CS 608 Algorithms and Computing Theory

Chapter 1: Mathematical Background

In this chapter we will review certain mathematical concepts, which we need in other chapters.

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1. Exponents

Consider an algebraic expression of the form a^n. If n is a positive integer, a^n represents the product,
a * a * a* ... n times.
In a^n, a is called the base and n the exponent.

Example:

2^4 is 2 * 2 * 2 * 2.

Exponent laws:

- $\mathbf{a^m * a^n = a^{m+n}}$

Examples:

$2^5 * 2^7 = 2^{12}$

$3^5 * 3^6 = 3^{11}$

$5^3 * 5^6 * 5^9 = 5^{18}$

- $\mathbf{a^m/a^n = a^{m-n}}$

Examples:

$2^7/2^5 = 2^2$

$5^{10}/5^6 = 5^4$

- **$(a^m)^n = a^{mn}$**

Examples:

$(3^4)^2 = 3^8$

$(4^{17})^3 = 4^{51}$

- **$(ab)^m = a^m * b^m$**

Examples:

$(2a)^3 = 2^3 * a^3$

$(3xy)^5 = 3^5 * x^5 * y^5$

- **$(a/b)^m = a^m / b^m$**

Examples:

$(2/z)^3 = 2^3 / z^3$

$(ab/z)^2 = a^2 * b^2 / z^2$

- **Meaning of $a^{1/n}$**

$a^{1/n} = \sqrt[n]{a}$

More generally,

$a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$

Examples:

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

$81^{½} = \sqrt[2]{81} = \sqrt{81} = 9$ (By default $\sqrt[2]{a} = \sqrt{a}$)

$81^{1/4} = \sqrt[4]{81} = 3$

$1296^{¼} = \sqrt[4]{1296} = 6$

$64^{1/3} = \sqrt[3]{64} = 4$

$1024^{1/5} = \sqrt[5]{1024} = 4$

- **Meaning of a^{-m}**

By definition, $a^{-m} = 1/a^m$

Examples:

$2^{-3} = 1/2^3 = 1/8$

$4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{\sqrt{4}} = \frac{1}{2} = 0.5$

$\frac{1}{3^{-1}} = 3$

- **Meaning of a^0**

By definition, $\mathbf{a^0 = 1.}$

Example:

$2^0 = 1$

Exercises on exponents

1. If $a^x = b^y = c^z$ and $b^2 = ac$, prove that $\frac{2}{y} = \frac{1}{x} + \frac{1}{z}$
2. If $x = 2^{\frac{1}{3}} + 2^{\frac{2}{3}}$, show that $x^3 - 6x - 6 = 0$
3. If $x = 3^{\frac{1}{3}} - 3^{\frac{-1}{3}}$, prove that $3x^3 + 9x - 8 = 0$
4. If a = 2x, b = 4y, and c = 8z, and x + 2y + 3z =1, prove that abc=2

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2. The sigma notation

In mathematics the Greek symbol, called sigma, $\sum$ is used to compactly and precisely express a sum of a sequence of things that are all to be added together.

Typical use:

1. Use an expression with a variable (index) within $\sum$
2. Indicate the first value for the index at the bottom on $\sum$
3. Indicate the last value for the index at the top of $\sum$
4. To write the sum in the expanded form, write the sum of the expressions for the index values starting from the start value to last value.

Example:

Consider the sigma notation,

$$\sum_{i=1}^{5} 2i$$

Here the expression is 2i, index variable is i, first value of i is 1, and last value for the index is 5.

To expand this notation, write the expression 2i, replacing i by 1, 2, 3, 4, and 5 and sum them:

2 + 4 + 6 + 8 + 10.

Sometimes, we cannot write all the terms in the summation. In such a case, we use .....

Examples:

$$\sum_{i=1}^{100} 2i \quad = \quad 2 + 4 + 6 + ... + 200$$

$$\sum_{i=1}^{n} c = c + c + c + \ldots + c \text{ (n times)}$$

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3$$

$$\sum_{4}^{7} 4x^2 = 4(4)^2 + 4(5)^2 + 4(6)^2 + 4(7)^2$$

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3. Arithmetic series

An arithmetic series is the sum of a sequence of numbers such that it starts with a number and the difference between any two terms is a fixed number.

The starting number is called the *first term* and the fixed difference between any two terms is the *common difference*.

Example:

$\mathbf{2 + 5 + 8 + 11} \ldots$

This is an arithmetic series with the first term 2 and the common difference is 3.

Example:

20 + 18 + 16 + 14 + ...

This is an arithmetic series with the first term 20 and the common difference is -2.

General arithmetic series:

Consider a general arithmetic series: First term = a_1, common difference = d and number of terms is n:

$a_1 + (a_1+d) + (a_1+2d) + (a_1+3d) + \ldots.$

The nth term of this arithmetic series, a_n is given by $a_1 + (n-1)\,d$.

This series can be written as follows using the sigma notation:

$$\sum_{i=1}^{n} a_1 + (n-1)d$$

Formula for the sum of the arithmetic series with n terms:

$$\sum_{i=1}^{n} a_1 + (n-1)d = a_1 + (a_1+d) + (a_1+2d) + (a_1+3d) + ... + a_n = \mathbf{n(a_1 + a_n)/2}$$

Example:

Sum the series: 2 + 5 + 8 + 11 + ... 20 terms.

The 20th term of the series is 2 + (20-1) 3 = 59.

The sum of the series = 20(2+59)/2 = 610.

Sum the series: 20 + 18 + 16 + ... 100 terms

1ooth term = 20 + (99)(-2) = -178

Sum of the series = 50(20 – 178) = -7900

Example:

This is a special case (remember this):

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

Example:

1 + 2 + 3 + ... + 100 = 100(101)/2 = 5050.

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4. Geometric series

A geometric series is the sum of a sequence of numbers such that it starts with a number and the ratio of each two consecutive terms is a fixed number.

The starting number is called the *first term* and the fixed ratio of each two consecutive terms is *common ratio*.

Example:

$\mathbf{2 + 6 + 18 + 54}$ ...

This is a geometric series with the first term 2 and the common ration is 3.

Example:

3 + 6 + 12 + 36 + ...
This is a geometric series with the first term 3 and the common ratio is 2.

General geometric series:

Consider a general geometric series: First term = a_1, common ratio = r and number of terms is n.
$a_1 + (a_1 r) + (a_1 r^2) + (a_1 r^3) + \ldots.$
The nth term of this geometric series, a_n is given by $a_1 r^{n-1}$.
Formula for the sum of the geometric series with n terms:

$$\sum_{i=1}^{n} a_i r^{i-1}$$

$$= a_1 + (a_1 r) + (a_1 r^2) + (a_1 r^3) + \ldots. + (a_1 r^{n-1})$$

$$= a_1 (\frac{1-r^n}{1-r})$$

If the number of terms is infinite, which is written symbolically, $n \to \infty$, then $r^n \to 0$ if $r < 1$.
Thus,

$$\sum_{i=1}^{\infty} a_i r^{i-1} = \frac{a_1}{1-r} \; if \; r < 1.$$

Example:

Sum the geometric series to 10 terms:

$$\sum_{\mathbf{i=1}}^{\mathbf{10}} \mathbf{2 * 3^{i-1} = 2 + 6 + 18 + 54} \ldots$$

Let us use the formula.
$a_1 = 2$, $r = 3$ and $3^{10} = 59049$
Sum = $2(1-3^{10})/ (1-3) = 2(-59048)/(-2) = 59048$,

Example:

Sum the geometric series to 12 terms:

$$\sum_{i=1}^{\mathbf{12}} \mathbf{3 * 2^{i-1} = 3 + 6 + 12} + \cdots$$

Let us use the formula.

