SMOOTH PROJECTIVE SURFACES WITH PSEUDO-EFFECTIVE TANGENT BUNDLES

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Abstract. Let S be a non-uniruled (i.e., non-birationally ruled) smooth projective surface. We show that the tangent bundle T_S is pseudo-effective if and only if the canonical divisor K_S is nef and the second Chern class vanishes, i.e., $c_2(S) = 0$. Moreover, we study the blow-up of a non-rational ruled surface with pseudo-effective tangent bundle.

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers. A smooth projective variety comes naturally equipped with the tangent bundle, which is the dual of its sheaf of Kähler differentials, and the related properties of such objects can be applied to classify algebraic varieties. A well-known theorem of Mori [Mor79] asserts that if the tangent bundle of a smooth projective variety is ample, then it is a projective space, which gives a solution to Hartshorne's conjecture. Since then, the study of smooth projective varieties whose tangent bundles admit some positivity properties has attracted a lot of attention and such properties are usually expected to impose strong restrictions on the geometry of the underlying varieties.

Definition 1.1. Let X be a smooth projective variety. Given a vector bundle E on X, we denote by $\mathbb{P}(E)$ the Grothendieck projectivisation of E with $\mathcal{O}_{\mathbb{P}(E)}(1)$ denoting the relative hyperplane section bundle. Recall that E is ample (resp. nef, big, pseudo-effective) if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample (resp. nef, big, pseudo-effective) on $\mathbb{P}(E)$.

Following the program of Campana and Peternell, a smooth Fano variety with nef tangent bundle is conjectured to be a rational homogeneous space, and this conjecture has been intensively studied (cf. [CP91, DPS94, MnOSC$^+$15, Kan17]). Starting from this aspect, it is natural to classify smooth projective varieties with other positivity properties, e.g., with big or pseudo-effective tangent bundles. In the past few years, there are many beautiful results in this direction, especially when X is a Fano manifold, i.e., the anti-canonical divisor $-K_X$ is ample. For example, Höring, Liu and Shao showed in [HLS22, Theorem 1.2] that the tangent bundle of a smooth del Pezzo surface (i.e., a Fano surface) of degree $d \coloneqq K_X^2$ is big (resp. pseudo-effective) if and only if $d \geq 5$ (resp. $d \geq 4$). We refer readers to [Hsi15, Sha20, Mal21, FL22, HL22, HLS22, KKL22, Kim22, Liu22] and the references therein for more information involving projective manifolds with big tangent bundles.

As smooth projective varieties with big tangent bundles are known to be uniruled (cf. [Miy87b, Corollary 8.6] and [Mal21, Proposition 7.1]), one may ask if there exist many non-uniruled projective varieties sitting in the "boundary", i.e., admitting a pseudo-effective but non-big tangent bundle. Apart from the trivial example of abelian varieties, a product of an abelian variety and any smooth projective variety becomes another example coming to our mind (cf. Lemma 2.5). To the best knowledge of ourselves, up to a finite étale cover, there seems no more other example which has been explored before. Therefore, we propose the following question to study whether such varieties of product type are basically the only possibilities.

Question 1.2. *Let X be a non-uniruled smooth projective variety. Are the following assertions equivalent?*

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(1) The tangent bundle T_X is pseudo-effective;
(2) The top Chern class $c_{\mathrm{top}}(X)$ vanishes, and the augmented irregularity $q^\circ(X)$ does not vanish.

Here, the *augmented irregularity* $q^\circ(X)$ of a smooth projective variety X is defined to be the supremum of $q(X') := h^1(X', \mathcal{O}_{X'})$ where $X' \to X$ runs over all the finite étale covers of X (cf. [NZ09, Definition 4.1]; see Remark 1.5). Note that the top Chern class of a product variety $X = Y \times Z$ satisfies $c_{\mathrm{top}}(X) = c_{\mathrm{top}}(Y) \times c_{\mathrm{top}}(Z)$. Hence, by a theorem of Lieberman [Lie78], if a non-uniruled smooth projective variety admits a global holomorphic vector field, then up to a finite étale cover, it will split into a product of an abelian variety and a projective variety admitting no global holomorphic vector field, in which case, the implication (1) $\Rightarrow$ (2) in Question 1.2 follows. However, in general, the pseudo-effectiveness does not necessarily imply the existence of any global sections of the tangent bundle; indeed, it is even not clear to us about the non-vanishing $H^0(X, \mathrm{Sym}^m T_X) \neq 0$ for some m-th symmetric power.

We also note that, different from the ampleness and the nefness of a line bundle, there is a lack of numerical characterisations on the bigness or pseudo-effectiveness, which makes our investigation a bit more difficult (cf. Remark 1.8).

The main result of our paper is to give a positive answer to Question 1.2 in dimension 2:

Theorem 1.3. *Let S be a non-uniruled (i.e., non-birationally ruled) smooth projective surface. Then the following assertions are equivalent.*
(1) The tangent bundle T_S is pseudo-effective;
(2) S is minimal and the second Chern class vanishes, i.e., $c_2(S) = 0$.
Moreover, if one of the above equivalent conditions holds, then the Kodaira dimension $\kappa(\mathbb{P}(T_S), \mathcal{O}(1)) = 1 - \kappa(S) \in \{0, 1\}$, and there is a finite étale cover $S' \to S$ such that S' is either an abelian surface or a product $E \times F$ where E is an elliptic curve and F is a smooth curve of genus ≥ 2.

From Theorem 1.3, the pseudo-effectiveness of the tangent bundle forces the surface to be minimal, i.e., the canonical divisor is nef. However, this is no longer true in the higher dimensional case (cf. Example 2.6). Besides, as the second Chern class of a smooth projective surface of general type is always positive (cf. [BHPVdV04, VII, (2.4) Proposition]), our result excludes the possibility of a general type surface having pseudo-effective tangent bundle; see Proposition 3.3 and Remark 3.4 for the case of higher dimensional varieties. As a consequence of Theorem 1.3, we obtain the following corollary.

Corollary 1.4. *Let S be a non-uniruled smooth projective surface. If the tangent bundle T_S is pseudo-effective, then there is some integer m such that $H^0(S, \mathrm{Sym}^m T_S) \neq 0$; in particular, the tautological line bundle of $\mathbb{P}(T_S)$ is $\mathbb{Q}$-linearly equivalent to an effective divisor.*

Before moving to the second part of this section, we give one remark as kindly pointed out by Höring.

