

# WILD AUTOMORPHISMS OF COMPACT COMPLEX SPACES OF LOWER DIMENSIONS

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**ABSTRACT.** An automorphism of a compact complex space is called wild in the sense of Reichstein–Rogalski–Zhang if there is no non-trivial proper invariant analytic subset. We show that a compact complex surface admitting a wild automorphism must be a complex torus or an Inoue surface of certain type, and this wild automorphism has zero entropy. As a by-product of our argument, we obtain new results about the automorphism groups of Inoue surfaces. We also study wild automorphisms of compact Kähler threefolds or fourfolds, and generalise the results of Oguiso–Zhang from the projective case to the Kähler case.

## 1. INTRODUCTION

Let  $X$  be a compact complex space. We will use the *analytic Zariski topology* on  $X$  whose closed sets are all analytic sets (cf. [GR84, Page 211]). An automorphism  $\sigma \in \text{Aut}(X)$  is called *wild* in the sense of Reichstein–Rogalski–Zhang ([RRZ06]) if for any non-empty analytic subset  $Z$  of  $X$  satisfying  $\sigma(Z) = Z$ , we have  $Z = X$ ; or equivalently, for every point  $x \in X$ , its orbit  $\{\sigma^n(x) \mid n \geq 0\}$  is Zariski dense in  $X$ .

The following two conjectures generalise [RRZ06, Conjecture 0.3] and [OZ22, Conjecture 1.4] from the projective case to the Kähler case.

**Conjecture 1.1** (cf. [RRZ06, Conjecture 0.3]). *Assume that a compact Kähler space  $X$  admits a wild automorphism. Then  $X$  is isomorphic to a complex torus.*

**Conjecture 1.2** (cf. [OZ22, Conjecture 1.4]). *Every wild automorphism  $\sigma$  of a compact Kähler space  $X$  has zero entropy.*

When  $X$  is a projective variety, wild automorphisms are related with the twisted homogeneous coordinate rings, which play a role in noncommutative algebraic geometry (see [RRZ06]). The study of wild automorphisms is also of interest from the viewpoint of dynamical systems (see [CPR20]).

For a compact Kähler surface  $X$  with a wild automorphism  $\sigma$ , it is well-known that  $X$  is a complex torus, and  $\sigma$  is some translation of  $X$ , a priori, of zero entropy (see [RRZ06, Theorem 6.5] and [CPR20, Theorem 6.10]).

In this article, we consider compact complex surfaces (not necessarily Kähler) with a wild automorphism. We give a characterisation of such surfaces and show that there do exist examples of non-Kähler surfaces that admit a wild automorphism.

**Theorem 1.3.** *Let  $X$  be a compact complex space of dimension  $\leq 2$ . Assume that  $X$  admits a wild automorphism  $\sigma$ . Then we have:*

- (1)  $X$  is either a complex torus or an Inoue surface of type  $S_M^{(+)}$ , and  $\sigma$  has zero entropy.
- (2) Both cases in (1) occur: there are pairs  $(X', \sigma')$  where  $X'$  is a complex torus or an Inoue surface of type  $S_M^{(+)}$  and  $\sigma'$  acts on  $X'$  as a wild automorphism.

We refer to Section 4 for the definition and the constructions of Inoue surfaces. In Section 4, we will construct examples of wild automorphisms of Inoue surfaces of type  $S_M^{(+)}$  (see Example 4.3). We remark that there are examples of wild automorphisms of complex abelian surfaces. More strongly, there are complex abelian surfaces with an automorphism of which all orbits are Euclidean dense (see [CPR20, Example 6.6 and Lemma 6.7]).

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**Question 1.4.** *Are there Inoue surfaces of type  $S_M^{(+)}$  with an automorphism of which all orbits are Euclidean dense?*

As a by-product of our argument, we obtain new results about the automorphism groups of Inoue surfaces, which might be of independent interest. Theorem 1.5 below generalises the results in [Jia22, Section 6]. We remark that Inoue surfaces are divided into three different types:  $S_M$ ,  $S_M^{(+)}$  and  $S_M^{(-)}$ .

**Theorem 1.5** (= Theorems 4.1, 4.2 and 4.4). *Let  $X$  be an Inoue surface.*

- (1) *If  $X$  is either of type  $S_M$  or  $S_M^{(-)}$ , then the (biholomorphic) automorphism group  $\text{Aut}(X)$  is finite.*
- (2) *If  $X$  is of type  $S_M^{(+)}$ , the neutral connected component  $\text{Aut}_0(X) \simeq \mathbb{C}^*$  and  $\text{Aut}(X)/\text{Aut}_0(X)$  is finite.*

We propose the following questions rather than conjectures due to the lack of evidence.

**Question 1.6.**

- (1) *Is a compact complex space in Fujiki's class  $\mathcal{C}$  admitting a wild automorphism a complex torus?*
- (2) *Is a compact complex space admitting a wild automorphism a solvmanifold?*
- (3) *Does every wild automorphism of a compact complex space have zero entropy?*

A compact complex space is called in *Fujiki's class  $\mathcal{C}$*  if it is bimeromorphic to a compact Kähler manifold. In dimension two, a compact complex manifold is in Fujiki's class  $\mathcal{C}$  if and only if it is Kähler, while starting from dimension three, the category of Fujiki's class  $\mathcal{C}$  is strictly larger. A *solvmanifold* is a compact homogeneous space of a connected solvable Lie group. Note that both complex tori and Inoue surfaces are solvmanifolds. In particular, the answers to all three questions are affirmative in dimension two due to Theorem 1.3.

In the rest of this article, we study Conjectures 1.1 and 1.2 in dimension three and four. We generalise results of Oguiso–Zhang ([OZ22, Theorems 1.2 and 1.5]) from the projective case to the Kähler case.

**Theorem 1.7.** *Let  $X$  be a compact Kähler space of dimension three, and let  $\sigma$  be a wild automorphism of  $X$ . Then  $X$  is either a complex torus or a weak Calabi–Yau threefold, and  $\sigma$  has zero entropy.*

Here a smooth complex projective variety  $V$  is called

- (1) a *weak Calabi–Yau manifold*, if  $K_V \sim_{\mathbb{Q}} 0$  and  $\pi_1(V)$  is finite;
- (2) a *Calabi–Yau manifold in the strict sense*, if  $V$  is simply connected,  $K_V \sim 0$  and  $H^j(V, \mathcal{O}_V) = 0$  for  $0 < j < \dim V$ .

Let us remark that,  $X$  in Theorem 1.7 could not be a weak Calabi–Yau threefold if one assumes the generalised non-vanishing conjecture which predicts that any nef Cartier divisor on a Calabi–Yau threefold is effective ([OZ22, Theorem 7.4], see also [Kir10, Theorem 4.7]). The following proposition provides further evidence.

**Proposition 1.8.** *Let  $X$  be a weak Calabi–Yau threefold, and let  $c_2(X)$  be the second Chern class of  $X$ . Assume that either*

- (1)  *$c_2(X) \cdot D > 0$  for every non-torsion nef Cartier divisor  $D$  on  $X$ ; or*
- (2) *there exists a non-torsion semi-ample Cartier divisor  $D$  on  $X$  such that  $c_2(X) \cdot D = 0$ .*

*Then  $X$  has no wild automorphism.*

**Theorem 1.9.** *Conjecture 1.2 is true in all three cases below.*

- (1)  $\dim X \leq 3$ .
- (2)  $\dim X = 4$ , and the Kodaira dimension  $\kappa(X) \geq 0$ ,
- (3)  $\dim X = 4$ , and the irregularity  $q(X) \neq 1, 2$ .

Theorems 1.7 and 1.9 demonstrate that the generalisations (Conjectures 1.1 and 1.2) from the projective case to the Kähler case are reasonable. We are unable to generalise our results to the case of Fujiki's class  $\mathcal{C}$ , due to the lack of some fundamental tools, such as the Aut-equivariant Kähler model, the Beauville–Bogomolov decomposition and the Gromov–Yomdin theorem. In authors' opinion, it would be more interesting if there exist examples of non-Kähler manifolds in Fujiki's class  $\mathcal{C}$  that admit a wild automorphism.

**Structure of the paper.** In Section 2, we collect some basic facts about wild automorphisms. In Section 3, we study wild automorphisms of compact complex surfaces. In Section 4, we study the automorphism groups of Inoue surfaces. The proofs of Theorems 1.3 and 1.5 will be completed in Sections 3 and 4. In Section 5, we prove Theorems 1.7 and 1.9 and Proposition 1.8.

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## 2. PRELIMINARIES

We refer to [BHPVdV04] and to [GR84, Uen75] for the knowledge about compact complex surfaces and for general theory of compact complex spaces. We also refer to [Ogu14, DS17] for an overview of topological entropy and dynamical degrees. When  $f: X \rightarrow X$  is a surjective holomorphic map of a compact Kähler manifold  $X$ , a corollary of the fundamental Gromov–Yomdin theorem says that  $f$  has zero entropy if and only if the first dynamical degree of  $f$  is 1 (see, e.g., [Ogu14, Corollary 3.8 (4)]).

In what follows, we collect some basic facts about wild automorphisms of compact complex spaces. We simply call a holomorphic map between complex spaces a morphism.

