AUTOMORPHISMS GROUPS OF COMPACT COMPLEX SURFACES: T-JORDAN PROPERTY, TITS ALTERNATIVE AND SOLVABILITY

JIA JIA

Abstract. Let X be a (smooth) compact complex surface. We show that the torsion subgroup of the biholomorphic automorphisms group $\mathrm{Aut}(X)$ is virtually nilpotent. Moreover, we study the Tits alternative of $\mathrm{Aut}(X)$ and virtual derived length of virtually solvable subgroups of $\mathrm{Aut}(X)$.

Contents

1. Introduction

1.1. **T-Jordan Property.** The below theorem is our starting point.

Theorem 1.1 ([Lee76])**.** *Let G be a connected Lie group. Then there is a constant $J = J(G)$ such that every torsion subgroup of G contains a (normal) abelian subgroup of index $\leq J$.*

Inspired by the *Jordan property* introduced by Popov [Pop11, Definition 2.1], we propose the following generalisation. A group G is called *T-Jordan* (alternatively, we say that G has the *T-Jordan* property) if there is a constant $J(G)$ such that every torsion subgroup H of G has an abelian subgroup H_1 with the index $[H : H_1] \leq J(G)$.

For a compact complex manifold X, we denote by $\mathrm{Aut}(X)$ the group of biholomorphic automorphisms. As in [Pop11], we propose the following conjecture.

Conjecture 1.2. *Let X be a compact complex manifold. Then $\mathrm{Aut}(X)$ is T-Jordan.*

Using the equivariant Kähler model, Meng and the author gave an affirmative answer to Conjecture 1.2 for compact complex manifolds in Fujiki's class $\mathcal{C}$ (for the definition, see [Fuj78, Definition 1.1]).

Theorem 1.3 (cf. [JM22, Corollary 1.5])**.** *Let X be a compact complex space which is in Fujiki's class $\mathcal{C}$. Then $\mathrm{Aut}(X)$ is T-Jordan.*

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In this paper, we are focusing on smooth compact complex surfaces, especially the non-Kähler (= non-Fujiki's class $\mathcal{C}$) surfaces. We use the following notation.

- Let Ξ be the set of smooth compact complex surface X in class VII with the algebraic dimension $a(X) = 0$ and the second Betti number $b_2(X) > 0$.
- Let $\Xi_0 \subseteq \Xi$ be those minimal surfaces which have no curve.

Here, surfaces of class VII are those smooth compact complex surfaces with the first Betti number $b_1 = 1$ and Kodaira dimension $\kappa = -\infty$ (cf. [Kod64, § 7]). By a *minimal surface* we mean a smooth compact complex surface that does not contain any (-1)-curve C (i.e., a smooth rational curve C with self-intersection $C^2 = -1$). It is known that a smooth compact complex surface X is minimal if and only if any bimeromorphic holomorphic map $X' \longrightarrow X$ from a smooth compact complex surface X' is an isomorphism.

Proposition 1.4. *Let X be a compact complex surface not in Ξ_0. Then* $\mathrm{Aut}(X)$ *is T-Jordan.*

Although we are not able to confirm Conjecture 1.2 for surfaces in Ξ_0, we have a slightly weaker result on torsion subgroups of $\mathrm{Aut}(X)$. Recall that a group G is *virtually* $\mathcal{P}$, if a finite-index subgroup of G has the property $\mathcal{P}$. We say that a group G is $\mathcal{P}$*-by-*$\mathcal{Q}$ if G fits into an short exact sequence

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$$

where K has the property $\mathcal{P}$ and H has the property $\mathcal{Q}$.

Proposition 1.5. *Let X be a smooth compact complex surface in Ξ_0. Let $G \leq$* $\mathrm{Aut}(X)$ *be a torsion subgroup. Then G is finite-by-abelian and hence virtually nilpotent and virtually metabelian.*

Since virtually abelian groups are virtually nilpotent, the next result is a direct consequence of Propositions 1.4 and 1.5.

Theorem 1.6. *Let X be a smooth compact complex surface. Then any torsion subgroup $G \leq$* $\mathrm{Aut}(X)$ *is virtually nilpotent.*

Note that by the global spherical shell (GSS) conjecture, $\Xi_0 = \emptyset$ (cf. [Nak90, Conjecture 1]).

Remark 1.7. Assume the global spherical shell conjecture. Let X be a smooth compact complex surface. Then $\mathrm{Aut}(X)$ is T-Jordan.

1.2. **Tits Alternative.** In group theory, the *Tits alternative* is an outstanding theorem about the structure of linear groups (cf. [Tit72]). Campana, Wang and Zhang studied the automorphism groups of compact Kähler manifolds and proved a Tits type alternative for such groups (cf. [CWZ13, Theorem 1.5]). We generalise their celebrated result to compact complex spaces in Fujiki's class $\mathcal{C}$ and smooth compact complex surfaces.

Theorem 1.8. *Let X be a compact complex space in Fujiki's class $\mathcal{C}$. Then* $\mathrm{Aut}(X)$ *satisfies the Tits alternative, that is, for any subgroup G of* $\mathrm{Aut}(X)$ *either it contains a non-abelian free subgroup, or it is virtually solvable, i.e., admits a solvable subgroup of finite index.*

For the definition and properties of Enoki surfaces and Inoue-Hirzebruch surfaces, see [Eno81] and [Ino77], respectively.

Theorem 1.9. *Let X be a smooth compact complex surface. Assume that either $X \notin \Xi$, or $X \in \Xi$ but its minimal model is an Enoki or Inoue-Hirzebruch surface. Then* $\mathrm{Aut}(X)$ *satisfies the Tits alternative.*

1.3. **Virtual Derived Length.** Once we have established the Tits type alternative for $\mathrm{Aut}(X)$, it is natural to study the properties of those virtually solvable subgroups.

We have the following result, which gives a uniform upper bound of the virtual derived length (cf. Definition 2.1) for virtually solvable subgroups of $\mathrm{Aut}(X)$.

Theorem 1.10. *Let X be a smooth compact complex surface. Assume that either $X \notin \Xi$, or $X \in \Xi$ but its minimal model is an Enoki or Inoue-Hirzebruch surface. Let $G \leq \mathrm{Aut}(X)$ be a virtually solvable subgroup. Then the virtual derived length $\ell_{\mathrm{vir}}(G) \leq 4$.*

Remark 1.11.

(1) Currently, we are not able to prove Theorems 1.9 and 1.10 in full generality for $X \in \Xi$.
(2) The GSS conjecture claims that any minimal surface in Ξ is a Kato surface (cf. [Nak90, Conjecture 1] and [DOT03, Main Theorem]).
(3) Kato surfaces are divided into four classes: Enoki surfaces (including parabolic Inoue surfaces), half Inoue surfaces, Inoue-Hirzebruch surfaces and intermediate surfaces (cf. [Tel19, §3.3.2]).
(4) When X is a parabolic Inoue surface (which is in Ξ), it has been proved that $\mathrm{Aut}(X)$ is virtually abelian (cf. [Fuj09, Theorem 1.1]). In particular, $\mathrm{Aut}(X)$ satisfies the Tits alternative and any subgroup G of $\mathrm{Aut}(X)$ is virtually solvable with virtual derived length $\ell_{\mathrm{vir}}(G) \leq 1$.

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2. Preliminaries

In this section, we gather some general preliminary results.

Definition 2.1. Recall that for a group G, its *n-th derived subgroup* is defined recursively by

$$G^{(0)} := G \quad \text{and} \quad G^{(n)} := [G^{(n-1)}, G^{(n-1)}],\ n \in \mathbb{N}.$$

By definition, the group G is *solvable* if and only if $G^{(n)} = 1$ for some non-negative integer n. The minimum of such n is the *derived length* of G (when G is solvable), denoted by $\ell(G)$.

If G is virtually solvable, we define the *virtual derived length* to be

$$\ell_{\mathrm{vir}}(G) := \min_{G'} \ell(G')$$

where G' runs through all finite-index subgroups of G.

A group G has *bounded torsion subgroups* if there is a constant $T = T(G)$ such that any torsion subgroup $G_1 \leq G$ has order $|G_1| \leq N$.

Lemma 2.2. *Let*

$$1 \longrightarrow N \longrightarrow G \longrightarrow H$$

be an exact sequence of groups. Then the following assertions hold:

(1) If N is T-Jordan and H has bounded torsion subgroups, then G is T-Jordan.
(2) Assume further that the sequence is also right exact. If N is a torsion group and G is T-Jordan, then H is T-Jordan.

Proof. (1) is clear. (2) Let $H' \leq H$ be a torsion subgroup, and $G' \leq G$ be its preimage. Since N is a torsion group, G' is also a torsion group and hence there exists an abelian subgroup $G'_1 \leq G'$ such that $[G' : G'_1] \leq J(G)$ for some constant $J(G)$. Let $H'_1 = G'_1/(G'_1 \cap N) \leq H'$, which is abelian with $[H' : H'_1] \leq J(G)$. Therefore, H is T-Jordan. □

The following two lemmas are adapted from [PS21, Corollaries 4.2, 4.4].

Lemma 2.3. *Let X be an irreducible Hausdorff reduced complex space, and let $\Gamma \leq \mathrm{Aut}(X)$ be a torsion subgroup. Suppose that Γ has a fixed point x on X. Then the natural representation*

$$\Gamma \longrightarrow \mathrm{GL}(T_{x,X})$$

is faithful.