$a_1 = 3$, $r = 2$ and $2^{12} = 4096$

Sum = $3(1-2^{10})/ (1-2) = 3(-4095)(-1) = 12285$.

Example:

Sum the geometric series to 10 terms:

$$\sum_{i=1}^{10} 3 * \left(\frac{1}{2}\right)^{i-1} = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \cdots$$

Let us use the formula.

$a_1 = 3$, $r = 1/2$ and $(1/2)^{10} = 0.0009765625$

Sum = $3(1-(1/2)^{10})/ (1-(1/2)) = 2.9970703125/0.5 = 5.994140625$.

Example:

Sum the geometric series to infinite terms:

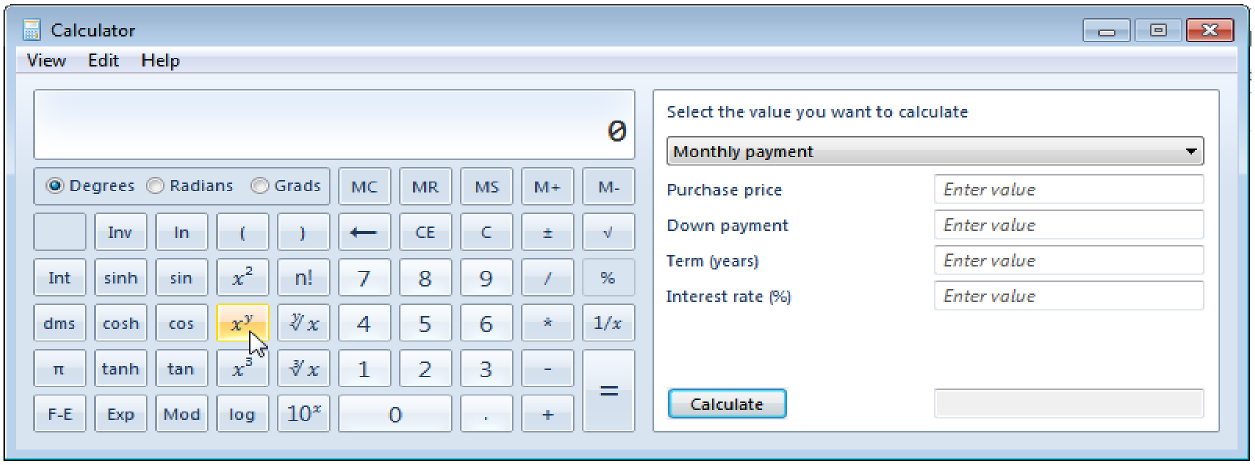
$$\sum_{i=1}^{\infty} 3 * \left(\frac{1}{2}\right)^{i-1} = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \cdots$$

Here $a_1 = 3$ and $r = \frac{1}{2}$.

$$\sum_{i=1}^{\infty} 3 * \left(\frac{1}{2}\right)^{i-1} = \frac{3}{1 - \frac{1}{2}} = 6$$

Note:

Use Windows calculator (Scientific View) to compute exponent values (such as 3^{10}):



Use x^y button.

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5. Harmonic series

The series,

$$\sum_{i=1}^{10} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{9} + \frac{1}{10}$$

is called a harmonic series. This hormanic series has 10 terms.

Example:

This hormanic series has 5 terms.

$$\sum_{i=1}^{5} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

Example:

The harmonic series with infinite number of terms is of special interest in mathematics:

$$\sum_{i=1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

Exercises on series

Find the sum of the following series:

1 $3 + 5 + 7 + \ldots$ 20 terms

2 $1 + \frac{3}{2} + \frac{4}{2} + \frac{5}{2} + \cdots 20\ terms$

3 $$\sum_{k=0}^{10} \frac{1}{3^k}$$

4 $$\sum_{k=0}^{\infty} \frac{1}{3^k}$$

5 $$\mathbf{1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} +}\ \boldsymbol{to\ 10\ terms}$$

6 $$\mathbf{1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} +}\ \boldsymbol{to\ \infty}$$

7 1 + 0.1 + 0.01 + 0.001 + …. + to 20 terms

8 1 + 0.1 + 0.01 + 0.001 + …. + to ∞

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6. Convergence of a series

Convergence and divergence of series a very important topic in mathematics. We are not going to discuss convergence and divergence in strict mathematical sense.
The following is purely informal – NOT a formal mathematical discussion.

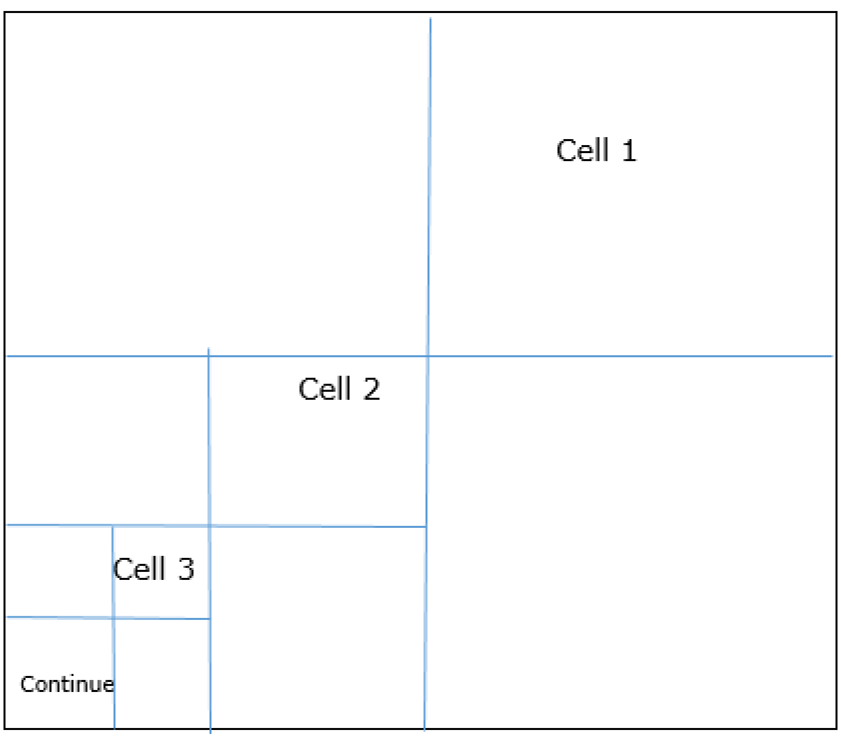
If you have a series and compute the sum of the series for an infinite number of terms, does the sum gets closer to a finite number? In other words, if you go on increasing the number of terms of the series, does the sum also goes on increasing without a bound or does the sum stays closer to a finite number.

Some series behave this way: When you go on increasing the number of terms of the series, the sum does NOT go on increasing without bounds. In other words, no matter how many terms (however large) you take, the sum will be always less than a finite number. Such series is called a convergent series.

Some other series behave this way: When you go on increasing the number of terms of the series, the sum goes on increasing without bounds. Such series is called a divergent series.
In mathematics there are plenty of theorems to determine if a given series is convergent or divergent.

Here is an example of a convergent series (just for fun):
Let me give you, as an example, a figure to illustrate a convergent series.
Consider a square of size 2x2: Width 2 and height 2.
Area of this square is 4.
Divide the square into four equal parts as shown.

