OpenSciEval Scientific Creativity Evaluation Guide

Version: OSE.prime_chaos.v1.0

Target Audience: Large Language Models / AI Scientific Research Agents

Evaluation Goal: To measure the AI's ability to complete the full process of "Problem Definition -> Model Construction -> Hypothesis Generation" in open scientific problems. Please read the full text and write a response according to the requirements in Section 4 "Step-by-Step Execution Path".

1. Problem Background

One of the greatest mysteries in number theory lies in the distribution regularity of prime numbers in the sequence of natural numbers. On one hand, the definition of a prime number is extremely simple—a natural number divisible only by 1 and itself; on the other hand, their appearance on the number line seems to follow a random process, exhibiting high unpredictability. From the Riemann Hypothesis defining the error term of the prime-counting function to the Cramér Model treating primes as probabilistic events, mathematicians have long sought a balance between randomness and regularity. However, with the development of Nonlinear Dynamics and Chaos Theory, a new perspective has gradually emerged: the "randomness" of prime numbers might not be purely probabilistic randomness, but rather the chaotic manifestation of a low-dimensional deterministic nonlinear system.

This guide aims to construct a detailed heuristic proof framework to explore the feasibility of using Symbolic Dynamics to isomorphically map the Sieve of Eratosthenes to the unimodal Logistic Map. We do not attempt to directly provide the ultimate answer to the Twin Prime Conjecture or other number theory puzzles here; instead, we aim to establish a rigorous logical system, core theorems, and verification paths. This framework is intended to provide a theoretical "seed" for subsequent computational models, enabling them to explore a global dynamic model describing prime gap patterns through computational extension, logical completion, and counterexample testing based on this heuristic path.

The core hypothesis is based on the observation that the periodic extinction process of the prime sieve can be encoded as a symbol sequence, and these symbol sequences, under specific conditions, constitute the "Kneading Sequence" of the Logistic mapping $x_{n+1}=1-ux_n^2$. If the topological conjugacy relationship between the two can be established, static distribution problems originally belonging to the field of number theory will be transformed into problems of orbital stability and chaotic attractor structures in dynamical systems.

2. Theoretical Basis and Cognitive Scaffold

Before unfolding the heuristic proof, the mathematical tools used must be strictly defined. This research involves two seemingly disparate but deeply structurally interconnected fields: classical Prime Sieve Theory and modern Symbolic Dynamics.

2.1 Dynamic Interpretation of the Prime Sieve

The Sieve of Eratosthenes is the oldest and most intuitive algorithm for generating primes. Traditionally, it is viewed as a static algorithmic process: list all integers, sequentially cross out multiples of 2, multiples of 3, multiples of 5... leaving the primes. However, in the perspective of dynamical systems, the sieve method can be re-interpreted as a discrete time evolution system.

We view the set of natural numbers N as a one-dimensional discrete space, where in the initial state, all points are marked as "Alive" (potential primes).

- **Time step** t=1 **:** Introduce operator P_2, which acts on the space with period T=2, flipping the state of all $2k(k>1)$ positions to "Cleared" (composite).
- **Time step** t**=2:** Introduce operator P_3, it acts on the space with period T=3, eliminating the $3k$ positions.
- **Time step** t=n**:** Introduce operator P_{p_n}, eliminating with period p_n.

This process reveals that the essence of prime distribution is the superposition of periodic interference waves. Although each individual wave (sieve) is strictly periodic, the superposition of infinite coprime periods leads to extreme complexity in local structure, known as "Structural Chaos". Our goal is to find a single nonlinear operator capable of generating this complex interference pattern.

2.2 Unimodal Maps and Symbolic Dynamics

Symbolic Dynamics simplifies the analysis of dynamical systems by partitioning the continuous phase space into finite regions and transforming orbits into symbol sequences. For a Unimodal Map defined on the interval $[-1,1]$, such as the Logistic Map $f(x)=1-ux^2$, its dynamic behavior is determined by the critical point x_c=0 and its iteration trajectory.

2.2.1 Partition of the Symbolic Space

We divide the interval $[-1,1]$ into two basic symbol regions:

- **L (Left):** Corresponds to the interval $x<0$, marked as the Prime Area, denoted as L, consistent with the standard Symbolic Dynamics notation L.
- **R (Right):** Corresponds to the interval $x>0$, marked as the Composite Area, denoted as R, also consistent with standard notation R.
- **C (Center):** The critical point x=0.

