Automorphism Groups of Compact Complex Surfaces

Recent Development in Algebraic Geometry

Jia Jia

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National University of Singapore

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T-Jordan Property

Definition

A group G is **Jordan** if it has "almost" abelian finite subgroups:

there is a constant J, such that every finite subgroup H of G has a (normal) abelian subgroup H_1 with the index $[H : H_1] \leq J$.

It is named after:

Theorem (C. Jordan, 1878)

The general linear group $\mathrm{GL}_n(\mathbb{C})$ *is Jordan.*

Jordan's theorem has been generalised to

Theorem (Boothby-Wang, 1964)

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- an algebraic manifold (variety)?
- a compact complex manifold (space)?

Known results:

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Theorem (Prokhorov-Shramov, 2021)

Let X be a smooth compact complex surface. Then the automorphism group $\mathrm{Aut}(X)$ of X is Jordan.

A compact complex space is in Fujiki's class $\mathcal{C}$ if it is the meromorphic image of a compact Kähler manifold.

Theorem (Meng-Perroni-Zhang, 2022)

Let X be a compact complex space in Fujiki's class $\mathcal{C}$. Then $\mathrm{Aut}(X)$ is Jordan.

Idea: $\mathrm{Aut}(X)^*|_{H^2(X,\mathbb{Q})}$ has bounded finite subgroups:

$$1 \longrightarrow \mathrm{Aut}_\tau(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(X)^*|_{H^2(X,\mathbb{Q})} \longrightarrow 1.$$

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$\mathrm{Aut}(X)$ *is Jordan iff so is* $\mathrm{Aut}_\tau(X)$.

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Let $\mathbf{Aut}_0(X)$ be the neutral component of $\mathbf{Aut}(X)$. Then

$$\mathbf{Aut}_0(X) \leq \mathbf{Aut}_\tau(X).$$

Fix a big $(1,1)$-class $[\alpha] \in H^{1,1}(X, \mathbb{R})$.

$$\mathbf{Aut}_{[\alpha]}(X) := \{g \in \mathbf{Aut}(X) \mid g^*[\alpha] = [\alpha]\} \geq \mathbf{Aut}_\tau(X).$$

Theorem (Meng-J, 2022)

$$[\mathbf{Aut}_{[\alpha]}(X) : \mathbf{Aut}_0(X)] < \infty.$$

So $\mathbf{Aut}(X)/\mathbf{Aut}_0(X)$ has bounded finite subgroups and hence

Lemma

$\mathbf{Aut}(X)$ *is Jordan iff so is* $\mathbf{Aut}_0(X)$.

Theorem (Lee, 1976)

Let G be a connected Lie group. Then there is a constant $T = T(G)$ such that every torsion subgroup H of G contains a (normal) abelian subgroup H_1 of index $[H : H_1] \leq T$.

For any group G satisfies the theorem above, we say that G has the **`T-Jordan`** property.

Using the equivariant Kähler model for Fujiki's class, we proved

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Lemma

Consider the exact sequence of groups:

$$1 \longrightarrow N \longrightarrow G \longrightarrow H.$$

- *If N is T-Jordan and H has bounded torsion subgroups, then G is T-Jordan.*
- *Assume that the exact sequence is also right exact. If N is a torsion group and G is T-Jordan, then H is T-Jordan.*

A smooth compact complex surface is called **minimal**, if it does not contain any (-1)-curve.

Theorem

Every smooth compact complex surface has a minimal model.

Proposition

Let X be a minimal surface. Suppose that X is neither rational nor ruled. Then X is the unique minimal model in its class of bimeromorphic equivalence, and $\mathrm{Bim}(X) = \mathrm{Aut}(X)$.

Corollary

Let X be a non-Kähler compact complex surface. Then there is a unique minimal model X' bimeromorphically equivalent to X and

$$\mathrm{Aut}(X) \leq \mathrm{Bim}(X) = \mathrm{Bim}(X') = \mathrm{Aut}(X').$$

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Table 1: Kähler minimal smooth compact complex surfaces

class of the surface X	$\kappa(X)$	$a(X)$	$b_1(X)$	$e(X)$
rational surfaces	$-\infty$	2	0	$3, 4$
ruled surfaces of genus $g \geq 1$	$-\infty$	2	$2g$	$4(1-g)$
complex tori	0	$0, 1, 2$	4	0
K3 surfaces	0	$0, 1, 2$	0	24
Enriques surfaces	0	2	0	12
bielliptic surfaces	0	2	2	0
properly elliptic surfaces	1	2	$\equiv 0 \mod 2$	≥ 0
surfaces of general type	2	2	$\equiv 0 \mod 2$	> 0

Table 2: non-Kähler minimal smooth compact complex surfaces

class of the surface X	$\kappa(X)$	$a(X)$	$b_1(X)$	$b_2(X)$	$e(X)$
surfaces of class VII	$-\infty$	$0, 1$	1	≥ 0	≥ 0
primary Kodaira surfaces	0	1	3	4	0
secondary Kodaira surfaces	0	1	1	0	0
properly elliptic surfaces	1	1	$\equiv 1 \bmod 2$		≥ 0

Let X be a compact complex surface of algebraic dimension $a(X) = 1$.