**Lemma 2.1** (cf. [OZ22, Lemma 2.5]). *Let  $X$  be a compact complex space and let  $\sigma$  be an automorphism on  $X$ .*

- (1) *If  $\sigma$  is wild then  $X$  is smooth.*
- (2)  *$\sigma$  is wild if and only if  $\sigma^m$  is wild for some  $m \geq 1$  (and hence for all  $m \geq 1$ ). In particular, a wild automorphism has infinite order.*
- (3) *Suppose that  $\sigma$  is wild and  $f: X \rightarrow Y$  (resp.  $g: W \rightarrow X$  with  $g(\text{Sing}(W)) \neq X$ ) is a  $\sigma$ -equivariant surjective morphism of compact complex spaces. Then  $f$  (resp.  $g$ ) is a smooth morphism.*
- (4) *Suppose that  $f: X \rightarrow Y$  is a  $\sigma$ -equivariant surjective morphism to a compact complex space  $Y$ . If the action  $\sigma$  on  $X$  is wild then so is the action of  $\sigma$  on  $Y$  (and hence  $Y$  is smooth).*
- (5) *Suppose that  $f: X \rightarrow Y$  is a  $\sigma$ -equivariant generically finite surjective morphism of compact complex spaces. Then the action of  $\sigma$  on  $X$  is wild if and only if so is the action of  $\sigma$  on  $Y$ . Further, if this is the case, then  $f: X \rightarrow Y$  is a finite étale morphism, and in particular, it is an isomorphism when  $f$  is bimeromorphic.*
- (6) *If  $X$  is a complex torus and  $\sigma$  is wild, then  $\sigma$  has zero entropy.*

*Proof.* (1) Note that  $\text{Sing } X$  is an analytic subset of  $X$  and stabilised by every automorphism. Since  $\sigma$  acts as a wild automorphism on  $X$ , the singular locus  $\text{Sing } X = \emptyset$  and  $X$  is smooth.

For (2), if  $\sigma^m$  stabilised an analytic subset  $Z$  of  $X$ , then  $\sigma$  stabilises the analytic subset  $\cup_{i=0}^{m-1} \sigma^i(Z)$  of  $X$ .

(3) Let  $X_1 \subseteq X$  be the subset consisting of points  $x \in X$  such that  $f: X \rightarrow Y$  is not flat at  $x$ . Then  $X_1$  is an analytic subset of  $X$  (cf. [Fri67, Théorème (VI, 9)]) and  $\sigma$ -stable. Hence  $X_1 = \emptyset$  and  $f$  is flat. Let  $X_2$  be the subset consisting of points  $x \in X$  such that  $X_{f(x)}$  is not smooth at  $x$ . Then  $X_2$  is an analytic subset of  $X$  and  $\sigma$ -stable. Hence  $X_2 = \emptyset$  and  $f$  is smooth.

The case of  $g$  is similar. Consider the subsets of  $W$  over which  $g$  is non-flat or singular. Then each of them has an analytic image in  $X$  by Remmert's proper mapping theorem (see e.g., [GR84, Page 213]), and is different from  $X$  by [Fri67, Proposition (VI, 14)] or our additional assumption that  $g(\text{Sing}(W)) \neq X$ .

(4) Suppose the induced automorphism  $\sigma_Y$  on  $Y$  is not wild. Then there is a proper analytic subset  $Y_1 \subsetneq Y$  satisfying  $\sigma_Y(Y_1) = Y_1$ . So  $f^{-1}(Y_1) \subsetneq X$  is a proper analytic subset satisfying  $\sigma(f^{-1}(Y_1)) = f^{-1}(Y_1)$ , which is a contradiction.

(5) is similar to (3) and (4).

(6) It follows from Theorem 2.4 below that  $\sigma = T_b \circ \alpha$  for some translation  $T_b$  and unipotent  $\alpha \in \text{End}(X)$ . In fact, the action of the unipotent  $\alpha$  on  $H^1(X, \mathbb{C})$  is also unipotent. Note that  $H^j(X, \mathbb{C}) \simeq \wedge^j H^1(X, \mathbb{C})$  for  $1 \leq j \leq 2 \dim X$  since  $X$  is a complex torus. Then the actions of  $\alpha$  on  $H^{2k}(X, \mathbb{C})$  are all unipotent and hence the dynamical degrees  $d_k(\alpha)$  are all equal to 1 for  $1 \leq k \leq \dim X$ . Therefore,  $d_k(\sigma) = 1$  and  $\sigma$  has zero entropy (cf. [Ogu14, Corollary 3.8 (4)]).  $\square$

**Proposition 2.2** (cf. [OZ22, Proposition 2.6]). *Let  $X$  be a compact complex space with a wild automorphism  $\sigma$ .*

- (1) *The Euler–Poincaré characteristic  $\chi(\mathcal{O}_X) = 0$ , and the topological Euler number  $e(X) = 0$ . In particular,  $X$  is not rationally connected.*
- (2) *Let  $D$  be a Cartier divisor on  $X$  such that  $\sigma^*D \sim D$ . Then either  $|D| = \emptyset$  or  $D \sim 0$ . In particular, the Kodaira dimension  $\kappa(X) \leq 0$ ; if  $\kappa(X) = 0$ , then  $K_X \sim_{\mathbb{Q}} 0$ .*
- (3) *Suppose that  $X$  is Kähler and  $\kappa(X) = 0$ . Then the Beauville–Bogomolov (minimal split) finite étale cover  $\tilde{X}$  of  $X$  is a product of a complex torus  $T$  and some copies of Calabi–Yau manifolds  $C_i$  in the strict sense; and a positive power of  $\sigma$  lifts to a diagonal action on  $\tilde{X} = T \times \prod_i C_i$  whose action on each factor is wild.*

*Proof.* (1) Since both Lefschetz fixed point theorem and holomorphic Lefschetz fixed point theorem hold for compact complex manifolds, the arguments of [RRZ06, Proposition 4.4, Remark 4.5] still hold.

(2) Note that  $X$  is smooth by Lemma 2.1 (1). So Cartier divisors and Weil divisors coincide on  $X$  (see [Uen75, Theorem 4.13]). Suppose that  $|D| \neq \emptyset$  and  $D \not\sim 0$ , then  $|D| \cong \mathbb{P}^N$  with  $N \geq 1$ . It follows that the action of  $\sigma$  on  $|D|$  has a fixed point, which gives a  $\sigma$ -invariant proper analytic subset of  $X$ . This is a contradiction and proves the first claim. In particular,  $\kappa(X) \leq 0$ .

If  $\kappa(X) = 0$ , then  $|mK_X| \neq \emptyset$  for some  $m > 0$  and there is a unique effective divisor  $E$  such that  $mK_X \sim E$ . Then  $\sigma(E) = E$  implies that  $E = 0$ , which shows  $K_X \sim_{\mathbb{Q}} 0$ .

(3) The argument of [OZ22, Proposition 2.6 (3)] using the Beauville–Bogomolov covering (cf. [Bea83]) still works.  $\square$

**Lemma 2.3** (cf. [RRZ06, Corollary 4.3]). *Let  $X$  be a complex torus, and suppose that  $Y \subseteq X$  is an irreducible subvariety of  $X$  such that  $Y$  admits a wild automorphism. Then  $Y$  is a translation of a subtorus of  $X$ .*

*Proof.* We know that the Kodaira dimension  $\kappa(Y) \leq 0$  by Proposition 2.2 (2). Then a theorem of Ueno ([Uen75, Lemma 10.1 and Theorem 10.3]) states that  $Y$  must be a translation of a subtorus of  $X$ .  $\square$

Let  $X$  and  $Y$  be complex tori. For any  $a \in X$  one has the *translation* automorphism  $T_a: x \mapsto x + a$ . Let  $f: X \rightarrow Y$  be a morphism. Then there is a unique (group) homomorphism  $\alpha \in \text{Hom}(X, Y)$  such that  $f = T_b \circ \alpha$ , where  $b \in Y$ .

**Theorem 2.4** (cf. [RRZ06, Theorem 7.2]). *Let  $X$  be a complex torus. Suppose that  $\sigma = T_b \circ \alpha$  is an automorphism of  $X$ , where  $\alpha \in \text{End}(X)$  is an automorphism and  $b \in X$ . Let  $\beta = \alpha - \text{id}$ , and set  $S = \{b, \beta(b), \beta^2(b), \dots\} \subseteq X$ . Then the following are equivalent:*

- (a)  *$\sigma$  is wild.*
- (b)  *$\alpha$  is unipotent and  $S$  generates  $X$ .*
- (c)  *$\alpha$  is unipotent and the image  $\bar{b}$  of  $b$  generates  $X/\beta(X)$ .*

*Proof.* The argument of [RRZ06, Theorem 7.2] is still available under our setting. Notice that for (b, c)  $\Rightarrow$  (a), one may use Lemma 2.3 instead of [RRZ06, Corollary 4.3].  $\square$

**Lemma 2.5** (cf. [OZ22, Lemma 2.7]). *Let  $X$  be a compact complex manifold with a wild automorphism  $\sigma$ , let  $T$  be a complex torus and let  $f: X \rightarrow T$  be a  $\sigma$ -equivariant morphism.*

Then the image  $Y := f(X)$  is a subtorus of  $T$  and  $f: X \rightarrow Y$  is a smooth surjective morphism. In particular, the Albanese map  $\text{alb}_X: X \rightarrow T = \text{Alb}(X)$  is a surjective smooth morphism with connected fibres.