Lemma 2.4. *Let X be an irreducible Hausdorff reduced complex space, and let $\Delta \leq \mathrm{Aut}(X)$ be a subgroup. Suppose that Δ has a fixed point x on X, and let*

$$\sigma\colon \Delta \longrightarrow \mathrm{GL}(T_{x,X})$$

be the natural representation. Suppose that there is a normal subgroup $\Gamma \leq \Delta$ with the quotient group Δ/Γ being torsion, such that the restriction $\sigma|_\Gamma$ is a group monomorphism. Then σ is an embedding as well.

Proof. Let Δ_0 be the kernel of σ. Since $\Delta_0 \cap \Gamma = \{\mathrm{id}\}$ and Δ/Γ is torsion, we see that Δ_0 is also torsion. Thus, Δ_0 is trivial by Lemma 2.3. □

Lemma 2.5. *Let $G \leq \mathrm{GL}_n(\mathbb{C})$ be a virtually solvable subgroup. Then $\ell_{\mathrm{vir}}(G) \leq n$*

Proof. By passing to a finite-index subgroup, we may assume that G itself is solvable. Then some subgroup G_1 of finite index in G can be put in triangular form (cf. [Bor12, Corollary, pp. 137]). It follows that the derived length of G_1 and hence the virtual derived length of G is at most n. □

Lemma 2.6. *Consider the short exact sequence of groups*

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1.$$

Then

(1) if N is solvable and H is virtually solvable, then G is virtually solvable with $\ell_{\mathrm{vir}}(G) \leq \ell(N) + \ell_{\mathrm{vir}}(H)$.
(2) if N is finite and H is virtually solvable, then G is virtually solvable with $\ell_{\mathrm{vir}}(G) \leq \ell_{\mathrm{vir}}(H) + 1$;
(3) the group G is virtually solvable if and only if both N and H are virtually solvable;
(4) if both N and H satisfy the Tits alternative, then so does G.

Proof. (1) Let $H' \leq H$ be a solvable subgroup of finite index (with $\ell(H') = \ell_{\mathrm{vir}}(H)$), and let $G' \leq G$ be its preimage. Since $[G : G'] = [G/N : G'/N] = [H : H'] < \infty$, we may replace G, H by G', H' and assume that H is solvable. Then it is clear that G is solvable, and $\ell_{\mathrm{vir}}(G) \leq \ell(G) \leq \ell(N) + \ell(H) \leq \ell(N) + \ell_{\mathrm{vir}}(H)$.

(2) As in (1), we may assume that H is solvable. Let $Z_G(N)$ be the centraliser of N in G. Since N is finite, $Z_G(N) \leq G$ is of finite index. Note that $Z_G(N) \cap N$ is abelian (in particular, solvable) and $Z_G(N)/(Z_G(N) \cap N) \leq G/N \simeq H$ is solvable. Hence, $Z_G(N)$ is also solvable with derived length $\ell(Z_G(N)) \leq \ell(H) + 1$. It follows that $\ell_{\mathrm{vir}}(G) \leq \ell_{\mathrm{vir}}(H) + 1$.

(3) The "only if" direction is clear. Suppose that N and H are virtually solvable. As in (2), we may assume that H is solvable. Let $N' \lhd N$ be a finite-index solvable (normal) subgroup of minimal index. Then N' is normal in G; otherwise, if N'' is another conjugate of N', then $N'N'' \lhd N$ is solvable with smaller index, a contradiction to the choice of N'. Consider the following short exact sequence

$$1 \longrightarrow N/N' \longrightarrow G/N' \longrightarrow G/N \simeq H \longrightarrow 1.$$

By assumption, the first term is finite and H is solvable. Thus, G/N' is virtually solvable by (2). Then using the short exact sequence

$$1 \longrightarrow N' \longrightarrow G \longrightarrow G/N' \longrightarrow 1,$$

we obtain that G is virtually solvable by (1).

(4) This follows from (3). $\square$

Using the lemma above, we can give a proof of Theorem 1.8.

Proof of Theorem 1.8. Let $\pi\colon X' \longrightarrow X$ be an $\mathrm{Aut}(X)$-equivariant resolution of singularities (cf. [BM97, Theorem 13.2]) with $\mathrm{Aut}(X)$ lifts to a (unique) subgroup of $\mathrm{Aut}(X')$ via π. Note that X' is still in Fujiki's class $\mathcal{C}$. Therefore, we may replace X by X' and assume that X is smooth.

Denote by

$$\mathrm{Aut}_\tau(X) \coloneqq \{g \in \mathrm{Aut}(X) \mid g^*|_{H^2(X,\mathbb{Q})} = \mathrm{id}\}.$$

Then we have the following short exact sequence

$$1 \longrightarrow G_\tau \longrightarrow G \longrightarrow G|_{H^2(X,\mathbb{Q})} \longrightarrow 1$$

where $G_\tau = G \cap \mathrm{Aut}_\tau(X)$. It suffices to prove that G_τ satisfies the Tits alternative by [Tit72, Theorem 1.1] and Lemma 2.6 (4). Let $\sigma\colon \widetilde{X} \longrightarrow X$ be a bimeromorphic holomorphic map from a compact Kähler manifold $\widetilde{X}$ such that $\mathrm{Aut}_\tau(X)$ lifts to $\widetilde{X}$ holomorphically via σ (cf. [JM22, Theorem 1.1]). Viewing $G_\tau \le \mathrm{Aut}_\tau(X)$ as a subgroup of $\mathrm{Aut}(\widetilde{X})$, we know that G_τ satisfies the Tits alternative (cf. [CWZ13, Theorem 1.5]). Therefore, G satisfies the Tits alternative. $\square$

The theorem below is due to [Pin84, §2, Theorem].

Theorem 2.7. *Let X be an Inoue-Hirzebruch surface. Then $\mathrm{Aut}(X)$ is finite. In particular, $\mathrm{Aut}(X)$ satisfies the Tits alternative and every subgroup $G \le \mathrm{Aut}(X)$ is finite and hence virtually solvable with virtual derived length $\ell_{\mathrm{vir}}(G) = 0$.*

The following lemmas will be used in Section 8.

Lemma 2.8 (cf. [Hal50, §2, 4])**.** *Let G be a finitely generated group. Then every finite-index subgroup of G contains a finite-index subgroup G' which is characteristic in G.*

Lemma 2.9. *Let G be a solvable group and $U \lhd G$ a non-trivial unipotent subgroup. Then, after replacing G by a finite-index subgroup, there is a subgroup $Z_1 \simeq \mathbb{G}_a \le U$ which is normal in G.*

Proof. Let $Z = Z(U)$ be the centre of U which is nontrivial and normal in G. Note that the commutative unipotent group $Z \simeq \mathbb{G}_a^n$, the additive affine algebraic group of dimension n for some $n \ge 1$. Consider the conjugation action of G on Z:

$$c_g\colon x \mapsto gxg^{-1} \quad \text{for } x \in Z \text{ and } g \in G.$$

One can show that this action is linear and induces a homomorphism $G \longrightarrow \mathrm{GL}_n(\mathbb{C})$. Replacing G by a finite-index subgroup, we may assume that the image G' of the solvable group G in $GL_n(\mathbb{C})$ is of triangular form (cf. [Bor12, Corollary, pp. 137]). Let $Z_1 \simeq \mathbb{G}_a$ be an (1-dimensional) eigenspace of G'. Then Z_1 is stable under the conjugation action of G and hence is normal in G. $\square$

3. Equivariant Fibrations

In this section we will consider smooth compact complex surfaces X with algebraic dimension $a(X) = 1$, or with Kodaira dimension $\kappa(X) = 1$. If $a(X) = 1$, there is a natural elliptic fibration, the *algebraic reduction* $\pi\colon X \longrightarrow Y$ (cf. [BHPV04, VI. (5.1) Proposition]), which is a holomorphic map. For $g \in \mathrm{Aut}(X)$, the image of a fibre F of π under g is another fibre; otherwise the self-intersection number of $g(F) + F$ is positive and hence X is projective (cf. [BHPV04, IV, Theorem 6.2]), a contradiction (cf. [BHPV04, IV, Corollary 6.5]). Thus, π is $\mathrm{Aut}(X)$-equivariant. When $\kappa(X) = 1$, the *pluricanonical fibration* gives a natural elliptic fibration as well, which is also $\mathrm{Aut}(X)$-equivariant.

In addition to the above, we will also study affine $\mathbb{A}^1$-bundle $\pi\colon A \longrightarrow E$ over an elliptic curve E. Note that the fibre of $\pi\colon A \longrightarrow E$ being an affine line $\mathbb{A}^1$ cannot dominate the base elliptic curve E. It follows that π is $\mathrm{Aut}(A)$-equivariant.

Let $\pi\colon X \longrightarrow Y$ be a morphism between complex spaces. Let G be a complex Lie group contained in $\mathrm{Aut}(X)$ such that $\pi \circ g = \pi$ for all $g \in G$, i.e., $G \le \mathrm{Aut}_Y(X)$. For any finite subset $S \subseteq Y$, denote by $G_S := \{g \in G \mid g|_{\pi^{-1}(S)} = \mathrm{id}\}$, i.e., G_S is the kernel of the natural group homomorphism $G \longrightarrow \prod_{y\in S} \mathrm{Aut}(X_y)$. Note that any Lie group is second countable and hence has only countably many connected components.

