CMSC 27200 - Spring 2024 — Lorenzo Orecchia
due 4/12/24 via Gradescope

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- The assignment is due at Gradescope on 4/12/24.
- A LaTeX template will be provided for each homework. You are strongly encouraged to type your homework into this template using LaTeX. If you are writing by hand, please fill in the solutions in this template, inserting additional sheets as necessary. This will help facilitate the grading.
- You are permitted to discuss the problems with up to 2 other students in the class (per problem); however, you must write up your own solutions, in your own words. Do not submit anything you cannot explain. If you do collaborate with any of the other students on any problem, please list all your collaborators in the appropriate spaces.
- Similarly, please list any other source you have used for each problem, including other textbooks or websites.
- Show your work. Answers without justification will be given little credit.
- Your homework is resubmittable. Please refer to the course syllabus on Canvas for a more detailed description of this. For any problem that you have not changed from your last submission, please make sure to indicate this in your submission to help our graders grade faster.
- Is anyone still reading these?

Problem 1 (Linear Algebra Practice) This question is on the basics of linear algebra, please write out all steps for computation and formal proofs:

(a) Let $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, compute the following:

   (i) A^{-1}

   (ii) A^T

   (iii) A^2

(b) Let A and B be $n \times n$ matrices. What's wrong with the equation $(A+B)^2 = A^2 + 2AB + B^2$? Prove that the above equation holds if and only if A and B commute.

(c) Let A be a $n \times m$ matrix. Consider function $f : \mathbb{R}^n \to \mathbb{R}^m$ where $f(x) = Ax$. Formally prove that f is a linear map (i.e. you need to show that $f(ax+by) = af(x) + bf(y)$ for all $x, y \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$).

Collaborators:

Solution: The solution to this problem is below.

(a) (i) $A^{-1} = \frac{1}{a \cdot d - b \cdot c} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{2x3 - 5x1} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

   (ii) $A^T = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$

   (iii) $A^2 = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2x2 + 5x1 & 2x5 + 5x3 \\ 1x2 + 3x1 & 1x5 + 3x3 \end{pmatrix} = \begin{pmatrix} 12 & 17 \\ 5 & 14 \end{pmatrix}$

(b) Assume A and B commute, then $AB = BA$. Then $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2$. Conversely, if $(A+B)^2 = A^2 + 2AB + B^2$, then $AB = BA$, which implies that A and B commute.

(c) To prove f is a linear map, we need to show the addicitivity and homogeneity properties. Consider $f(x+y) = A(x+y)$, we get $f(x+y) = Ax + Ay = f(x) + f(y)$. Consider $f(ax) = A(ax)$, we can pull the scaler a out and get $f(ax) = a(Ax) = af(x)$. Therefore, f is a linear map.

Problem 2 (Modular FFT) This question is based on problem 2.30 from [DPV].

This problem illustrates how to do the Fourier Transform (FT) in modular arithmetic, for example, modulo 7.

(a) There is at least one number ω such that all the powers ω, ω^2,..., ω^6 are distinct (modulo 7). Find one such ω, and show that $\omega + \omega^2 + \cdot \cdot \cdot + \omega^6 \equiv 0 \mod 7$. (Interestingly, for any prime modulus there is such a number.)

(b) Using the matrix form of the FT, produce the transform of the sequence $(0, 1, 1, 1, 5, 2)$ modulo 7; that is, multiply this vector by the 6×6 matrix $M_6(\omega)$, for the value of ω you found earlier. In the matrix multiplication, all calculations should be performed modulo 7.

(c) Write down the matrix necessary to perform the inverse FT. Recall that dividing by a number amounts to multiplying by its inverse. Show that multiplying by this matrix returns the original sequence. (Again all arithmetic should be performed modulo 7.)

(d) Now show how to multiply the polynomials $x^2 + x + 1$ and $x^3 + 2x - 1$ using the FT modulo 7.

Collaborators:

Solution: The solution to this problem is below.

(a) $\omega = 3$ is one such number. $\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 3 + 2 + 6 + 4 + 5 + 1 = 21 \equiv 0 \mod 7$.

(b) Continue the calculations from part (a).