*Proof.* By Lemma 2.3, after choosing a new origin for  $T$ , the subvariety  $Y$  is a subtorus of  $T$ . It follows from Lemma 2.1 (3) that the surjective morphism  $X \rightarrow Y = f(X)$  is smooth.

For the last assertion, note that  $\text{alb}_X: X \rightarrow \text{Alb}(X)$  is  $\text{Aut}(X)$ - and hence  $\sigma$ -equivariant, and  $\text{Alb}(X)$  is generated by the image  $\text{alb}_X(X)$  (which is a complex torus), so  $\text{alb}_X(X) = \text{Alb}(X)$ . Let  $g: X \rightarrow W$  be the Stein factorisation of  $\text{alb}_X$ . Since  $\text{alb}_X$  is  $\sigma$ -equivariant, so is  $g$ . By Lemma 2.1, the induced morphisms  $\sigma_W$  and  $\sigma_{\text{Alb}(X)}$  are both wild,  $W$  is smooth, and the induced morphism  $X \rightarrow W$  is smooth while  $W \rightarrow \text{Alb}(X)$  is étale. Hence  $W$  is also a complex torus. In fact, the surjective étale morphism  $W \rightarrow \text{Alb}(X)$  is an isomorphism by the universality of the Albanese morphism. This completes the proof of the Lemma.  $\square$

A maximal rational connected (MRC) fibration on a uniruled compact Kähler manifold has general fibres  $F$  rationally connected, and the base is not uniruled (cf. [CH20, Remark 6.10]).

**Lemma 2.6** (cf. [OZ22, Lemma 2.8]). *Let  $X$  be a uniruled compact Kähler manifold of dimension  $\geq 1$ , with a wild automorphism  $\sigma$ . Then we can choose the maximal rationally connected (MRC) fibration  $X \rightarrow Y$  to be a well-defined  $\sigma$ -equivariant surjective smooth morphism with  $0 < \dim Y < \dim X$ . Further, the action of  $\sigma$  on  $Y$  is also wild.*

*Proof.* By [Zho21, Proof of Theorem 1.3(1)], we can choose the MRC fibration  $X \dashrightarrow Y$  to be  $\sigma$ -equivariant. Moreover, the natural (surjective) bimeromorphic morphism  $\pi: \Gamma \rightarrow X$  from the graph of  $X \dashrightarrow Y$ , is  $\sigma$ -equivariant, and hence  $\pi$  is an isomorphism by Lemma 2.1 (5). Thus, we may assume that  $X = \Gamma \rightarrow Y$  is a well-defined surjective holomorphic map.

Since  $X$  is not rationally connected by Proposition 2.2 (1),  $Y$  is not a point. Since  $X$  is uniruled,  $\dim Y < \dim X$ . The action of  $\sigma$  on  $Y$  is wild by Lemma 2.1 (4). This proves the lemma.  $\square$

**Proposition 2.7** (cf. [RRZ06, Proposition 3.2]). *Let  $X$  be a compact complex space in Fujiki's class  $\mathcal{C}$ , and let  $G \leq \text{Aut}(X)$  be a Lie group with finitely many components which acts biholomorphically on  $X$ . Assume that there exists an element  $\sigma \in G$  acting on  $X$  as a wild automorphism. Then  $X$  is a complex torus and  $\sigma$  is a translation automorphism when a group structure in  $X$  is chosen properly.*

*Proof.* Since  $\sigma^m$  is wild by Lemma 2.1 (2), we may replace  $\sigma$  by  $\sigma^m$  and  $G$  by the neutral connected component  $G_0$ , and so we may assume that  $G \leq \text{Aut}_0(X)$ . Let  $H$  be the (Zariski) closure of the subgroup  $\{\sigma^i\}_{i \in \mathbb{Z}}$  in  $G$ . Then  $H$  is a closed Lie subgroup of  $G$ . Since  $H$  is a closure of an abelian group,  $H$  itself is abelian. Without loss of generality, we may assume from now on that  $G = H$ ; in particular,  $G$  is abelian and closed. Note that now  $G$  is not assumed to have only finitely many components.

By [Fuj78, Theorem 5.5], there is a short exact sequence

$$1 \rightarrow L(X) \rightarrow \text{Aut}_0(X) \rightarrow T(X) \rightarrow 1$$

where  $L(X)$  is a linear algebraic group and  $T(X)$  is a complex torus. The image of  $G$  in  $T(X)$  is closed and hence compact. In particular, the image of  $G$  in  $T(X)$  has finitely many components. Replacing  $\sigma$  by a power and replacing  $G$  accordingly, we may assume that the image of  $G$  in  $T(X)$  is a complex subtorus. Note that  $G \cap L(X)$  is an algebraic group and hence has finitely many components. It is not hard to see that  $G$  has finitely many components as well. After replacing  $\sigma$  by some power and replacing  $G$  accordingly, we may assume that  $G \leq \text{Aut}_0(X)$  is irreducible.

Assume that the neutral component  $U$  of the linear part of  $G$  is non-trivial. Then  $U$  being a commutative affine algebraic group, is isomorphic to  $\mathbb{G}_m^s \times \mathbb{G}_a^t$  for some  $s, t \geq 0$ . Note that  $U \triangleleft G$  is a meromorphic group acting biholomorphically and meromorphically on  $X$  (see [Fuj78, Definition 2.1]). Let  $\pi: X \dashrightarrow Y = X/U$  be the quotient map to the cycle space (cf. [Fuj78, Lemma 4.2]), so that  $G$  acts biholomorphically on  $Y$  and the map is  $G$ -equivariant.



The set of points of indeterminacy  $S(\pi)$  is an analytic subset. Since  $S(\pi)$  is  $\sigma$ -invariant, one has  $S(\pi) = \emptyset$  and  $\pi$  is a holomorphic map. The general fibre of  $\pi$  is the closure of some orbit of  $U$  with respect to the natural action of  $U$  on  $X$ . In particular, the (multi-)section at infinity is fixed by  $G$ . But  $\sigma \in G$  is wild, a contradiction. Thus, the linear part of  $G$  is trivial and hence  $G$  is a complex torus.

Choose any  $x \in X$ . The rule  $g \mapsto gx$  defines a morphism  $f: G \rightarrow X$ , which is proper. By Remmert's proper mapping theorem (see e.g., [GR84, Page 213]), the image of  $f(G)$  is an analytic subset of  $X$ . Since  $f(G)$  is  $\sigma$ -stable and  $\sigma$  is wild,  $f(G) = X$ .

Next, let  $G_0$  be the stabiliser of  $x$ . Since  $G$  is abelian,  $G_0$  is the stabiliser of every point in  $Gx = X$ . Since  $G$  is a subgroup of  $\text{Aut}(X)$  and automorphisms of  $X$  are determined by their action on each point, the group  $G_0$  is trivial. Hence, the morphism  $f: G \rightarrow X$  is bijective and hence biholomorphic.

Then  $X$  is a complex torus. The isomorphism  $f$  transforms the translation automorphism  $T_\sigma$  of  $G$  to the translation automorphism  $T_{f(\sigma)} = \sigma$  of  $X$ . Since we replaced  $\sigma$  by  $\sigma^n$  during the proof, we see that some power of the original  $\sigma$  is a translation automorphism of  $X$ . Write  $\sigma = T_b \circ \alpha$  for some translation  $T_b$  and unipotent  $\alpha \in \text{End}(X)$ . Then  $\alpha^n = \text{id}$  for some  $n \geq 1$ . It follows that  $\alpha$  is conjugate to  $\alpha^n = \text{id}$  (cf. [RRZ06, Remark 8.2 (a)]), so  $\alpha = \text{id}$  and  $\sigma$  itself is a translation, as desired.  $\square$

**Proposition 2.8** ([JM22, Corollary 1.3]). *Let  $X$  be a compact complex space in Fujiki's class  $\mathcal{C}$ , and we fix a big  $(1, 1)$ -class  $[\alpha] \in H^{1,1}(X, \mathbb{R})$ . Then the group*

$$\text{Aut}_{[\alpha]}(X) := \{g \in \text{Aut}(X) \mid g^*[\alpha] = [\alpha]\}$$

*is a finite extension of the identity component  $\text{Aut}_0(X)$  of  $\text{Aut}(X)$ , that is,  $[\text{Aut}_{[\alpha]}(X) : \text{Aut}_0(X)] < \infty$ .*

**Corollary 2.9.** *Let  $X$  be a compact complex space in Fujiki's class  $\mathcal{C}$ . Assume the  $X$  admits a wild automorphism  $\sigma$ . Suppose that a positive power of  $\sigma$  fixes a big  $(1, 1)$ -class of  $X$ . Then  $X$  is a complex torus.*

*Proof.* Assume that for some positive integer  $s$ , the wild automorphism  $\sigma^s$  fix a big  $(1, 1)$ -class  $[\alpha] \in H^{1,1}(X, \mathbb{R})$ . Then  $\sigma^s \in \text{Aut}_{[\alpha]}(X)$ , which is a Lie group with finitely many components by Proposition 2.8. Now we may apply Proposition 2.7 with  $G = \text{Aut}_{[\alpha]}(X)$ .  $\square$

The following is a partial generalisation of [OZ22, Lemma 2.11] from the projective case to the Kähler case, which is sufficient for our purpose.

