The question we want to answer now is the following:

> If A is not similar to a diagonal matrix, then what is the simplest matrix that A is similar to?

Before we can provide the answer, we will have to introduce a few definitions.

Definition: A square matrix A is **block diagonal** if A has the form

$$A = \begin{bmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_k \end{bmatrix},$$

where each A_i is a square matrix, and the diagonals of each A_i lie on the diagonal of A. Each O is a zero matrix of appropriate size. Each A_i is called a **block** of A.

Technically, every square matrix is a block diagonal matrix. But we only use the terminology when there are at least two blocks in the matrix. Here is an example of a 'typical' block diagonal matrix:

$$A = \left[\begin{array}{ccc|c|cc} 1 & 3 & 2 & 0 & 0 & 0 \\ 7 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 6 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 & 3 \end{array}\right]$$

This matrix has blocks of size 3, 1 and 2 as we move down the diagonal. The three blocks in this matrix are

$$A_1 = \begin{bmatrix} 1 & 3 & 2 \\ 7 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}, \qquad A_2 = [6], \qquad A_3 = \begin{bmatrix} 2 & 1 \\ 3 & 3 \end{bmatrix}.$$

The lines are just drawn to illustrate the blocks.

Definition: A **Jordan block** with value λ is a square, upper triangular matrix whose entries are all λ on the diagonal, all 1 on the entries immediately above the diagonal, and 0 elsewhere:

$$J(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}.$$

Here's what the Jordan blocks of size 1, 2 and 3 look like:

$$[\lambda], \qquad \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Definition: A **Jordan form** matrix is a block diagonal matrix whose blocks are all Jordan blocks.

For example, every diagonal $p \times p$ matrix is a Jordan form, with p 1×1 Jordan blocks. Here are some more interesting examples (again, lines have been drawn to illustrate the blocks):

$$\left[\begin{array}{cc|cc|c|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array}\right] \qquad \left[\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

Now, here's the big theorem that answers our first question:

Theorem 1 *Let A be a $p \times p$ matrix. Then there is a Jordan form matrix J that is similar to A.*

In fact, we can be more specific than that:

Theorem 2 *Let A be a $p \times p$ matrix, with s distinct eigenvalues $\lambda_1, \cdots, \lambda_s$. Let each λ_i have algebraic multiplicity m_i and geometric multiplicity μ_i. Then A is similar to a Jordan form matrix*

$$J = \begin{bmatrix} J_1 & O & \cdots & O \\ O & J_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & J_\mu \end{bmatrix},$$

where

1. $\mu = \mu_1 + \mu_2 + \cdots + \mu_s$,
2. *For each λ_i, the number of Jordan blocks in J with value λ_i is equal to μ_i,*
3. *λ_i appears on the diagonal of J exactly m_i times.*

Further, the matrix J is unique, up to re-ordering the Jordan blocks on the diagonal.

This is a pretty complicated theorem, and we aren't going to try to prove it here. But we will learn a method for finding the Jordan form of a matrix A, and also finding the nonsingular matrix Q such that $J = Q^{-1}AQ$.

Algorithm for the Jordan Form of A

1. Compute the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$, along with the associated algebraic multiplicities $m_1, m_2, \ldots, m_s$ and geometric multiplicities $\mu_1, \mu_2, \ldots, \mu_s$.

2. Treat each eigenvalue in turn. For a given eigenvalue λ of algebraic multiplicity m and geometric multiplicity μ, we start computing the **E-spaces** and their dimensions. The k-th E-space is

$$E_\lambda^k = \left\{ \mathbf{x} : (A - \lambda I)^k \mathbf{x} = \mathbf{0} \right\}$$

   So E_λ^1 is just E_λ, and we build from there. We stop when we get to an E_λ^k that has dimension m, the algebraic multiplicity of λ.

3. We make a diagram of boxes as follows. Compute the numbers

$$\begin{aligned} d_1 &= \dim E_\lambda^1, \\ d_2 &= \dim E_\lambda^2 - \dim E_\lambda^1, \\ \vdots \quad & \quad \vdots \\ d_k &= \dim E_\lambda^k - \dim E_\lambda^{k-1}. \end{aligned}$$

   Now make a diagram with d_1 boxes in the first row, d_2 boxes in the second row, and so on. For example, if $d_1 = 4, d_2 = 2, d_3 = 2, d_4 = 1$, then we get a diagram

$$\begin{array}{l} \square\square\square\square \\ \square\square \\ \square\square \\ \square \end{array}$$

   We are going to 'fill in' the diagram with vectors as follows.

4. Start at the bottom of the diagram, and fill the boxes in row k with linearly independent vectors that belong to E_λ^k **but not** E_λ^{k-1}. Anytime you have a vector v in a box, the box immediately above it gets filled with the vector $(A - \lambda I)v$. If a box is the lowest in its column, and belongs to row i, fill that box with a new vector from E_λ^i, which is linearly independent to both E_λ^{i-1} and all the other vectors in row i.

5. Repeat steps 2 through 4 for each distinct eigenvalue. You will get a diagram full of vectors for each one.

6. Make a matrix Q as follows. For each eigenvalue, consider the associated diagram. The vectors in the boxes become the columns of Q as follows. Start at the top of the leftmost column, and use the vectors as you go down the column. When you reach the end of a column, go to the next column. When you finish one diagram, go to first column of the next diagram. This gives the matrix Q.