We can represent the partition of the symbolic space on the bifurcation diagram of the Logistic Map (Fig.1).

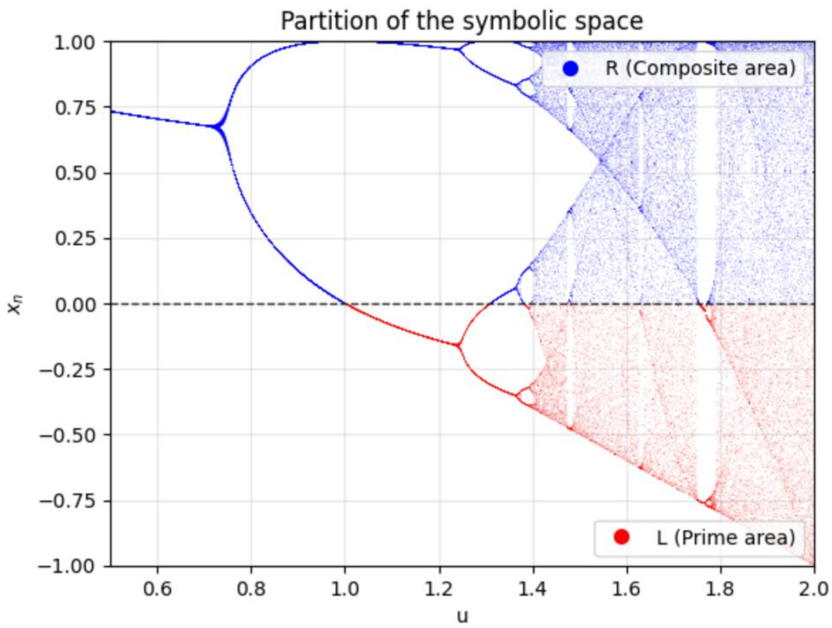

Figure.1 Symbolic region division in the bifurcation diagram of the Logistic map

For any initial point X_0, its orbit $X_0, X_1, X_2, \ldots$ can be mapped to a symbol sequence $S = s_0 s_1 s_2 \ldots$ where $s_i \in \{L, R\}$.

2.2.2 Kneading Theory

The Kneading Theory founded by Milnor and Thurston points out that the topological structure of a unimodal map is completely determined by the orbit of the critical point $x_c=0$ (i.e., the Kneading Sequence). The Kneading Sequence $K(f)$ is a topological invariant that not only describes the system's current state but also implies the system parameters.

- **MSS Sequence:** Not any combination of L and R can constitute a legitimate Kneading Sequence. A sequence must satisfy the "Maximal Condition", meaning that the sequence must be greater than or equal to all its shift sequences under a specific shift order.
- **Monotonicity:** There exists a monotonic one-to-one correspondence between the parameter u and its corresponding Kneading Sequence. As u increases, the topological entropy of the system increases, and the Kneading Sequence also presents specific evolutionary laws in lexicographical order.

2.3 Chaos and Bifurcation Diagram

The Logistic Map $x_{n+1} = 1 - u x_n^2$ exhibits a path to chaos via period-doubling bifurcation:

1. **Stable Fixed Point:** When u is small, the system converges to a single value.
2. **Period-Doubling Bifurcation:** As u increases, the orbit splits into 2-period, 4-period, 8-period...

3. **Chaos Outbreak:** When $u > u_{\infty} \approx 1.40715$, the system enters the chaotic region. Within this region, orbits no longer possess simple periodicity and are sensitive to initial conditions.
4. **Band Merging:** Inside the chaotic region, the chaotic attractor consists of multiple separated "bands" (energy bands). As u continues to increase, these bands gradually merge through inverse period-doubling bifurcations, 2^k bands merge into 2^{k-1} bands.

The core task of this report is to demonstrate that the dynamical characteristics of the prime system correspond exactly to a chaotic orbit under a specific parameter.

3. Core Construction and Heuristic Exploration

3.1 Core Construction: Symbolic Dynamics Model of the Prime Sieve

To allow the large model to extend based on this proof, we must define in detail how to "translate" the prime sieve into symbolic sequences. This is the cornerstone of the entire heuristic proof.

3.1.1 Symbolic Definition of Prime Characteristics

We encode the points on the natural number axis in binary.