Remark 1.5. Different from Theorem 1.3 in the surface case, the implication (2) $\Rightarrow$ (1) in Question 1.2 would have a negative answer in $\dim X \geq 3$ if we drop the assumption on the non-vanishing of the augmented irregularity. Indeed, there do exist a few smooth (strict) Calabi-Yau threefolds which have the vanishing top Chern classes (cf. [KS00, Fig 1]). On the other hand, as proved in [HP19, Theorem 1.6] (cf. [Dru18, Corollary 6.5]), the tangent bundle of a (strict) Calabi-Yau manifold is never pseudo-effective.

Kim proved in [Kim22] that a projective bundle $\mathbb{P}_C(\mathcal{E})$ over a smooth curve C has a big tangent bundle if and only if either $C \cong \mathbb{P}^1$ or $\mathcal{E}$ is not semi-stable. Indeed, according to the proof in [Kim22], any (relatively minimal) ruled surface (over any smooth curve), or any projective bundle (of arbitrary rank) over an elliptic curve always have pseudo-effective tangent bundles (cf. Lemmas 6.1 and 6.2). We slightly summarise such results in Section 6. In particular, we ask the following question.

Question 1.6. *Let $X = \mathbb{P}_C(\mathcal{E})$ be a projective bundle over a smooth curve C of genus $g(C) \geq 2$. Is the tangent bundle T_X always pseudo-effective?*

Note that, according to [Kim22, Proof of Proposition 3.2], when $g(C) \geq 2$, we have $h^0(X, \mathrm{Sym}^m T_X) = h^0(X, \mathrm{Sym}^m T_{X/C})$ for any m. Hence, we can reduce Question 1.6 to the study of the pseudo-effectiveness of the relative tangent bundle $T_{X/C}$. In particular, Question 1.6 has a positive answer when $\operatorname{rank}\mathcal{E} = 2$ (cf. [Kim22] and Lemma 6.1).

In terms of the non-rational ruled surface, by analysing the position of a single blow-up, we show the following proposition. In particular, compared with the non-uniruled surfaces, there does exist a non-relatively-minimal non-rational uniruled surface with pseudo-effective but non-big tangent bundle.

Proposition 1.7. *Let $f\colon S = \mathbb{P}_C(\mathcal{E}) \to C$ be a $\mathbb{P}^1$-bundle over a smooth curve C with the genus $g(C) \geq 1$. Suppose the tangent bundle T_S is pseudo-effective but not big. Then the blow-up of S along a point p has a pseudo-effective tangent bundle if and only if there exist some positive integer m and some line bundle $\mathcal{L}$ which is numerically equivalent to the relative tangent bundle $T_{S/C}$ such that $H^0(S, \mathfrak{m}_p^m \otimes \mathcal{L}^{\otimes m}) \neq 0$, where $\mathfrak{m}_p$ is the maximal ideal of the local ring $\mathcal{O}_{S,p}$.*

Making Proposition 1.7 as an initial point, we would like to give a rather clean description of non-rational uniruled projective surfaces admitting pseudo-effective but non-big tangent bundles.

We summarise the organisation of our paper. In Section 2, we prepare some preliminary results for the convenience of later use. In Section 3, we prove our Theorem 1.3 for the cases $\kappa(S) = 0$ and 2. In Section 4, we study the case $\kappa(S) = 1$ when S is minimal, and the minimality of such S with pseudo-effective tangent bundle will be shown in Section 5. Finally, we give a positive answer to Question 1.6 when $g(C) = 1$ (Lemma 6.2) and prove Proposition 1.7 in Section 6.

Let us end up the introduction with the following remark.

Remark 1.8 (Comparison with previous papers)**.** In [Mat22, Theorem 1.1] (cf. [HIM22]), Matsumura nicely establishes the minimal model program of a projective klt variety with strongly pseudo-effective (reflexive) tangent sheaf and ends up with a quasi-étale quotient of an abelian variety. Here, a vector bundle E on a smooth projective variety X is said to be strongly pseudo-effective if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is pseudo-effective and the restricted base locus does not dominate X (cf. [BDPP13, Definition 7.1]). Since our assumption is weaker than the assumption in [Mat22], even in the surface case, we could not expect to obtain a similar result like [Mat22, Theorem 1.1]. The main reason is that, when running the minimal model program, although our (weak) pseudo-effectiveness of the tangent bundle descends along any birational contraction or flips (cf. Lemma 2.5), it could not be preserved under a Fano contraction any more (see [Kim22] and Proposition 1.7; cf. [Mat22, Proposition 3.2]).

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2. Preliminary

First of all, we fix the following notation throughout the paper.

Notation 2.1. Let X be a projective variety.

(1) The symbol $\sim$ (resp. $\sim_{\mathbb{Q}}$, $\equiv$) denotes the *linear equivalence* (resp. $\mathbb{Q}$-*linear equivalence*, *numerical equivalence*) on Cartier divisors (resp. $\mathbb{Q}$-Cartier divisors). Let $f\colon X \to Y$ be a morphism of projective varieties. We denote by $\sim_f$ the *relative (or f-) linear equivalence* of Cartier divisors, i.e., for two Cartier divisors D_1 and D_2 on X, $D_1 \sim_f D_2$ if and only if there is some Cartier divisor E on Y such that $D_1 - D_2 \sim f^*E$.

(2) Denote by $\mathrm{NS}(X)$ the *Néron-Severi group* of X. Let $\mathrm{N}^1(X) \coloneqq \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ be the space of $\mathbb{R}$-Cartier divisors modulo numerical equivalence and $\rho(X) \coloneqq \dim_{\mathbb{R}} \mathrm{N}^1(X)$ the *Picard number* of X. Let $\mathrm{N}_1(X)$ be the dual space of $\mathrm{N}^1(X)$ consisting of 1-cycles. Denote by $\mathrm{Nef}(X)$ (resp. $\mathrm{PE}(X)$) the cone of *nef divisors* (resp. *pseudo-effective divisors*) in $\mathrm{N}^1(X)$ and $\overline{\mathrm{NE}}(X)$ the dual cone consisting of *pseudo-effective 1-cycles* in $\mathrm{N}_1(X)$. In particular, when X is a smooth projective surface, we have $\mathrm{N}_1(X) = \mathrm{N}^1(X)$ and $\overline{\mathrm{NE}}(X) = \mathrm{PE}(X)$.