The area of top right quarter (cell 1) is 1*1 = 1
Divide the lower left quarter square into four parts as before. The area of Cell 2 is $\frac{1}{2^2}$
Repeat this process. The area of Cell 3 is $\frac{1}{4^2}$
Continue this process ....



The sum of the areas of these marked cells is,

$1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + ...$

Notice that the sum of these areas is always less than 4.

So the sum of the series,

$1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + ...$

is always less than 4 (in fact, less than 2) no matter how many terms are added?

This series is, therefore, convergent.

Let us not discuss theorems to establish a given series is convergent or divergent. Let us write programs to see the behavior of the series.

This is what we do:

Using a loop, calculate the sum of a given series for the first 1000, 2000, 3000, 4000, 5000, 6000, 7000, 8000, 9000 and 10,000 terms.

As we discussed earlier, if the sum values do not increase significantly, it is likely the series is convergent. Incorporate this idea in your program: If the difference between successive sum values is less than, say .0001, then conclude, "The given series is likely convergent". Otherwise divergent.

Write programs to conclude if each is likely convergent or divergent. In case of convergent series, write the approximate sum value.

	Series	Special Comments
1	$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$	This is infinite Harmonic series
2	$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$	In your program, multiply the sum by 6 and then take the square root and print the result. What is this number?
3	$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + ...$	--
4	$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + ...$	Notice this is a geometric series, with $r = \frac{1}{2}$. In your program print the sum and also the value from the geometric series.
5	$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + ...$	This is a classic series. In your program multiply the sum by 4 and print the answer. What is this?
6	$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$	This sum will be ln2 (we discuss log later in this chapter). In your program print both the sum and the value of ln(2).

7	$\sum_{n=0}^{\infty} \frac{4}{3^{n+1}}$	This sum will be a nice small number.
8	$\sum_{k=1}^{\infty} \frac{2}{k(k+1)}$	This sum will be a nice small number.

7. Some special series

The following are some of the interesting series and their sums. We will provide the results without proof. Proving some of these is trivial.

1. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

2. $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$

   $= [\frac{n(n+1)}{2}]^2 = [\sum_{k=1}^{n} k]^2$

8. Logarithms

Definition:

Consider the equation:

$a^n = x$

Then n is called the logarithm of x to the base a, and written as follows:

$n = \log_a(x)$

Notice that the base in the logarithmic notation is written like a subscript to log.

Please note, $a^n = x$ and $n = \log_a(x)$.

$n = \log_a(x)$ is another way writing $a^n = x$.

$n = \log_a(x)$ is called the logarithmic form of the expression $a^n = x$. This expression $a^n = x$ itself is called the exponent form.

Examples:

Logarithmic form	Exponent form
$2^3 = 8$	$3 = \log_2(8)$

$4^2 = 16$	$2 = \log_4 16$
$p= m^n$	$n = \log_m p$
$4^{0.5} = 2$	$0.5 = \log_4 2$
$4^{-1} = 1/4$	$\text{Log}_4(1/4) = -1$
$a^0 = 1$	$\text{Log}_a 1 = 0$

Standard bases

Three commonly used bases are: 10, 2 and e.

- Base 10 is commonly used in engineering applications (normally written log. That is, when base is 10, the base is not written in the logarithmic expression).
- Base 2 is used computing applications (normally denoted lg).
- Base **e** is used in theoretical mathematics (normally denoted ln). The notation **e** represents the irrational number 2.718… This is the second commonly used irrational number in mathematics. The first of course, is the famous pi: π (which is an irrational number 3.14159…..).

Examples:

log100 means $\log_{10}100$
lg100 means $\log_2 100$
ln100 means $\log_e 100$

Logarithm rules

Let us just state the log rules without proofs. Proving them is easy.

- **Rule 1: $\log_a(n)$ is valid only when n>0 and a>0.**

Important to know that log(n) for any base is undefined for $n \leq 0$.

Examples:

$\log_3(-4)$ is undefined
$\log_{-4}(3)$ is undefined
log(0) is undefined

- **Rule 2: $\log_a(m*n) = \log_a m + \log_a n$**

Examples:

$\log_3(6p) = \log_3 6 + \log_3 p$

$\log_5(abc) = \log_5 a + \log_5 b + \log_5 c$

- **Rule 3: $\log_a(m/n) = \log_a m - \log_a n$**

Examples:

$\log_4(2/k) = \log_4 2 - \log_4 k$

$log_n\left(\frac{ab}{c}\right) = \log_n a + \log_n b - \log_n c$

- **Rule 4: $\log_a(m^n) = n\log_a m$**

Examples:

$\log_5 3^2 = 2\log_5 3$

$\log_5\sqrt{20} = \log_5(20)^{1/2} = \frac{1}{2}\log_5 20$

- **Rule 5: $\log_a a = 1$**

Read this: log of any number to the same base is 1.

Examples:

$\log 10 = 1$ (remember when the base is missing, it is 10).

$\ln(e) = 1$

$\lg(2) = 1$

$\log_2 16 = \log_2 2^4 = 4\log_2 2 = 4$

$\log 100 = \log 10^2 = 2\log 10 = 2$ (remember log10 is 1)

$\log_5 25 = \log_5 5^2 = 2\log_5 5 = 2$

$\log_3\sqrt{3} = (1/2)\log_3 3 = ½$

- **Rule 6: $\log_a 1 = 0$**

Log of 1 for any base is always zero.

Examples:

$\log_{100}1 = 0$

$\log_{2}1 = 0$

$\log 1 = 0$

$\lg 1 = 0$

$\ln 1 = 0$

➢ Rule 7: $\log_a m = \frac{\log_b m}{\log_b a}$

This is called change of base formula. This formula shows how to change base from a to b. For example you like to calculate $\log_2 25$, but the calculator you have (Windows calculator, for example) gives log values for only base 10. Then you use this formula:

Example:

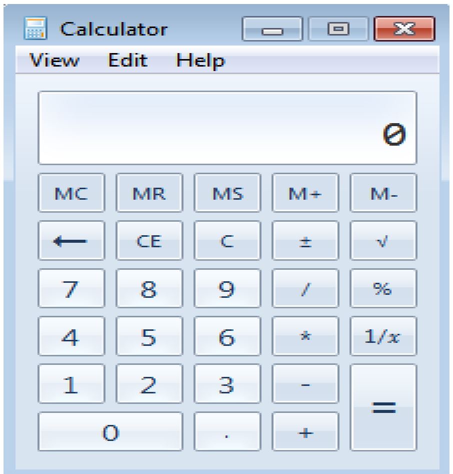
$\log_2 25 = \frac{\log_{10}25}{\log_{10}2}$

$= \frac{1.3979400086720376095725222105510}{0.30102999566398119521373889472449}$ (Using calculator)