**Lemma 2.10.** *Let  $X$  be a compact Kähler manifold with a wild automorphism  $\sigma$ , let  $A$  be a complex torus and let  $f: X \rightarrow A$  be a  $\sigma$ -equivariant surjective projective morphism with connected fibres of positive dimension. Assume general fibres of  $f$  are isomorphic to  $F$ . Suppose that a positive power  $\sigma_A^s$  of  $\sigma_A$  fixes some big  $(1, 1)$ -class  $\alpha$  on  $A$  in  $H^{1,1}(A)$  (this holds if  $\dim A = 1$  or a positive power of  $\sigma_A$  is a translation on  $A$ ). Then  $-K_F$  is not a big divisor.*

Let us explain the claim in the bracket of the statement of Lemma 2.10. In fact, if  $\dim A = 1$ , then  $\sigma_A^{12}$  is a translation on  $A$ ; if a positive power  $\sigma_A^s$  of  $\sigma_A$  is a translation on  $A$ , then it fixes every big  $(1, 1)$ -class on  $A$ .

*Proof.* Note that the fibres of  $f$  are analytically isomorphic. In fact, let  $A_1$  be the set of points of  $A$  over which the fibre is not isomorphic to  $F$ . Then  $A_1$  is  $\sigma$ -invariant and is disjoint with a non-empty open subset. Since  $\sigma_A$  is wild,  $A_1$  is empty. By a theorem of Grauert–Fischer (see, e.g., [BHPvdV04, I.10.1]),  $f$  is locally trivial, i.e.,  $f$  is a holomorphic fibre bundle.

Suppose that  $-K_F = (-K_X)|_F$  is big. Then  $c_1(F)$  is a big  $(1, 1)$ -form on  $F$ . Let  $\omega_F, \omega_A$  be Kähler forms on  $F$  and  $A$ , respectively. Since both  $c_1(F)$  and  $\alpha$  are big (cf. [Bou02, § 2.3]), there are plurisubharmonic functions  $\varphi, \psi$  on  $F$  and  $A$  respectively such that

$$\begin{aligned} c_1(F) + dd^c \varphi &\geq \delta \omega_F \quad \text{for some } \delta > 0, \\ \alpha + dd^c \psi &\geq \varepsilon \omega_A \quad \text{for some } \varepsilon > 0. \end{aligned}$$

For each  $x \in X$ , take an open neighbourhood  $U$  of  $a = f(x)$  in  $A$  such that  $f$  is trivial on  $U$ . On  $U$ , let  $q: f^{-1}(U) \simeq F \times U \rightarrow F$  be the first projection. Then  $c_1(X) - q^*c_1(F) = f^*\beta$  for some  $(1, 1)$ -class  $\beta$  on  $U$ . Take  $n_U \gg 0$  such that  $\beta + n_U\varepsilon(\omega_A)|_U$  is a Kähler form on  $U$ . Then on  $f^{-1}(U)$ ,

$$\begin{aligned} & c_1(X) + n_U f^* \alpha + dd^c(n_U \psi \circ f + \varphi \circ q) \\ &= c_1(X) + n_U f^* \alpha + n_U f^* dd^c \psi + q^* dd^c \varphi \\ &= c_1(X) - q^* c_1(F) + q^* c_1(F) + q^* dd^c \varphi + n_U f^* \alpha + n_U f^* dd^c \varphi \\ &= f^* \beta + n_U f^* (\alpha + dd^c \varphi) + q^* (c_1(F) + dd^c \varphi) \\ &\geq f^* (\beta + n_U \varepsilon \omega_A) + \delta q^* \omega_F. \end{aligned}$$

Note that by the construction,  $f^*(\beta + n_U \varepsilon \omega_A) + \delta q^* \omega_F$  is a Kähler form on  $f^{-1}(U)$ . It follows that  $(c_1(X) + n_U f^* (\alpha))|_{f^{-1}(U)}$  is a big form on  $f^{-1}(U)$  as  $n_U \psi \circ f + \varphi \circ q$  is a plurisubharmonic function on  $f^{-1}(U)$ . Since  $A$  is compact, by taking a uniform  $n$ , we can ensure that  $E := c_1(X) + n f^* \alpha$  is a big form on  $X$ .

Now  $\sigma^s$  fixes the big class  $[E]$ . Thus, by Corollary 2.9,  $X$  is a complex torus and hence  $K_F = K_X|_F = 0$ , a contradiction.  $\square$

**Lemma 2.11** (cf. [Kir10, Lemma 2.8]). *Let  $\sigma$  be a wild automorphism of a complex torus  $T$ . Let  $H \leq \text{Aut}(T)$  be a finite subgroup centralised by  $\sigma$ :  $\sigma h = h \sigma$  for all  $h \in H$ . Then  $H$  consists of translations of  $T$ .*

*Proof.* The argument of [Kir10, Lemma 2.8] is still valid for complex torus.  $\square$

A compact complex space  $X$  is called a  $Q$ -torus if it has a complex torus  $T_1$  as an étale finite cover; or equivalently, it is the quotient of a complex torus  $T_2$  by a finite group acting freely on  $T_2$ .

**Proposition 2.12** (cf. [OZ22, Proposition 2.13]). *Let  $X$  be a  $Q$ -torus with a wild automorphism  $\sigma$ . Then  $X$  is a complex torus.*

*Proof.* Let  $T \rightarrow X$  be the minimal splitting cover of  $X$  in [Bea83, § 3]. The  $\sigma$  lifts to an automorphism on  $T$ , also denoted as  $\sigma$ . Note that the  $\sigma$  on  $T$  normalises  $H := \text{Gal}(T/X)$ . Hence  $\sigma^{r!}$  centralises every element of  $H$ , where  $r := |H|$ . Since  $\sigma^{r!}$  is still wild by Lemma 2.1 (2),  $H$  consists of translations by Lemma 2.11. Hence  $H = \{\text{id}_T\}$  by the minimality of  $T \rightarrow X$ . Therefore,  $X = T$  and  $X$  is a complex torus.  $\square$

In the rest of this section, we give two results from number theory, which will be used in the study of the automorphism groups of Inoue surfaces.

**Lemma 2.13.** *Let  $f(x) := x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 \in \mathbb{Z}[x]$  with  $a_0 = \pm 1$ . If  $\alpha$  is a real root of  $f(x)$ , then either  $\alpha$  is irrational, or  $\alpha = \pm 1$ .*

*Proof.* The proof is elementary. Assume that  $\alpha = c/d$  is rational with  $c, d$  coprime integers. Then

$$c \cdot (c^{n-1} + a_{n-1}c^{n-2}d + \dots + a_1d^{n-1}) = -a_0d^n.$$

This implies  $c = \pm 1$ , since otherwise, a prime factor  $p$  of  $c$  satisfies  $p \mid d$ , which is a contradiction. A similar argument yields  $d = \pm 1$ . This shows the assertion.  $\square$

**Lemma 2.14.** *Let  $M \in \text{GL}_n(\mathbb{Z})$  be a diagonalisable matrix where  $n = 2$  or  $3$ . Assume that  $M$  has either*

- *two real eigenvalues  $\alpha (\neq \pm 1)$  and  $1/\alpha$  or  $-1/\alpha$ , when  $n = 2$ ; or*
- *three eigenvalues  $\alpha (\neq \pm 1)$ ,  $\beta$  and  $\bar{\beta}$  ( $\beta \neq \bar{\beta}$ ), when  $n = 3$ .*

*Denote*

$$\Gamma := \{N \in \text{GL}_n(\mathbb{Z}) \mid N \text{ and } M \text{ are simultaneously diagonalisable}\}.$$

*Then  $\Gamma \simeq U \times \mathbb{Z}$  where  $U$  is a finite group. In particular, if we denote by  $M^{\mathbb{Z}}$  the subgroup of  $\Gamma$  generated by  $M$ , then the quotient  $\Gamma/M^{\mathbb{Z}}$  is finite.*

*Proof.* We adapt this proof from MathOverflow<sup>1</sup>.

It is known that the set of  $n \times n$  rational matrices which commute with  $M$  is a rational vector space spanned by  $\text{id}, M, \dots, M^{n-1}$  over  $\mathbb{Q}$ . Notice that the  $\mathbb{Q}$ -span of the powers of  $M$  is isomorphic, as a ring, to  $\mathbb{Q}[x]/p(x)$ , where  $p(x)$  is the characteristic polynomial of  $M$ . Since  $p(x)$  is irreducible over  $\mathbb{Q}$  (cf. Lemma 2.13),  $\mathbf{K} := \mathbb{Q}[x]/p(x)$  is a number field, and the matrices in  $\text{GL}_n(\mathbb{Q})$ , which commute with  $M$  are isomorphic to  $\mathbf{K}^*$ . The set of the matrices whose entries are in  $\mathbb{Z}$  forms an order  $\mathcal{O}$  in  $\mathbf{K}$ . Those matrices of  $\mathcal{O}$  which are in  $\text{GL}_n(\mathbb{Z})$ , is the unit group  $\mathcal{O}^*$  of  $\mathcal{O}$ . Then by Dirichlet's unit theorem, in each case,  $\mathcal{O}^* \simeq \mu(\mathcal{O}) \times \mathbb{Z}$ , where  $\mu(\mathcal{O})$  is the finite cyclic group of roots of unity in  $\mathcal{O}$ . Since the subgroup  $M^{\mathbb{Z}} \leq \Gamma$  is isomorphic to  $\mathbb{Z}$ , the quotient  $\Gamma/M^{\mathbb{Z}}$  is a finite group, which proves the last claim.  $\square$

### 3. WILD AUTOMORPHISMS IN DIMENSION TWO

In this section, we study wild automorphisms of compact complex surfaces and prove Theorem 1.3. We distinguish two cases: Kähler (Proposition 3.1) and non-Kähler (Proposition 3.2).