We get the matrix $M_6(\omega)$ is $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 6 & 4 & 5 \\ 1 & 2 & 4 & 1 & 2 & 4 \\ 1 & 6 & 1 & 6 & 1 & 6 \\ 1 & 4 & 2 & 1 & 4 & 2 \\ 1 & 5 & 4 & 6 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ 41 \\ 25 \\ 30 \\ 31 \\ 31 \end{pmatrix} \equiv \begin{pmatrix} 3 \\ 6 \\ 4 \\ 2 \\ 3 \\ 3 \end{pmatrix} \mod 7$

(c) $M_6(\omega^{-1}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 4 & 6 & 3 & 2 \\ 1 & 4 & 2 & 1 & 4 & 2 \\ 1 & 6 & 1 & 6 & 1 & 6 \\ 1 & 2 & 4 & 1 & 2 & 4 \\ 1 & 3 & 2 & 6 & 4 & 5 \end{pmatrix}$ $\textit{Inverse FT} = 6 \cdot M_6(\omega^{-1}) \cdot \begin{pmatrix} 3 \\ 6 \\ 4 \\ 2 \\ 3 \\ 3 \end{pmatrix} = 6 \cdot M_6(\omega^{-1}) \cdot \begin{pmatrix} 3 \\ 6 \\ 4 \\ 2 \\ 3 \\ 3 \end{pmatrix} =$

$\begin{pmatrix} 21 \\ 76 \\ 55 \\ 76 \\ 51 \\ 68 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 5 \\ 2 \end{pmatrix} \mod 7$

(d) To multiply the polynomials $x^2 + x + 1$ and $x^3 + 2x - 1$, we first represent them as vectors $(1, 1, 1, 0, 0, 0)$ and $(6, 2, 0, 1, 0, 0)$. FFT of the vector $(1, 1, 1, 0, 0, 0)$ is $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 6 & 4 & 5 \\ 1 & 2 & 4 & 1 & 2 & 4 \\ 1 & 6 & 1 & 6 & 1 & 6 \\ 1 & 4 & 2 & 1 & 4 & 2 \\ 1 & 5 & 4 & 6 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 3 \\ 6 \\ 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} \mod 7.$

Similarly, FFT of the vector $(6, 2, 0, 1, 0, 0)$ is $\begin{pmatrix} 2 \\ 4 \\ 4 \\ 3 \\ 1 \\ 1 \end{pmatrix}$. The product of the two polynomials is the product of the two FFTs, which is $\begin{pmatrix} 3 \\ 6 \\ 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 4 \\ 3 \\ 1 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 6 \\ 3 \\ 0 \\ 3 \\ 0 \\ 3 \end{pmatrix}$. We need to calculate the final result by inversing the FFT, this gives us $6 \cdot \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 4 & 6 & 3 & 2 \\ 1 & 4 & 2 & 1 & 4 & 2 \\ 1 & 6 & 1 & 6 & 1 & 6 \\ 1 & 2 & 4 & 1 & 2 & 4 \\ 1 & 3 & 2 & 6 & 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 3 \\ 0 \\ 3 \\ 0 \\ 3 \end{pmatrix} \equiv \begin{pmatrix} -1 \\ 1 \\ 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} \mod 7$. Therefore, the final result is $-x^5 + x^4 + x^3 + 3x^2 + x + 1$.

Problem 3 (Bipartite Graphs) This question is similar to question 3.7 in [DPV].

A bipartite graph is an undirected graph $G = (V, E)$ whose vertex set V can be partitioned into two sets V_1 and V_2 (i.e. $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$) such that there are no edges between vertices of the same set. In other words, all edges are of the form (u, v), where $u \in V_1$ and $v \in V_2$.

(a) Sketch a proof that a graph is bipartite if and only if it contains no odd cycle.

(b) Give an $O(|V| + |E|)$ time algorithm with pseudocode to check if a graph is bipartite, and formally prove that it is correct and that it runs in $O(|V| + |E|)$ time.

Collaborators:

Solution: The solution goes below:

(a) To prove the if part, we can divide the graph into two sets V_1 and V_2 such that there are no edges between vertices of the same set. If there is an odd cycle, then there must be an edge between two vertices in the same set, which contradicts the definition of a bipartite graph. To prove the only if part, we can prove by contradiction. Assume there is an odd cycle in the graph, then there must be an edge between two vertices in the same set, which contradicts the definition of a bipartite graph.

(b) To achieve $O(|V| + |E|)$, we could use a BFS algorithm as following:

Algorithm 1 Check if a graph is bipartite

```
Input: Graph G = (V, E)
Output: True if the graph is bipartite, False otherwise
Adjecency lists adj = V*[[ ]]
for all (u, v) in E: adj[u].append(v), adj[v].append(u)
colors = [V*-1]
queue = []
for i in V:
if i is not colored then
    color i with 0
    add i to queue
    while queue is not empty do
        currentVertex = queue.pop(0)
        for v in adj[currentVertex] do
            if v is not colored then
                color v with the opposite color of currentVertex
                add v to queue
            else if v has the same color as currentVertex then
                return False
            end if
        end for
    end while
end if
return True
```

The proof of correctness goes as following: The algorithm checks if a graph is bipartite by coloring the vertices with two colors. For each uncolored vertex, the algorithm assigns a color and adds it to the queue, then proceeds with a BFS to color each adjacent vertex with the opposite color. If the algorithm encounters an adjacent vertex that is already colored with the same color as the current vertex, then the graph is not bipartite and therefore returns false. For the adjecent verteces with the opposite color, it skips over and proceeds to the next vertex in queue. If the algorithm successfully colors all vertices with the two colors without a conflict, it returns True.