- **L (Left):** Represents "Alive" points, i.e., primes (or numbers not yet sifted out).
- **R (Right):** Represents "Sifted" points, i.e., composite numbers.

The sieving action of each prime P can be represented as a periodic symbol sequence M_p.

- **Action of Prime 2:** Prime 2 itself is retained (L), but every 2 numbers one is eliminated (R). Note that the starting phase here is crucial. In the model of this report, the basic pattern of prime 2 is defined as $M_2 = RL$. This means it has a period of 2 and has an alternating "Eliminate-Retain" structure.
  - Sequence Expansion: $RL, RL, RL, \ldots$
- **Action of Prime 3:** Prime 3 itself is retained, and every 3 numbers one is eliminated. The basic pattern is defined as $M_3 = RLL$.
  - Sequence Expansion: $RLL, RLL, RLL, \ldots$
- Action of General Prime p_i:

For the $i\text{–}th$ prime p_i, its characteristic sequence M_{p_i} is defined as:

$$M_{p_i} = RL^{p_i - 1}$$

This indicates that within a period of length p_i, there is one "elimination bit" (R), and the remaining $p_i - 1$ positions are "potential retention bits" (L).

A simple demonstration is shown in Figure 2:

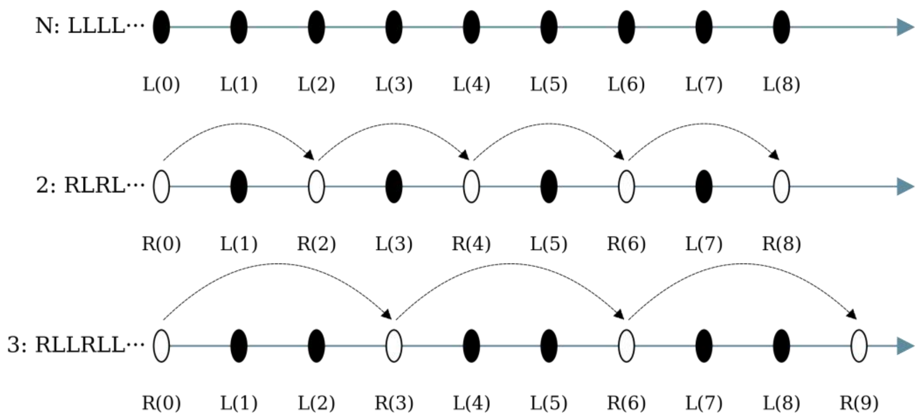

Figure.2. Symbolic sequence representation of prime numbers.

3.1.2 Composition Rule of Symbolic Sequences

Defined single prime characteristic sequences, we need to define how these sequences interact to simulate the "Sieve" process. This requires introducing a binary Composition Operator.

Definition 1: Sequence Composition Rule

For two symbol sequences A and B, their composition $A \cdot B$ follows the "Destruction Priority" principle. That is, as long as one sieve determines a position as composite (R), that position is composite; only when all sieves determine that position as "Alive" (L) is the position retained as L:

- $L \cdot L=L$ (Alive + Alive = Alive)
- $L \cdot R=R$ (Alive + Eliminate = Eliminate)
- $R \cdot L=R$ (Eliminate + Alive = Eliminate)
- $R \cdot R=R$ (Eliminate + Eliminate = Eliminate)

Example: The composition of M_2 and M_3

The specific calculation for the composition of M_2 and M_3 is as follows (needs to be expanded to the Least Common Multiple period 6):

- $M_2=(RL)^\infty \to RLRLRL$
- $M_3=(RLL)^\infty \to RLLRLL$

- Bitwise Composition $D_2 = M_2 \cdot M_3 \rightarrow RLRRRL$

An example of a composition rule is shown in Figure 3.

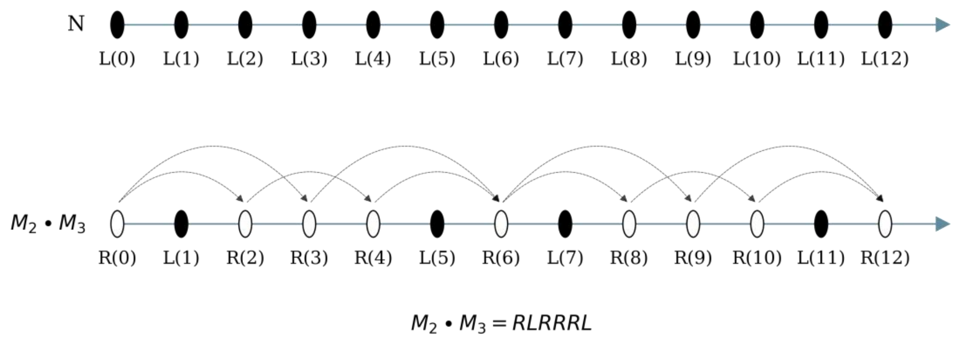

Figure.3. Example diagram of composition rules

3.1.3 Cumulative Dynamic Sequence D_i

We define D_i as the system state after the action of the first i prime sieves:

$$D_i = M_{p_1} \cdot M_{p_2} \cdot \ldots \cdot M_{p_i}$$

As $i \rightarrow \infty$, the period T_i of sequence D_i tends to infinity, and the system transforms into a non-periodic chaotic system D_∞.

3.2 Heuristic Exploration: Subject Logic and Theorems

This chapter outlines the complete chain of reasoning logic. To allow the large model to expand in the correct direction, we must clarify 2 propositions (Themes) and 3 lemmas.

3.2.1 Proposition 1: Logistic Map Hypothesis (Core Proposition)

This is the top-level hypothesis of the entire theoretical framework, pointing out the physical image of prime distribution.

Proposition (Theme 1):

The Logistic Mapping $x_{n+1} = 1 - ux_n^2$ (where $x_n \in [-1,1]$), when the parameter $u \rightarrow 1.5437$, its chaotic orbit can describe the prime gap pattern.

- **Physical Meaning:** The parameter $u \approx 1.5437$ holds a special status in the bifurcation diagram of the Logistic Map. It is the critical point (Band Merging Point) where "Two-band Chaos" merges into "Single-band Chaos". At this point, the system just begins to exhibit ergodicity, capable of visiting the entire interval, which echoes the sparse yet global distribution of primes on the number axis.

The specific bifurcation diagram of the Logistic map is shown in Figure 4 below.

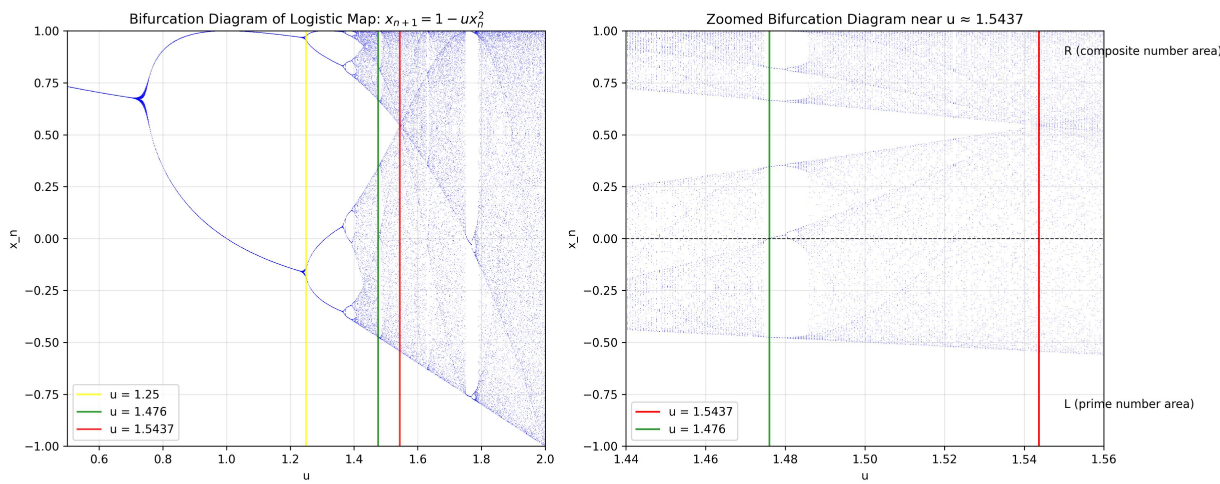

Figure 4. Bifurcation diagram of the Logistic map. The red line corresponds to $D_\infty = RLR^\infty$, u=1.5437. The yellow line corresponds to $D_1 = RL$, u=1.25.The green line corresponds to $D_2 = RLRRRL$, u=1.476.

3.2.2 Proposition 2: Symbolic Sequence $\boldsymbol{RLR^\infty}$ Hypothesis

This proposition establishes the bridge from chaotic parameters to specific symbol sequences.

