Introduction and Exploration of Three-Dimensional Numbers

Adding a Third Perplex Identity to Complex Numbers to extend the Complex Numbers into a Field of Percomplex Numbers.

Scientific Work for Fun

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Contents

1 Abstract

This work aims to find a way of exploring a well known problem:

The idea of having three-dimensional numbers.

Till this day only two-dimensional (complex), four-dimensional (quaternions) and eight-dimensional (octonion) numbers have been introduced to have the property to be invertible (division) [1].

2 Introduction

Everyone that is familiar with complex numbers and likes them might encounter the question if it's possible to add more dimensions.

In this case the set of complex numbers ($\mathbb{C}$) is defined by:

A complex number z	$z = a + bi$
with	$i^2 = -1$
where the real part	$Re(z) = a$
and the imaginary part	$Im(z) = b$

A complex number is also called a two-dimensional number because it has two parts:
the real part $Re(z) = a$, and the imaginary part $Re(z) = b$, that can be plotted to a two-dimensional field (see Figure 1). In this work I will not explain the properties of complex numbers and we will therefore move on with studying three-dimensional numbers.

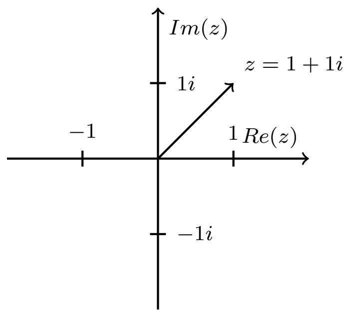

Figure 1: Complex Plane

3 Exploration

To explore which properties are needed to have one or more satisfying algebraic groups of three-dimensional numbers, let's define a three-dimensional number as followed:

Percomplex Numbers	
$\mathfrak{p} = a + bi + cp$	$a, b, c \in \mathbb{R}$ and i, p not real (undefined for now ...)
$\mathfrak{p} \in \mathfrak{P}$	The set of percomplex numbers is defined as $\mathfrak{P}$

3.1 Algebraic Group Definiton

An algebraic group can be categorized if it follows at least one or more of the following rules in the same order.

1. Group is Closed Each illustration $\circ$ in a Group G results in a member of the Group:

$$\forall x, y \in G \Rightarrow x \circ y = g \text{ with } g \in G$$

2. Associative The illustration is associative

$$\forall x, y, z \in G \Rightarrow (x \circ y) \circ z = x \circ (y \circ z)$$

3. Neutral Element It exists a neutral element $e \in G$ for all elements in G such that:

$$\forall x \in G : x \circ e = x$$

4. Inverse Element There is an inverse Element $x^{-1} \in G$ for all $x \in G$

$$\forall x \in G : x \circ x^{-1} = e$$

5. Abelian A group is abelian if the commutative law is applicable

$$\forall x, y \in G : x \circ y = y \circ x$$

3.2 Thoughts

The strucure $(\mathfrak{P}, +)$ is an infinite abelian group. I will leave the proof open as it's trivial. I will focus more on the strucuture $(\mathfrak{P}, \cdot)$ in the following.

Percomplex Multiplication $(\mathfrak{P}, \cdot)$ is not necessarily closed.

$$\begin{aligned} \mathfrak{p}_1 \cdot \mathfrak{p}_2 &= (x_1 + x_2 i + x_3 p) \cdot (y_1 + y_2 i + y_3 p) && x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R} \\ &= (x_1 \cdot y_1 + x_1 \cdot y_2 i + x_1 \cdot y_3 p) \quad + \\ &\quad (x_2 i \cdot y_1 + x_2 i \cdot y_2 i + \underline{x_2 i \cdot y_3 p}) \quad + \\ &\quad (x_3 p \cdot y_1 + \underline{x_3 p \cdot y_2 i} + x_3 p \cdot x_3 p) \end{aligned}$$

It depends on how we define the multiplication of i and p.