(3) For a smooth projective variety X, we denote by K_X the *canonical divisor* and $\kappa(X) = \kappa(X, K_X)$ the *Kodaira dimension* of X.

(4) Let $f\colon X \to C$ be a surjective morphism between normal projective varieties. We say that f is a *fibration* if $f_*\mathcal{O}_X = \mathcal{O}_C$ or equivalently, the general fibre of f is connected. A fibration is said to be *isotrivial* if all the smooth fibres are isomorphic to each other; otherwise, we say that it is *non-isotrivial*. We say that f is *trivial* if there exist another projective variety F and an isomorphism $X \cong F \times C$ such that f is the natural projection. We say that f is *locally trivial* (or a *fibre bundle*) if each point $c \in C$ is contained in a small open neighbourhood U having the property that $f^{-1}(U)$ is trivial over U. See Notation 2.2 for a more detailed description on surface fibrations.

(5) Let H be a nef and big divisor on X. Let $\mathcal{F}$ be a torsion free coherent sheaf on X. The *slope* of $\mathcal{F}$ with respect to H is defined to be the rational number

$$\mu_H(\mathcal{F}) \coloneqq \frac{c_1(\mathcal{F}) \cdot H^{\dim(X)-1}}{\operatorname{rank}(\mathcal{F})}$$

where c_1 is the first Chern class. A torsion free coherent sheaf $\mathcal{E}$ is said to be *μ-semistable* if for any non-zero subsheaf $\mathcal{F} \subseteq \mathcal{E}$, the slopes satisfy the inequality $\mu_H(\mathcal{F}) \le \mu_H(\mathcal{E})$.

Next, we recall and develop some basic properties on the fibration of surfaces. Most of them come from [Ser96, Section 3] and [HP20, Section 5].

Notation 2.2 (Surface fibration)**.**

(1) Let S be a smooth projective surface and $f\colon S \to C$ a fibration with F a general fibre of f.

(2) For a closed point $c \in C$, we denote by S_c its fibre over c.

(3) Let $W \coloneqq \mathbb{P}(T_S)$ be the Grothendieck projectivisation of the tangent bundle of S with $\pi\colon W \to S$ the natural projection, and let $\xi \coloneqq \mathcal{O}_W(1)$ be the corresponding tautological class.

(4) Let $\{\nu_i E_i\}_{i\in I}$ be the set of all components of non-multiple fibres of f, where the ν_i's denote their corresponding multiplicities within their fibres. Let $\{m_j F_j\}_{j\in J}$ be the set of multiple fibres of f. Clearly, for each $j \in J$ and $c \in C$, we have $m_j F_j \equiv \sum_{i\in I, f(E_i)=c} \nu_i E_i \equiv F$.

(5) Let

$$E = E_S \coloneqq \sum_{c\in C} f^*c - (f^*c)_{\mathrm{red}} = \sum_{i\in I}(\nu_i - 1)E_i + \sum_{j\in J}(m_j - 1)F_j = E_0 + \sum_{j\in J}(m_j-1)F_j$$

where $E_0 \coloneqq \sum_{i\in I}(\nu_i - 1)E_i$ comes from the non-multiple non-reduced fibres. This decomposition is indeed the Zariski decomposition where $\sum_{j\in J}(m_j-1)F_j$ is the nef part and E_0 is the fixed part.

(6) Applying [Ser96, Section 3], we have an exact sequence

$$0 \longrightarrow T_{S/C} \longrightarrow T_S \longrightarrow f^*T_C \longrightarrow \mathcal{F} \longrightarrow 0,$$

where T_S, T_C are the tangent bundles of S and C respectively, $\mathcal{F}$ is a torsion sheaf and $T_{S/C} \coloneqq \Omega_{S/C}^\vee$ is the relative tangent sheaf of f, which is locally free. Let $J_{S/C}$ be the torsion-free image of $T_S \to f^*T_C$.

(7) By [Ser96, (3.0.3) and Proposition 3.1], we have

$$J_{S/C} = J_{S/C}^{\vee\vee} \otimes \mathcal{I}_\Gamma = K_S^{-1} \otimes T_{S/C}^{-1} \otimes \mathcal{I}_\Gamma$$

where $\mathcal{I}_\Gamma$ is an ideal sheaf and the support of Γ consists of points $s \in S$ such that the reduced structure $f^{-1}(f(s))_{\mathrm{red}}$ is singular at s. In addition, we have $T_{S/C} = -K_S + f^*K_C + E$ where E is defined in (5). In particular, we have the following short exact sequence

$$0 \longrightarrow T_{S/C} \longrightarrow T_S \longrightarrow (-f^*K_C - E) \otimes \mathcal{I}_\Gamma \longrightarrow 0. \tag{†}$$

(8) Let $Y \coloneqq \mathbb{P}_S((-f^*K_C - E)\otimes \mathcal{I}_\Gamma) \cong \mathbb{P}_S(\mathcal{I}_\Gamma) \subseteq W$ which is a prime divisor on W. Note that Y is isomorphic to the blow-up of S along the ideal sheaf $\mathcal{I}_\Gamma$ since Γ is locally generated by a regular sequence (cf. [Ser96, Proposition 3.1 (iii)] and [HP20, § 3.10]). However, since the length$(\mathcal{O}_{\Gamma,s})$ at each point $s \in \Gamma$ is not necessarily 1, such Y in general is not necessarily smooth.

(9) Denote by $\mathrm{Exc}(\pi|_Y)$ the exceptional divisor of $\pi|_Y$. By the short exact sequence (†) and [Har77, Chapter II, Lemma 7.9 and Proposition 7.13], we have $\xi|_Y = \mathcal{O}_W(1)|_Y = \mathcal{O}_Y(1)$ where

$$\mathcal{O}_Y(1) = \mathcal{O}_Y(-\mathrm{Exc}(\pi|_Y)) \otimes (\pi|_Y)^*(-f^*K_C - E).$$

Moreover, from [BHPVdV04, IV, (10.5) Lemma] and its proof, we know that $\xi - Y \sim \pi^* T_{S/C}$ and $(\pi|_Y)_* \mathcal{O}_Y(1) = (-f^* K_C - E) \otimes \mathcal{I}_\Gamma$.