The ideas of Lemma 3.1 and Proposition 3.2 come from Doctor Sheng Meng.

Lemma 3.1. *Suppose that* $\dim G = 0$. *Then* G_y *is trivial for very general* $y \in Y$.

Proof. Let $Y_g := \{y \in Y \mid g|_{X_y} = \mathrm{id}\}$. Then Y_g is a proper closed subspace of Y for $g \neq \mathrm{id}$. Let $Z := \bigcup_{\mathrm{id}\neq g\in G} Y_g$. Then any $y \in Y \setminus Z$ is a very general choice. Here we use the condition that G is countable since $\dim G = 0$. □

A very general finite subset $S \subseteq Y$ means that every element of S is very generally chosen in Y.

Proposition 3.2. *For very general finite subset* $S \subseteq Y$, G_S *is trivial.*

Proof. First, we show by induction on $\dim G$ that $\dim G_S = 0$ for very general finite subset $S \subseteq Y$. When $\dim G = 0$, the claim follows from Lemma 3.1. Now assume that $\dim G \geq 1$. Denote by $T := \{y \in Y \mid G_0|_{X_y} = \mathrm{id}\}$, where G_0 is the neutral component of G. Note that $T \subsetneq Y$ is a closed subspace. Then for $y \notin T$, we have $\dim G_y < \dim G$. By induction, $\dim(G_y)_S = 0$ for any $y \notin T$ and very general finite subset $S \subseteq Y$ (depending on y). Note that $(G_y)_S = G_{\{y\}\cup S}$. So the claim is proved.

Then by Lemma 3.1, we see that $G_{S\cup\{y\}} = (G_S)_y$ is trivial for very general $y \in Y$. □

Lemma 3.3. *Let* X *be either a (primary or secondary) Kodaira surface, or a minimal smooth compact complex surface of Kodaira dimension* 1. *Then there is an* $\mathrm{Aut}(X)$*-equivariant elliptic fibration* $\pi\colon X \longrightarrow Y$ *to a smooth curve* Y. *Also, the natural image of* $\mathrm{Aut}(X)$ *in* $\mathrm{Aut}(Y)$ *is finite.*

Proof. We may take π to be either the algebraic reduction or the pluricanonical fibration (see the first paragraph of this section). Then our claim follows from [Shr20, Corollary 3.4, Lemma 3.5] and [PS20b, Proposition 1.2]. □

Theorem 3.4. *Let* X *be either a (primary or secondary) Kodaira surface or a minimal smooth compact complex surface of Kodaira dimension* 1. *Then* $\mathrm{Aut}(X)$ *is virtual abelian. In particular,* $\mathrm{Aut}(X)$ *satisfies T-Jordan property, the Tits alternative and every subgroup* $G \le \mathrm{Aut}(X)$ *is virtually abelian and hence virtually solvable with virtual derived length* $\ell_{\mathrm{vir}}(G) \le 1$.

Proof. By Lemma 3.3, there is a short exact sequence

$$1 \longrightarrow \mathrm{Aut}_Y(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow \Gamma \longrightarrow 1,$$

where $\pi\colon X \longrightarrow Y$ is an elliptic fibration and Γ is a finite subgroup of $\mathrm{Aut}(Y)$. By Proposition 3.2, there exists a finite subset S of Y such that the natural map $\mathrm{Aut}_Y(X) \hookrightarrow \prod_{y\in S} \mathrm{Aut}(X_y)$ is an injection. Since X_y is an elliptic curve, $\prod_{c\in S} \mathrm{Aut}(X_y)$ and hence $\mathrm{Aut}_Y(X)$ are virtually abelian. Then it is clear that $\mathrm{Aut}(X)$ is virtually abelian. □

Theorem 3.5. *Let $X \in \Xi$ be an Enoki surface. Then $\mathrm{Aut}(X)$ is virtually solvable (and hence $\mathrm{Aut}(X)$ satisfies the Tits alternative) and every subgroup $G \le \mathrm{Aut}(X)$ is virtually solvable with $\ell_{\mathrm{vir}}(G) \le 3$.*

Proof. By [Eno81, Main Theorem], X has a cycle of rational curves C such that $A := X \setminus C$, the complement of C in X, is an affine $\mathbb{A}^1$-bundle $\pi\colon A \longrightarrow E$ over an elliptic curve E. Since the algebraic dimension $a(X) = 0$, the surface X contains finitely many (irreducible) curves (cf. [BHPV04, IV. Theorem 8.2]). Replacing $\mathrm{Aut}(X)$ by a finite-index subgroup, we may assume that C is $\mathrm{Aut}(X)$-invariant. Then $\mathrm{Aut}(X)$ acts on $A = X \setminus C$ biholomorphically and faithfully and π is $\mathrm{Aut}(X)$-equivariant (see the second paragraph of this section).

Consider the short exact sequence

$$1 \longrightarrow K \longrightarrow \mathrm{Aut}(X) \longrightarrow \Gamma \longrightarrow 1,$$

where $K \le \mathrm{Aut}_E(A)$ and $\Gamma \le \mathrm{Aut}(E)$. Since E is an elliptic curve, $\mathrm{Aut}(E)$ and hence Γ are virtually abelian, or equivalently, virtually solvable with virtual derived length ≤ 1. By Proposition 3.2, there exists a finite subset S of E such that the natural map $K \hookrightarrow \prod_{e\in S} \mathrm{Aut}(A_e)$ is an injection. Since $A_e \simeq \mathbb{A}^1$, the group $\prod_{e\in S} \mathrm{Aut}(A_e)$ and hence K are metabelian, or equivalently, solvable with derived length ≤ 2. By Lemma 2.6 (1), $\mathrm{Aut}(X)$ is virtually solvable with virtual derived length ≤ 3. It is clear that any subgroup $G \le \mathrm{Aut}(X)$ is also virtually solvable and its virtual derived length $\ell_{\mathrm{vir}}(G) \le 3$. □

4. Surfaces with Invariant Curves

Lemma 4.1. *Let X be a smooth compact complex surface. Suppose that there is a finite non-empty $\mathrm{Aut}(X)$-invariant set Σ of (irreducible) curves on X. Then $\mathrm{Aut}(X)$ is T-Jordan.*

Proof. Let C be one of the curves from Σ. Then the group $\mathrm{Aut}(X, C)$ of automorphisms of X that preserves the curve C has finite index in $\mathrm{Aut}(X)$. We only need to show that $\mathrm{Aut}(X, C)$ is T-Jordan.

Assume first that C is singular. Then $\mathrm{Aut}(X, C)$, after replacing by a subgroup of finite index, fixes some point on C. Now Lemma 2.3 implies that $\mathrm{Aut}(X, C)$ is embedded into $\mathrm{GL}_2(\mathbb{C})$. Therefore, the group $\mathrm{Aut}(X, C)$ is T-Jordan (cf. Theorem 1.1).

Now we assume that C is smooth and let $G \le \mathrm{Aut}(X, C)$ be a torsion subgroup. Let $\mathcal{N}_{C/X}$ be the normal bundle of C in X. Consider the natural group homomorphism

$$\mathcal{N}\colon\ \mathrm{Aut}(X, C) \longrightarrow \mathrm{Aut}(\mathcal{N}_{C/X} \to C)$$

via $\sigma \mapsto \mathcal{N}_\sigma$ (cf. [MPZ22, Notation 2.2]). By [MPZ22, Lemma 2.3], $\ker \mathcal{N} \cap G = \{\mathrm{id}\}$. Moreover, by the proof of [MPZ22, Theorem 3.1], there is a monomorphism

$$\mathrm{Aut}(\mathcal{N}_{C/X} \to C) \longrightarrow \mathrm{Aut}(\mathbb{P}_C(\mathcal{N}_{C/X} \oplus \mathcal{O})).$$

So we may view G as a subgroup of $\mathrm{Aut}(\mathbb{P}_C(\mathcal{N}_{C/X} \oplus \mathcal{O}))$. Note that C is projective. So $\mathbb{P}_C(\mathcal{N}_{C/X} \oplus \mathcal{O})$ is also a projective manifold. Then the theorem follows from Theorem 1.3. □

Corollary 4.2. *Let X be a compact complex surface with $a(X) = 0$. Assume that X contains at least one curve. Then $\mathrm{Aut}(X)$ is T-Jordan.*

Proof. It follows from Lemma 4.1, since X contains at most a finite number of (irreducible) curves (cf. [BHPV04, IV. Theorem 8.2]). □

Proposition 4.3. *Let X be a compact complex surface with $e(X) \neq 0$ and $a(X) = 1$. Then $\mathrm{Aut}(X)$ is T-Jordan.*

Proof. Let $\pi\colon X \longrightarrow Y$ be the algebraic reduction (cf. Section 3), so that Y is a smooth curve and π is an $\mathrm{Aut}(X)$-equivariant elliptic fibration. Since $e(X) \neq 0$, the fibration π has at least one fibre X_y such that $F = (X_y)_{\mathrm{red}}$ is not a smooth elliptic curve by Suzuki's formula (cf. [BHPV04, III, Proposition 11.4]). Note that π has only finitely many singular fibres, whose union is $\mathrm{Aut}(X)$-invariant. Then the assertion follows from Lemma 4.1. □