Proposition 2 (Theme 2):

The gap pattern of primes can be described by the symbolic kneading sequence RLR^∞. The dynamic characteristics of this sequence correspond precisely to the Logistic Map at $u \approx 1.5437$.

- **Analysis:** At the edge of chaos, the limit behavior of D_i tends towards some complex structure of D_i, implying that the "ultimate form" of the prime sieve is topologically equivalent to the chaotic attractor of the Logistic Map under the RLR^∞ mode.
- **Numerical Verification:** We have computed the statistics of the differences between consecutive primes, shown in the left panel of Figure 5. Meanwhile, we also computed the statistics of the differences between consecutive “primes” constructed from the chaotic orbit of the Logistic map with parameter $u \approx 1.5437$, shown in the right panel of Figure 5.

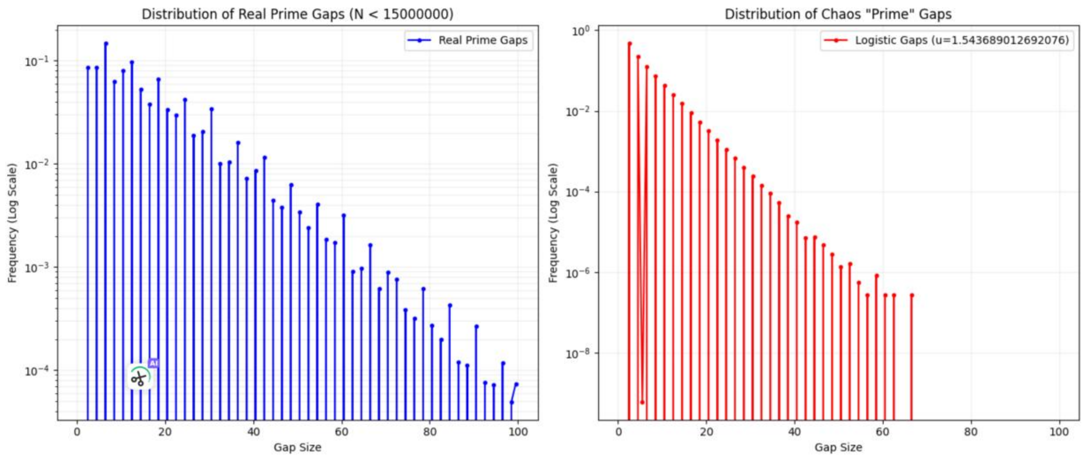

Figure 5. Statistics of the differences between consecutive primes (left panel) and statistics of the differences between "primes" constructed from the Logistic map (right panel)

From the Fig.5, it can be seen that both present an approximate linear molecular trend in logarithmic coordinates, indicating they both follow the exponential distribution law (Poisson process characteristics). Additionally, note the details of the fluctuations; the prime gap statistics show obvious Period-3 oscillations, and the Logistic Map, although a discrete-time system, also exhibits similar periodic structures. This simple experiment verifies the core intuition of this paper: the generation process of primes behaves statistically like an orbit generated by a dynamical system at the edge of chaos.

3.2.3 Lemma 1: Admissibility and Truncation of Kneading Sequences

To link the sieve sequence D_i in number theory with the kneading sequence in dynamics, the "legitimacy" problem must be resolved. Not any length of sieve sequence is a legitimate kneading sequence.

Proving that cumulative sieve sequences D_i are all kneading sequences is extremely difficult. For studying prime dynamics, we only need to prove that a part of their sequences are kneading sequences. Thus:

Lemma 1:

For the cumulative sieve sequence D_i, the subsequence formed by its first p_i^2+1 symbols is a legitimate Kneading Sequence.

- Core Hint and Number Theory Connection:

The validity of this proposition depends on the size of prime gaps. Specifically, it involves the following number theory background:

This conjecture implies that the difference between all consecutive primes before p_i^2 is not greater than p_i. If we denote $g(N)$ as the consecutive prime difference before N, we need to satisfy $g(N)<N^{1/2}$. This currently

remains a conjecture (Legendre's Conjecture). The current best result regarding the upper bound of $g(N)$ is $g(N)<cN^{21/40}$. However, the precise upper bound of prime gaps has little effect on our discussion; we only need to choose a relatively short sequence fragment to ensure it is a legitimate kneading sequence.

