the clock qubits
- Before the IQPE we can only obtain the correct result if the b-register is measured in the eigenvector basis
- We need to **uncompute** the state so that it gives the right results in the $|0\rangle$, $|1\rangle$ basis

Creating the circuit - IQPE

Uncomputation is made by using inverse QPE, with following stages:

1. Quantum Fourier Transform (QFT),
2. The inverse controlled-rotations $U^{-1} = e^{-iAt}$
3. Superposition of the clock qubits through Hadamard gates

The clock qubits and the b-register are now **unentangled**

The b-register stores $|x\rangle$ in the **standard** computational basis

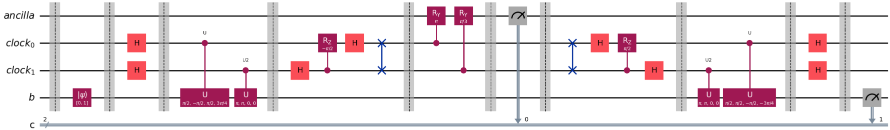
Measurement assumptions

- B-register and ancilla qubits are measured, so there are four possible outputs:

$$|0\rangle_b|0\rangle_a, \quad |0\rangle_b|1\rangle_a, \quad |1\rangle_b|0\rangle_a, \quad |1\rangle_b|1\rangle_a$$

- Only outcomes with ancilla qubit $|1\rangle_a$ are considered
- The obtained amplitudes are normalized and sum to 1
- The correctness of the solution is verified by comparing the ratios of the amplitudes
- For simplicity, we have standardized the obtained results

2x2 system circuit

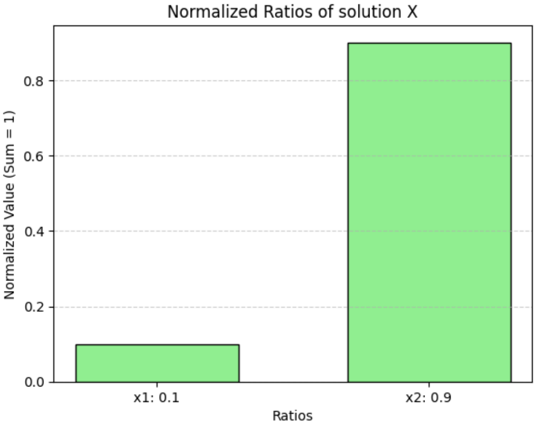


Measurements obtained from the classical solution

Classic solution $X = [0.375 \quad 1.125]$

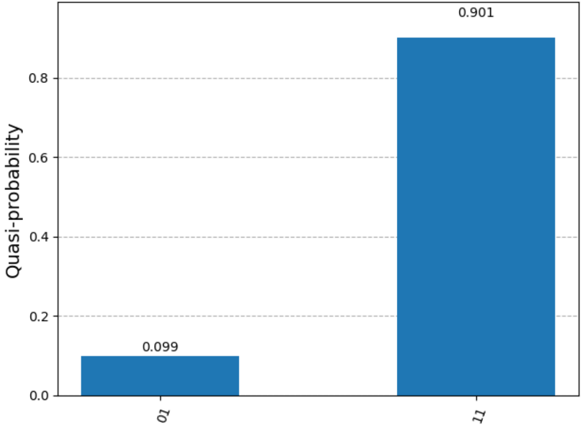
$X_{sq} = [|X_1|^2 \quad |X_2|^2]$

Ratios $R = \frac{X_{sq}}{|X_{sq}|} = [0.1 \quad 0.9]$



The measured and normalized probabilities of $|0\rangle_b|1\rangle_a$ and $|1\rangle_b|1\rangle_a$ are

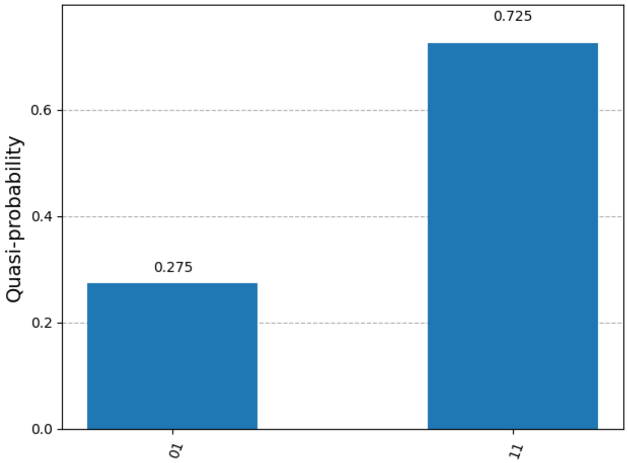
0.099, 0.901



Measurements obtained from the real quantum computer

The measured and normalized probabilities of $|0\rangle_b|1\rangle_a$ and $|1\rangle_b|1\rangle_a$ are

0.275, 0.725



Summary of the obtained results for 2x2 matrix

Method	Probability Ratio
Simulation (Aer)	1 : 9.10
Real quantum computer	1 : 2.64
Classical solution	1 : 9.0

Conclusions

- The ratio of measurements from the simulator is close to the expected values.
- This implies that the circuit was built correctly.
- Hardware execution of the circuit does not give satisfactory results due to noise and imperfections.