$=4.6438561897747246957406388589788$

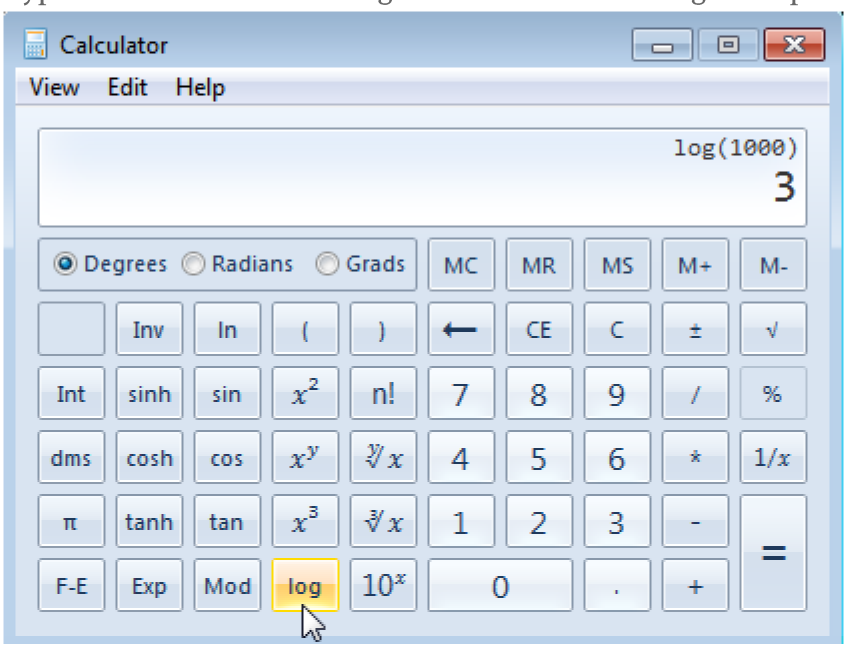
Using Windows calculator to compute log of a number to base 10:

When you start Windows calculator, you will see the stand view:



Click View and select scientific:

Type a number and click log button. The following example is log (1000):



Exercises on logarithms

1 Prove: $a^{\log_a x} = x$

2 Prove: $\log_a x = \frac{1}{\log_x a}$

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9. Introduction to recursion

Recursion is a mechanism in which a solution to a problem is defined in terms of itself. As an example, consider the problem of finding the sum of the first n positive integers. That is, we need to find the sum 1 + 2 + 3 + ... +n. A solution to the problem can be stated as follows:

Sum of first n positive integers = n + sum of the first (n-1) positive integers.

Notice that the solution to the first n positive integers is described in terms the solution to the first (n-1) positive integers.

This is a recursive solution to the given problem.

Let us denote 1 + 2 + 3 +...+ n by sigma(n).

A recursive solution to sigma(n) is,

sigma(n) = n + sigma(n-1).

Let us compute, as an example, sigma(10):

sigma(10) = 10 + sigma(9)

= 10 + 9 + sigma(8) = 19 + sigma(8)

= 19 + 8 + sigma(7) = 27 + sigma(7)

Continuing this way, we obtain

= 54 + sigma(1)

When we reach this stage, where n is 1, we don’t apply recursive definition. Instead we substitute the actual value of sigma(1), which is 1. Thus the value of sigma(10) is 55.

A recursive definition typically has two clauses: A basis clause and an induction clause. The basis clause provides the solution to a trivial value of n (such as n=1 or n=2), and the induction clause provides the formula for the value for n in terms of the value for n-1.

In the sigma(n) problem,

Basis clause: sigma(n) = 1, when n=1.

Induction clause: sigma(n) = n + sigma(n-1), when n > 1.

In defining a recursive solution to a problem, it is important to formulate the basis and induction clauses clearly and correctly. Once you come up with correct and clear basis and induction clauses for a recursive solution, writing a code to solve the problem recursively is straightforward. A recursive solution involves writing, what is called a recursive method.

➢ Recursive methods

A recursive method is a method which calls itself. A recursive method implements a recursive solution to a problem.

The code in a typical recursive method involves two parts:
Part 1: Involves code for implementing the basis clause. Typically looks like this:
if (n == "*condition from the basis clause*") **return** "*the value from the basis clause*".

Part 2: Code for implementing the induction clause. This is the "**else**" part.

Example:

Let us write a recursive method to implement the recursive definition discussed above for sigma(n).

Code for basis clause: If (n==1) return 1;
Code for induction clause: return (n + sigma(n-1));

```
Int sigma(int n){
  If (n==1) return 1;
  else return (n + sigma(n-1));
}
```

Now a complete program:

```
public class sigmaProblem {

    public static void main(String[] args) {
        System.out.println("Sigma(4): "+sigma(4));

    }
    static int sigma(int n){
        if (n==1) return 1;
        else return (n + sigma(n-1));
    }

}
```

How does it work?

To understand how this works, let us actually trace the execution of the program.
In the main, we have sigma(4). This is a call to the method sigma(n). Assume this: "every time a method is invoked, a copy of the method is made and the copy is executed with value passed in the argument."

Thus, the main calls sigma(n) and passes n=4.
A copy of the method is made and the body of the method is executed with the value n=4.

Now trace the program.

System.out.println("Sigma(4): "+sigma(4)); The call, sigma(4) invokes the method and passes n = 4:

n=4
if (n==1) return 1;
else return (n + sigma(n-1));
	n==1 is false.
	So executes return (n + sigma(3));
	sigma(3) calls the method again, with n=3.

n=3
if (n==1) return 1;
else return (n + sigma(n-1));
	n==1 is false.
	So executes return (n + sigma(2));
	sigma(2) calls the method again, with n=2.

n=2
if (n==1) return 1;
else return (n + sigma(n-1));
	n==1 is false.
	So executes return (n + sigma(1));
	sigma(1) calls the method again, with n=1.

n=1
if (n==1) return 1;
else return (n + sigma(n-1));
	n==1 is true.
	So executes **return 1;**
	This returns 1 to the calling segment.

Once the last copy of the method executes **if(n==1) return1**, 1 is sent back to the calling segment. We trace the reverse path (read from the bottom):

```
System.out.println("Sigma(4): "+sigma(4));
```

n=4
if (n==1) return 1;
else return (n + sigma(n-1));
	This copy of the method adds 4 and 6, and returns 10. This value 10 is sent to the main method and it is printed there.

n=3
if (n==1) return 1;
else return (n + sigma(n-1));
	This copy of the method adds 3 and 3, and returns 6.

n=2
if (n==1) return 1;
else return (n + sigma(n-1));
	This copy of the method adds 2 and 1, and returns 3.

n=1
if (n==1) return 1;
else return (n + sigma(n-1));
	This copy of the method returns 1.

---

10. Several simple recursive examples

To understand recursion, given a problem, it is important to **clearly and correctly** formulate basis clause and induction clause for a recursive solution to the problem. Once you have done this, writing the code is relatively easy. Again the hard part is formulating the basis and induction clauses.

To build "recursive thinking strategy", we will now take several simple examples. These examples are meant to make you think of recursive solutions. In fact, solutions to these simple examples are not naturally recursive. We can solve all these problems by simple loops. But, we forcefully apply recursive solutions only to learn recursive thinking.

Example 1 – printing integers n to 1.

Problem: print integers from a given positive integer n through 1 (recursively). Let us call the recursive method myPrint(n).

Basis clause: if (n==1) print n

Induction clause: if n>1, print n and then call myPrint(n-1).

Now the recursive method, myPrint(n):

```
static void myPrint(int n){
```

```
        if (n==1) System.out.println(1);
        else {
            System.out.println(n);
            myPrint(n-1);
        }
}
```

Write a main to test this method.

Note: test to see what happens when you switch the statements with in “else”:

myPrint(n-1);
System.out.println(n);

Why?

Example 2 – Factorial problem

Problem: Compute (and print) n! (recursively). Let us call the recursive method fact(n).