The following result is well-known (see e.g., [CPR20, Theorem 6.10]). Here we give a proof for the sake of completeness.

**Proposition 3.1.** *Let  $X$  be a compact Kähler space of dimension  $\leq 2$ . Assume that  $X$  admits a wild automorphism  $\sigma$ . Then  $X$  is a complex torus, and  $\sigma$  has zero entropy.*

*Proof.* Here we follow the argument of [OZ22, Theorem 3.1]. Note that  $X$  is smooth and  $\kappa(X) \leq 0$  by Lemma 2.1 (1) and Proposition 2.2 (2). Thus, when  $\dim X = 1$ ,  $X$  is an elliptic curve by Proposition 2.2 (1). Let us consider the case where  $\dim X = 2$ .

If  $\kappa(X) = -\infty$ , by Proposition 2.2 (1) and Lemma 2.6,  $X$  admits a smooth fibration  $f: X \rightarrow Y$  with fibres  $F$  smooth rational curve and  $Y$  an elliptic curve. But then  $F$  has ample  $-K_F$ , which contradicts with Lemma 2.10.

If  $\kappa(X) = 0$ , then  $X$  is either a complex torus or a hyperelliptic surface by Proposition 2.2 (2) and (1) together with the classification of minimal Kähler surfaces of Kodaira dimension 0. Hence  $X$  is a complex torus by Proposition 2.12.

The second claim follows from the first one and Lemma 2.1 (6).  $\square$

Now we deal with the non-Kähler case.

**Proposition 3.2.** *Let  $X$  be a compact complex surface which is not Kähler. Suppose that  $X$  has a wild automorphism  $\sigma$ . Then  $X$  is an Inoue surface of type  $S_M^{(+)}$ , and  $\sigma$  has zero entropy.*

*Proof.* Notice that, by a result of Cantat ([Can99, Proposition 1]), any automorphism of a non-Kähler surface has zero entropy. It remains to show that  $X$  is an Inoue surface of type  $S_M^{(+)}$ .

Note that any non-Kähler compact complex surface  $X$  has a unique minimal model  $X_m$  and  $\text{Aut}(X) \leq \text{Bim}(X) \simeq \text{Bim}(X_m) = \text{Aut}(X_m)$  (cf. [PS21, Proposition 3.5]), where  $\text{Bim}(X)$  denotes the group of bimeromorphic transformations of  $X$ . So there is a  $\sigma$ -equivariant surjective bimeromorphic morphism  $X \rightarrow X_m$  and hence  $X \rightarrow X_m$  is an isomorphism by Lemma 2.1 (5). Therefore,  $X$  is minimal; see Table 1 below.

TABLE 1. non-Kähler minimal smooth compact complex surfaces

class of the surface $X$	$\kappa(X)$	$a(X)$	$b_1(X)$	$b_2(X)$	$e(X)$
surfaces of class VII	$-\infty$	0, 1	1	$\geq 0$	$\geq 0$
primary Kodaira surfaces	0	1	3	4	0
secondary Kodaira surfaces	0	1	1	0	0
properly elliptic surfaces	1	1			$\geq 0$

First we claim that the algebraic dimension  $a(X) = 0$ . Suppose that the algebraic dimension  $a(X) = 1$ . Let  $\pi: X \rightarrow Y$  be the algebraic reduction with  $Y$  a complex projective curve, which is  $\text{Aut}(X)$ -equivariant (cf. [Uen75, Chapter I, Definition 3.3]). In particular,  $\sigma$  descends to a

<sup>1</sup><https://mathoverflow.net/q/55646>



wild automorphism  $\sigma_Y$  on the base  $Y$  (cf. Lemma 2.1 (4)) and hence  $Y$  is an elliptic curve and  $\sigma_Y$  is of infinite order. As  $X$  cannot be a Kodaira surface or a properly elliptic surface by [PS20, Lemma 2.4 and Proposition 1.2], we conclude that  $\kappa(X) = -\infty$ . So  $X$  is a Hopf surface with  $a(X) = 1$ . But then the base  $Y$  of the algebraic reduction of  $X$  is  $\mathbb{P}^1$ , which is a contradiction. Therefore, the algebraic dimension  $a(X) = 0$ , a priori,  $\kappa(X) = -\infty$ . In particular,  $X$  is a minimal surface of class VII.

Now we claim that the second Betti number  $b_2(X) = 0$ . Otherwise, the Euler number  $e(X) = b_2(X) > 0$  as  $b_0(X) = b_1(X) = 1$ , which is a contradiction to Proposition 2.2 (1). Thus  $b_2(X) = 0$  and  $X$  is a minimal surface of class VII. In other words,  $X$  is either a Hopf surface with  $a(X) = 0$  or an Inoue surface (cf. [Tel94, Theorem 2.1]; see also [Bog76]). Note that there are finitely many (and at least one) curves on a Hopf surface with  $a(X) = 0$  (see [BHPVdV04, Theorems IV 8.2 and V 18.7]), which are  $\text{Aut}(X)$ -invariant. Therefore,  $X$  cannot be a Hopf surface. This proves that  $X$  is an Inoue surface.

Finally, by Theorems 4.1 and 4.4 in the next section, we conclude that  $X$  must be an Inoue surface of type  $S_M^{(+)}$ .  $\square$

#### 4. AUTOMORPHISM GROUPS OF INOUE SURFACES

In this section, we study the automorphism groups of Inoue surfaces. Recall that an *Inoue surface*  $X$  is a compact complex surface obtained from  $W := \mathbb{H} \times \mathbb{C}$  as a quotient by an infinite discrete group, where  $\mathbb{H}$  is the upper half complex plane. Inoue surfaces are minimal surfaces in class VII, contain no curve, and have the following numerical invariants:

$$a(X) = 0, \quad b_1(X) = 1, \quad b_2(X) = 0.$$

There are three families of Inoue surfaces:  $S_M$ ,  $S_M^{(+)}$ , and  $S_M^{(-)}$  (cf. [Ino74]), and we will study them separately.

**4.1. Type  $S_M$ .** Let  $M = (m_{i,j}) \in \text{SL}_3(\mathbb{Z})$  be a matrix with eigenvalues  $\alpha, \beta, \bar{\beta}$  such that  $\alpha > 1$  and  $\beta \neq \bar{\beta}$ . Take  $(a_1, a_2, a_3)^T$  to be a real eigenvector of  $M$  corresponding to  $\alpha$ , and  $(b_1, b_2, b_3)^T$  an eigenvector corresponding to  $\beta$ . Let  $G_M$  be the group of automorphisms of  $W$  generated by

$$\begin{aligned} g_0(w, z) &= (\alpha w, \beta z), \\ g_i(w, z) &= (w + a_i, z + b_i), \quad i = 1, 2, 3, \end{aligned}$$

which satisfy these conditions

$$\begin{aligned} g_0 g_i g_0^{-1} &= g_1^{m_{i,1}} g_2^{m_{i,2}} g_3^{m_{i,3}}, \\ g_i g_j &= g_j g_i, \quad i, j = 1, 2, 3. \end{aligned}$$

Note that  $G_M = G_1 \rtimes G_0$  where

$$G_1 = \{g_1^{n_1} g_2^{n_2} g_3^{n_3} \mid n_i \in \mathbb{Z}\} \simeq \mathbb{Z}^3 \quad \text{and} \quad G_0 = \langle g_0 \rangle \simeq \mathbb{Z}.$$

It can be shown that the action of  $G_M$  on  $W$  is free and properly discontinuous. The quotient  $X := W/G_M$  is an Inoue surface of type  $S_M$  (cf. [Ino74, § 2]). Thus, there is a short exact sequence of groups

$$1 \longrightarrow G_M \longrightarrow \widetilde{\text{Aut}}(X) \longrightarrow \text{Aut}(X) \longrightarrow 1$$

where  $\widetilde{\text{Aut}}(X)$  acts biholomorphically on  $W$ , which is the normaliser of  $G_M$  in  $\text{Aut}(W)$ .

As in [Jia22, § 6.1],  $\widetilde{\text{Aut}}(X) = K \rtimes \Gamma$ , where

$$K = \left\{ (w, z) \mapsto \left( w + \frac{1}{1-\alpha} \sum_{j=1}^3 n_j a_j, z + \frac{1}{1-\beta} \sum_{j=1}^3 n_j b_j \right) \mid n_i \in \mathbb{Z} \right\} \simeq \mathbb{Z}^3$$

and

$$\Gamma := \{N \in \text{GL}_3(\mathbb{Z}) \mid N \text{ and } M \text{ are simultaneously diagonalisable}\}.$$

Now we have the following commutative diagram by the snake lemma

$$(4.1) \quad \begin{array}{ccccccc} & & 1 & & 1 & & 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & G_1 & \xrightarrow{\phi} & K & \longrightarrow & F \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & G_M & \longrightarrow & \widetilde{\text{Aut}}(X) & \longrightarrow & \text{Aut}(X) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & G_0 & \xrightarrow{\psi} & \Gamma & \longrightarrow & \Gamma/G_0 \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 1 & & 1 & & 1 \end{array}$$

where  $\phi$  is given by

$$\phi(\mathbf{n}) = \mathbf{n} \cdot (I - M) \quad \text{with } \mathbf{n} = (n_1, n_2, n_3)$$

and  $\psi$  is defined via  $g_0 \mapsto M$ . Since  $I - M$  is invertible, the group  $F$  being the cokernel of  $\phi$ , is finite.

