Homework 9

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1.i

True.

$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$

$$\|\vec{u}\|^2 + \|\vec{v}\|^2 = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$

Therefore,

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 \rightarrow 2\vec{u}\cdot\vec{v} = 0$$

Thus, $\vec{u}$ and $\vec{v}$ are orthogonal if $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$.

1.ii

False. Counterexample: Let $\vec{u} = \begin{vmatrix}1\\0\end{vmatrix}$, and let $\vec{v} = \begin{vmatrix}0\\2\end{vmatrix}$.

$$\vec{u}+\vec{v} = \begin{vmatrix}1\\2\end{vmatrix}$$

$$\vec{u}-\vec{v} = \begin{vmatrix}1\\-2\end{vmatrix}$$

$$\|\vec{u}+\vec{v}\| = \|\vec{u}-\vec{v}\| = \sqrt{5}$$

$$\|\vec{u}\| = 1 \neq \|\vec{v}\|$$

1.iii

True. Any $\vec{w} \in W^{\perp}$ must be in $V^{\perp}$, because any $\vec{w} \in W^{\perp}$ is orthogonal to all vectors in W, including all vectors in V.

1.iv

True. Any $\vec{w} \in W$ must be in V, because any $\vec{w} \in W$ is orthogonal to all vectors in $W^\perp$, which includes all vectors in $V^\perp$, which means that all $\vec{w} \in W$ are in $(V^\perp)^\perp$, which is just V.

1.v

False. Counterexample: Let V be the span of $\begin{vmatrix} 1 \\ 0 \end{vmatrix}$, and let W be the span of $\begin{vmatrix} 0 \\ 1 \end{vmatrix}$, with V and W in $\mathbf{R}^2$. Then $V^\perp$ is the span of $\begin{vmatrix} 0 \\ 1 \end{vmatrix}$, and $W^\perp$ is the span of $\begin{vmatrix} 1 \\ 0 \end{vmatrix}$. Thus, $V^\perp \cap W^\perp = \{\vec{0}\}$. So $(V^\perp \cap W^\perp)^\perp$ is all of $\mathbf{R}^2$. All of $\mathbf{R}^2$ is not contained in $V \cup W$, so the statement is false.

2.i

$\vec{v_1} = \begin{vmatrix} 1 \\ 2 \\ 0 \\ 2 \end{vmatrix}$, so $\|\vec{v_1}\|$ is 3. Thus, $\vec{u_1} = \begin{vmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{vmatrix}$

$\vec{v_2}^\perp = \vec{v_2} - (\vec{v_2} \cdot \vec{u_1})\vec{u_1}$, so:

$$\begin{aligned} \vec{v_2}^\perp &= \begin{vmatrix} 2 \\ 2 \\ -1 \\ 6 \end{vmatrix} - \left(\begin{vmatrix} 2 \\ 2 \\ -1 \\ 6 \end{vmatrix} \cdot \begin{vmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{vmatrix} \right) \begin{vmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{vmatrix} \\ &= \begin{vmatrix} 0 \\ -2 \\ -1 \\ 2 \end{vmatrix} \end{aligned}$$

$$\vec{u_2} = \vec{v_2}^\perp / \|\vec{v_2}^\perp\| = \vec{v_2}^\perp/3 = \begin{vmatrix} 0 \\ -2/3 \\ -1/3 \\ 2/3 \end{vmatrix}$$

$$
\begin{aligned}
\vec{v_3}^{\perp} &= \vec{v_3} - (\vec{v_3} \cdot \vec{u_1})\vec{u_1} - (\vec{v_3} \cdot \vec{u_2})\vec{u_2} \\
&= \begin{vmatrix} 5 \\ 3 \\ 1 \\ -1 \end{vmatrix} - \left(\begin{vmatrix} 5 \\ 3 \\ 1 \\ -1 \end{vmatrix} \cdot \begin{vmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{vmatrix} \right) \begin{vmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{vmatrix} - \left(\begin{vmatrix} 5 \\ 3 \\ 1 \\ -1 \end{vmatrix} \cdot \begin{vmatrix} 0 \\ -2/3 \\ -1/3 \\ 2/3 \end{vmatrix} \right) \begin{vmatrix} 0 \\ -2/3 \\ -1/3 \\ 2/3 \end{vmatrix} \\
&= \begin{vmatrix} 4 \\ -1 \\ 0 \\ -1 \end{vmatrix}
\end{aligned}
$$

$$
\vec{u_3} = \vec{v_3}^{\perp} / \|\vec{v_3}^{\perp}\| = \vec{v_3}^{\perp} / \sqrt{18} = \begin{vmatrix} 2\sqrt{2}/3 \\ -\sqrt{2}/6 \\ 0 \\ -\sqrt{2}/6 \end{vmatrix}
$$