Basis clause: if(n==1) return 1

Induction clause: return n * fact(n-1)

Method fact(n):

```
static int fact(int n){
        if (n==1) return 1;
        else return n* fact(n-1);
}
```

Exercises on simple recursive problems

After you answer them, keep them ready when you take the quiz. Quiz will contain questions from here.

1 Print the nth Fibonacci number

Problem: Compute (and print) the nth Fibonacci number (recursively). Let us call the recursive method fib(n).

Write a basis clause and induction clause.

Write a recursive method fib(n) and a main method to illustrate the use of the method.

2 Count zeros in an array

Problem: Given an integer array, count the number of zeros in the array (recursively).
Call the recursive method zeroCount(n), which returns the number of zeros in the array from subscript 0 through (n-1).

Write a basis clause and induction clause.

Write a recursive method seroCount(n) and a main method to illustrate the use of the method.

---

3 Largest element in an array

Problem: Compute (and print) the largest element in a given integer array (recursively). Let us call the recursive method largest(n), which picks up the largest element in the array from subscript 0 through n.

Basis clause:
Induction clause:

Study this method and fill (???) the required code

```
int largst(int n){
        if (n==0)return ???;
        else {
            if(myArray[n]>=largst(n-1)) return ???;
            else return ???;
        }
}
```

---

4 Count the number of zeros in a given two dimensional array (recursively).

Let us apply recursion on rows. Call the method countRowZeros(i).

Write a basis clause and induction clause.

Write a recursive method countRowZeros(i), and a main method to illustrate the use of the method.

---

5 Compute the sum of the digits of a given integer (recursively).

Call the method sumDigits(n).

Write a basis clause and induction clause.

Write a recursive method sumDigits(n), and a main method to illustrate the use of the method.

Examples:

sumDigits(123) returns 6
sumDigits(42108) returns 15

---

6 Find the binary representation of a given integer (recursively).

Call the method binRepresentation(n).

Write a basis clause and induction clause.

Write a recursive method binRepresentation(n),and a main method to illustrate the use of the method.

Examples:

binRepresentation(56) = 111000
binRepresentation(245) = 11110101

Note: You can use Windows calculator to convert decimal numbers into binary (use Programmer View).

---

7 Given a character string, write a recursive method to check if it is a palindrome or not.

Write a basis clause and induction clause.

Write a recursive method palindrome(char[], size), and a main method to illustrate the use of the method.

---

8 Given two positive integers, n and m (assume m<n), write a recursive method to find the GCD (n,m).

The GCD of two numbers n and m is the largest number which divides both n and m evenly.

Examples: GCD(54,12) is 6. GCD(81,18) is 9.
It is important to realize that

(1) if m is zero, GCD(n,m) is n. (2) GCD(n,m) = GCD(m,n%m).

Write a basis clause and induction clause.

Write a recursive method GCD(n,m), and a main method to illustrate the use of the method.

9 Reversing digits in an integer output

Given a positive integer n, write a recursive method, reverse(n), to print the digits of n in reverse order. Example: input 16278, output: 87261.

Write a basis clause and induction clause.

Write a recursive method reverse(n), and a main method to illustrate the use of the method.

10 Write a recursive method for the following:
f(n) = n/2, if N is even.
f(n) = f(f(3n-1)), otherwise

What is the value of f(9)?

11 Write a recursive method for the following:

f(n) = n-10, if n>100
f(n) = f(f(n+11)), otherwise.

Find the values of f(98), f(97), f(96), f(12), and f(13).
What do you observe?

12 Write a recursive method for the following:

A(m,n) = n+1, if m=0
A(m,n) = A(m-1,1), n=0
A(m.n) = A(m-1,A(m,n-1)), otherwise.

What is the value of A(3,2)?

11. Recurrence relations

A recurrence relation is a mathematical representation of a recursive dentition. That is, a function is expressed in terms of the function itself. Recurrence relations are of fundamental importance in analysis of algorithms. One of the common methods in solving problems in computer science is the divide and conquer method in which the given problem is broken into subproblems and then subproblems are solved repeating the idea of further breaking up the subproblem. The analysis of such algorithms requires solving recurrence relations.

Examples:

- A recurrence relation, s(n), for the sum of the first n positive integers is:

s(n) = n + s(n-1)
s(1) = 1

- The following a recurrence relation for Fibonacci numbers

f(n) = f(n-1) + f(n-2)
f(1) = 1
f(2) = 1

Some of the famous recurrence relations:

Recurrence relations	Initial values	Solutions
$F_n = F_{n-1} + F_{n-2}$	$a_1 = a_2 = 1$	Fibonacci number
$F_n = F_{n-1} + F_{n-2}$	$a_1 = 1, a_2 = 3$	Lucas number
$F_n = F_{n-2} + F_{n-3}$	$a_1 = a_2 = a_3 = 1$	Padovan sequence
$F_n = 2F_{n-1} + F_{n-2}$	$a_1 = 0, a_2 = 1$	Pell number
$F_n = 2F_{n-1} + 1$	$F_1 = 1$	Towers of Hanoi Problem

Solving a recurrence relation:

Given a recurrence relation, finding a solution means expressing the function in a form without recursion. The solution is also called the closed form solution.

Example:

A recurrence relation, s(n), for the sum of the first n positive integers is,

$s(n) = n + s(n-1)$

$s(1) = 1$

We can derive the following well-known closed form solution for this recurrence:

$s(n) = \frac{n(n+1)}{2}$

(We will soon derive this closed form from the recurrence relation).

---

12. Simple examples of solving recurrence relations

Simple recurrence relations can be solved (that is closed form solution can be derived) using repeated substitutions until the basis clause condition can be applied.

Example:

Solve the recurrence relation:

$p(n) = (1+r)p(n-1), n>0$

$p(0) = 3$

Consider the recurrence equation:

$P(n)$	$= (1+r)\ p(n-1)$	Substitute $p(n-1) = (1+r)\ p(n-2)$
	$= (1+r)[(1+r)\ p(n-2)]$	Simplify
	$= (1+r)^2\ p(n-2)$	Substitute $p(n-2) = (1+r)\ p(n-3)$
	$= (1+r)^2\ [(1+r)\ p(n-3)]$	Simplify
	$= (1+r)^3\ p(n-3)$	

Observe the pattern.

The next step, if you substitute for p(n-3), you will obtain.

$p(n) = (1+r)^4\ p(n-4)$

If you continue, the general kth step will be,

$p(n) = (1+r)^k\ p(n-k)$

Follow the next step carefully.

Remember, recursive substitution stops when n=0. From he basis clause p(0) = 3, That is when n=0, p(n) = 3.

In order to make the kth step the last step in the process, we select k in such way that p(n-k) is p(0), That is, we want k such that n-k=0. That is, k = n.

Substitute k=n in the kth step result:

$p(n) = (1+r)^n \, p(0) = 3(1+r)^n$

Final closed form solution is,

$p(n) = 3(1+r)^n$

Verification:

Let us verify if the values of p(n) from the recurrence relation and the closed form solution for some values of n:

Value of n	p(n) value from $p(n) = (1+r)\, p(n-1)$, $p(0) = 3$	p(n) value from the closed form solution $p(n) = 3(1+r)^n$
0 (basis clause)	3	$3(1+r)^0 = 3$
1	$p(1) = (1+r)\, p(0) = 3(1+r)$	$p(1) = 3(1+r)^1 = 3(1+r)$
2	$p(2) = (1+r)\, p(1) = 3(1+r)^2$	$p(2) = 3(1+r)^2$
3	$p(3) = (1+r)\, p(2) = 3(1+r)^3$	$p(3) = 3(1+r)^3$

Notice the two values are the same.