Now, we recall Maruyama's elementary transformation which will be heavily used in our proofs.

Notation 2.3 (Elementary transformation for projective bundles over surface blow-ups)**.**

(1) Let $h\colon S_2 \to S_1$ be a blow-up between smooth projective surfaces with the exceptional (-1)-curve $i\colon D \hookrightarrow S_2$. Let ξ_i be the tautological divisor of the projective bundle $\mathbb{P}(T_{S_i})$ and $\pi_i\colon \mathbb{P}(T_{S_i}) \to S_i$ the natural projection.

(2) There is a natural short exact sequence

$$0 \longrightarrow T_{S_2} \longrightarrow h^* T_{S_1} \longrightarrow i_* T_D(D) \longrightarrow 0.$$

Here, $T_D(D) \simeq \mathcal{O}_D(1)$ and $\widetilde{D} \coloneqq \mathbb{P}_D(T_D(D))$ is a projective subbundle of $D' \coloneqq \widetilde{\pi_1}^* D = \mathbb{P}_D(h^* T_{S_1})$ defined by the restriction of the following exact sequence

$$0 \to K = \mathcal{O}_D(-1) \to h^* T_{S_1}|_D \to \mathcal{O}_D(1) \to 0,$$

where the kernel $K = \det(h^* T_{S_1}|_D) \otimes \mathcal{O}_D(1)^\vee = \mathcal{O}_D(-1)$ by the projection formula.

(3) Applying Maruyama's elementary transformation (cf. [Mar82, Theorem 1.4 and Proposition 1.6]), we have the following commutative diagram, where β is the blow-up of $\mathbb{P}(h^* T_{S_1})$ along $\widetilde{D}$ and α is the blow-down of $\mathrm{Bl}_{\widetilde{D}}(\mathbb{P}(h^* T_{S_1}))$ along the β-strict transform of $D' = \widetilde{\pi_1}^* D$.

$$\begin{tikzcd}
 & \mathrm{Bl}_{\widetilde{D}}(\mathbb{P}(h^*T_{S_1})) \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\
\mathbb{P}(T_{S_1}) \arrow[d, "\pi_1"] & \mathbb{P}(h^*T_{S_1}) \arrow[l, "\widetilde{h}"'] \arrow[d, "\widetilde{\pi_1}"] \arrow[r, dashed] & \mathbb{P}(T_{S_2}) \arrow[d, "\pi_2"] \\
S_1 & S_2 \arrow[l, "h"'] \arrow[r, equal] & S_2
\end{tikzcd}$$

(4) By [Mar82, Theorem 1.4], we have $\beta^* \widetilde{h}^* \xi_1 \sim \alpha^* \xi_2 + G$ where G is the β-exceptional divisor.

(5) The β-blown-up section $\widetilde{D}$ satisfies $\widetilde{D} \sim c_0 + e$ in D' where $c_0 = \widetilde{h}^* \xi_1|_{D'}$ is a tautological divisor of $D' \to D$ and e is a fibre (cf. [Har77, Chapter V, Proposition 2.6]). In particular, $\widetilde{D} \not\subseteq \widetilde{h}^* Y_1$ for any section $Y_1 \in |\xi_1|$ by the above linear equivalence.

In what follows, we collect several results to be used in the subsequent sections.

Lemma 2.4 ([HLS22, Lemma 2.2]; cf. [Dru18, Lemma 2.7])**.** *Let X be a projective variety, $\mathcal{E}$ a vector bundle on X, and H a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then $\mathcal{E}$ is pseudo-effective if and only if for all $c > 0$ there exist sufficiently divisible integers $i, j \in \mathbb{N}$ such that $i > cj$ and*

$$H^0(X, \mathrm{Sym}^i \mathcal{E} \otimes \mathcal{O}_X(jH)) = 0.$$

Lemma 2.5. *Let $\mathcal{E} \subseteq \mathcal{F}$ be an injection between two vector bundles over a projective variety X. If $\mathcal{E}$ is pseudo-effective, then so is $\mathcal{F}$.*

Proof. Fix a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor H on X. By Lemma 2.4, we only need to show for all $c > 0$, there exist sufficiently divisible integers $i, j \in \mathbb{N}$ such that $i > cj$ and

$$H^0(X, \mathrm{Sym}^i \mathcal{F} \otimes \mathcal{O}_X(jH)) = 0.$$

This follows from the injection $\mathrm{Sym}^i \mathcal{E} \subseteq \mathrm{Sym}^i \mathcal{F}$ and the pseudo-effectiveness of $\mathcal{E}$ (cf. Lemma 2.4). □

As a consequence of the above lemma, a typical difference between nefness and pseudo-effectiveness is that the quotient bundle of a pseudo-effective vector bundle is not pseudo-effective any more. For example, a rank two vector bundle $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-1)$ over a smooth rational curve is pseudo-effective by Lemma 2.5 while its quotient $\mathcal{O}(-1)$ is not (cf. [Mat22, Proposition 3.4]).

The following example shows that we could not expect the variety equipped with pseudo-effective tangent bundle to be minimal in the higher dimensional case.

Example 2.6. Let $X := E \times S$ be a product of an elliptic curve E and a non-minimal smooth projective surface S which contains some (-1)-curve. Let $p\colon X \to E$ be the natural projection. Applying Lemma 2.5 and considering the natural injection $0 \to p^*\mathcal{O}_E \to T_X$, we see that T_X is pseudo-effective. However, it is clear that K_X is not nef.

Lemma 2.7 (cf. [HLS22, Corollary 2.4])**.** *Let $\pi\colon X' \to X$ be a birational morphism between smooth projective varieties. If the tangent bundle $T_{X'}$ is pseudo-effective, then so is T_X.*

We end up this section with the following lemma, which will be used to confirm the existence of sections of some symmetric power $\mathrm{Sym}^m T_S$ in Theorem 1.3 by taking an étale base change.

Lemma 2.8 (cf. [Uen75, Theorem 5.13])**.** *Let $f\colon X \to Y$ be a surjective morphism of projective varieties and D a Cartier divisor on Y. Then $\kappa(Y, D) = \kappa(X, f^*D)$.*

3. The case $\kappa(S) = 0$ or 2

In this section, we prove Theorem 1.3 when the Kodaira dimension $\kappa(S) = 0$ or 2. Let us begin with the minimal surface of Kodaira dimension zero.