There are a few options that need to be viewed to keep any percomplex number $\mathfrak{p}$ three-dimensional

$$\begin{array}{lll} 1) & i \cdot p = i & \\ 2) & i \cdot p = p & \\ 3) & i \cdot p = n & n \in \mathbb{R} \\ 4) & i \cdot p \neq p \cdot i & \text{not abelian} \end{array}$$

We consider the first three options (1-3) to be abelian, meaning that $i \cdot p = p \cdot i$.
The first two properties are coherent and we can proof that these can not be used as it's possible to create an contradiction. For this it's not even necessary to define the properties of i and p.

$$\begin{aligned} & i \cdot p = i && | : i \\ \Longleftrightarrow \quad & p = 1 \rightarrow \perp && p \text{ can not be real} \end{aligned}$$

The same is true for the second option. Therefore it remains to look at the third and fourth option, which need to be viewed with a definition of i and p.

An important property for such a group is to be invertible, which means that each element $\mathfrak{p} \neq 0$ has an inverse $\frac{1}{\mathfrak{p}}$.

With complex number this can be solved by the *complex conjugate* $\overline{z} = a - bi$, as this can be used to create an inverse.

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{a+bi} \\
&= \frac{1}{a+bi} \cdot \frac{\overline{z}}{\overline{z}} \\
&= \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} \\
&= \frac{a-bi}{(a+bi)\cdot(a-bi)} \\
&= \frac{a-bi}{a^2+b^2} \\
&= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} \cdot i
\end{aligned}$$

In other words this property depends on the crucial fact that $z \cdot \overline{z} = a^2 + b^2$ which is a real number. Therefore I will spend my effort on finding a way to multiply two percomplex numbers to get a real number as well.

$$\begin{aligned}
\mathfrak{p}_1 \cdot \mathfrak{p}_2 &= (x_1 + x_2 i + x_3 p) \cdot (y_1 + y_2 i + y_3 p) && x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R} \\
&= x_1 \cdot y_1 + x_1 \cdot y_2 i + \cdots + x_3 p \cdot y_2 i + x_3 p \cdot x_3 p \\
&\stackrel{!}{=} n && n \in \mathbb{R}
\end{aligned}$$

It makes sense to try the same approach as with complex numbers ...

$$\begin{aligned}
\mathfrak{p} \cdot \overline{\mathfrak{p}} &= (x_1 + x_2 i + x_3 p) \cdot (x_1 - x_2 i - x_3 p) && x_1, x_2, x_3 \in \mathbb{R} \\
&= x_1 \cdot x_1 + \cancel{x_1 \cdot (-x_2 i)} + \cancel{x_1 \cdot (-x_3 p)} && + \\
&\quad \cancel{x_2 i \cdot x_1} + x_2 i \cdot (-x_2 i) + x_2 i \cdot (-x_3 p) && + \\
&\quad \cancel{x_3 p \cdot x_1} + x_3 p \cdot (-x_2 i) + x_3 p \cdot (-x_3 p) \\
&= x_1^2 - x_2^2 \cdot i \cdot i - x_2 x_3 \cdot i \cdot p - x_2 x_3 \cdot p \cdot i - x_3^2 \cdot p \cdot p
\end{aligned}$$

I will try to produce a similar outcome as with the complex numbers, namely that $\mathfrak{p} \cdot \overline{\mathfrak{p}} = x^2 + y^2 + z^2$. Not only is this a real number, but also it's the squared distance of a percomplex number plotted into the three-dimenisional plane.