5. Hopf Surfaces

In this section, we study the automorphism groups of Hopf surfaces. Be definition, a *Hopf surface* X is a smooth compact complex surface with universal covering being isomorphic to $\mathbb{C}^2 \setminus \{0\}$. Then, X can be obtained from $\mathbb{C}^2 \setminus \{0\}$ as a quotient by a free action of a discrete group $\Gamma \simeq \pi_1(X)$. If $\pi_1(X) \simeq \mathbb{Z}$, we call such X a *primary* Hopf surface. In this case, after an appropriate choice of coordinates of $\mathbb{C}^2$, the generator of Γ has the form

$$(z, w) \longmapsto (az + \lambda w^m, bw), \tag{5.1}$$

where $m \in \mathbb{Z}_{>0}$ and a, b, λ are complex constants subject to the restrictions

$$(a - b^m)\lambda = 0, \quad 0 < |a| \leq |b| < 1.$$

Any Hopf surface is either primary or a finite étale quotient of a primary Hopf surface, which are called the Hopf surface (cf. [Kod66, Theorem 30])

By above, Γ contains $\Lambda \simeq \mathbb{Z}$ as a subgroup of finite index. In particular, Γ is virtually solvable. After replacing Λ by a suitable subgroup $\Lambda_0 \simeq \mathbb{Z}$, we may assume that Λ is a characteristic subgroup of Γ (cf. [PS21, Lemma 2.10]), with the generator of Λ having the form Eq. (5.1). Now there is a short exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \widetilde{\mathrm{Aut}}(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow 1 \tag{5.2}$$

where $\widetilde{\mathrm{Aut}}(X)$ acts on $\mathbb{C}^2 \setminus \{0\}$ biholomorphically. It follows from Hartogs extension theorem, $\widetilde{\mathrm{Aut}}(X)$ can be extended to biholomorphic actions on $\mathbb{C}^2$, which fixes the origin $0 \in \mathbb{C}$. By assumption, $\Lambda \lhd \widetilde{\mathrm{Aut}}(X)$ is a normal subgroup.

Theorem 5.1. *Let X be a Hopf surface. Then the group $\mathrm{Aut}(X)$ satisfies the Tits alternative.*

Proof. Assume first that X is a primary Hopf surface. Then we may identify $\widetilde{\mathrm{Aut}}(X)$ with a subgroup of $\mathrm{GL}_2(\mathbb{C}) \times \mathbb{C}$ by [Nam74, § 2] and [Weh81, pp. 24]. Since $\mathrm{GL}_2(\mathbb{C}) \times \mathbb{C}$ satisfies the Tits alternative, so do $\widetilde{\mathrm{Aut}}(X)$ and $\mathrm{Aut}(X)$ (cf. Lemma 2.6 (4)).

Now let X be a secondary Hopf surface. Then either $\widetilde{\mathrm{Aut}}(X) \leq \mathrm{GL}_2(\mathbb{C})$, or $\widetilde{\mathrm{Aut}}(X) \simeq \mathbb{C} \rtimes \mathbb{C}^*$ or $\widetilde{\mathrm{Aut}}(X) \simeq \mathbb{C} \rtimes (\mathbb{C}^*)^2$ (cf. [MN00, Theorem 1]). In the first case, $\widetilde{\mathrm{Aut}}(X)$ and hence $\mathrm{Aut}(X)$ satisfies the Tits alternative as $\mathrm{GL}_2(\mathbb{C})$ does; in the latter two cases, $\widetilde{\mathrm{Aut}}(X)$ and hence $\mathrm{Aut}(X)$ are already solvable. In a word, $\mathrm{Aut}(X)$ satisfies the Tits alternative. □

Theorem 5.2. *Let X be a Hopf surface and $G \leq \mathrm{Aut}(X)$ a virtually solvable subgroup. Then $\ell_{\mathrm{vir}}(G) \leq 2$.*

Proof. Let $\widetilde{G} \leq \widetilde{\mathrm{Aut}}(X)$ be the preimage of G. Then $\widetilde{G}$ is virtually solvable by Lemma 2.6 (3) as Γ is virtually solvable. Therefore, $\ell_{\mathrm{vir}}(G) \leq \ell_{\mathrm{vir}}(\widetilde{G}) \leq \ell_{\mathrm{vir}}(\widetilde{\mathrm{Aut}}(X)) \leq 2$ by Lemma 2.5 and proof of Theorem 5.1. □

The lemma below is a simple linear algebra, which is taken from [PS21, Lemma 6.3].

Lemma 5.3. *Let*

$$M = \begin{pmatrix} a & \lambda \\ 0 & b \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

be an upper triangular matrix, and $Z \le \mathrm{GL}_2(\mathbb{C})$ the centraliser of M. Then the following assertions hold:

(i) If $a = b$ and $\lambda = 0$, then $Z = \mathrm{GL}_2(\mathbb{C})$.
(ii) If $a \neq b$ and $\lambda = 0$, then $Z \simeq (\mathbb{C}^*)^2$.
(iii) If $a = b$ and $\lambda \neq 0$, then $Z \simeq \mathbb{C}^* \times \mathbb{C}^+$.

Theorem 5.4. *Let X be a Hopf surface. Then $\mathrm{Aut}(X)$ is T-Jordan.*

Proof. The proof is adapted from [PS21, Lemma 6.4].

Consider the short exact sequence Eq. (5.2). The image of the generator of Λ is mapped by the natural homomorphism

$$\sigma\colon \widetilde{\mathrm{Aut}}(X) \longrightarrow \mathrm{GL}(T_{0,\mathbb{C}^2}) \simeq \mathrm{GL}_2(\mathbb{C})$$

to the matrix

$$M = \begin{pmatrix} a & \lambda\delta_1^m \\ 0 & b \end{pmatrix}$$

where δ is the Kronecker symbol.

Let $G \le \mathrm{Aut}(X)$ be a torsion subgroup, and $\widetilde{G}$ its preimage in $\widetilde{\mathrm{Aut}}(X)$. Thus, one has $G \simeq \widetilde{G}/\Gamma$. By Lemma 2.4, $\sigma|_{\widetilde{G}}$ is an embedding since σ_Λ is a group monomorphism and $\widetilde{G}/\Lambda$ is torsion. Let Ω be the normaliser of $\sigma(\Lambda)$ in $\mathrm{GL}_2(\mathbb{C})$. By construction $\sigma(\widetilde{G})$ is contained in the normaliser of $\sigma(\Gamma)$ in $\mathrm{GL}_2(\mathbb{C})$, which in turn is contained in Ω because Λ is a characteristic subgroup of Γ. Hence, every torsion subgroup of $\mathrm{Aut}(X)$ is contained in the group $\Omega/\sigma(\Gamma)$. On the other hand, $\Omega/\sigma(\Gamma)$ is a quotient of $\Omega/\sigma(\Lambda)$ by a finite subgroup isomorphic to $\sigma(\Gamma)/\sigma(\Lambda)$. Thus, by Lemma 2.2, it is sufficient to show that the group $\Omega/\sigma(\Lambda)$ is T-Jordan.

Since $\sigma(\Lambda) \simeq \mathbb{Z}$, the group Ω has a (normal) subgroup Ω' of index at most 2 that coincides with the centraliser of the matrix M. It remains to check that the group $\Omega'/\sigma(\Lambda)$ is T-Jordan. If $\lambda = 0$ and $a = b$, it follows from Lemma 5.3(i) that $\Omega'/\sigma(\Lambda)$ is a connected Lie group and hence T-Jordan by Theorem 1.1. If either $\lambda = 0$ and $a \neq b$, or $\lambda \neq 0$ and $m \ge 2$, then this follows from Lemma 5.3(ii) that $\Omega'/\sigma(\Lambda)$ is abelian. If $\lambda \neq 0$ and $m = 1$, then this follows from Lemma 5.3(iii) that $\Omega'/\sigma(\Lambda)$ is abelian. □

6. Inoue Surfaces

An *Inoue surface* X is a compact complex surface obtained from $W := \mathbb{H} \times \mathbb{C}$ as a quotient by an infinite discrete group, where $\mathbb{H}$ is the upper half complex plane. Inoue surfaces are minimal surfaces in class VII, contain no curve, and have the following numerical invariants:

$$a(X) = 0, \quad b_1(X) = 1, \quad b_2(X) = 0.$$

There are three families of Inoue surfaces: S_M, $S^{(+)}$, and $S^{(-)}$ (cf. [Ino74]), and we will study their automorphisms separately.

