Introduction into
Three-Dimensional Numbers

Adding a Third Perplex Identity to Complex Numbers to extend the Complex Numbers into a Field of Percomplex Numbers.

Scientific Work for Fun

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1 Abstract

This work aims to find a way of exploring a well known problem:

The idea of having three-dimensional numbers.

Till this day only two-dimensional (complex), four-dimensional (quaternions) and eight-dimensional (octonion) numbers have been introduced to have the property to be invertible (division) [1].

2 Introduction

Everyone that is familiar with complex numbers and likes them might encounter the question if it's possible to add more dimensions.

In this case the set of complex numbers ($\mathbb{C}$) is defined by:

A complex number z	$z = a + bi$
with	$i^2 = -1$
where the real part	$Re(z) = a$
and the imaginary part	$Im(z) = b$

A complex number is also called a two-dimensional number because it has two parts:
the real part $Re(z) = a$, and the imaginary part $Re(z) = b$, that can be plotted to a two-dimensional field (see Figure 1). In this work I will not explain the properties of complex numbers and we will therefore move on with studying three-dimensional numbers.

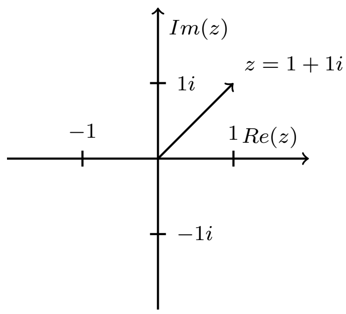

Figure 1: Complex Plane

There might be many ways of how to define three-dimensional numbers, but in this case we will focus the most on the property to invert the multiplication of two three-dimensional numbers.

3 Exploration

To explore which properties are needed to have the most satisfying algebraic group of three-dimensional numbers, let's define a three-dimensional number as followed:

Percomplex Numbers

$$\mathfrak{p} = a + bi + cp \qquad a, b, c \in \mathbb{R} \text{ and } i, p \text{ are undefined for now ...}$$
$$\mathfrak{p} \in \mathfrak{P}$$

The strucure $(\mathfrak{P}, +)$ is an infinate abelian group. I will leave the proof open as it's trivial. I will focus more on the strucuture $(\mathfrak{P}, \cdot)$ in the following.

Percomplex Multiplication is not necessarily closed.

$$\begin{aligned}\mathfrak{p}_1 \cdot \mathfrak{p}_2 &= (x_1 + x_2 i + x_3 p) \cdot (y_1 + y_2 i + y_3 p) && x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R} \\ &= (x_1 \cdot y_1 + x_1 \cdot y_2 i + x_1 \cdot y_3 p) \quad + \\ &\quad (x_2 i \cdot y_1 + x_2 i \cdot y_2 i + x_2 i \cdot y_3 p) \quad + \\ &\quad (x_3 p \cdot y_1 + x_3 p \cdot y_2 i + x_3 p \cdot x_3 p)\end{aligned}$$

It depends on how we define the multiplication of i and p.

There are a few options that need to be viewed to keep any percomplex number $\mathfrak{p}$ three-dimensional

$$\begin{aligned} &1) && i \cdot p = i \\ &2) && i \cdot p = p \\ &3) && i \cdot p = n, && n \in \mathbb{R} \\ &4) && i \cdot p \neq p \cdot i \end{aligned}$$

An important property for such a group is to be invertible, which means that each element $\mathfrak{p} \neq 0$ has an inverse $\frac{1}{\mathfrak{p}}$.

With complex number this can be solved by the *complex conjugate* $\overline{z} = a - bi$, as this can be used to create an inverse.

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{a+bi} \\
&= \frac{1}{a+bi} \cdot \frac{\overline{z}}{\overline{z}} \\
&= \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} \\
&= \frac{a-bi}{(a+bi)\cdot(a-bi)} \\
&= \frac{a-bi}{a^2+b^2} \\
&= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} \cdot i
\end{aligned}$$

In other words this property depends on the crucial fact that $z \cdot \overline{z} = a^2 + b^2$ which is a real number. Therefore I will spend my effort on finding a way to multiply two percomplex numbers to get a real number as well.

$$\begin{aligned}
\mathfrak{p}_1 \cdot \mathfrak{p}_2 &= (x_1 + x_2 i + x_3 p) \cdot (y_1 + y_2 i + y_3 p) && x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R} \\
&= x_1 \cdot y_1 + x_1 \cdot y_2 i + \cdots + x_3 p \cdot y_2 i + x_3 p \cdot x_3 p \\
&\overset{!}{=} n && n \in \mathbb{R}
\end{aligned}$$

It makes sense to try the same approach as with complex numbers.

$$\begin{aligned}
\mathfrak{p} \cdot \overline{\mathfrak{p}} &= (x_1 + x_2 i + x_3 p) \cdot (x_1 - x_2 i - x_3 p) && x_1, x_2, x_3 \in \mathbb{R} \\
&= (x_1 \cdot x_1 + x_1 \cdot x_2 i + x_1 \cdot x_3 p) \qquad + \\
&\quad\; (x_2 i \cdot x_1 + x_2 i \cdot x_2 i + x_2 i \cdot x_3 p) \qquad + \\
&\quad\; (x_3 p \cdot x_1 + x_3 p \cdot x_2 i + x_3 p \cdot x_3 p)
\end{aligned}$$

Obviously if we multiply two percomplex numbers, we will get the quantity $i \cdot p$ or $p \cdot i$, which is outside of the set if we don't apply a rule for that to be inside the set as well.

However if we apply rules to make $i \cdot p$ or $p \cdot i$ a part of the set again

4 Literature