Lemma 3.1. *Let S be a smooth minimal projective surface with $\kappa(S) = 0$. Then the tangent bundle T_S is pseudo-effective if and only if S is a Q-abelian surface (and thus the second Chern class $c_2(S) = 0$), i.e., S is a finite étale quotient of an abelian surface. In this case, $\kappa(\mathbb{P}(T_S), \mathcal{O}(1)) = 1$.*

Proof. From the abundance, we know that $K_S \equiv 0$. Applying [BHPVdV04, VI, Table 10] to S, we see that S has to be one of the following: an Enriques surface, a bi-elliptic surface, a K3 surface or an abelian surface. Since the tangent bundles of Enriques surfaces and K3 surfaces are not pseudo-effective (cf. [Nak04, Chapter VI, Theorem 4.15], or more generally, [HP19, Theorem 1.6]), our S is covered by an abelian surface. Conversely, for a finite étale morphism $\pi\colon A \to S$ from an abelian surface A, we have $\pi^*T_S = T_A = \mathcal{O}_A^{\oplus 2}$. Then T_S is pseudo-effective (and even nef), noting that there is an induced étale morphism $\widetilde{\pi}\colon \mathbb{P}(\pi^*T_S = T_A) \to \mathbb{P}(T_S)$ such that $\widetilde{\pi}^*\mathcal{O}_{\mathbb{P}(T_S)}(1) = \mathcal{O}_{\mathbb{P}(T_A)}(1)$ is effective; in particular, $\mathcal{O}_{\mathbb{P}(T_S)}(1)$ and hence T_S are pseudo-effective (cf. e.g. [Nak04, Chapter II, Lemma 5.6]). So the first half our lemma is proved. The second half of our lemma follows immediately from Lemma 2.8 and the fact that $h^0(A, \mathrm{Sym}^m T_A) = m + 1$ for each positive integer m. □

Now we show the minimality of a smooth projective surface which is of Kodaira dimension zero and has pseudo-effective tangent bundle.

Proposition 3.2. *Let S be a smooth projective surface of $\kappa(S) = 0$. If T_S is pseudo-effective, then K_S is nef, i.e., S is minimal; in particular, S is an étale quotient of an abelian surface.*

Proof. In the view of Lemma 2.7, we only need to exclude the case when $S := S_2$ is a blow-up of a smooth minimal surface S_1 with $K_{S_1} \equiv 0$. Suppose to the contrary that the tangent bundle T_{S_2} is pseudo-effective. By Lemma 3.1, there is an étale cover $S_1' \to S_1$ from an abelian surface S_1'. Let $S_2' := S_1' \times_{S_1} S_2$ be the fibre product, which is also a smooth projective surface with $\kappa(S_2') = 0$. Note that $T_{S_2'}$ (as the pullback of T_{S_2}) is also pseudo-effective. Replacing S_i with S_i', we may assume that S_1 is an abelian surface and thus $T_{S_1} \simeq \mathcal{O}_{S_1}^{\oplus 2}$.

Let us consider the elementary transformation in Notation 2.3 and use the notation therein. Note that the tautological divisors ξ_1 and $\widetilde{h}^*\xi_1$ can be taken as irreducible horizontal sections Y_1 and $\widetilde{Y_1}$, and both of them are nef. Note also that $Y_1|_{Y_1} \equiv 0$ and $\widetilde{Y_1}|_{\widetilde{Y_1}} \equiv 0$, since Y_1 (resp. $\widetilde{Y_1}$) is a fibre of $\mathbb{P}(T_{S_1}) \cong S_1 \times \mathbb{P}^1 \to \mathbb{P}^1$ (resp. $\mathbb{P}(h^*T_{S_1}) \cong S_2 \times \mathbb{P}^1 \to \mathbb{P}^1$). Moreover, ξ_2 is pseudo-effective if and only if so is $\alpha^*\xi_2 = \beta^*\widetilde{h}^*\xi_1 - G$ (cf. e.g. [Nak04, Chapter II, Lemma 5.6]). On the other hand, since $\beta^*\widetilde{h}^*\xi_1 - G$ is pseudo-effective and $\widetilde{h}^*\xi_1$ is nef, by taking a sufficiently ample divisor A on $\mathbb{P}(h^*T_{S_1})$ such that $\beta^*A - G$

is ample (cf. [KM98, Proposition 1.45]), we have

$$\begin{aligned}0 &\le (\beta^*\widetilde{h}^*\xi_1 - G)\cdot(\beta^*A - G)\cdot\beta^*\widetilde{h}^*\xi_1\\ &= (\beta^*\widetilde{h}^*\xi_1\cdot\beta^*A - \beta^*\widetilde{h}^*\xi_1\cdot G - G\cdot\beta^*A + G^2)\cdot\beta^*\widetilde{h}^*\xi_1\\ &= G^2\cdot\beta^*\widetilde{h}^*\xi_1 = \beta_*(G|_G)\cdot\widetilde{h}^*\xi_1.\end{aligned}$$

Recall that $G \to \widetilde{D}$ is a ruled surface and $G|_G \sim_\beta \mathcal{O}_G(1) \sim_\beta -C_0$ where C_0 is some horizontal section, noting that $G|_G$ may contain some β-fibre component. In particular, $G^2\cdot\beta^*\widetilde{h}^*\xi_1 = -\widetilde{h}^*\xi_1\cdot\widetilde{D} = -1 < 0$ (cf. Notation 2.3), which gives us the desired contradiction. The second half follows from Lemma 3.1. □

In the higher dimensional case, applying the Beauville-Bogomolov decomposition, we can easily see that for a projective manifold X with $K_X \equiv 0$, the tangent bundle T_X is pseudo-effective if and only if the augmented irregularity $q^\circ(X) > 0$ (cf. Question 1.2). However, the minimality is not preserved any longer as we observed in Example 2.6.

In the second part of this section, we deal with the case when the variety is of general type.

Proposition 3.3. *Let S be a smooth projective surface of general type. Then T_S is not pseudo-effective.*

Proof. By Lemma 2.7, we only need to exclude the case when S is minimal (and hence K_S is nef) and has pseudo-effective tangent bundle. By the semi-stability of the cotangent bundle Ω_S (cf. [Bog79, Section 13.1]), the tangent bundle T_S is also semi-stable with respect to K_S (cf. [Eno87, Corollary 1.2]). We claim that

$$H^0(S, \operatorname{Sym}^i T_S \otimes \mathcal{O}_S(jK_S)) = 0,$$

for any $i > 2j$. Suppose the claim for the time being. Then it follows from Lemma 2.4 that T_S is not pseudo-effective which concludes our proposition.