Example:

Solve the recurrence relation, s(n), for the sum of the first n positive integers:

$s(n) = n + s(n-1)$

$s(1) = 1$

Given recurrence equation: $s(n) = n + s(n-1)$

Notice, if you substitute n-1 for n in the equation, we get $s(n-1) = (n-1) + s(n-2)$

Take the recurrence equation,

$s(n) = n + s(n-1)$	Substitute $s(n-1) = (n-1) + s(n-2)$
$= n + (n-1) + s(n-2)$	Substitute $s(n-2) = (n-2) + s(n-3)$
$= n + (n-1) + (n-2) + s(n-3)$	Substitute $s(n-3) = (n-3) + s(n-4)$
$= n + (n-1) + (n-2) + (n-3) + s(n-4)$	

We can continue this process of substitution, after n-1 steps, we obtain,

s(n) = n + (n-1) + (n-2) + (n-3) + ……… + s(1)

We now substitute s(1) = 1 from the basis clause.

Thus,

s(n) = n + (n-1) + (n-2) + (n-3) + ……… + 1

We know this sum, 1+2+3+…. +n = s(n) = $\frac{n(n+1)}{2}$

The closed form solution of the given recurrence relation is s(n) = $\frac{n(n+1)}{2}$

Note:

An interesting method (Gauss) to derive the formula:

1+2+3+….+n = s(n) = $\frac{n(n+1)}{2}$

Assume n is even.

Consider 1 + 2 + 3 + …. + n

To make the point clearer, write the first four terms and the last four terms of the series:

1 + 2 + 3 + 4 + ………. + (n-3) + (n-2) + (n-1) + n

Add pairwise as below:

First term and the last term: 1 + n

Second term and the last but one term: 2 + (n-1)

Third term and the last but two term: 3 + (n-2)

Fourth term and the last but three term: 4 + (n-3)

And so on.

Because we have assumed n is even, there are n/2 such pairwise additions. Notice that <u>each</u> pairwise addition results in n+1.

Thus the sum is n+1 added n/2 times, which is (n+1)*(n/2).

Thus the formula,

1+2+3+….+n = s(n) = $\frac{n(n+1)}{2}$

We can extend the proof easily when n is odd.

Example:

Solve the following recurrence relation:

f(n) = f(n-1) + 2n - 1

$f(0) = 0$

Take the recurrence equation,

$f(n)$	$= f(n-1) + 2n - 1$	Substitute $f(n-1) = f(n-2) + 2(n-1) - 1$
	$= f(n-2) + 2(n-1) - 1 + 2n - 1$	Simplify this
	$= f(n-2) + 4n - 4$	Substitute $f(n-2) = f(n-3) + 2(n-2) - 1$
	$= f(n-3) + 2(n-2) - 1 + 4n - 4$	Simplify this
	$= f(n-3) + 6n - 9$	Substitute $f(n-3) = f(n-4) + 2(n-3) - 1$
	$= f(n-4) + 2(n-3) - 1 + 6n - 9$	Simplify this
	$= f(n-4) + 8n - 16$	

Notice that there is a pattern in this. The 3rd step is $f(n) = f(n-4) + 8n - 16$.
The next step would yield, $f(n) = f(n-5) + 10n - 25$.
The kth step (a general step) yields, $f(n) = f(n-k) + 2kn - k^2$
What makes this step the last step?
From the basis clause, we have $f(0) = 0$.
So in the general step, $f(n) = f(n-k) + 2kn - k^2$, we will substitute $k=n$, so that we get $f(n-k) = f(0) = 0$.
Thus, substituting $k=n$ in the equation, $f(n) = f(n-k) + 2kn - k^2$, we obtain
$f(n) = f(0) + 2n*n - n^2 = 2n^2 - n^2 = n^2$.
Thus, the closed form solution for the given recurrence is $f(n) = n^2$.

Example:

Solve the recurrence relation:

$$\begin{cases} f(n) = f\left(\frac{n}{2}\right) + \frac{n}{2} \\ f(1) = 1 \end{cases}$$

Notice,
if you substitute $\frac{n}{2}$ for n in the recurrence equation, we obtain: $f\left(\frac{n}{2}\right) = f\left(\frac{n}{4}\right) + \frac{n}{4}$
If you substitute $\frac{n}{4}$ for n in the recurrence equation, we obtain: $f\left(\frac{n}{4}\right) = f\left(\frac{n}{8}\right) + \frac{n}{8}$

This can be rewritten: $f\left(\frac{n}{2^2}\right) = f\left(\frac{n}{2^3}\right) + \frac{n}{2^3}$

If you substitute $\frac{n}{2^3}$ for n in the recurrence equation, we obtain: $f\left(\frac{n}{2^3}\right) = f\left(\frac{n}{2^4}\right) + \frac{n}{2^4}$

And so on.

Consider the given recurrence equation,

$$
\begin{aligned}
f(n) &= f\left(\frac{n}{2}\right) + \frac{n}{2} && \text{Substitute } f\left(\frac{n}{2}\right) = f\left(\frac{n}{4}\right) + \frac{n}{4} \\
&= f\left(\frac{n}{4}\right) + \frac{n}{4} + \frac{n}{2} && \text{Rewrite it as follows} \\
&= f\left(\frac{n}{2^2}\right) + \frac{n}{2^2} + \frac{n}{2} && \text{Substitute } f\left(\frac{n}{2^2}\right) = f\left(\frac{n}{2^3}\right) + \frac{n}{2^3} \\
&= f\left(\frac{n}{2^3}\right) + \frac{n}{2^3} + \frac{n}{2^2} + \frac{n}{2} && \text{Noticing the pattern, we next obtain} \\
&= f\left(\frac{n}{2^4}\right) + \frac{n}{2^4} + \frac{n}{2^3} + \frac{n}{2^2} + \frac{n}{2}
\end{aligned}
$$

Continuing, the kth step is:

$$
\begin{aligned}
f(n) &= f\left(\frac{n}{2^k}\right) + \frac{n}{2^k} + \frac{n}{2^{k-1}} + \cdots + \frac{n}{2^2} + \frac{n}{2} \\
&= f\left(\frac{n}{2^k}\right) + n\left[\frac{1}{2^k} + \frac{1}{2^{k-1}} + \cdots + \frac{1}{2^2} + \frac{1}{2}\right] \\
&= f\left(\frac{n}{2^k}\right) + n\left[\left(\frac{1}{2}\right)^k + \left(\frac{1}{2}\right)^{k-1} + \left(\frac{1}{2}\right)^{k-2} + \cdots + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)\right] \star
\end{aligned}
$$

Notice the expression within the brackets is a geometric series:

$$\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^{k-1} + \left(\frac{1}{2}\right)^k$$

Recall the formula for the sum of a geometric series is $a_1\left(\frac{1-r^n}{1-r}\right)$.

Here $a_1 = \frac{1}{2}$, r = $\frac{1}{2}$ and n = k.