2.ii

Q is $\begin{bmatrix} \vec{u_1} & \vec{u_2} & \vec{u_3} \end{bmatrix}$, or $\begin{bmatrix} 1/3 & 0 & 2\sqrt{2}/3 \\ 2/3 & -2/3 & -\sqrt{2}/6 \\ 0 & -1/3 & 0 \\ 2/3 & 2/3 & -\sqrt{2}/6 \end{bmatrix}$.

R is $\begin{bmatrix} \|\vec{v_1}\| & \vec{u_1} \cdot \vec{v_2} & \vec{u_1} \cdot \vec{v_3} \\ 0 & \|\vec{v_2}^{\perp}\| & \vec{u_2} \cdot \vec{v_3} \\ 0 & 0 & \|\vec{v_3}^{\perp}\| \end{bmatrix}$.

$\|\vec{v_1}\| = 3$. $\|\vec{v_2}^{\perp}\| = 3$. $\|\vec{v_3}^{\perp}\| = \sqrt{18}$.

$$
\vec{u_1} \cdot \vec{v_2} = \begin{vmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{vmatrix} \cdot \begin{vmatrix} 2 \\ 2 \\ -1 \\ 6 \end{vmatrix} = 6
$$

$$
\vec{u_1} \cdot \vec{v_3} = \begin{vmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{vmatrix} \cdot \begin{vmatrix} 5 \\ 3 \\ 1 \\ -1 \end{vmatrix} = 3
$$

$$
\vec{u_2} \cdot \vec{v_3} = \begin{vmatrix} 0 \\ -2/3 \\ -1/3 \\ 2/3 \end{vmatrix} \cdot \begin{vmatrix} 5 \\ 3 \\ 1 \\ -1 \end{vmatrix} = -3
$$

$$R = \begin{bmatrix} 3 & 6 & 3 \\ 0 & 3 & -3 \\ 0 & 0 & 3\sqrt{2} \end{bmatrix}$$

2.**iii**

$$proj_V(\vec{x}) = (\vec{u_1} \cdot \vec{x})\vec{u_1} + (\vec{u_2} \cdot \vec{x})\vec{u_2} + (\vec{u_3} \cdot \vec{x})\vec{u_3}$$
$$A = \begin{bmatrix} proj_V(\vec{e_1}) & proj_V(\vec{e_2}) & proj_V(\vec{e_3}) & proj_V(\vec{e_4}) \end{bmatrix}$$
$$proj_V(\vec{e_1}) = 1/3\vec{u_1} + 2\sqrt{2}/3\vec{u_3} = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$
$$proj_V(\vec{e_2}) = 2/3\vec{u_1} - 2/3\vec{u_2} - \sqrt{2}/6\vec{u_3} = \begin{vmatrix} 0 \\ 17/18 \\ 2/9 \\ 1/18 \end{vmatrix}$$
$$proj_V(\vec{e_3}) = -1/3\vec{u_2} = \begin{vmatrix} 0 \\ 2/9 \\ 1/9 \\ -2/9 \end{vmatrix}$$
$$proj_V(\vec{e_4}) = 2/3\vec{u_1} + 2/3\vec{u_2} - \sqrt{2}/6\vec{u_3} = \begin{vmatrix} 0 \\ 1/18 \\ -2/9 \\ 17/18 \end{vmatrix}$$
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 17/18 & 2/9 & 1/18 \\ 0 & 2/9 & 1/9 & -2/9 \\ 0 & 1/18 & -2/9 & 17/18 \end{bmatrix}$$