To prove the claim, suppose to the contrary that $H^0(S, \operatorname{Sym}^i T_S \otimes \mathcal{O}_S(jK_S)) \ne 0$ for some $i > 2j$. Fix a non-zero section s of $\operatorname{Sym}^i T_S \otimes \mathcal{O}_S(jK_S)$ which defines an injection $0 \to \mathcal{O}_S \to \operatorname{Sym}^i T_S \otimes \mathcal{O}_S(jK_S)$. On the other hand, we have

$$\begin{aligned}c_1(\operatorname{Sym}^i T_S \otimes \mathcal{O}_S(jK_S)) &= c_1(\operatorname{Sym}^i T_S) + \operatorname{rank}(\operatorname{Sym}^i T_S)\cdot c_1(\mathcal{O}_S(jK_S))\\ &= \frac{i(i+1)}{2}\cdot c_1(T_S) - (i+1)\cdot j\cdot c_1(T_S)\\ &= (i+1)\cdot\left(\frac{i}{2} - j\right)c_1(S)\end{aligned}$$

and hence $c_1(\operatorname{Sym}^i T_S \otimes \mathcal{O}_S(jK_S))\cdot c_1(K_S) < 0$. This contradicts the semi-stability of T_S and our claim is thus proved. □

Indeed, with the same argument as above, we obtain the following result in higher dimensional cases.

Remark 3.4. Let X be a normal ($\mathbb{Q}$-factorial) projective variety which is of general type and has at worst klt singularities. We shall show that the reflexive tangent sheaf $T_X := \Omega_X^\vee$ is not pseudo-effective in the sense of [HP20, Definition 3.5]. In the view of [HP20, Corollary 4.3] and Lemma 2.4, after running the special minimal model program with scaling (cf. [BCHM10]), we may assume that X is minimal, i.e., K_X is nef and big. By the semi-stability of the tangent sheaf T_X with respect to K_X (cf. [Eno87, Corollary 1.2]), we have

$$H^0(X, \operatorname{Sym}^{[i]} T_X \otimes \mathcal{O}_X(jK_X)) = 0,$$

for $i > dj$, where $d := \dim(X)$ and $\operatorname{Sym}^{[i]} T_X$ is the reflexive hull of the symmetric power. By [HP20, Definition 3.5] and a similar calculation as in Proposition 3.3, T_X is not pseudo-effective. Consequently, the tangent bundle of any hyperbolic smooth projective variety is never pseudo-effective.

4. Relatively minimal elliptic fibration, the case $\kappa(S) = 1$

In this section, we study Theorem 1.3 for the case $\kappa(S) = 1$. With the further assumption that S is minimal, we obtain the following theorem as our main result of this section.

Theorem 4.1. *Let $f\colon S \to C$ be a relatively minimal elliptic fibration from a smooth projective surface S of Kodaira dimension $\kappa(S) = 1$. Then the following assertions are equivalent.*

(1) The tangent bundle T_S is pseudo-effective;

(2) The second Chern class vanishes, i.e., $c_2(S) = 0$;

(3) f is almost smooth, i.e., the only singular fibres are multiples of smooth elliptic curves.

We stick to Notation 2.2 and the following additional notation throughout this section.

Notation 4.2.

(1) We use the same notation as in Notation 2.2. In addition, S is further assumed to be a smooth projective surface of Kodaira dimension $\kappa(S) = 1$ and f is a relatively minimal elliptic fibration, i.e., free of (-1)-curves among the fibres of f. In fact, such S is a minimal surface, i.e., K_S is nef and $c_1(S)^2 = 0$; see Lemma 4.3.

(2) For each point $c \in C$, we define the *normalised fibre* $\widetilde{S_c}$ over c as follows

$$\widetilde{S_c} := \frac{1}{12} \sum_{j\in J, f(F_j)=c} e(F_j)m_jF_j + \sum_{i\in I, f(E_i)=c} \Big(1 - \Big(1 - \frac{1}{12}e(S_{f(E_i)})\Big)\nu_i\Big)E_i,$$

where $e(\ell)$ is the Euler number of a curve ℓ. It is clear that if S_c is a multiple of a smooth elliptic curve, then the normalised multiple fibre will vanish since $e(S_c) = 0$ in this case. We shall see in Lemma 4.6 that all of the normalised fibre $\widetilde{S_c}$ are $\mathbb{Q}$-effective (cf. Remark 4.9 for a more detailed description).

(3) By the canonical bundle formula for relatively minimal elliptic fibrations [BHPVdV04, V, (12.1) Theorem], we have

$$\omega_S = f^*(\omega_C \otimes (R^1f_*\mathcal{O}_S)^\vee) \otimes \mathcal{O}_S\big(\sum_j (m_j - 1)F_j\big),$$

where $\deg(R^1f_*\mathcal{O}_S)^\vee = \chi(\mathcal{O}_S)$ (cf. [BHPVdV04, V, (12.2) Proposition]). By Noether's formula, $\chi(\mathcal{O}_S) = c_2(S)/12$, noting that $c_1(S)^2 = 0$.

(4) For the relatively minimal elliptic fibration $f\colon S \to C$, recall the following invariant (cf. [BHPVdV04, V, (12.5) Proposition])

$$\delta(f) := \chi(\mathcal{O}_S) + (2g(C) - 2 + \sum_{j\in J}(1 - m_j^{-1})).$$

Then $\kappa(S) = 1$ is equivalent to $\delta(f) > 0$.

Lemma 4.3. *Let $S \to C$ be an elliptic fibration which is relatively minimal. Suppose that $\kappa(S) \geq 0$. Then S is minimal, i.e., the canonical divisor K_S is nef.*

Proof. Note that $\kappa(S) \leq 1$ since $S \to C$ is an elliptic fibration. Let S_{m} be the minimal model of S. By Notation 4.2 (3), we know that $K_S^2 = 0$. If $S \to S_1$ contracts some (-1)-curve, then $K_{S_1}^2 > 0$. Inductively, $K_{S_{\mathrm{m}}}^2 > 0$ and thus $K_{S_{\mathrm{m}}}$ is nef and big, a contradiction to $\kappa(S) \leq 1$. □

We recall the following lemma for the convenience of later proofs.