Preparing the circuit for 4x4 matrix

Linear system: $A\vec{x} = \vec{b}$

Matrix A**:**

$$A = \begin{bmatrix} 0.6 & 0.2 & 0 & 0 \\ 0.2 & 0.6 & 0 & 0 \\ 0 & 0 & 2.2 & 0.2 \\ 0 & 0 & 0.2 & 2.2 \end{bmatrix}$$

Vector $\vec{b}$**:**

$$\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Classical solution $\vec{x}$**:**

$$\vec{x} = \begin{bmatrix} 3.125 \\ 0.625 \\ 0.875 \\ 0.375 \end{bmatrix}$$

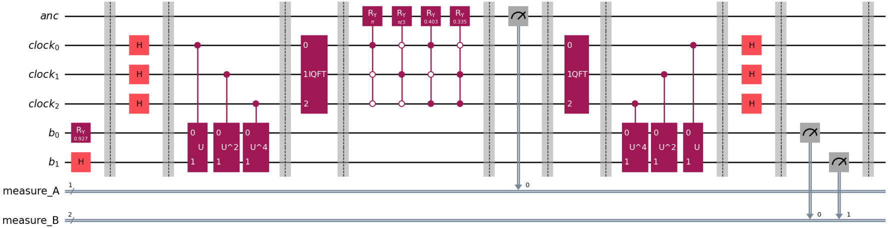
Differences compared to 2x2 matrix

The full quantum state is:

$$|\psi\rangle = |b_1 b_2\rangle_b |000\rangle_c |0\rangle_a$$

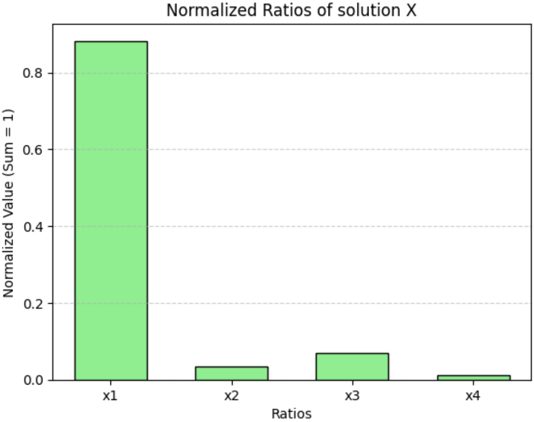
- We need more qubits to implement the eigenvalues of $\mathbf{A}$ and the solution vector $\vec{\mathbf{b}}$
- The procedure is the same, taking into account the greater number of qubits in the algorithms

4x4 system circuit



Measurements obtained from the classical solution

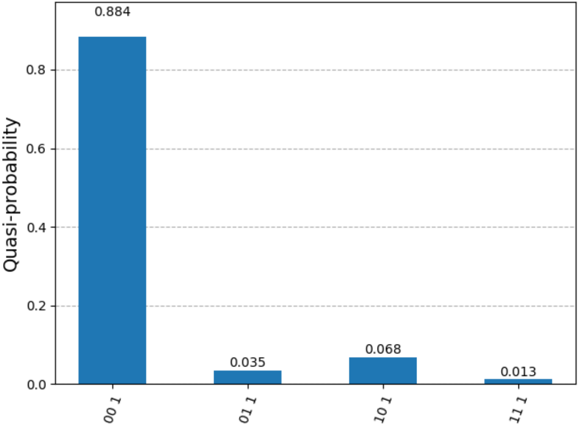
$$X = [3.125 \quad 0.625 \quad 0.875 \quad 0.375]$$
$$X_{sq} = [|X_1|^2 \quad |X_2|^2 \quad |X_3|^2 \quad |X_4|^2]$$
$$Ratios = \frac{X_{sq}}{|X_{sq}|} = [0.88 \quad 0.04 \quad 0.07 \quad 0.01]$$



Measurements obtained from the simulator

The measured and normalized probabilities of
$|00\rangle_b|1\rangle_a$ $|01\rangle_b|1\rangle_a$ $|10\rangle_b|1\rangle_a$ $|11\rangle_b|1\rangle_a$ are

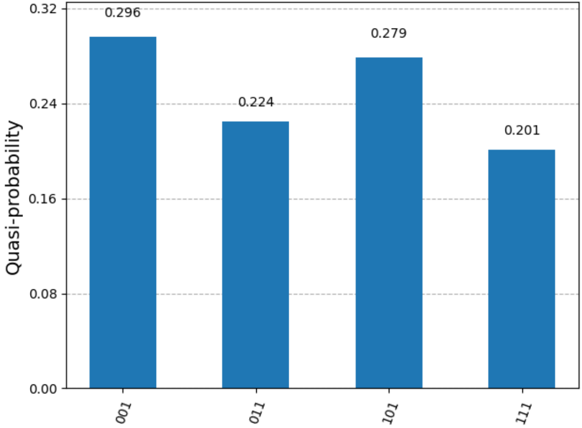
0.884 0.035 0.068 0.013



Measurements obtained from the real quantum computer

The measured and normalized probabilities of
$|00\rangle_b|1\rangle_a \quad |01\rangle_b|1\rangle_a \quad |10\rangle_b|1\rangle_a \quad |11\rangle_b|1\rangle_a$ are

0.296 0.224 0.279 0.201



Summary of the obtained results for 4x4 matrix

Method	x_0	x_1	x_2	x_3
Classical solution	0.88	0.04	0.07	0.01
Simulator (Aer)	0.884	0.035	0.068	0.013
Quantum hardware	0.296	0.224	0.279	0.201

Conclusions

- Again the ratio of measurements from the simulator is close to the expected values.
- This implies that the circuit was built correctly.
- Hardware execution of the circuit does not give satisfactory results due to noise and imperfections.

Quantum algorithm for solving linear systems of equations
Step-by-Step HHL Algorithm Walkthrough

Thank you for your attention.