Since every holomorphic map from $\mathbb{C}$ to $\mathbb{H}$ is constant, any automorphism u of W has the form

$$u(w,z) = (s(w), t(w,z)) \tag{6.1}$$

where

$$s(w) = \frac{aw+b}{cw+d}, \quad a,b,c,d \in \mathbb{R} \text{ and } ad - bc > 0.$$

We may expand t in the power series of w and z at $(\sqrt{-1}, 0)$:

$$t(w,z) = \sum_{p \ge 0} C_{p,0}(w - \sqrt{-1})^p + C_{p,1}(w - \sqrt{-1})^p z.$$

6.1. **Type S_M.** Let $M = (m_{i,j}) \in \mathrm{SL}_3(\mathbb{Z})$ be a matrix with eigenvalues $\alpha, \beta, \overline{\beta}$ such that $\alpha > 1$ and $\beta \neq \overline{\beta}$. Take $(a_1, a_2, a_3)^T$ to be a real eigenvector of M corresponding to α, and $(b_1, b_2, b_3)^T$ an eigenvector corresponding to β. Let G_M be the group of automorphisms of W generated by

$$\begin{aligned} g_0(w, z) &= (\alpha w, \beta z), \\ g_i(w, z) &= (w + a_i, z + b_i), \quad i = 1, 2, 3, \end{aligned}$$

which satisfy these conditions

$$\begin{aligned} g_0 g_i g_0^{-1} &= g_1^{m_{i,1}} g_2^{m_{i,2}} g_3^{m_{i,3}}, \\ g_i g_j &= g_j g_i, \quad i, j = 1, 2, 3. \end{aligned}$$

It can be shown that the action of G_M on W is free and properly discontinuous. The quotient $X \coloneqq W/G_M$ is an Inoue surface of type S_M. Thus, there is a short exact sequence of groups

$$1 \longrightarrow G_M \longrightarrow \widetilde{\mathrm{Aut}}(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow 1$$

where $\widetilde{\mathrm{Aut}}(X)$ acts biholomorphically on W, which is the normaliser of G_M in $\mathrm{Aut}(W)$.

We will study $\widetilde{\mathrm{Aut}}(X)$ in details. Assume that $u \in \widetilde{\mathrm{Aut}}(X)$. Since $G_M \lhd \widetilde{\mathrm{Aut}}(X)$, we have $ugu^{-1} \in G_M$ for all $g \in G_M$. Indeed, we only need to verify this for the generators of G_M.

By the commutative relations of G_M, we may assume that

$$u g_i u^{-1} = g_1^{n_{i,1}} g_2^{n_{i,2}} g_3^{n_{i,3}} g_0^{k_i}, \quad 0 \leq i \leq 3 \tag{6.2}$$

for some $k_i \in \mathbb{Z}$, $\mathbf{n} = (n_{0,1}, n_{0,2}, n_{0,3}) \in \mathbb{Z}^3$ and $N \coloneqq (n_{i,j})_{1 \leq i,j \leq 3} \in \mathrm{Mat}_{3\times 3}(\mathbb{Z})$. Then

$$g_1^{n_{i,1}} g_2^{n_{i,2}} g_3^{n_{i,3}} g_0^{k_i} u(w, z) = (\alpha^{k_i} s(w) + \sum_{j=1}^{3} n_{i,j} a_j, \beta^{k_i} t(w, z) + \sum_{j=1}^{3} n_{i,j} b_j).$$

Note that

$$\begin{aligned} u g_0(w, z) &= u(\alpha w, \beta z) = (s(\alpha w), t(\alpha w, \beta z)), \\ u g_i(w, z) &= u(w + a_i, z + b_i) = (s(w + a_i), t(z + b_i)). \end{aligned}$$

Then, Eq. (6.2) implies

$$s(\alpha w) = \alpha^{k_0} s(w) + \sum_j n_{0,j} a_j, \tag{6.3}$$

$$t(\alpha w, \beta z) = \beta^{k_0} t(w, z) + \sum_j n_{0,j} b_j, \tag{6.4}$$

$$s(w + a_i) = \alpha^{k_i} s(w) + \sum_j n_{i,j} a_j, \tag{6.5}$$

$$t(z + b_i) = \beta^{k_i} t(w, z) + \sum_j n_{i,j} b_j. \tag{6.6}$$

By Eq. (6.3), we have

$$\begin{cases} ac(1 - \alpha^{k_0}) = c^2 \sum_j n_{0,j} a_j, \\ ad(\alpha - \alpha^{k_0}) + bc(1 - \alpha^{k_0+1}) = cd(1 + \alpha) \sum_j n_{0,j} a_j, \\ bd(1 - \alpha^{k_0}) = d^2 \sum_j n_{0,j} a_j. \end{cases} \tag{6.7}$$

If $cd \neq 0$, then $a(1 - \alpha^{k_0}) = c \sum_j n_{0,j} a_j$ and $b(1 - \alpha^{k_0}) = d \sum_j n_{0,j} a_j$ by the first and the third equalities of Eq. (6.7). Using the fact that $ad - bc > 0$, the second equality of Eq. (6.7) implies $k_0 = 0$ and $\sum_j n_{0,j} a_j = 0$, which contradicts the middle equality above. Therefore, either $c = 0$ or $d = 0$.

From Eq. (6.5) one deduces that

$$\begin{cases} ac(1-\alpha^{k_i}) = c^2 \sum_j n_{i,j} a_j, \\ (aca_i + ad + bc)(1-\alpha^{k_i}) = (c^2 a_i + 2cd) \sum_j n_{i,j} a_j, \\ (ad - bc\alpha^{k_i}) a_i + bd(1-\alpha^{k_i}) = (cda_i + d^2) \sum_j n_{i,j} a_j, \end{cases} \quad \text{for all } i. \tag{6.8}$$

If $d = 0$, then $bc\alpha^{k_i} a_i = 0$ for all i by the last equality of Eq. (6.8). Since $bc \neq 0$ now and $\alpha > 1$, the third equality of Eq. (6.8) implies $a_i = 0$ for all i, which contradicts the assumption that (a_1, a_2, a_3) is an eigenvector of M. Hence, $c = 0$ and we may rewrite s as $s(w) = aw + b$ with $a > 0$ and $b \in \mathbb{R}$ (i.e., $d = 1$). Then, by Eq. (6.7), $k_0 = 1$ and $(1-\alpha)b = \sum_j n_{0,j} a_j$ and Eq. (6.8) gives $k_i = 0$ and $aa_i = \sum_j n_{i,j} a_j$.

Comparing the coefficients of z in Eq. (6.4), we have

$$\begin{aligned} \sum_p C_{p,0}(\alpha w - \sqrt{-1})^p &= \beta \sum_p C_{p,0}(w - \sqrt{-1})^p + \sum_j n_{0,j} b_j, \\ \beta \sum_p C_{p,1}(\alpha w - \sqrt{-1})^p &= \beta \sum_p C_{p,1}(w - \sqrt{-1})^p. \end{aligned} \tag{6.9}$$

It follows that $C_{p,0} = C_{p,1} = 0$ for all $p \geq 1$. Thus, we may rewrite t as $t(w,z) = Az + B$ for some $A \neq 0, B \in \mathbb{C}$. Then Eq. (6.9) becomes $(1-\beta)B = \sum_j n_{0,j} b_j$, and Eq. (6.6) gives $Ab_i = \sum_j n_{i,j} b_j$.

We conclude that $u(w,z) = (aw + b, Az + B)$ where a, b, A, B satisfy

$$\mathbf{n} \cdot \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} = \begin{pmatrix} (1-\alpha)b & (1-\beta)B \end{pmatrix}$$

and

$$N \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad N \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = A \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \tag{6.10}$$

It follows that M and N are simultaneously diagonalisable. So M and N commute. Moreover, the matrix associated to u^{-1} is $N^{-1} \in \mathrm{Mat}_{3\times 3}(\mathbb{Z})$, so $\det N = \pm 1$ and $N \in \mathrm{SL}_3(\mathbb{Z})$.

Consider the following subgroups of $\widetilde{\mathrm{Aut}}(X)$:

$$K = \left\{ (w,z) \longmapsto \left(w + \frac{1}{1-\alpha} \sum_{j=1}^{3} n_i a_i, z + \frac{1}{1-\beta} \sum_{j=1}^{3} n_i b_i \right) \mid n_i \in \mathbb{Z} \right\} \simeq \mathbb{Z}^3$$

and

$$\Gamma \coloneqq \{ N \in \mathrm{SL}_3(\mathbb{Z}) \mid N \text{ and } M \text{ are simultaneously diagonalisable} \},$$

which are abelian groups. By the construction, $\widetilde{\mathrm{Aut}}(X) = K \rtimes \Gamma$. Note also that $G_M = G_1 \rtimes G_0$ where $G_1 = \{ g_1^{n_1} g_2^{n_2} g_3^{n_3} \mid n_i \in \mathbb{Z} \} \simeq \mathbb{Z}^3$ and $G_0 = \langle g_0 \rangle \simeq \mathbb{Z}$. Now we have the following commutative diagram

(by the snake lemma)

$$\begin{tikzcd} & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\ 1 \arrow[r] & G_1 \arrow[r, "\phi"] \arrow[d] & K \arrow[r] \arrow[d, "\iota"] & F \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & G_M \arrow[r] \arrow[d] & \widetilde{\mathrm{Aut}}(X) \arrow[r] \arrow[d] & \mathrm{Aut}(X) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & G_0 \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & \Gamma/G_0 \arrow[r] \arrow[d] & 1 \\ & 1 & 1 & 1 & \end{tikzcd} \tag{6.11}$$

where ϕ is given by

$$\phi(\mathbf{n}) = \mathbf{n} \cdot (I - M) \quad \text{with } \mathbf{n} = (n_1, n_2, n_3).$$