Lemma 4.4 ([BHPVdV04, III, (18.2) Theorem and V, (12.1) Theorem])**.** *For a relatively minimal elliptic fibration $f\colon S \to C$, one has $\chi(S, \mathcal{O}_S) \geq 0$ with the equality holds if and only if f is isotrivial and all the singular fibres are multiple of smooth elliptic curves (i.e., f is almost smooth).*

Lemma 4.5. *We have the following linear and numerical equivalences*

$$\xi \sim Y + \pi^*T_{S/C} \equiv Y + \pi^*(E_0 - \frac{c_2(S)}{12}F).$$

In particular, if $c_2(S) = 0$, then the tangent bundle T_S is pseudo-effective.

Proof. By the canonical bundle formula and Notation 2.2 (5), we have

$$\omega_S = f^*(\omega_C \otimes (R^1 f_* \mathcal{O}_S)^\vee) \otimes \mathcal{O}_S\big(\sum_{j\in J}(m_j-1)F_j\big) = f^*(\omega_C \otimes (R^1 f_* \mathcal{O}_S)^\vee) \otimes \mathcal{O}_S(E-E_0)$$

where $\deg((R^1 f_*\mathcal{O}_S)^\vee) = c_2(S)/12$. By Notation 2.2 (5) and (7), we have $T_{S/C} \equiv E_0 - c_2(S)F/12$. Together with Notation 2.2 (9), our lemma is thus proved. □

In what follows, we show the normalised fibres defined in Notation 4.2 (2) are all $\mathbb{Q}$-effective. We refer readers to Remark 4.9 for a more detailed description.

Lemma 4.6. *The divisor $-T_{S/C} \equiv c_2(S)F/12 - E_0$ is pseudo-effective. In particular, the normalised fibres defined in Notation 4.2 (2) are all $\mathbb{Q}$-effective.*

Proof. Applying [BHPVdV04, III, (11.4) Proposition], we have

$$\begin{aligned}
\frac{c_2(S)}{12}F - E_0 &= \Big(\frac{1}{12}\sum_{c\in C} e(S_c)\Big)F - E_0 \equiv \frac{1}{12}\sum_{c\in C} e(S_c)S_c - \sum_{i\in I}(\nu_i-1)E_i \\
&= \frac{1}{12}\sum_{j\in J} e(F_j)m_jF_j + \frac{1}{12}\sum_{i\in I} e(S_{f(E_i)})\nu_iE_i - \sum_{i\in I}(\nu_i-1)E_i \\
&= \frac{1}{12}\sum_{j\in J} e(F_j)m_jF_j + \sum_{i\in I}\Big(1-\Big(1-\frac{1}{12}e(S_{f(E_i)})\Big)\nu_i\Big)E_i = \sum_{c\in C}\widetilde{S_c}.
\end{aligned}$$

By the non-negativity of $e(F_j) \geq 0$, we only need to verify that for each non-multiple non-reduced singular fibre $S_{f(E_i)}$ with $i \in I$, we have

$$1-\Big(1-\frac{1}{12}e(S_{f(E_i)})\Big)\nu_i \geq 0.$$

We refer to the Kodaira's table [BHPVdV04, p. 201] and check case by case, noting that the Euler number for a singular fibre of Types I_0^*, I_b^*, II^*, III^* or IV^* is $6, b+6, 10, 9$ or 8, respectively, and the multiplicity of an irreducible component in a singular fibre of Types I_0^*, I_b^*, II^*, III^* or IV^* is no more than 2, 2, 6, 4 or 3, respectively (cf. [Mir89, Table IV.3.1]). □

To prove Theorem 4.1, we first treat the case when f is non-isotrivial. The proof of the following proposition is inspired by [HP20, Proposition 5.4].

Proposition 4.7. *Suppose that f is non-isotrivial. Then the tangent bundle T_S is not pseudo-effective.*

Proof. Suppose to the contrary that T_S is pseudo-effective. Since f is non-isotrivial, there is a non-zero Kodaira-Spencer class which induces the following unique non-trivial extension on an elliptic curve

$$0 \longrightarrow \mathcal{O}_F \longrightarrow T_S|_F \cong \mathcal{F}_2 \longrightarrow \mathcal{O}_F \longrightarrow 0.$$

Note that the above short exact sequence is the restriction of

$$0 \longrightarrow T_{S/C} \longrightarrow T_S \longrightarrow f^*T_C \longrightarrow \mathcal{F} \longrightarrow 0$$

to a general fibre F (cf. Notation 2.2). Let $W_F := \pi^{-1}(F) = \mathbb{P}(T_S|_F)$, and $C_F := Y \cap W_F$ which is the section of $\pi|_{W_F}$ associated with the surjection $T_S|_F \to \mathcal{O}_F \to 0$. Since ξ is pseudo-effective, by the divisorial Zariski decomposition (cf. [Bou04, Theorem 3.12]), we have

$$\xi \equiv \sum a_iY_i + P \tag{1}$$

where Y_i are finitely many prime divisors, $a_i > 0$, and P is a modified nef $\mathbb{R}$-divisor (in the sense that $P|_D$ is pseudo-effective for every prime divisor D on W) (cf. [Bou04, Proposition 2.4]). Restricting the decomposition (1) to W_F, we have

$$C_F \sim \xi|_{W_F} \equiv \sum_i a_iY_i|_{W_F} + P|_{W_F}.$$

Here, we can choose sufficiently general F such that $Y_i \cap W_F$ (if non-zero) are mutually distinct, noting that some of Y_i come from the pull-back of components of E on S and hence have no intersection with

W_F. Since the tautological section C_F is extremal in W_F but $P|_{W_F}$ is still numerically movable by noting that $\{W_F\}_F$ is a free family, we have $P|_{W_F} = 0$. Without loss of generality, we may assume that $a_1 = 1$, $Y_1 \cap W_F = C_F$ and $Y_i \cap W_F = 0$ for $i \neq 1$. Now that $Y_1 \cap W_F = C_F = Y \cap W_F$ for every sufficiently general fibre F of f, we have $Y_1 = Y$. In particular, $\pi^* T_{S/C} \equiv \xi - Y \equiv \sum_{i \geq 2} a_i Y_i + P$ is pseudo-effective (cf. Lemma 4.5). By Lemma 4.6 and the Zariski decomposition on the surface S, we have $E_0 = 0$ and hence $c_2(S) = 0$, a contradiction to Lemma 4.4. $\square$

In the remaining part of this section, we shall deal with the isotrivial case, which is more troublesome. Let us first recall the following lemma.