Now by Lemma 2.14,  $\Gamma/G_0$  is finite. Consequently, the diagram (4.1) implies:

**Theorem 4.1.** *Let  $X$  be an Inoue surface of type  $S_M$ . Then the automorphism group  $\text{Aut}(X)$  is finite. In particular,  $X$  does not have any wild automorphism.*  $\square$

**4.2. Type  $S_M^{(+)}$ .** Let  $M \in \text{SL}_2(\mathbb{Z})$  be a matrix with two real eigenvalues  $\alpha$  and  $1/\alpha$  with  $\alpha > 1$ . Let  $(a_1, a_2)^T$  and  $(b_1, b_2)^T$  be real eigenvectors of  $M$  corresponding to  $\alpha$  and  $1/\alpha$ , respectively, and fix integers  $p_1, p_2, r$  ( $r \neq 0$ ) and a complex number  $\tau$ . Define  $(c_1, c_2)^T$  to be the solution of the following equation

$$(I - M) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \frac{b_1 a_2 - b_2 a_1}{r} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

where

$$e_i = \frac{1}{2} m_{i,1} (m_{i,1} - 1) a_1 b_1 + \frac{1}{2} m_{i,2} (m_{i,2} - 1) a_2 b_2 + m_{i,1} m_{i,2} b_1 a_2, \quad i = 1, 2.$$

Let  $G_M^{(+)}$  be the group of analytic automorphisms of  $W = \mathbb{H} \times \mathbb{C}$  generated by

$$\begin{aligned} g_0 &: (w, z) \mapsto (\alpha w, z + \tau), \\ g_i &: (w, z) \mapsto (w + a_i, z + b_i w + c_i), \quad i = 1, 2, \\ g_3 &: (w, z) \mapsto \left( w, z + \frac{b_1 a_2 - b_2 a_1}{r} \right). \end{aligned}$$

We have the following relations between these generators

$$\begin{aligned} g_3 g_i &= g_i g_3 \quad \text{for } i = 0, 1, 2, \quad g_1^{-1} g_2^{-1} g_1 g_2 = g_3^r, \\ g_0 g_j g_0^{-1} &= g_1^{m_{j,1}} g_2^{m_{j,2}} g_3^{p_j} \quad \text{for } j = 1, 2. \end{aligned}$$

The action of  $G_M^{(+)}$  is free and properly discontinuous. The quotient space  $X := W/G_M^{(+)}$  is an Inoue surface of type  $S_M^{(+)}$  (cf. [Ino74, § 3]). Note that  $G_M^{(+)} \simeq H(r) \rtimes \mathbb{Z}$  as an abstract group, where

$$H(r) = \langle g_1, g_2, g_3 \mid g_3 g_i = g_i g_3, g_1^{-1} g_2^{-1} g_1 g_2 = g_3^r \rangle$$

and  $\mathbb{Z}$  is generated by  $g_0$ . Further,

$$(4.2) \quad Z(H(r)) = \langle g_3 \rangle \simeq \mathbb{Z} \quad \text{and} \quad H(r)/Z(H(r)) \simeq \mathbb{Z}^2,$$

where  $Z(H(r))$  is the centre of  $H(r)$ .

Similarly, there is a short exact sequence of groups

$$1 \longrightarrow G_M^{(+)} \longrightarrow \widetilde{\text{Aut}}(X) \longrightarrow \text{Aut}(X) \longrightarrow 1,$$

where  $\widetilde{\text{Aut}}(X) \leq \text{Aut}(W)$  is the normaliser of  $G_M^{(+)}$ .

Let

$$\Gamma := \{N \in \text{GL}_2(\mathbb{Z}) \mid N \text{ and } M \text{ are simultaneously diagonalisable}\},$$

which is an abelian group, and let  $\tau: \widetilde{\text{Aut}}(X) \rightarrow \Gamma$  be the homomorphism (not necessarily surjective) defined by  $u \mapsto N$ , where  $N$  is the matrix associated with  $u$  as constructed in [Jia22, § 6.2]. Let  $K$  be the kernel of this homomorphism, with image  $\Gamma' \leq \Gamma$ . It is clear that any automorphism in  $K$  has the form

$$(w, z) \mapsto (w + b, z + Aw + B)$$

for some  $b \in \mathbb{R}$  and  $A, B \in \mathbb{C}$  satisfying some conditions ([Jia22, (6.21) and (6.22)]). An explicit calculation gives that the centre

$$(4.3) \quad Z(K) \simeq \mathbb{C}$$

which is generated by  $(w, z) \mapsto (w, z + B)$ . Further,

$$(4.4) \quad K/Z(K) \simeq \mathbb{Z} \rtimes \mathbb{Z}$$

which is generated by the images of

$$(w, z) \mapsto (w + b, z) \quad \text{and} \quad (w, z) \mapsto (w, z + Aw).$$

Combining these, the snake lemma shows the diagrams below are commutative with exact rows and columns.

$$\begin{array}{ccccccc} & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\ 1 \rightarrow & H(r) & \rightarrow & K & \rightarrow & F & \rightarrow & 1 \\ & \downarrow & & \downarrow & & \downarrow & & \\ 1 \rightarrow & G_M^{(+)} & \rightarrow & \widetilde{\text{Aut}}(X) & \rightarrow & \text{Aut}(X) & \rightarrow & 1 \\ & \downarrow & & \downarrow \tau & & \downarrow & & \\ 1 \rightarrow & \mathbb{Z} & \rightarrow & \Gamma' & \rightarrow & \Gamma'/\mathbb{Z} & \rightarrow & 1 \\ & \downarrow & & \downarrow & & \downarrow & & \\ & 1 & & 1 & & 1 & & \end{array} \quad \begin{array}{ccccccc} & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\ 1 \rightarrow & Z(H(r)) & \rightarrow & H(r) & \rightarrow & H(r)/Z(H(r)) & \rightarrow & 1 \\ & \downarrow & & \downarrow & & \downarrow & & \\ 1 \rightarrow & Z(K) & \rightarrow & K & \rightarrow & K/Z(K) & \rightarrow & 1 \\ & \downarrow & & \downarrow & & \downarrow & & \\ 1 \rightarrow & F' & \rightarrow & F & \rightarrow & F'' & \rightarrow & 1 \\ & \downarrow & & \downarrow & & \downarrow & & \\ & 1 & & 1 & & 1 & & \end{array}$$

Note that  $F' \simeq \mathbb{C}^*$  by Equations (4.2) and (4.3), and that  $F''$  is finite by Equations (4.2) and (4.4). Similarly to Section 4.1,  $\Gamma'/\mathbb{Z} \leq \Gamma/\mathbb{Z}$  is also a finite group by Lemma 2.14. In conclusion,

**Theorem 4.2.** *Let  $X$  be an Inoue surface of type  $S_M^{(+)}$ . Then the neutral connected component  $\text{Aut}_0(X)$  of the automorphism group  $\text{Aut}(X)$  is isomorphic to  $\mathbb{C}^*$  and the group of components  $\text{Aut}(X)/\text{Aut}_0(X)$  is finite.*  $\square$

**Example 4.3.** Let  $X$  be an Inoue surface of type  $S_M^{(+)}$  defined by  $G_M^{(+)}$  and keep the notation in Section 4.2. It is known that, for  $n_1, n_2, l, k \in \mathbb{Z}$ ,

$$g_1^{n_1} g_2^{n_2} g_3^l g_0^k(w, z) = \left( \alpha^k w + \sum_j n_j a_j, z + k\tau + l \frac{b_1 a_2 - b_2 a_1}{r} + \left( \sum_j n_j b_j \right) \alpha^k w + \sum_j n_j c_j + e(n_1, n_2) \right),$$

where

$$e(n_1, n_2) = \frac{1}{2} n_1 (n_1 - 1) a_1 b_1 + \frac{1}{2} n_2 (n_2 - 1) a_2 b_2 + n_1 n_2 a_2 b_1.$$

Pick an element  $B \in \mathbb{C} \simeq Z(K)$  such that  $\{nB \mid n \in \mathbb{Z}_{>0}\}$  is disjoint with the following set

$$T := \left\{ k\tau + l \frac{b_1 a_2 - b_2 a_1}{r} + \left( \sum_j n_j b_j \right) \left( \sum_j n_j \alpha_j \right) \frac{\alpha^k}{1 - \alpha^k} + \sum_j n_j c_j + e(n_1, n_2) \mid n_1, n_2, l, k \in \mathbb{Z} \right\}.$$

This is possible because only  $\tau$  in the definition of the set  $T$  could be a non-real number. Consider the automorphism

$$\tilde{\sigma} \in \widetilde{\text{Aut}}(X), \quad (w, z) \mapsto (w, z + B)$$

and the corresponding automorphism  $\sigma \in \text{Aut}(X)$ . We claim that  $\sigma$  is a wild automorphism.