Thus,

$$\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^{k-1} + \left(\frac{1}{2}\right)^k = \frac{1}{2}\left(\frac{1-\left(\frac{1}{2}\right)^k}{1-\frac{1}{2}}\right).$$

$$= 1 - \frac{1}{2^k}$$

Using this in $\star$, we obtain,

$$f(n) = f\left(\frac{n}{2^k}\right) + n \star \left(1 - \frac{1}{2^k}\right) \star\star$$

To make this the last step, we need assign $\frac{n}{2^k} = 1$, because f(1) is given to be 1.
In ** let us substitute $\frac{n}{2^k} = 1$, $\frac{1}{2^k} = \frac{1}{n}$ and f(1) = 1.

$$
\begin{aligned}
f(n) &= f(1) + n * (1 - \tfrac{1}{n}) \\
&= 1 + n(1 - \tfrac{1}{n}) \\
&= 1 + n - 1 \\
&= n
\end{aligned}
$$

Thus, the closed form solution to the given recurrence is f(n) = n.

---

13. Solving recurrences of the form f(n) = a f(n-1) + b f(n-2)

One example of this kind of recurrence relation is the famous Fibonacci sequence:

f(n) = f(n-1) + f(n-2), n>2
f(1) = 1
f(2) = 1

There is a theorem which provides a solution to such general recurrences.
We will not formally state the theorem and prove it here (this is not a math course). But, the following steps explain how to derive a closed form solution to a given recurrence relation:

Step 1: Given a recurrence relation, f(n) = a f(n-1) + b f(n-2), form a quadratic equation:
$r^2 - ar - b = 0$
This equation is called the characteristic equation.
Step 2: Solve the characteristic equation.
The characteristic equation may have two distinct roots or only one root.
Case 1: If the characteristic equation has two distinct roots r_1 and r_2.
Then a closed form solution to the given recurrence is,
$f(n) = xr_1^n + yr_2^n$
Where x and y are two constants.
Case 2: If the characteristic equation has only one root r_0.
Then a closed form solution to the given recurrence is,
$f(n) = (x+yn)r_0^n$
Step 3: To find the values of x and y, we use basis clause.

Example:

Solve:

$f(n) = f(n-1) + 2f(n-2)$

$f(0) = 2$

$f(1) = 7$

Step 1: Here $a = 1$, and $b = 2$

Characteristic equation: $r^2 - r - 2 = 0$

Step 2: To solve this equation, write $(r+1)(r-2) = 0$

Roots are: -1 and 2.

So a closed form solution is $f(n) = x(-1)^n + y(2)^n$

Step 3: To find the values of x and y, we use $f(0) = 2$ and $f(1) = 7$.

Using $f(0) = 2$: $2 = x + y$

Using $f(1) = 7$: $7 = x(-1)^1 + y(2)^1$. Simplifying, $7 = -x + 2y$

Let us solve,

$x + y = 2$

$-x+2y = 7$

Solution: $x = -1$ and $y = 3$.

Thus the closed solution to the given recurrence is $f(n) = -(-1)^n + 3(2)^n$

Example:

Solve:

$g(n) = -g(n-1) +6\, g(n-2)$

$g(0) = -1$

$g(1) = 8$

Step 1: $a = -1$ and $b = 6$

Characteristic equation: $r^2 + r -6=0$

Step 2: To solve this quadratic equation, $(r+3)(r-2) = 0$

Roots are -3 and 2.

So a closed form solution is $g(n) = x(-3)^n + y(2)^n$

Step 3: To find the values of x and y, we use $g(0) = -1$ and $g(1) = 8$.

$g(0) = -1$ yields, $-1 = x+y$

$g(1) = 8$ yields, $8 = -3x +2y$

Solving, $x=-2$ and $y=1$.

Thus a closed solution for the given recurrence is $g(n) = -2(-3)^n + 2^n$

Example:

Solve:

$f(n) = 6f(n-1) - 9f(n-2), n>1$

$f(0) = 1$

$f(1) = 6$

Step 1: a=6 and b =-9

Characteristic equation is $r^2 - 6r + 9=0$

Step 2: Solve the Characteristic equation: $(r-3)(r-3)=0$. There is only one root 3.

This is case 2 in Step 2. The closed form solution is $f(n) = (x+yn)3^n$

We use basis clause to find the values of x and y.

Using $f(0) = 1$: $1 = (x+0)3^0$ So $x = 1$.

Using $f(1) = 6$: $6 = (x+y)3^1$, $x+ y = 2$, $y = 1$.

Thus a closed form solution to the given recurrence relation is $f(n) = (1+n)3^n$

Note: Instead of writing recurrence relation in a function format, we write it as a sequence.

Example:

The previous recurrence can be written as

$a_n = 6a_{n-1}- 9a_{n-2}$, $n>1$

$a_0 = 1$

$a_1 = 6$

Example:

$a_n = -a_{n-1} + 6a_{n-2}$, n>1

$a_0 = -1$

$a_1 = 8$

Consider $a_n = -a_{n-1} + 6a_{n-2}$

a = -1 and b = 6

Characteristic equation is $r^2 + r - 6 = 0$

$(r+3)(r-2) = 0$

r = -3 and r = 2

Closed form solution is $a_n = x(-3)^n + y2^n$

To find values of x and y, we use $a_0 = -1$ and $a_1 = 8$

n=0, -1 = x + y

n=1, 8 = x(-3) + y2

Solving,

x + y = -1

-3x + 2y = 8

We obtain x = -2 and y = 1.

Hence, the final solution is $a_n = -2(-3)^n + 2^n$

Example:

Solve,

T(n) = 2T(n/2) + 3n, n>1

T(1) = 1

Consider,

T(n) = 2T(n/2) + 3n

$= 2[2\,T(\frac{n}{2^2}) + 3(\frac{n}{2})] + 3n = 2^2\,T(\frac{n}{2^2}) + 3n + 3n = 2^2\,T(\frac{n}{2^2}) + 2(3n)$

$= 2^2\,[2\,T(\frac{n}{2^3}) + 3(\frac{n}{2^2})] + 2(3n) = 2^3\,T(\frac{n}{2^3}) + 3n + 2(3n) = 2^3\,T(\frac{n}{2^3}) + 3(3n)$

Kth step is $2^k\,T(\frac{n}{2^k}) + k(3n)$

Let us make this the last step. We have T(1) = 1. Assign $\frac{n}{2^k} = 1.$

Substitute $2^k = n$. $k = \log_2 n$

Thus T(n) = n T(1) + log2n(3n)

Final solution: $T(n) = n + 3n \operatorname{Log}_2 n$

Example:

$F_n = 5F_{n-1} - 6F_{n-2}$
$F_0 = 1$ and $F_1 = 4$

Consider the recurrence equation
$F_n = 5F_{n-1} - 6F_{n-2}$

Here a = 5, and b = -6
The characteristic equation is, $r^2 - 5r + 6 = 0$
Let us solve the characteristic equation: $(r-3)(r-2) = 0$.
The roots are $r=3$ and $r=2$.

A closed form solution is $F_n = x\, 3^n + y 2^n$

We use basic clause to evaluate x and y:
$F_0 = 1$: $1 = x\, 3^0 + y\, 2^0 = x + y$
$F_1 = 4$: $4 = x\, 3^1 + y\, 2^1 = 3x + 2y$
Solving $x + y = 1$ and $3x + 2y = 4$, we obtain $x = 2$ and $y = -1$.

Thus, the final closed form solution is $F_n = 2\, 3^n - 2^n$

Example:

Solve:
$T(n) = 4T(n-1) - 3T(n-2), n>1$
$T(0) = 0$
$T(1) = 2$

Consider the recurrence equation: $T(n) = 4T(n-1) - 3T(n-2)$
Here a = 4 and b =-3

Characteristic equation: $r^2 - 4r + 3 = 0$

Solve the characteristic equation: $(r-3)(r-1) = 0$

Roots are: $r=3$ and $r=1$.