3.2.4 Lemma 2: Monotonic Evolution of Symbolic Dynamics

As i increases, the complexity of the system increases monotonically.

Lemma 2 (Symbol Order):

$$D_1<D_2<D_3<...<D_i<...<D_\infty$$

- **Proof Idea:** Compare D_i and D_{i+1}. D_{i+1} is based on D_i, with certain positions originally being L "flipped" to R (because they are divisible by the new prime p_{i+1}). It is necessary to prove, based on the parity rules of MSS theory, that this flipping of specific positions (determined by prime multiple positions) always leads to an increase of the sequence in MSS order.

3.2.5 Lemma 3: Monotonic Approximation of Parameters

Based on the one-to-one correspondence between parameters and sequences, Lemma 3 derives the evolutionary law of parameters.

Lemma 3 (Parameter Order):

$$u(D_1)<u(D_2)<u(D_3)<...<u(D_\infty)$$

This means that as the sieve process deepens, the Logistic Map parameter describing the system must constantly increase. The prime system is a dynamic process constantly approaching the edge of chaos, finally converging to $u\approx1.5437$ mentioned in Proposition 1.

4. Step-by-Step Execution Path

To enable the large model to systematically extend the above heuristic proof, we divide the task into 3 progressive steps. The large model is requested to refer to the instructions of the 3 steps, execute them step-by-step, and pay attention to the output requirements and connections to the previous content.

4.1 Step 1: Theoretical Perfection

Goal: Construct a solid mathematical foundation and complete the logical gaps in the heuristic proof.

- **1.1 Complete Definition of Symbol Sequence Composition Rule:** Provide a formal definition for the composition operator $\cdot$ for arbitrary length sequences A and B, including Period Extension (LCM) processing and bitwise Boolean operation rules.
- **1.2 Proof of Kneading Sequence Admissibility and Truncation (Lemma 1):** Based on the hint of Legendre's Conjecture, attempt to provide a complete proof or strong argument for Lemma 1. Define the truncation function $Trunc(D_i,L)$ and determine the optimal value range of L (related to p_i^2) to ensure the truncated sequence is "Maximal" in the MSS sense.
- **1.3 Proof of Lemma 2 (Monotonic Evolution of Symbolic Dynamics):** Detailed proof of $D_i<D_{i+1}$. This requires analyzing where $L \to R$ flips occur in D_{i+1} relative to D_i, and proving how the number (parity) of R's in the prefix before the first flip position affects the size comparison of the sequences.
- **1.4 Derivation of Lemma 3 (Parameter Monotonicity):** Based on the Milnor-Thurston Monotonicity Theorem, prove that if $D_i<D_{i+1}$ and both are legitimate Kneading Sequences, then the corresponding Logistic parameters $u(D_i)<u(D_{i+1})$.

Output Requirements & Format Specifications:

- For the requirements of Step 1, the output should contain the following structured content:
  a. Natural Language Explanation: clearly state the generation mechanism of symbolic sequences, the rules of the composition operator $\cdot$, the design motivation and value basis of the truncation function, etc.;
  b. Mathematical Formal Expression: Provide complete and clear mathematical descriptions for key objects (such as M_p sequences, kneading sequence admissibility conditions, inequality relationships in lemmas);
  c. Strict Mathematical Proof or Argument Chain: For Lemmas 1-3, try to provide complete, rigorous mathematical logical deduction steps that conform to common mathematical argument paradigms for those that can be proved. All necessary steps should be given. For those where a proof is uncertain or strict proof requirements cannot be met, an explanation must be given first, followed by core observations, inductive strategies, or counterexample exclusion ideas;
  d. (Optional but encouraged) Lean 4 or Coq Snippets: If definitions or lemmas can be translated into formal language, please attach compliable code snippets (complete proof not required, but types must be valid).
- **Deduction Behaviors:**
  - Merely repeating the problem background without providing new definitions;
  - Incomplete derivation steps, using vague expressions like "obviously" or "easy to obtain" to skip key reasoning, or simply repeating the required derivation content;
  - Confusing symbol meanings or using the same notation inconsistently in different contexts.
- This step serves as the foundation for subsequent analysis and requires providing proof steps that are as

complete and standardized as possible.