3.**i**

$T(\vec{x} - T(\vec{x})) = T(\vec{x}) - T(T(\vec{x}))$, because T is a linear transformation. It is given that $T(T(\vec{x})) = T(\vec{x})$. So $T(\vec{x} - T(\vec{x})) = T(\vec{x}) - T(\vec{x}) = \vec{0}$. Therefore, $(\vec{x} - T(\vec{x})) \in Ker(T)$.

$$\begin{aligned} \vec{x} &= \vec{x} + T(\vec{x}) - T(\vec{x}) \\ &= T(\vec{x}) + (\vec{x} - T(\vec{x})) \end{aligned}$$

But $T(\vec{x}) \in Im(T)$, and $(\vec{x} - T(\vec{x})) \in Ker(T)$. So any $\vec{x}$ can be written as the sum of some $\vec{y} \in Ker(T)$, and some $\vec{z} \in Im(T)$.

3.ii

$$\begin{aligned} f(t) &= \|t\vec{y} + \vec{z}\|^2 \\ &= (t\vec{y} + \vec{z}) \cdot (t\vec{y} + \vec{z}) \\ &= t^2(\vec{y} \cdot \vec{y}) + 2t(\vec{y} \cdot \vec{z}) + \vec{z} \cdot \vec{z} \\ &= t^2\|\vec{y}\|^2 + 2t(\vec{y} \cdot \vec{z}) + \|\vec{z}\|^2 \end{aligned}$$

3.iii

Because T is a linear transformation and $\vec{y}$ is in the kernel of T:

$$T(t\vec{y} + \vec{z}) = tT(\vec{y}) + T(\vec{z}) = \vec{0} + T(\vec{z}) = T(\vec{z})$$

Because $\|T(\vec{x})\| \leq \|\vec{x}\|$ for all $\vec{x}$, $\|T(\vec{z})\| \leq \|t\vec{y} + \vec{z}\|$. Therefore, since sizes are non-negative, $\|T(\vec{z})\|^2 \leq \|t\vec{y} + \vec{z}\|^2$. Because $\vec{z} = T(\vec{x})$ for some $\vec{x}$, $T(\vec{z}) = T(T(\vec{x})) = T(\vec{x}) = \vec{z}$. So $T(\vec{z}) = \vec{z}$. Therefore, $\|\vec{z}\|^2 \leq \|t\vec{y} + \vec{z}\|^2$.

But $f(t) - f(0) = \|t\vec{y} + \vec{z}\|^2 - \|\vec{z}\|^2$, since $f(0) = \|\vec{z}\|^2$. So $f(t) - f(0)$ is always non-negative, meaning that $f(0)$ is a global minimum.

3.iv

$f(t)$ is a quadratic function, because $f(t) = at^2 + bt + c$, where $a = \|\vec{y}\|^2$, $b = 2(\vec{y} \cdot \vec{z})$, and $c = \|\vec{z}\|^2$. So because there is a global minimum at 0, $f'(0) = 0$. $f'(t) = 2\|\vec{y}\|^2 t + 2(\vec{y} \cdot \vec{z})$, so $f'(0) = 2(\vec{y} \cdot \vec{z})$. This means that $\vec{y} \cdot \vec{z} = 0$, so $\vec{y}$ and $\vec{z}$ are orthogonal.

3.v

An orthogonal projection onto vector subspace V is a function $L(\vec{x})$ such that $L(\vec{x}) = \vec{x}^{\parallel}$, where $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$, $\vec{x}^{\perp}$ is orthogonal to all $\vec{v} \in V$, and $\vec{x}^{\parallel} \in V$, for any $\vec{x}$ in the domain.

We have shown that any $\vec{y} \in Ker(T)$ is orthogonal to any $\vec{z} \in Im(T)$, and that $\vec{x} - T(\vec{x}) \in Ker(T)$. Therefore, $\vec{x} - T(\vec{x})$ is orthogonal to any

$\vec{v} \in Im(T)$. We know that any $\vec{x} = (\vec{x} - T(\vec{x})) + T(\vec{x})$. $T(\vec{x}) \in Im(T)$ by definition of the image. We know that the image of T is a linear subspace, because it is the image of a linear transformation.

Thus, T is the orthogonal projection onto $Im(T)$, where $T(\vec{x}) = \vec{x}^{\parallel}$ and $\vec{x} - T(\vec{x}) = \vec{x}^{\perp}$.