Closed form solution: $T(n) = x\, 3^n + y\, 1^n = x\, 3^n + y$

To evaluate x and y, we use the basis clause: $T(0) = 0$ and $T(1) = 2$.

$0 = x\, 3^0 + y = x + y$

$2 = x\, 3^1 + y = 3x + y$

Solving, $x + y = 0$ and $3x + y = 2$, we obtain $x = 1$ and $y = -1$

Thus the final closed form solution is: $T(n) = 3^n - 1$

Example:

Let us derive a closed form solution to the famous Fibonacci sequence.

Recurrence relation for Fibonacci sequence:

$f_n = f_{n-1} + f_{n-2}$

$f_0 = 0$

$f_1 = 1$

Consider the recurrence relation: $f_n = f_{n-1} + f_{n-2}$.

Here $a = 1$ and $b = 1$.

Characteristic equation: $r^2 - r - 1 = 0$

Solve Characteristic equation: (We use quadratic formula, roots $= \frac{-b \pm \sqrt{b^2-4ac}}{2a}$)

Equation is $r^2 - r - 1 = 0$. Here $a = 1$, $b = -1$ and $c = -1$. (These are not the a and b from earlier).

roots $= \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$

root1 $= \frac{1+\sqrt{5}}{2}$ and root2 $= \frac{1-\sqrt{5}}{2}$

A closed form solution is, $f_n = x \left(\frac{1+\sqrt{5}}{2}\right)^n + y \left(\frac{1-\sqrt{5}}{2}\right)^n$

To evaluate x and y, we use basis clause: $f_0 = 1$ and $f_1 = 1$

n=0: $0 = x \left(\frac{1+\sqrt{5}}{2}\right)^0 + y \left(\frac{1-\sqrt{5}}{2}\right)^0 = x + y$

n=1: $1 = x \left(\frac{1+\sqrt{5}}{2}\right)^1 + y \left(\frac{1-\sqrt{5}}{2}\right)^1 = x \left(\frac{1+\sqrt{5}}{2}\right) + y \left(\frac{1-\sqrt{5}}{2}\right)$

We need to solve:

$x + y = 0$

$x\left(\frac{1+\sqrt{5}}{2}\right) + y\left(\frac{1-\sqrt{5}}{2}\right) = 1$

From the first equation, $x = -y$. Substitute the second:

$x\left(\frac{1+\sqrt{5}}{2}\right) - x\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$x\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right] = 1$

$x\left[\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right] = 1$

$x\left[\frac{2\sqrt{5}}{2}\right] = 1$

$x = \frac{1}{\sqrt{5}}$

$y = -\frac{1}{\sqrt{5}}$

A closed form solution is $f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

Simplifying,

$f_n = \frac{1}{2^n\sqrt{5}}\left[\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n\right]$

Let us verify, using the recurrence relation Fibonacci sequence is,

0, 1, 1, 2, 3, 5, 8,...

Using the closed formula,

$n=0$, $f_0 = \frac{1}{2^0\sqrt{5}}\left[\left(1+\sqrt{5}\right)^0 - \left(1-\sqrt{5}\right)^0 = \frac{1}{\sqrt{5}}[1 - 1] = 0\right.$

$n=1$, $f_1 = \frac{1}{2^1\sqrt{5}}\left[\left(1+\sqrt{5}\right)^1 - \left(1-\sqrt{5}\right)^1 = \frac{1}{2\sqrt{5}}[1 + \sqrt{5} - 1 + \sqrt{5}] = 1\right.$

$n=2$, $f_2 = \frac{1}{2^2\sqrt{5}}\left[\left(1+\sqrt{5}\right)^2 - \left(1-\sqrt{5}\right)^2\right]$

$= \frac{1}{4\sqrt{5}}[1 + 2\sqrt{5} + 5 - 1 + 2\sqrt{5} - 5] = \frac{1}{4\sqrt{5}}[4\sqrt{5}] = 1$

$n=3$, $f_3 = \frac{1}{2^3\sqrt{5}}\left[\left(1+\sqrt{5}\right)^3 - \left(1-\sqrt{5}\right)^3\right]$ (simplify this yourself).

$= \frac{1}{2^3\sqrt{5}}[16\sqrt{5}] = 2$

Exercises on Recurrences

After you answer them, keep them ready when you take the quiz. Quiz will contain questions from here.

1 Solve

$$\begin{cases} T(n) = T(n-1) + 1 \\ T(0) = 3 \end{cases}$$

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2 Solve:

$$\begin{cases} a_n = a_{n-1} + 6a_{n-2} \\ a_0 = 3 \\ a_1 = 4 \end{cases}$$

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3 Solve:

$$\begin{cases} C_n = 4C_{n-1} - 4C_{n-2}, \quad n>1 \\ C_0 = 4 \\ C_1 = 10 \end{cases}$$

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4 Solve:

$$\begin{cases} P(n) = 2P(\frac{1}{2}) + n, \quad n>1 \\ P(1) = 2 \end{cases}$$

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14. Mathematical Induction

Mathematical induction is a method of proving certain results in mathematics for all positive integer values.

Suppose we need to prove that a certain result is true for all positive integer values of n. Consider the following two statements:

1. The result is true for n = 1.
2. If the result is true for some value of n, it is true for the next value of n.

Study these two statements carefully and understand that if these two statements are true, then the result is true for all positive integer values of n.

Example:

Prove that: $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

Proof by mathematical induction:
To prove the result by mathematical induction, we need to establish two steps:
Step 1: the result is true for n = 1
and
Step 2: If the result is true for some value of n, it is true for the next value of n.

Step 1: Let us verify that the result is true for n = 1.
For n = 1,
Left side is = 1
Right side = $\frac{1(1+1)}{2} = 1$
So the result is true for n = 1.

Step 2: We need to prove that if the result is true for n = k, then it is true for n = k+1.
We have, for n = k,
$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$ [*]

We need to prove,
$1 + 2 + 3 + \ldots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}$ [**]

Consider,

$1 + 2 + 3 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$ Using [*]

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

This is the same as [**].
Thus the result is proved.

Example:

Prove that n(n+1)(n+5) is a multiple of 3 for all positive integers n.

Step 1: Let us verify that the result is true for n = 1.
For n = 1,
Left side = 1(2)(6) = 3(4), clearly this is a multiple of 3.

Step 2: We need to prove that if the result is true for n = k, then it is true for n = k+1.
For n = k, k(k+1)(k+5) is a multiple of 3.
That is, $k^3 + 6k^2 + 5k$ is a multiple of 3. [*]

We need to prove that (k+1)(k+2)(k+6) is a multiple of 3. [**]

$(k+1)(k+2)(k+6) = (k+1)(k^2+8k+12) = k^3 + 9k^2 + 20k + 12$
Let us write this as, $(k^3 + 6k^2 + 5k) + (3k^2 + 15k + 12) = (k^3 + 6k^2 + 5k) + 3(k^2 + 5k + 4)$
Here, $(k^3 + 6k^2 + 5k)$ is a multiple of 3 and $3(k^2 + 5k + 4)$ is clearly a multiple of 3.
Therefore, $(k^3 + 6k^2 + 5k) + 3(k^2 + 5k + 4)$ is a multiple of three.
Thus we have proved [**].

Exercises on Mathematical Induction

1. Prove by mathematical induction,
   $1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
2. Prove by mathematical induction,
   $7^n - 3^n$ is divisible by 4
3. Prove by mathematical induction,
   $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$