Lemma

Any compact complex surface of algebraic dimension 1 is elliptic.

This elliptic fibration $\pi : X \longrightarrow Y$ is called the **algebraic reduction** of X.

Lemma

The algebraic reduction $\pi : X \longrightarrow Y$ of X is $\mathrm{Aut}(X)$-equivariant.

Proof.

For $g \in \mathrm{Aut}(X)$, the image of a fibre F of π under g is another fibre; otherwise the self-intersection number of $g(F) + F$ is positive and hence X is projective. A compact complex surface is projective iff its algebraic dimension is 2. □

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Recall that surfaces of class VII are those smooth compact complex surfaces with the first Betti number $b_1 = 1$ and Kodaira dimension $\kappa = -\infty$.

Class VII surfaces with $b_2 = 0$ are classified:

Theorem (F. A. Bogomolov, 1970; A. Teleman, 1994)

Any class VII_0 surface with $b_2 = 0$ is biholomorphic to either a Hopf or an Inoue surface.

A **Hopf** surface is a quotient of the form $\mathbb{C}^2 \setminus \{0\}/\Gamma$, where Γ acts properly and discontinuously on $\mathbb{C}^2 \setminus \{0\}$.

An **Inoue surface** is a quotient of the form $\mathbb{H} \times \mathbb{C}/\Gamma$, where $\mathbb{H}$ is the upper half plane, and Γ is a solvable group of affine transformations of the complex plane leaving invariant and acting properly and discontinuously on $\mathbb{H} \times \mathbb{C}$.

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We use the following notation:

- Let Σ be the set of smooth compact complex surface X in class VII with the algebraic dimension $a(X) = 0$ and the second Betti number $b_2(X) > 0$.
- Let $\Sigma_0 \subseteq \Sigma$ be those minimal surfaces which have no curve.

Proposition 1

Let X be a smooth compact complex surface not in Σ_0. Then $\mathbf{Aut}(X)$ is T-Jordan.

Proposition 2

Let X be a smooth compact complex surface in Σ_0. Let $G \leq \mathbf{Aut}(X)$ be a torsion subgroup. Then G is virtually abelian.

Combine the two propositions above:

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Sketch Proof of Proposition 2:

We only consider the case that X does not have any curves.

$$1 \longrightarrow \mathbf{Aut}^*(X) \longrightarrow \mathbf{Aut}(X) \longrightarrow \mathbf{Aut}(X)|_{H^*(X,\mathbb{Q})} \longrightarrow 1.$$

Let $G \leq \mathbf{Aut}(X)$ be an infinite torsion subgroup. The image of G in $\mathbf{GL}(H^*(X,\mathbb{Q}))$ is finite.

By passing to a finite index subgroup, may assume $G \leq \mathbf{Aut}^*(X)$.

Pick $\mathbf{id} \neq g \in G$, and let G' be the centraliser of $\langle g \rangle$ in G.

Since g has finite order, $[G : G']$ is finite.

Replacing G by the finite-index subgroup G', may assume $g \in Z(G)$.

The fixed point set $\mathbf{Fix}(g)$ of g is finite with cardinality $|\,\mathbf{Fix}(g)| = b_2(X)$.

Consider the action of G on the finite set $\mathbf{Fix}(g)$.

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Sketch Proof of Proposition 2:

We only consider the case that X does not have any curves.

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Conjecture

For an arbitrary minimal class VII surface with b_2 positive the following are equivalent:

1. *It has a cycle of rational curves;*
2. *It has at least b_2 rational curves;*
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Theorem (Tits)

For any subgroup $G \leq \mathbf{GL}_n(\mathbb{C})$, either

- *G contains a free non-abelian subgroup, or*
- *G contains a solvable subgroup of finite index.*

Known results:

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Theorem 2

Let X be a compact complex space in Fujiki's class $\mathcal{C}$. Then $\mathbf{Aut}(X)$ satisfies the Tits alternative.

Sketch proof: $\mathbf{Aut}(X)^*|_{H^2(X,\mathbb{Q})} \leq \mathbf{GL}(H^2(X,\mathbb{Q}))$ satisfies the Tits alternative.

$$1 \longrightarrow \mathbf{Aut}_\tau(X) \longrightarrow \mathbf{Aut}(X) \longrightarrow \mathbf{Aut}(X)^*|_{H^2(X,\mathbb{Q})} \longrightarrow 1$$

Then $\mathbf{Aut}(X)$ satisfies the Tits alternative iff so does $\mathbf{Aut}_\tau(X)$.

There is a bimeromorphic holomorphic map $\widetilde{X} \longrightarrow X$ from a compact Kähler manifold $\widetilde{X}$ such that $\mathbf{Aut}_\tau(X)$ lifts to $\widetilde{X}$ holomorphically.

View $\mathbf{Aut}_\tau(X) \leq \mathbf{Aut}(X')$ as a subgroup. Note that X' is kähler and $\mathbf{Aut}(X')$ satisfies the Tits alternative.