7. The Jordan form of A is given by $J = Q^{-1}AQ$. But the nice part of the algorithm is that you can compute J **without finding** Q! In fact J will have one Jordan block for each column of each diagram. The value of the block is given by the eigenvalue, and the size of the block is equal to the number of squares in the column. You put the blocks down the diagonal of J in the same order you chose the vectors in Q.

Examples

1. $A = \begin{bmatrix} 2 & -3 \\ 3 & -4 \end{bmatrix}$. For this matrix, the characteristic polynomial is $(1+\lambda)^2$, so there is one eigenvalue, $\lambda = -1$, with $m = 2$. Now, we compute E-spaces:

   E^1_{-1}: Solving $(A+I)\mathbf{x} = \mathbf{0}$ gives:

   $$\left[\begin{array}{cc|c} 3 & -3 & 0 \\ 3 & -3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

   $$E^1_{-1} = \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

   So $d_1 = \mu = 1$. Since $\dim E^1_{-1} < m$, we have to compute another E-space.

   E^2_{-1}: Solving $(A+I)^2\mathbf{x} = \mathbf{0}$:

   $$(A+I)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

   so $E^2_{-1} = \text{span}\,(\mathbf{e}_1, \mathbf{e}_2)$, and $d_2 = 2 - 1 = 1$. Since $\dim E^2_{-1} = m$, we don't need any more E-spaces. Since we have $d_1 = 1, d_2 = 1$, our diagram looks like:

   $$\begin{array}{c} \square \\ \square \end{array}$$

   We put a vector in the lower box. It has to be a vector in E^2_{-1} that is linearly independent to E^1_{-1}. That's easy enough – how about $v_1 = [1 \quad 0]^T$. Above v_1, we have to put $(A+I)v_1$, which is

   $$v_2 = \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

   Hence, our diagram is

   $$\begin{array}{c} \boxed{v_2} \\ \boxed{v_1} \end{array},$$

   So

   $$Q = [v_2 \; v_1] = \begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix}.$$

   Finally, without computing $Q^{-1}AQ$, we still know what J looks like. There is only one column, so J is just one Jordan block, of size 2, with value $\lambda = -1$:

   $$J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

2. $A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & 2 \end{bmatrix}$.

We skip the computation to show that A has only one eigenvalue, $\lambda = 2$, of multiplicity 3. Computing E_2^1:

$$\left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So E_2^1 is spanned by the vector $[0 \quad 0 \quad 1]^T$. So $d_1 = 1$. Turning to E_2^2, we solve $(A - 2I)^2\mathbf{x} = \mathbf{0}$:

$$\left[\begin{array}{rrr|r} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$$

So E_2^2 is spanned by $[-1 \quad -1 \quad 0]^T$ and $[0 \quad 0 \quad 1]^T$. So $d_2 = 2 - 1$. We need to compute the E-space. But computation shows that $(A - 2I) = O$, the zero matrix, so E_2^3 is spanned by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. So $d_3 = 1$ and we can stop here. Our diagram is one column of three boxes.

The bottom box gets filled with a vector from E_2^3 that is linearly independent of E_2^2. The vector $v_1 = \mathbf{e}_1$ will work. Above there goes $v_2 = (A - 2I)v_1 = [1 \quad -1 \quad 3]^T$, and above that goes $v_3 = (A - 2I)v_2 = [0 \quad 0 \quad 1]^T$. Since our diagram now looks like

$$\begin{array}{c} \boxed{v_3} \\ \boxed{v_2} \\ \boxed{v_1} \end{array},$$

we get the transition matrix

$$Q = [v_3 \quad v_2 \quad v_1] = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 3 & 0 \end{bmatrix}.$$

Again, there is only one column, so only one Jordan block, which has value 2 and size 3. We get the Jordan form matrix

$$J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

3. $A = \begin{bmatrix} 2 & 4 & -8 \\ 0 & 0 & 4 \\ 0 & -1 & 4 \end{bmatrix}$.

You can check that this matrix also has only the eigenvalue 2, with multiplicity 3. We compute the E-spaces. First, for E_2^1,

$$[A - 2I|O] \to \left[\begin{array}{ccc|c} 0 & 4 & -8 & 0 \\ 0 & -2 & -4 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So E_2^1 is spanned by the two vectors $[1 \quad 0 \quad 0]^T$ and $[0 \quad 2 \quad 1]^T$. Also, $d_2 = 2 < 3$, so we need another E-space. Computing E_2^2, we see that $(A - 2I)^2 = O$, so E_2^2 is spanned by the standard basis, and $d_2 = 3 - 2 = 1$. We can stop, since E_2^2 has dimension 3.

Our diagram looks like

$$\begin{array}{ll} \boxed{v_2} & \boxed{v_3} \\ \boxed{v_1} & \end{array}$$

where v_1 is a vector in E_2^2 linearly independent of E_2^1. We get $v_2 = (A - 2I)v_1$, and we finally choose $v_3 \in E_2^1$ linearly independent of v_2. If we start by choosing $v_1 = \mathbf{e}_2$, we wind up getting

$$Q = [v_2 \quad v_1 \quad v_3] = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Finally, the diagram tells us that we get 2 Jordan blocks this time. Both have value 2, but one is of size 2 and one is of size 1. So

$$J = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 2 \end{array}\right].$$

We drew the lines just to illustrate the blocks. You can check in this example, and in all of the previous ones, that indeed $J = Q^{-1}AQ$.