Since $I - M$ is invertible, the group F being the cokernel of ϕ, is finite (and abelian). Then, by the last column of the commutative diagram, $\mathrm{Aut}(X)$ is finite-by-abelian and metabelian. By Eq. (6.11), we have:

Theorem 6.1. *Let X be an Inoue surface of type S_M. Then $\mathrm{Aut}(X)$ is solvable. Moreover, for any $G \le \mathrm{Aut}(X)$, one has $\ell_{\mathrm{vir}}(G) \le 2$.*

Next, we claim that any torsion subgroup of $\Gamma/\mathbb{Z}$ is finite. Let n_0 be the largest integer such that, for some n_0-th root of M, $M^{\frac{1}{n_0}} \in \mathrm{SL}_3(\mathbb{Z})$. Let $N \in \Gamma$ be an element such that its image in $\Gamma/\mathbb{Z}$ is torsion. Then there is a pair of coprime integers (m, n) such that $N^n = M^m$ and hence $M^{\frac{m}{n}} \in \mathrm{SL}_3(\mathbb{Z})$. Since m and n are coprime, there are integers a and b such that $am + bn = 1$. Then

$$\frac{1}{n} = a\frac{m}{n} + b.$$

It follows that $M^{\frac{1}{n}} \in \mathrm{SL}_3(\mathbb{Z})$ and hence $n \le n_0$. Therefore,

$$\{(m, n) \mid M^{\frac{m}{n}} \in \mathrm{SL}_3(\mathbb{Z}), m, n \text{ coprime}, m < n\}$$

is a finite set. It follows that the set of torsion elements of $\Gamma/\mathbb{Z}$ is finite (with the bound depending on X), which proves the claim. This claim and Eq. (6.11) imply:

Theorem 6.2. *Let X be an Inoue surface of type S_M. Then any torsion subgroup $G \le \mathrm{Aut}(X)$ is finite. In particular, $\mathrm{Aut}(X)$ is T-Jordan.*

6.2. **Types $S^{(+)}$ and $S^{(-)}$.** Since the constructions of Inoue surfaces of type $S^{(+)}$ and $S^{(-)}$ are almost parallel, we will only focus on type $S^{(+)}$ in this subsection. The same argument works for Inoue surfaces of type $S^{(-)}$.

Let $M \in \mathrm{SL}_2(\mathbb{Z})$ be a matrix with two real eigenvalues α and $1/\alpha$ with $\alpha > 1$. Let $(a_1, a_2)^T$ and $(b_1, b_2)^T$ be real eigenvectors of M corresponding to α and $1/\alpha$, respectively, and fix integers p_1, p_2, r ($r \neq 0$) and a complex number τ. Define $(c_1, c_2)^T$ to be the solution of the following equation

$$(I - M)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \frac{b_1a_2 - b_2a_1}{r}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

where

$$e_i = \frac{1}{2}m_{i,1}(m_{i,1} - 1)a_1b_1 + \frac{1}{2}m_{i,2}(m_{i,2} - 1)a_2b_2 + m_{i,1}m_{i,2}b_1a_2, \quad i = 1, 2.$$

Let $G_M^{(+)}$ be the group of analytic automorphisms of $W = \mathbb{H} \times \mathbb{C}$ generated by

$$\begin{aligned} g_0 &\colon (w,z) \mapsto (\alpha w, z+\tau), \\ g_i &\colon (w,z) \mapsto (w+a_i, z+b_i w + c_i), \quad i=1,2, \\ g_3 &\colon (w,z) \mapsto \Big(w, z+\frac{b_1a_2-b_2a_1}{r}\Big). \end{aligned}$$

We have the following relations between these generators

$$\begin{gathered} g_3g_i = g_ig_3 \quad \text{for } i=0,1,2, \quad g_1^{-1}g_2^{-1}g_1g_2 = g_3^r, \\ g_0g_jg_0^{-1} = g_1^{m_{j,1}}g_2^{m_{j,2}}g_3^{p_j} \quad \text{for } j=1,2. \end{gathered}$$

The action of $G_M^{(+)}$ is free and properly discontinuous. The quotient space $X \coloneqq W/G_M^{(+)}$ is an Inoue surface of type $S^{(+)}$. Note that $G_M^{(+)} \simeq H(r) \rtimes \mathbb{Z}$ as an abstract group, where $H(r) = \langle g_1, g_2, g_3 \mid g_3g_i = g_ig_3, g_1^{-1}g_2^{-1}g_1g_2 = g_3^r\rangle$ and $\mathbb{Z}$ is generated by g_0. In fact, the centre $Z(H(r)) \simeq \mathbb{Z}$ is generated by g_3 and $H(r)/Z(H(r)) \simeq \mathbb{Z}^2$.

Similarly, there is a short exact sequence of groups

$$1 \longrightarrow G_M^{(+)} \longrightarrow \widetilde{\mathrm{Aut}}(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow 1,$$

where $\widetilde{\mathrm{Aut}}(X) \le \mathrm{Aut}(W)$ is the normaliser of $G_M^{(+)}$. Now suppose that $u \in \widetilde{\mathrm{Aut}}(X)$ as in Eq. (6.1) and

$$ug_iu^{-1} = g_1^{n_{i,1}}g_2^{n_{i,2}}g_3^{l_i}g_0^{k_i}, \quad 0 \le i \le 3,\ j=1,2 \tag{6.12}$$

for some $l_i, k_i \in \mathbb{Z}$, $\mathbf{n}_0 = (n_{0,1}, n_{0,2}), \mathbf{n}_3 = (n_{3,1}, n_{3,2}) \in \mathbb{Z}^2$ and $N = (n_{i,j})_{i,j=1,2} \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$. Then,

$$\begin{aligned} &g_1^{n_{i,1}}g_2^{n_{i,2}}g_3^{l_i}g_0^{k_i}u(w,z) \\ &= \Big(\alpha^{k_i}s(w) + \sum_j n_{i,j}a_j, t(w,z) + k_i\tau + l_i\frac{b_1a_2-b_2a_1}{r} + \Big(\sum_j n_{i,j}b_j\Big)\alpha^{k_i}s(w) + \sum_j n_{i,j}c_j + e_i(n)\Big), \end{aligned}$$

where

$$e_i(n) = \frac{1}{2}n_{i,1}(n_{i,1}-1)a_1b_1 + \frac{1}{2}n_{i,2}(n_{i,2}-1)a_2b_2 + n_{i,1}n_{i,2}a_2b_1.$$

As in the previous subsection, Eq. (6.12) implies

$$s(\alpha w) = \alpha^{k_0}s(w) + n_{0,1}a_1 + n_{0,2}a_2, \tag{6.13}$$

$$t(\alpha w, z+t) = t(w,z) + k_0\tau + l_0\frac{b_1a_2-b_2a_1}{r} + \Big(\sum_j n_{0,j}b_j\Big)\alpha^{k_0}s(w) + \sum_j n_{0,j}c_j + e_0(n), \tag{6.14}$$

$$s(w+a_i) = \alpha^{k_i}s(w) + n_{i,1}a_1 + n_{i,2}a_2, \tag{6.15}$$

$$t(w+a_i, z+b_iw+c_i) = t(w,z) + k_i\tau + l_i\frac{b_1a_2-b_2a_1}{r} + \Big(\sum_j n_{i,j}b_j\Big)\alpha^{k_i}s(w) + \sum_j n_{i,j}c_j + e_i(n), \tag{6.16}$$

$$s(w) = \alpha^{k_3}s(w) + n_{3,1}a_1 + n_{3,2}a_2, \tag{6.17}$$

$$t\Big(w, z+\frac{b_1a_2-b_2a_1}{r}\Big) = t(w,z) + k_3\tau + l_3\frac{b_1a_2-b_2a_1}{r} + \Big(\sum_j n_{3,j}b_j\Big)\alpha^{k_3}s(w) + \sum_j n_{3,j}c_j + e_3(n). \tag{6.18}$$

By Eq. (6.13), using the assumption that $\alpha > 1$, we have

$$\begin{cases} ac(1-\alpha^{k_0}) = c^2(n_{0,1}a_1 + n_{0,2}a_2), \\ ad(\alpha - \alpha^{k_0}) + bc(1-\alpha^{k_0+1}) = cd(1+\alpha)(n_{0,1}a_1+n_{0,2}a_2), \\ bd(1-\alpha^{k_0}) = d^2(n_{0,1}a_1+n_{0,2}a_2). \end{cases} \tag{6.19}$$

If $c \neq 0$, then $k_0 = -1$ by the first two equalities of Eq. (6.19) since $ad - bc > 0$. Similarly, if $d \neq 0$, then one has $k_0 = 1$ by the last two equalities of Eq. (6.19). Therefore, either $c = 0$ and $k_0 = 1$, or $d = 0$ and $k_0 = -1$.