The time complexity of the algorithm is $O(|V| + |E|)$. The algorithm initializes the adjacency list in $O(|E|)$ time, colors the vertices in $O(|V|)$ time, as each vertex is added to queue once and colored once. Therefore, the total time complexity is $O(|V| + |E|)$.

Problem 4 (Verifying Binary Search Trees) For this problem you may assume that a binary tree node is defined as follows:

```
class node:
    left: node = None
    right: node = None
    value
```

You can assume that the values are distinct and the value type supports comparison. A binary tree T is called a binary search tree (BST) if and only if for every node V in the tree

$$\max_{U \in V.left} U.val < V.val < \min_{W \in V.right} W.val$$

(a) Write pseudocode for a recursive algorithm that takes as input the root T of a binary tree and returns whether it is a BST. Your algorithm can optionally return more than just a boolean value. You are allowed to define helper functions if you want. Your algorithm should run in $O(n)$ time, where n is the number of nodes in tree T.

(b) Prove the correctness of your algorithm.

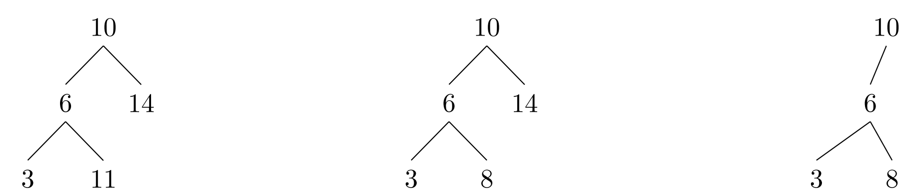
The height of a tree is defined as

$$height(T) = \begin{cases} 0 & T = None \\ 1 + \max\left(height(T.left), height(T.right)\right) & \text{o.w.} \end{cases}$$

We would like all our Binary Search Trees operations (look-up, insertion, deletion) to have $O(\log n)$ time complexity. To achieve that, we require the BST to also be balanced. A BST is balanced if and only if for every node V in the tree

$$\left|height(V.left) - height(V.right)\right| \leq 1$$

(c) Modify below your pseudocode from part (a) to check if the binary tree T is a balanced BST. Your modified algorithm should also have $O(n)$ time complexity.



(a) This is not a BST since 11 > 10 (b) This is a balanced BST (c) This is a BST, but not balanced

Collaborators:

Solution: The solution goes below:

(a) The algorithm is as follows:

Algorithm 2 Check if a binary tree is a BST

```
Input: Root node T of a binary tree
Output: True if the tree is a BST, False otherwise
function isBST(node T, Min, Max) (helper function)
    if T is None then
        return True
    end if
    if (Min is not None and T.value ≤ Min) or (Max is not None and T.value ≥ Max) then
        return False
    end if
    return isBST(T.left, Min, T.value) and isBST(T.right, T.value, Max)
end function
function verifyBST(node T)
    return isBST(T, None, None)
end function
```

(b) The algorithm uses a helper function isBST to check if the tree is a BST. The function takes in the root node T and two values Min and Max. If the tree is empty, the function returns True. If the value of the current node is less than Min or greater than Max, the function returns False. Otherwise, the function recursively checks the left and right subtrees with the updated Min and Max values. The main function verifyBST calls the helper function with Min and Max set to None. The correctness of the algorithm is proved by the fact that the algorithm checks if the tree is a BST by comparing the value of the current node with the Min and Max values recursively, which enables the algorithm to examine the values from both ends and get back to the root. Since each root is traversed only once, the time complexity of the algorithm is $O(n)$, where n is the number of nodes in the tree. The algorithm visits each node once and performs a constant amount of work at each node.

(c) The modified algorithm is as follows:

Algorithm 3 Check if a binary tree is a BST

```
Input: Root node T of a binary tree
Output: True if the tree is a Balanced BST, False otherwise
function isBalancedBST(node T) (helper function)
    if T is None then
        return True, 0
    end if
    leftIsBalanced, leftHeight = isBalancedBST(T.left)
    if not leftIsBalanced then
        return False, 0
    end if
    rightIsBalanced, rightHeight = isBalancedBST(T.right)
    if not rightIsBalanced then
        return False, 0
    end if
    return True, 1 + max(leftHeight, rightHeight)
end function
function verifyBalancedBST(node T)
    isBalanced, height = isBalancedBST(T)
    return isBalanced
end function
```