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Main Result

Another point of view: By the result of [Meng-J, 2022],

$$[\mathrm{Aut}_\tau(X) : \mathrm{Aut}_0(X)] < \infty.$$

Then $\mathrm{Aut}(X)$ satisfies the Tits alternative iff so does $\mathrm{Aut}_0(X)$.

$$\mathrm{Ad}\colon\ \mathrm{Aut}_0(X) \longrightarrow \mathrm{GL}(T_{\mathrm{Aut}_0(X),e}), \quad \ker \mathrm{Ad} = Z(\mathrm{Aut}_0(X)).$$

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A **spherical shell** in a complex surface X is an open subset $U \subseteq X$ which is biholomorphic to a standard neighbourhood of S^3 in $\mathbb{C}^2$. A spherical shell $U \subseteq X$ is called global if $X \setminus U$ is connected.

A **Kato** surface is a minimal class VII surface with $b_2 > 0$ which contains a global spherical shell.

By a result of Dloussky, Oeljeklaus and Toma, the GSS conjecture implies that every minimal class VII surface with $b_2 > 0$ is a Kato surface.

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Theorem (Enoki, 1980/81)

An Enoki surface is biholomorphic to a compactification of a holomorphic affine line bundle over an elliptic curve.

Let X be a $\mathbb{P}^1$-bundle over an elliptic curve with an infinity section C_∞ (but possibly with no zero section) with $C_\infty^2 = -n$. Then the complement of C_∞ in X can be uniquely compactified into a class VII surface S with $b_2(S) = n$ by replacing C_∞ with a cycle of n-rational curves. This S is an **Enoki surface**.

If X also has the zero section, then S has an elliptic curve. In the second case we call the surface a **parabolic Inoue surface**.

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Theorem 3

Let X be a smooth compact complex surface.

Assume that either $X \notin \Sigma$, or $X \in \Sigma$ but its minimal model is an Enoki surface or Inoue-Hirzebruch surface.

Then $\mathbf{Aut}(X)$ satisfies the Tits alternative.

Virtual Derived Length

Given a group G, its p**-th derived subgroups** are inductively defined by

$$G^{(0)} = G, G^{(1)} = [G, G], \cdots, G^{(i+1)} = [G^{(i)}, G^{(i)}].$$

By definition, $G^{(p)} = 1$ for some integer $p \geq 0$ if and only if G is **solvable**. We call the minimum of such p the **derived length** of G (when G is solvable) and denote it by $\ell(G)$. If G is not solvable, we set $\ell(G) = \infty$.

If G is virtually solvable, we then define the **virtual derived length** to be

$$\ell_{\mathrm{vir}}(G) = \min_{G'} \ell(G')$$

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Lemma

Consider the short exact sequence of groups:

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1.$$

- *If N is solvable and H is virtually solvable, then G is virtually solvable with $\ell_{\mathrm{vir}}(G) \leq \ell(N) + \ell_{\mathrm{vir}}(H)$.*
- *If N is finite and H is virtually solvable, then G is virtually solvable with $\ell_{\mathrm{vir}}(G) \leq \ell_{\mathrm{vir}}(H) + 1$.*
- *G is virtually solvable iff both N and H are virtually solvable.*
- *If both N and H satisfy the Tits alternative, then so does G.*

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Let X be a compact Kähler manifold. For a subgroup G of $\mathrm{Aut}(X)$, define $G^0 := G \cap \mathrm{Aut}_0(X)$.

Theorem (Dinh-Lin-Oguiso-Zhang, 2022)

Let X be a compact Kähler manifold of dimension $n \geq 1$. Then every subgroup $G \leq \mathrm{Aut}(X)$ of zero entropy has a finite index subgroup $G' \leq G$ such that $\ell(G'/G'^0) \leq n-1$.

The invariant $\ell(G'/G'^0)$ does not depend on the choice of G', and it is called the **essential derived length** of the subgroup $G \leq \mathrm{Aut}(X)$.

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Assume that either $X \notin \mathbf{\Sigma}$, or $X \in \mathbf{\Sigma}$ but its minimal model is an Enoki surface or Inoue-Hirzebruch surface.

Let $G \leq \mathbf{Aut}(X)$ be a virtually solvable subgroup. Then the virtually derived length $\ell_{\mathrm{vir}}(G) \leq 4$.

Remark

1. Currently, we are not able to prove Theorems 3 & 4 in full generality for $X \in \mathbf{\Sigma}$.
2. Kato surfaces consist of four subclasses: Enoki surfaces (including parabolic Inoue surfaces), half Inoue surfaces, Inoue-Hirzebruch surfaces and intermediate surfaces.
3. Fix $b > 0$. The moduli space of framed Enoki surfaces with $b_2 = b$ is an open subset of the moduli space of framed Kato surfaces with $b_2 = b$.
4. When X is a parabolic Inoue surface, it has been proved that $\mathbf{Aut}(X)$ is virtually abelian.

Questions?

Thank you!