By Eq. (6.15), we have

$$\begin{cases} ac(1-\alpha^{k_i}) = c^2(n_{i,1}a_1 + n_{i,2}a_2), \\ (aca_i + bc + ad)(1-\alpha^{k_i}) = (c^2 a_i + 2cd)(n_{i,1}a_1 + n_{i,2}a_2), \\ ada_i + bd(1-\alpha^{k_i}) = \alpha^{k_i} bca_i + (cda_i + d^2)(n_{i,1}a_1 + n_{i,2}a_2). \end{cases} \tag{6.20}$$

If $d = 0$, we have $\alpha^{k_i} bca_i = 0$ for $i = 1, 2$ by the last equality of Eq. (6.20). This implies that $a_i = 0$ $(i = 1, 2)$ since $\alpha > 1$ and $ad - bc > 0$, which contradicts the fact that $(a_1, a_2)^T$ is an eigenvector of M. So, $c = 0$ and $k_0 = 1$. Also, the second equality of Eq. (6.20), which becomes $ad = \alpha^{k_i} ad$, implies that $k_i = 0$ for $i = 1, 2$. Now we may rewrite s as $s(w) = aw + b$ with $a > 0$ and $b \in \mathbb{R}$. Then Eqs. (6.19) and (6.20) imply that

$$\begin{cases} (1-\alpha)b = n_{0,1}a_1 + n_{0,2}a_2, \\ aa_i = n_{i,1}a_1 + n_{i,2}a_2, \quad i = 1, 2. \end{cases} \tag{6.21}$$

Comparing the coefficients of z of Eq. (6.14), we have

$$\sum_p C_{p,1}(\alpha w - \sqrt{-1})^p = \sum_p C_{p,1}(w - \sqrt{-1})^p,$$

which implies that $C_{p,1} = 0$ for all $p \geq 1$. For the z-constant part of Eq. (6.14), one has the following equality

$$\begin{aligned} &\sum_p C_{p,0}(\alpha w - \sqrt{-1})^p + C_{0,1}\tau \\ = &\sum_p C_{p,0}(w - \sqrt{-1})^p + (n_{0,1}b_1 + n_{0,2}b_2)\alpha a w + \tau + l_0 \frac{b_1 a_2 - b_2 a_1}{r} + n_{0,1}c_1 + n_{0,1}c_2 + e_0(n). \end{aligned}$$

Similarly, we conclude that $C_{p,0} = 0$ for $p \geq 2$. Hence, we may rewrite t as $t(w, z) = Aw + B + Cz$ for some $A, B, C \in \mathbb{C}$ with $C \neq 0$. Now comparing the coefficients of w of Eqs. (6.14) and (6.16), we get

$$\begin{cases} \left(1 - \frac{1}{\alpha}\right)\frac{A}{a} = n_{0,1}b_1 + n_{0,2}b_2, \\ (C-1)\tau = l_0 \frac{b_1 a_2 - b_2 a_1}{r} + (n_{0,1}b_1 + n_{0,2}b_2)\alpha b + n_{0,1}c_1 + n_{0,2}c_2 + e_0(n), \\ Cb_i = a(n_{i,1}b_1 + n_{i,2}b_2), \\ Aa_i + Cc_i = l_i \frac{b_1 a_2 - b_2 a_1}{r} + (n_{i,1}b_1 + n_{i,2}b_2)b + n_{i,1}c_1 + n_{i,2}c_2 + e_i(n). \end{cases} \tag{6.22}$$

Finally, Eqs. (6.17) and (6.18) simply give that $k_3 = 0$, $l_3 = C$ and $n_{3,1} = n_{3,2} = 0$.

Therefore, N has two eigenvalues a and C/a with eigenvectors $(a_1, a_2)^T$ and $(b_1, b_2)^T$, respectively. It is not hard to see that the matrix associated to u^{-1} is N^{-1}. Then $\det N = \pm 1$ and hence $N \in \mathrm{SL}_2(\mathbb{Z})$. Moreover, by Eqs. (6.21) and (6.22), u (if exists) is determined by $\mathbf{n}_0$, N and B.

Let

$$\Gamma \coloneqq \{N \in \mathrm{SL}_2(\mathbb{Z}) \mid N \text{ and } M \text{ are simultaneously diagonalisable}\},$$

which is an abelian group, and let $\tau\colon \widetilde{\mathrm{Aut}}(X) \longrightarrow \Gamma$ be the homomorphism (not necessarily surjective) defined by $u \mapsto N$, where N is the matrix associated with u as constructed above. Let K be the kernel of this homomorphism, with image $\Gamma' \leq \Gamma$. It is clear that any automorphism in K has the form

$$(w, z) \longmapsto (w + b, z + Aw + B)$$

for some $b \in \mathbb{R}$ and $A, B \in \mathbb{C}$ satisfying Eqs. (6.21) and (6.22). An explicit calculation gives that the centre $Z(K) \simeq \mathbb{C}$, which is generated by

$$(w, z) \longmapsto (w, z + B).$$

Further, $K/Z(K) \simeq \mathbb{Z} \rtimes \mathbb{Z}$ is generated by the images of

$$(w, z) \longmapsto (w + b, z) \quad \text{and} \quad (w, z) \longmapsto (w, z + Aw).$$

Combining these, the snake lemma shows the diagrams below are commutative with exact rows and columns.

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & H(r) \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & G_M^{(+)} \arrow[r] \arrow[d] & \widetilde{\mathrm{Aut}}(X) \arrow[r] \arrow[d, "\tau"] & \mathrm{Aut}(X) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \Gamma' \arrow[r] \arrow[d] & \Gamma'/\mathbb{Z} \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & Z(H(r)) \arrow[r] \arrow[d] & H(r) \arrow[r] \arrow[d] & H(r)/Z(H(r)) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & Z(K) \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K/Z(K) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & F' \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & F'' \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

Note that F' is abelian and F'' is finite and metabelian. Consequently, F, as the quotient of K by $H(r)$, is abelian-by-finite and abelian-by-metabelian.

By the above diagrams:

Theorem 6.3. *Let X be an Inoue surface of type $S^{(+)}$ or $S^{(-)}$. Then $\mathrm{Aut}(X)$ is solvable. Moreover, for any subgroup $G \le \mathrm{Aut}(X)$, one has $\ell_{\mathrm{vir}}(G) \le 4$.*

Similar to the argument before Theorem 6.2, any torsion subgroup of $\Gamma'/\mathbb{Z}$ is finite. It follows from the diagram above that each torsion subgroup of $\mathrm{Aut}(X)$ has an abelian subgroup with bounded index depending on X.

Theorem 6.4. *Let X be an Inoue surface of type $S^{(+)}$ or $S^{(-)}$. Then $\mathrm{Aut}(X)$ is T-Jordan.*

7. Class VII Surfaces with $b_2 > 0$

We will prove Proposition 1.5 in the section.

Proof of Proposition 1.5. We may assume that G is an infinite group. By Corollary 4.2, we only need to consider such X that does not have any curves. Consider the action of $\mathrm{Aut}(X)$ on $H^*(X, \mathbb{Q})$, and let $\mathrm{Aut}^*(X)$ be the kernel of $\mathrm{Aut}(X) \longrightarrow \mathrm{GL}(H^*(X, \mathbb{Q}))$. Note that the image of G in $\mathrm{GL}(H^*(X, \mathbb{Q}))$ is finite (cf. [JM22, Lemma 5.2]). By passing to a finite index subgroup, we may assume that $G \le \mathrm{Aut}^*(X)$.

Pick $\mathrm{id} \ne g \in G$, and let G' be the centraliser of $\langle g \rangle$ in G. Since g has finite order, $[G : G']$ is finite. Replacing G by the finite-index subgroup G', we may assume that $g \in Z(G)$, the centre of G. Note that g has $b_2(X)$ fixed points (cf. [PS20a, Lemma 5.4]). Let $Y = X/\langle g \rangle$, which is singular, and let $\widetilde{Y} \longrightarrow Y$ be the minimal $\mathrm{Aut}(Y)$-equivariant resolution. Then $\widetilde{Y} \notin \Xi_0$ since it contains some curves.

By the construction, G acts on Y and hence on $\widetilde{Y}$ (but may not be faithful):

$$1 \longrightarrow G'' \longrightarrow G \longrightarrow G|_Y \longrightarrow 1.$$

It follows from Proposition 3.2 that the natural map $G'' \hookrightarrow \prod_{y\in S} \mathrm{Aut}(X_y)$ is an injection for some very general finite subset S of Y. Since $X \longrightarrow Y$ is a finite morphism, G'' is a finite group. Note that $G|_Y \simeq G|_{\widetilde{Y}}$ is a torsion group. By Proposition 1.4, $G|_{\widetilde{Y}}$ and hence $G|_Y$ are virtually abelian. Then G is virtually nilpotent (cf. Proof of Lemma 2.6 (2)) and hence virtually solvable with $\ell_{\mathrm{vir}}(G) \le 2$. □

8. Compact Kähler Surfaces

In this section, we deal with smooth compact surfaces which are Kähler. The idea of this section comes from Professor De-Qi Zhang.

Let X be a compact Kähler manifold. For a subgroup G of $\mathrm{Aut}(X)$, define $G^0 \coloneqq G \cap \mathrm{Aut}_0(X)$ and denote by $N(G)$ the set of all elements in G with zero entropy.

Lemma 8.1. *Let X be a smooth compact Kähler surface, and $G \le \mathrm{Aut}(X)$ a virtually solvable subgroup. Then $\ell_{\mathrm{vir}}(G) \le 1 + \ell_{\mathrm{vir}}(N(G))$.*

Proof. By taking a finite-index subgroup, we may assume that G is a solvable, $N(G)$ is a normal subgroup of G and $G/N(G) \simeq \mathbb{Z}^r$ for some $r = 0, 1$ (cf. [CWZ13, Theorem 1.5]). Take a finite-index subgroup $N' \le N(G)$ such that $\ell(N') = \ell_{\mathrm{vir}}(N(G))$ and $\ell(N'/N'^0) \le 1$ as in [DLOZ22, Theorem 1.2 and Proposition 2.6].