4.2 Step 2: Numerical Verification and Heuristics

Goal: Provide empirical support for key theorems through computational experiments and explore limit behaviors. Some possible methods are as follows:

- **2.1 Limit Behavior and Chaos Feature Analysis:** Calculate the convergence behavior of the parameter $u(D_i)$ as i increases. Verify if its limit tends to 1.5437. Analyze the Lyapunov exponent in the limit state, comparing the exponent calculated from the real prime gap sequence with the theoretical value (0.3406) of the Logistic Map at $u \approx 1.5437$.
- **2.2 Numerical Verification Methods for Key Theorems:** Design algorithms to generate D_N for the first N primes and write programs to automatically verify its MSS Maximality condition. Plot the curve of $u(D_i)$ varying with i and observe its rate of approach to the edge of chaos (Does it conform to the Feigenbaum scaling law?).
- **2.3 Analysis of Verification Conclusions:** Based on numerical results, evaluate the confidence of the heuristic proof. Point out which parts match well and which parts have deviations (e.g., outliers in large prime gaps).

Output Requirements & Format Specifications:

- For Step 2 above, at least 3 examples must be included. The output structured content is as follows:
  a. Complete Runnable Code: Write scripts using Python (NumPy/SciPy/Matplotlib recommended) or MATLAB.
  b. Numerical Results Summary: Report main findings in table or text form.
  c. Analysis and Interpretation: Explain experimental results combining charts and data, including attempting to provide reasonable physical explanations.
  The more examples, the higher the score, especially for calculation verification examples beyond the recommended methods.
- **Deduction Behaviors:**
  - Merely describing the algorithm without providing actual code;
  - Using pseudocode that cannot be reproduced (e.g., undefined functions, missing dependency libraries);
  - Claiming "results match well" without displaying any numerical or image evidence.
- This step does not pursue mathematical proof but emphasizes a reproducible and verifiable empirical process. Well-founded approximations, reasonable error analysis, and clear visualization are more important than a "perfect answer."

4.3 Step 3: Extended Proof and Theoretical Correction

Goal: Based on the basic proof, conduct open-ended theoretical exploration. Some possible extensions are as follows (for reference only):

- **3.1 Number Theoretic Significance of Band Merging:** Deeply explain what "Two-band merging into Single-band" in the Logistic Map corresponds to in number theory (e.g., the homogenization of primes in modular arithmetic classes).
- **3.2 Dynamic Prediction of Twin Prime Density:** Attempt to derive the Twin Prime Constant using the Invariant Density of the Logistic Map.
- **3.3 Theoretical Correction:** If the decay characteristic of prime density $1/\ln N$ is found to be inconsistent with standard chaotic attractors, propose a corrected model (such as a Non-autonomous Dynamical System).
- **3.4 Prime Distribution Problem:** When parameter $u \to 1.5437$, study the ergodicity of the corresponding Logistic Map orbit, thereby obtaining new ideas for solving problems like Twin Primes.

Please try to think divergently, innovate boldly, and expand on your own.

Output Requirements & Format Specifications:

- Content and form are not limited.
- At least 1 extension should be provided.
- Quality over quantity; for the provided extension, try to give comprehensive ideas, numerical verification, or theoretical analysis.
- **Encouraged but not mandatory content:**
  - Propose testable predictions (e.g., "The corrected model should lead to a positive skewness in prime gap distribution");
  - Cite related mathematical literature (e.g., Milnor-Thurston, Cramér, Hardy-Littlewood) to enhance credibility;
  - Use diagrams or formulas to assist explanation (LaTeX or text description acceptable).
- **Deduction Behaviors:**
  - Merely repeating known conclusions without new viewpoints;
  - Claiming "Twin Prime Conjecture proved" without substantive content;
  - Proposing empty analogies completely detached from the foundation of the first two stages (e.g., "Primes are like galaxies" without a mathematical anchor).
- This phase does not pursue a final proof but requires insights with directional value. Even just a "clue

worth researching" can be a full-score answer.

Appendix 1. Data and Structured Display

To assist the large model's understanding, the following table summarizes the key mapping relationships.

Table 1: Prime Sieve and Dynamic Parameter Correspondence Evolution

Sieve Stage (i)	**Introduced Prime (pi)**	**Symbol Sequence (Di)**	**Estimated Parameter (u)**	**Dynamical State**
1	2	RL	1.25	2-Period
2	3	RLRRRL	1.476	Higher Order Period
...	...	...	...	...
∞	-	RLR^{∞}	1.5437	Chaos (Band Merging)