To achieve this it's necessary to let $i^2 = -1$ as well as $p^2 = -1$.

$$
\begin{aligned}
\mathfrak{p} \cdot \overline{\mathfrak{p}} &= (x_1 + x_2 i + x_3 p) \cdot (x_1 - x_2 i - x_3 p) & x_1, x_2, x_3 \in \mathbb{R} \\
&= x_1 \cdot x_1 + \cancel{x_1 \cdot (-x_2 i)} + \cancel{x_1 \cdot (-x_3 p)} \qquad + \\
&\quad \cancel{x_2 i \cdot x_1} + x_2 i \cdot (-x_2 i) + x_2 i \cdot (-x_3 p) \qquad + \\
&\quad \cancel{x_3 p \cdot x_1} + x_3 p \cdot (-x_2 i) + x_3 p \cdot (-x_3 p) \\
&= x_1^2 - x_2^2 \cdot i \cdot i - x_2 x_3 \cdot i \cdot p - x_2 x_3 \cdot p \cdot i - x_3^2 \cdot p \cdot p \\
&= x_1^2 + x_2^2 - \underline{x_2 x_3 \cdot i \cdot p} - \underline{x_2 x_3 \cdot p \cdot i} + x_3^2
\end{aligned}
$$

Obviously we still have to deal with the quantities $i \cdot p$ or $p \cdot i$ as it's outside of the group. If we think back to the options above there are only two possible ways to define the behavior of the multiplication of i and p to get rid of the terms $x_2 x_3 \cdot i \cdot p$ and $x_2 x_3 \cdot p \cdot i$

$$
\begin{aligned}
&1) \qquad && i \cdot p = 0 \\
&2) \qquad && i \cdot p = -p \cdot i
\end{aligned}
$$

The first option is obviously not working as it leads to many contradictions

$$i \cdot p = 0 \iff i \cdot i \cdot p = i \cdot 0 \iff (i \cdot i) \cdot p = 0 \iff -p = 0 \perp p \text{ is not real}$$

Therefore only one option is left namely the definition that $i \cdot p = -p \cdot i$.
This does not necessarily solve the issue that the multiplication is closed. We need to add an additional condition in order to create a closed multiplication for a subset of numbers. We can derive that condition by looking back at the multiplication of two distinct percomplex numbers with the definition of i and p that we have so far.

$$
\begin{aligned}
\mathfrak{p}_1 \cdot \mathfrak{p}_2 &= (x_1 + x_2 i + x_3 p) \cdot (y_1 + y_2 i + y_3 p) \\
&= x_1 \cdot y_1 + x_1 \cdot y_2 i + x_1 \cdot y_3 p \\
&\quad x_2 i \cdot y_1 + x_2 i \cdot y_2 i + x_2 i \cdot y_3 p \\
&\quad x_3 p \cdot y_1 + x_3 p \cdot y_2 i + x_3 p \cdot x_3 p \\
&= x_1 \cdot y_1 + (x_1 y_2 + x_2 y_1) \cdot i + (x_1 y_3 + x_3 y_1) \cdot p + \underline{(x_2 y_3 - x_3 y_2)} \cdot i \cdot p \Rightarrow x_2 y_3 - x_3 y_2 = 0 \iff x_2 y_3 = x_3
\end{aligned}
$$

From this equation it follows that for any multiplication of two percomplex numbers the following condition must hold for all percomplex numbers $\forall \mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P}$ in order to remove

$i \cdot p$ from the equation:

$$\mathfrak{p}_1 = x_1 + x_2 i + x_3 p$$
$$\mathfrak{p}_2 = y_1 + y_2 i + y_3 p$$
$$\text{with} \qquad x_2 \cdot y_3 = x_3 \cdot y_2$$

This leads to the following definiton of percomplex numbers:

Percomplex Numbers

percomplex number	$\mathfrak{p} = a + bi + cp$	$a, b, c \in \mathbb{R}$
imaginary unit	$i^2 = -1$	
perplex unit	$p^2 = -1$	
unreal product	$i \cdot p = -p \cdot i$	
condition	$\forall_{\substack{\mathfrak{p}_1 = x_1 + x_2 i + x_3 p \\ \mathfrak{p}_2 = y_1 + y_2 i + y_3 p}} \in \mathfrak{P} : x_2 \cdot y_3 = x_3 \cdot y_2$	$x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$

3.3 Proof

With the previous definition of three-dimensional numbers, namely percomplex

4 Literature