Note that $N(G)|_{H^2(X,\mathbb{C})}$ is finitely generated (cf. [CWZ13, Theorem 2.2(1)]). Since $N'|_{H^2(X,\mathbb{C})}$ has finite index in $N(G)|_{H^2(X,\mathbb{C})}$, after replacing N' by a finite-index subgroup, we may assume that $N'|_{H^2(X,\mathbb{C})}$ is characteristic in $N(G)|_{H^2(X,\mathbb{C})}$ by Lemma 2.8. In particular, $N'|_{H^2(X,\mathbb{C})} \lhd G|_{H^2(X,\mathbb{C})}$.

Let $N \lhd N(G)$ be the preimage of $N'|_{H^2(X,\mathbb{C})}$. Then $N \lhd G$, $N(G)/N$ is finite, $\ell(N) = \ell_{\mathrm{vir}}(N(G))$ and $\ell(N/N^0) \le 1$. Since $(G/N)/(N(G)/N) \simeq \mathbb{Z}^r$, replacing G by a finite-index subgroup, we may assume $G/N \simeq \mathbb{Z}^r$ (cf. [Zha13, Lemma 2.4]). It follows that

$$\ell_{\mathrm{vir}}(G) \le \ell(G) \le \ell(G/N) + \ell(N) \le 1 + \ell_{\mathrm{vir}}(N(G)).$$ □

Indeed, it follows from the above proof that

$$\ell_{\mathrm{vir}}(G) \le 1 + \ell(N) \le 1 + \ell(N/N^0) + \ell(N^0) \le 2 + \ell(N^0).$$

When $\kappa(X) \ge 0$, $\mathrm{Aut}_0(X)$ is a complex torus (cf. [Fuj78, Corollary 5.11]). In particular, $N^0 = N \cap \mathrm{Aut}_0(X)$ is abelian and hence of derived length (at most) 1. Thus, we have

Proposition 8.2. *Let X be a smooth compact Kähler surface with Kodaira dimension $\kappa(X) \ge 0$. Let $G \le \mathrm{Aut}(X)$ be virtually solvable. Then $\ell_{\mathrm{vir}}(G) \le 3$.*

Suppose that now X is a compact Kähler surface with $\kappa(X) = -\infty$. Then X is projective.

Proposition 8.3. *Let X be a smooth projective surface. Suppose that $G \le \mathrm{Aut}(X)$ is virtually solvable, and $U \le \mathrm{Aut}_0(X)$ is non-trivial unipotent with $U \lhd G$. Then the derived length $\ell(U) \le 2$ and after replacing by a finite-index subgroup, we have $G = G^0$.*

Proof. Replacing by closure or finite-index subgroup, we may assume that $G \le \mathrm{Aut}(X)$ is closed and solvable, $G|_{\mathrm{NS}(X)_{\mathbb{C}}}$ is connected and closed, and $U \le \mathrm{Aut}_0(X)$ is closed and unipotent (cf. [Ogu06, Proof of Lemma 2.1(2)]).

By Lemma 2.9, after replacing G by a finite-index subgroup, there is a subgroup $Z_1 \simeq \mathbb{G}_a \le U$ which is normal in G. Let $X \dashrightarrow Y = X/Z_1$ be the quotient map to the cycle space (cf. [Fuj78, § 4]). Then G acts biregularly on Y and the map is G-equivariant. Replacing the map by the graph and taking equivariant resolutions, we may assume that both X and Y are smooth, and the map $X \longrightarrow Y$ is a morphism, a

$\mathbb{P}^1$-fibration, whose general fibre is a curve with dense orbit $Z_1 \cdot x \simeq \mathbb{A}^1$ and whose section at infinity is fixed by G. Consider the exact sequence

$$1 \longrightarrow K \longrightarrow U \longrightarrow U|_Y \longrightarrow 1.$$

Then both K and $U|_Y$ are unipotent. Now the restriction to generic fibre $K \longrightarrow K|_{X_{\overline{k(Y)}}}$ is injective with image contained in a closed unipotent group acting faithfully on $X_{\overline{k(Y)}} = \mathbb{P}^1_{\overline{k(Y)}}$, and hence is abelian. It is clear that either $U|_Y = \{1\}$, or $U|_Y \simeq \mathbb{G}_a$ and $Y = \mathbb{P}^1$. Hence $\ell(U) \leq \ell(U|_Y) + 1 \leq 2$.

Note that G fixes a big class H_Y on the curve Y. Take the class H to be the sum of the pullback of H_Y and the section at infinity of the $\mathbb{P}^1$-fibration $X \longrightarrow Y$. Then G fixes the big class H on X and hence replaced by a finite-index subgroup, G is contained in $\mathrm{Aut}_0(X)$ (cf. [Fuj78, Theorem 4.8], [Lie78, Proposition 2.2] and [DHZ15, Corollary 2.2]). Thus $G = G^0$. □

Proposition 8.4. *Let G be a smooth projective surface. Suppose that $G \leq \mathrm{Aut}(X)$ is virtually solvable. Then $\ell_{\mathrm{vir}}(G) \leq 4$.*

Proof. Replacing by finite-index subgroup, we may assume that G is solvable.

First assume that $G = G^0$ (and connected). Let U be the unipotent radical of (the linear part of) $G = G^0$ so that G/U is a semi-torus and hence commutative (cf. [Iit76, Lemma 4]). Thus, $\ell_{\mathrm{vir}}(G) \leq 1 + \ell(U) \leq 3$ by Proposition 8.3.

Now we may assume that $G \neq G^0$ even after replacing G by a finite-index subgroup. Let $G^0_0 \lhd G^0$ be the identity component, Then by Proposition 8.3, the unipotent radical of (the linear part of) G^0_0 is trivial and hence G^0_0 itself is a semi-torus and hence commutative.

If G is of zero entropy, by [DLOZ22, Theorem 1.2] one has $\ell(G/G^0) \leq 1$ after replacing G by a finite-index subgroup. Consider the short exact sequence

$$1 \longrightarrow G^0/G^0_0 \longrightarrow G/G^0_0 \longrightarrow G/G^0 \longrightarrow 1.$$

Since the first term is a finite group, its centraliser in G/G^0_0 is of finite index (say of index 1, after replacing G by a finite index subgroup). Thus,

$$\ell(G/G^0_0) \leq 1 + \ell(G/G^0) = 2,$$
$$\ell_{\mathrm{vir}}(G) \leq \ell(G) \leq \ell(G/G^0_0) + \ell(G^0_0) \leq 3.$$

If G is not of zero entropy, using Lemma 8.1 and the argument above, $\ell_{\mathrm{vir}}(G) \leq 1 + \ell_{\mathrm{vir}}(N(G)) \leq 1 + 3 = 4$. □

9. Summary

In the Kähler setting, Proposition 1.4 follows from Theorem 1.3; Theorem 1.9 has been proved in [CWZ13, Theorem 1.5]; Theorem 1.10 follows from Propositions 8.2 and 8.4.

Let X be a smooth compact complex surface, which is not Kähler. In particular, X is not rational or ruled. Thus, there is a unique minimal surface X' bimeromorphic to X such that

$$\mathrm{Aut}(X) \subseteq \mathrm{Bim}(X) \simeq \mathrm{Bim}(X') = \mathrm{Aut}(X').$$

See [PS21, Proposition 3.5]. It suffices for us to prove Proposition 1.4 and Theorems 1.9 and 1.10 for minimal surfaces (in the non-Kähler setting). Note that non-Kähler minimal surfaces are surfaces of class VII, (primary or secondary) Kodaira surfaces and some properly elliptic surfaces (cf. [BHPV04, IV. Theorem 3.1 and VI. Theorem 1.1]). Since every minimal surface of class VII with vanishing b_2 is either a Hopf surface or an Inoue surface (cf. [Bog76]) and minimal surface of class VII with algebraic dimension 1 is a Hopf surface (cf. [BHPV04, V. Theorem 18.6]), minimal surfaces of class VII not in Ξ are exactly Hopf surfaces or Inoue surfaces.

TABLE 1. non-Kähler minimal smooth compact complex surfaces

class of the surface X	$\kappa(X)$	$a(X)$	$b_1(X)$	$b_2(X)$	$e(X)$
surfaces of class VII	$-\infty$	$0, 1$	1	≥ 0	≥ 0
primary Kodaira surfaces	0	0	3	4	0
secondary Kodaira surfaces	0	0	1	0	0
properly elliptic surfaces	1	1			≥ 0

Then Proposition 1.4 follows from Theorems 3.4, 5.4, 6.2 and 6.4 and Corollary 4.2; Theorem 1.9 follows from Theorems 2.7, 3.4, 3.5, 5.1, 6.1 and 6.3; and Theorem 1.10 follows from Theorems 2.7, 3.4, 3.5, 5.2, 6.1 and 6.3. Finally, Proposition 1.5 has been proved in Section 7.

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National University of Singapore, Singapore 119076, Republic of Singapore

Email address: `jia_jia@u.nus.edu`