It follows from the construction of  $\sigma$  that it does not have any periodic points. Indeed, if  $(w, z) \in W$  corresponds to a  $\sigma$ -periodic point on  $X$ , then  $\tilde{\sigma}^n(w, z) = g_1^{n_1} g_2^{n_2} g_3^l g_0^k(w, z)$  for some  $n_1, n_2, l, k \in \mathbb{Z}$  and  $n \in \mathbb{Z}_{>0}$ . Now we have

$$\begin{cases} w = \alpha^k w + \sum_j n_j a_j, \\ z + nB = z + k\tau + l \frac{b_1 a_2 - b_2 a_1}{r} + \left( \sum_j n_j b_j \right) \alpha^k w + \sum_j n_j c_j + e(n_1, n_2), \end{cases}$$

which implies that  $nB \in T$ , a contradiction. Since  $X$  does not contain any curve ([Ino74, Proposition 3.i] in page 277),  $X$  cannot have any non-trivial proper  $\sigma$ -invariant analytic subset. Hence  $\sigma$  acts on  $X$  as a wild automorphism.

**4.3. Type  $S_M^{(-)}$ .** Let  $M \in \text{GL}_2(\mathbb{Z})$  be a matrix with two real eigenvalues  $\alpha$  and  $-1/\alpha$  with  $\alpha > 1$ . Let  $(a_1, a_2)^T$  and  $(b_1, b_2)^T$  be real eigenvectors of  $M$  corresponding to  $\alpha$  and  $1/\alpha$ , respectively, and fix integers  $p_1, p_2, r$  ( $r \neq 0$ ) and a complex number  $\tau$ . Define  $(c_1, c_2)^T$  to be the solution of the following equation

$$-(I + M) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \frac{b_1 a_2 - b_2 a_1}{r} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

where

$$e_i = \frac{1}{2} m_{i,1} (m_{i,1} - 1) a_1 b_1 + \frac{1}{2} m_{i,2} (m_{i,2} - 1) a_2 b_2 + m_{i,1} m_{i,2} b_1 a_2, \quad i = 1, 2.$$

Let  $G_M^{(-)}$  be the group of analytic automorphisms of  $W = \mathbb{H} \times \mathbb{C}$  generated by

$$\begin{aligned} g_0 &: (w, z) \mapsto (\alpha w, -z), \\ g_i &: (w, z) \mapsto (w + a_i, z + b_i w + c_i), \quad i = 1, 2, \\ g_3 &: (w, z) \mapsto \left( w, z + \frac{b_1 a_2 - b_2 a_1}{r} \right). \end{aligned}$$

We have the following relations between these generators

$$\begin{aligned} g_3 g_i &= g_i g_3 \quad \text{for } i = 1, 2, \quad g_1^{-1} g_2^{-1} g_1 g_2 = g_3^r, \\ g_0 g_j g_0^{-1} &= g_1^{m_{j,1}} g_2^{m_{j,2}} g_3^{p_j} \quad \text{for } j = 1, 2, \quad g_0 g_3 g_0^{-1} = g_3^{-1}. \end{aligned}$$

The action of  $G_M^{(-)}$  is free and properly discontinuous. The quotient space  $X := W/G_M^{(-)}$  is an Inoue surface of type  $S_M^{(-)}$  (cf. [Ino74, § 4]). Note that  $G_M^{(-)} \simeq H(r) \rtimes \mathbb{Z}$  as an abstract group, where  $H(r) = \langle g_1, g_2, g_3 \mid g_3 g_i = g_i g_3, g_1^{-1} g_2^{-1} g_1 g_2 = g_3^r \rangle$  and  $\mathbb{Z}$  is generated by  $g_0$ . We have

$$(4.5) \quad Z(H(r)) = \langle g_3 \rangle \simeq \mathbb{Z} \quad \text{and} \quad H(r)/Z(H(r)) \simeq \mathbb{Z}^2.$$

Similarly, there is a short exact sequence of groups

$$1 \longrightarrow G_M^{(-)} \longrightarrow \widetilde{\text{Aut}}(X) \longrightarrow \text{Aut}(X) \longrightarrow 1,$$

where  $\widetilde{\text{Aut}}(X) \leq \text{Aut}(W)$  is the normaliser of  $G_M^{(-)}$ .

Let

$$\Gamma := \{N \in \text{GL}_2(\mathbb{Z}) \mid N \text{ and } M \text{ are simultaneously diagonalisable}\},$$

which is an abelian group, and let  $\tau: \widetilde{\text{Aut}}(X) \rightarrow \Gamma$  be the homomorphism (not necessarily surjective) defined by  $u \mapsto N$ , where  $N$  is the matrix associated with  $u$  as constructed in [Jia22, § 6.2]. Let  $K$  be the kernel of this homomorphism, with image  $\Gamma' \leq \Gamma$ . It is clear that any automorphism in  $K$  has the form

$$(w, z) \mapsto (w + b, z + Aw + B)$$

for some  $b \in \mathbb{R}$  and  $A, B \in \mathbb{C}$  satisfying some conditions. An explicit calculation gives that the centre

$$(4.6) \quad Z(K) \simeq \mathbb{Z}$$

which is generated by

$$(w, z) \mapsto \left( w, z + \frac{1}{2} \cdot \frac{b_1 a_2 - b_2 a_1}{r} \right).$$

Further,

$$(4.7) \quad K/Z(K) \simeq \mathbb{Z} \rtimes \mathbb{Z}$$

which is generated by the images of

$$(w, z) \mapsto (w + b, z) \quad \text{and} \quad (w, z) \mapsto (w, z + Aw).$$

Combining these, the snake lemma shows the diagrams below are commutative with exact rows and columns.

$$\begin{array}{ccccccc}
 & 1 & & 1 & & 1 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 1 \rightarrow & H(r) & \rightarrow & K & \rightarrow & F & \rightarrow 1 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 1 \rightarrow & G_M^{(-)} & \rightarrow & \widetilde{\text{Aut}}(X) & \rightarrow & \text{Aut}(X) & \rightarrow 1 \\
 & \downarrow & & \downarrow \tau & & \downarrow & \\
 1 \rightarrow & \mathbb{Z} & \rightarrow & \Gamma' & \rightarrow & \Gamma'/\mathbb{Z} & \rightarrow 1 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 1 & & 1 & & 1 & 
 \end{array}
 \quad
 \begin{array}{ccccccc}
 & 1 & & 1 & & 1 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 1 \rightarrow & Z(H(r)) & \rightarrow & H(r) & \rightarrow & H(r)/Z(H(r)) & \rightarrow 1 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 1 \rightarrow & Z(K) & \rightarrow & K & \rightarrow & K/Z(K) & \rightarrow 1 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 1 \rightarrow & F' & \rightarrow & F & \rightarrow & F'' & \rightarrow 1 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 1 & & 1 & & 1 & 
 \end{array}$$

Note that both  $F'$  and  $F''$  are finite by Equations (4.5)–(4.7). Consequently,  $F$  is also a finite group. Also, as before, the quotient  $\Gamma'/\mathbb{Z} \leq \Gamma/\mathbb{Z}$  is finite by Lemma 2.14.

**Theorem 4.4.** *Let  $X$  be an Inoue surface of type  $S_M^{(-)}$ . Then the automorphism group  $\text{Aut}(X)$  is finite. In particular,  $X$  does not have any wild automorphism.*  $\square$

*Proof of Theorem 1.3.* Part (1) is a consequence of Propositions 3.1 and 3.2. Part (2) follows from [CPR20, Lemma 6.7] and Example 4.3 immediately.  $\square$

*Remark 4.5.* Let  $V$  be a compact complex manifold. Then the automorphism group  $\text{Aut}(V)$  is a complex Lie group whose Lie algebra is  $\Gamma(V, \Theta_V)$ . Here  $\Theta_V$  is the holomorphic tangent bundle of  $V$ .

By [Ino74, Propositions 2.ii), 3.ii) and 5], we have  $\dim \Gamma(X, \Theta_X) = 0, 1, 0$  for  $X$  an Inoue surface of type  $S_M, S_M^{(+)}, S_M^{(-)}$ , respectively, with which our results are compatible.

## 5. WILD AUTOMORPHISMS IN DIMENSION THREE AND FOUR

In this section, we study wild automorphisms of compact Kähler manifolds, and prove Theorems 1.7 and 1.9 following the strategy of [OZ22].

**Lemma 5.1.** *Let  $X$  be a compact Kähler space of dimension  $n$  with a wild automorphism  $\sigma$ . Then the Albanese morphism  $\text{alb}_X: X \rightarrow \text{Alb}(X)$  is a smooth surjective morphism with connected fibres and hence  $0 \leq q(X) = \dim \text{Alb}(X) \leq n$ . Moreover, if  $q(X) = n$ , then  $X = \text{Alb}(X)$ , a complex torus; if  $q(X) \geq n - 1$ , then  $\sigma$  has zero entropy.*

*Proof.* The same proof of [OZ22, Proposition 2.15] works under our setting. Notice that the product formula used there holds for compact Kähler manifolds (see [DNT12, Theorem 1.1]). Moreover, we use Lemmas 2.5 and 2.1 (6) instead of [OZ22, Lemmas 2.7 and 2.5 (6)]  $\square$



**Proposition 5.2** (cf. [OZ22, Proposition 3.2]). *Let  $X$  be a compact Kähler space of dimension three with a wild automorphism  $\sigma$ . Suppose that  $X$  is neither a complex torus, nor a weak Calabi–Yau manifold.*

*Then  $X$  is uniruled and the maximal rationally connected (MRC) fibration can be chosen to be a  $\sigma$ -equivariant surjective smooth morphism  $f: X \rightarrow Y$  and coincides with the Albanese map  $\text{alb}_X: X \rightarrow \text{Alb}(X) = \text{Alb}(Y)$ , such that  $\sigma$  acts on  $Y$  as a wild automorphism and one of the following cases occurs.*

- (1) *The irregularity  $q(X) = 1$ . In this case,  $Y$  is an elliptic curve,  $\sigma$  acts on  $Y$  as a translation of infinite order and every closed fibre  $X_y$  over  $y \in Y$  is a smooth rational surface.*
- (2) *The irregularity  $q(X) = 2$ . In this case,  $Y$  is a complex torus of dimension two,  $\sigma$  acts on  $Y$  as a wild automorphism, and every closed fibre  $X_y$  over  $y \in Y$  is a smooth rational curve:  $X_y \cong \mathbb{P}^1$ .*

*Proof.* Since  $X$  is neither a complex torus, nor a weak Calabi–Yau threefold, the Kodaira dimension  $\kappa(X) < 0$  by Propositions 2.2 (2) and 2.12. Hence  $X$  is uniruled by [Bru06, Corollary 1.2]. Moreover, by Lemma 2.6, the MRC fibration  $f: X \rightarrow Y$  is a  $\sigma$ -equivariant surjective smooth morphism with  $1 \leq \dim Y \leq 2$ . Therefore,  $Y$  is a complex torus of dimension 1 or 2, and every fibre  $F$  of  $f$  is a smooth rational variety of dimension 2 or 1, respectively. Here we use the fact that a rationally connected variety of dimension at most two is a rational variety.

Since  $Y = \text{Alb}(Y)$ , there is a morphism  $g: \text{Alb}(X) \rightarrow Y$  such that  $g \circ \text{alb}_X = f$  by the universal property of the Albanese map  $\text{alb}_X$ . Since  $f$  is surjective with connected fibres, so is  $g$ . Let  $X_y$  ( $y \in Y$ ) be any closed fibre of  $f$ . Then  $\text{alb}_X(X_y)$  is a point for each  $X_y$ , because  $X_y$  is rationally connected, and any complex torus contains no rational curve. Thus,  $g$  is a finite morphism (with connected fibres), so it is an isomorphism by the normality of  $\text{Alb}(X)$  and  $Y$ . This proves the proposition.  $\square$

In what follows, we show that none of these two cases in the above proposition occurs, so that Theorem 1.7 follows.

**Lemma 5.3** (cf. [OZ22, Proposition 4.1]). *The case  $q(X) = 2$  in Proposition 5.2 does not occur.*

*Proof.* Suppose that the case  $q(X) = 2$  in Proposition 5.2 occurs. Consider the  $\sigma$ -equivariant fibration  $f: X \rightarrow Y$  with  $X_y \cong \mathbb{P}^1$  the fibre over  $y \in Y$ . Note that  $f$  is a locally trivial projective morphism. By Theorem 2.4, we can write the wild automorphism  $\sigma_Y = T_b \circ \alpha$  on  $Y$  where  $T_b$  is a translation on the complex torus  $Y$  and  $\alpha: Y \rightarrow Y$  is a group automorphism such that the endomorphism  $\beta = \alpha - \text{id}_Y$  is nilpotent.

If  $\beta = 0$ , then  $\sigma_Y = T_b$  and it is a translation. By Lemma 2.10, the anti-canonical divisor  $-K_{X_y}$  of a general fibre  $X_y$ , is not ample, which contradicts that  $X_y \cong \mathbb{P}^1$  has ample anti-canonical divisor. Thus we may assume that  $\beta \neq 0$ .

Let  $B$  be a connected component of  $\text{Ker } \beta$ . Then  $B$  is a non-trivial proper subtorus of the 2-dimensional complex torus  $Y$ , and hence an elliptic curve. Thus,  $\sigma_Y$  permutes the cosets of  $E := Y/B$ , an elliptic curve. Hence the quotient map  $Y \rightarrow E$  is  $\sigma_Y$ -equivariant. Since the action of  $\sigma_Y$  on  $Y$  is wild, so is the induced action  $\sigma_E$  on  $E$  (cf. Lemma 2.1(4)). Hence  $\sigma_E$  is a translation of infinite order (see Proposition 2.7).

For the rest, the identical proof of [OZ22, Proposition 4.1] still works here, with their Lemma 2.5 and Theorem 3.1 replaced by our Lemma 2.1 and Proposition 3.1, respectively.  $\square$

*Proof of Theorem 1.7.* Suppose that  $X$  is neither a complex torus nor a weak Calabi–Yau threefold. Then by Proposition 5.2 and Lemma 5.3,  $X$  is as in Case (1) in Proposition 5.2. So  $X$  admits a rationally connected fibration to an elliptic curve. This implies that  $X$  is projective (cf. [Pro21, 16.3.1 Proposition]), which contradicts [OZ22, Proposition 5.2]. Therefore,  $X$  is either a complex torus or a weak Calabi–Yau threefold. Furthermore, by Lemma 2.1 (6) and [OZ22, Lemma 6.1],  $\sigma$  has zero entropy.  $\square$

Recall that a weak Calabi–Yau manifold  $V$  is a complex projective manifold with torsion canonical divisor and finite fundamental group. In particular,  $H^1(\mathcal{O}_V) = 0$  and  $\text{Pic}^0(V)$  is trivial.

So on  $V$  the Picard group coincides with the Néron–Severi group, which is a finitely generated abelian group.

Now we consider a weak Calabi–Yau threefold  $X$ . By a result of Miyaoka ([Miy87, Theorem 6.6]), we have  $c_2(X) \cdot D \geq 0$  for each nef Cartier divisor  $D$  on  $X$ . Moreover, by [Kob14, Corollary IV.4.15],  $c_2(X) \neq 0$ , and thus,  $c_2(X) \cdot H > 0$  for every ample Cartier divisor  $H$  on  $X$ .

*Proof of Proposition 1.8.* (1) Assume that  $c_2(X) \cdot D > 0$  for every non-torsion nef Cartier divisor  $D$  on  $X$ . Then it is well-known that the automorphism group  $\text{Aut}(X)$  is finite (see e.g., [Wil94, Lemma 3.1]). Thus,  $X$  admits no wild automorphism in this case.

(2) Assume that there is a non-torsion semi-ample Cartier divisor  $D$  such that  $c_2(X) \cdot D = 0$ . Then some multiple of  $D$  induces a  $c_2$ -contraction (in the sense that  $c_2(X) \cdot D = 0$ ; cf. [OS01, Page 45]). Consider the maximal  $c_2$ -contraction  $\phi: X \rightarrow Y$  on  $X$  (see [OS01, Lemma-Definition 4.1]). Then  $\phi$  is not an isomorphism and  $1 \leq \dim Y \leq 3$ .

Suppose that  $X$  admits a wild automorphism  $\sigma$ . Then by the uniqueness of the maximal  $c_2$ -contraction ([OS01, Lemma-Definition 4.1]),  $\sigma$  descends to an automorphism on  $Y$  which is wild by Lemma 2.1 (4). If  $\dim Y = 3$ , then  $\phi$  is birational and hence an isomorphism by Lemma 2.1 (5), a contradiction. If  $\dim Y \leq 2$ , then by Theorem 1.3,  $Y$  is a complex torus. On the other hand, since  $\pi_0(X)$  is finite,  $H^1(\mathcal{O}_X) = 0$ , which implies  $H^1(\mathcal{O}_Y) = 0$ . This is a contradiction. Therefore,  $X$  admits no wild automorphism.  $\square$

We refer to [OS01] for classification results and concrete examples of Calabi–Yau threefolds satisfying assumption (2) in Proposition 1.8.

*Proof of Theorem 1.9.* Claim (1) follows from Theorems 1.3 and 1.7. For Claim (2), first note that by Proposition 2.2 (2),  $K_X \sim_{\mathbb{Q}} 0$ ; moreover, we may assume that  $X$  coincides with its Beauville–Bogomolov minimal split cover and that each factor  $X_i$  of  $X$  is stable under  $\sigma$  with  $\sigma_{X_i}$  wild. By Proposition 2.2 (1),  $\chi(\mathcal{O}_{X_i}) = 0$ , so  $X_i$  is either a complex torus or a Calabi–Yau threefold in the strict sense. Note that  $\sigma$  has zero entropy if and only if  $\sigma_{X_i}$  has zero entropy for every factor  $X_i$ . Now Claim (2) follows from Lemma 2.1 (6) and Theorem 1.7.

Let us finally show Claim (3). Note that  $X$  is smooth. By assumption and Lemma 5.1, we may assume  $q(X) = 0$ . It is equivalent to show the first dynamical degree  $d_1(\sigma) = 1$ . Now the argument of [OZ22, Lemma 6.3] is still available. In fact, the proof of [OZ22, Lemma 6.3] involves Hodge theory which holds for compact Kähler manifolds (see [Voi02, Chapter II, § 6]).  $\square$

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