Lemma 4.8 (cf. e.g. [PS20, Lemma 3.2])**.** *Let $f\colon S \to C$ be a relatively minimal isotrivial elliptic fibration. Then the singular fibre of f is either a multiple of smooth elliptic curves (of Type ${}_m\mathrm{I}_0$) or a non-multiple fibre not being of Types I_b or I_b^* for $b \geq 1$.*

Before moving into the proof of Theorem 4.1 for the isotrivial case, we do some preparations. We give the following remark, which gives a detailed computation in terms of the normalised fibres and their local equations.

Remark 4.9. For every point $c \in C$, one can calculate the normalised singular fibre $\widetilde{S_c}$ (cf. Notation 4.2, Lemma 4.6 and [Mir89, Table IV.3.1]) as follows.

$$\widetilde{S_c} = \sum_{j \in J, f(F_j)=c} \frac{1}{12} e(F_j) m_j F_j + \sum_{i \in I, f(E_i)=c} \Big(1 - \Big(1 - \frac{1}{12} e(S_{f(E_i)})\Big)\nu_i\Big) E_i$$

$$= \begin{cases} \frac{1}{6} e_1 & \mathrm{II} \\ \frac{1}{4}(e_1 + e_2) & \mathrm{III} \\ \frac{1}{3}(e_1 + e_2 + e_3) & \mathrm{IV} \\ \frac{1}{2}(e_2 + e_3 + e_4 + e_5) & \mathrm{I}_0^* \\ \frac{5}{6} e_1 + \frac{2}{3}(e_2 + e_9) + \frac{1}{2}(e_3 + e_7) + \frac{1}{3}(e_4 + e_8) + \frac{1}{6} e_5 & \mathrm{II}^* \\ \frac{3}{4}(e_1 + e_8) + \frac{1}{2}(e_2 + e_5 + e_7) + \frac{1}{4}(e_3 + e_6) & \mathrm{III}^* \\ \frac{2}{3}(e_1 + e_3 + e_5) + \frac{1}{3}(e_2 + e_4 + e_6) & \mathrm{IV}^* \end{cases}$$

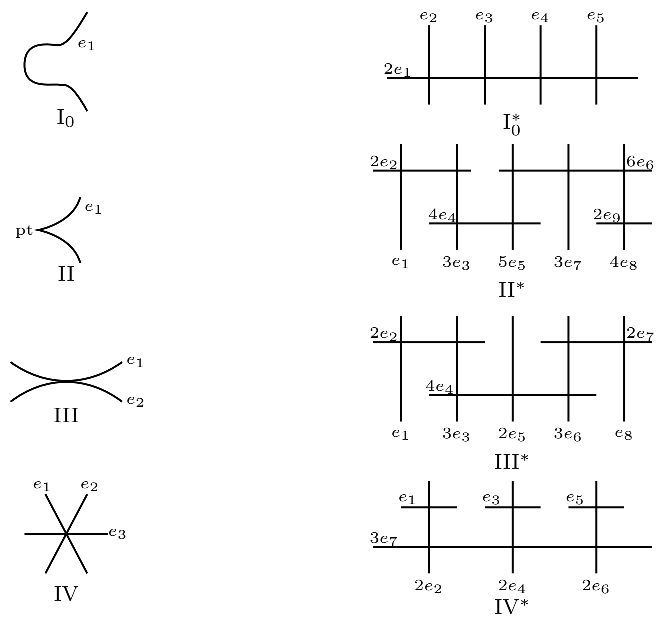

FIGURE 1. Singular fibre of an isotrivial elliptic fibration

For every point $s \in \Gamma$, we may use x, y as regular parameters of $\mathcal{O}_{S,s}$. The ideal sheaf of the singular points on each fibre could be chosen as below (cf. [Ser96, Proof of Proposition 3.1]).

TABLE 1. Local equation of each fibre

type of S_c	II	III	IV	I_0^*	II^*	III^*	IV^*
local equation of S_c	$x^3 = y^2$	$x^4 = y^2$	$x^3 = y^3$	$x^{\nu_i} y^{\nu_j} = 0$			
$\mathcal{I}_{\Gamma,s}$	(x^2, y)	(x^3, y)	(x^2, y^2)	(x, y)			

The following observation plays a significant role to the proof of Theorem 4.1. Once f is not almost smooth and if we assume ξ is pseudo-effective, then the prime divisor Y would appear in the negative part of the divisorial Zariski decomposition of ξ.

Lemma 4.10. *Let f be an isotrivial fibration. If $c_2(S) > 0$, then $\xi|_Y$ is not pseudo-effective.*

Proof. By Notation 2.2 (9), $\xi|_Y \sim -\operatorname{Exc}(\pi|_Y) - (\pi|_Y)^*(f^*K_C + E)$. If the genus $g = g(C) \geq 1$, then $f^*K_C + E \equiv (2g-2)F + E \geq 0$ is effective. Since F is free and $(\pi|_Y)^*E$ is the sum of the proper transform and some $(\pi|_Y)$-exceptional curve, it follows that $(\pi|_Y)^*(f^*K_C + E)$ is effective. In particular, $\xi|_Y$ is anti-pseudo-effective. Suppose that $\xi|_Y$ is pseudo-effective. Then we have $\xi|_Y \equiv 0$, which in turn implies $g(C) = 1$ and $\operatorname{Exc}(\pi|_Y) = 0$. In other words, $\Gamma = \emptyset$, i.e., the reduced structure of every fibre is smooth. This leads to a contradiction to $\chi(S, \mathcal{O}_S) = c_2(S)/12 > 0$ (cf. Lemma 4.4).

In the following, we may assume that $C \simeq \mathbb{P}^1$. By [HP20, Corollaries 3.13 and 3.19], $\xi|_Y$ being not pseudo-effective is equivalent to the vanishing

$$H^0(S, \mathcal{I}_\Gamma^k \otimes \mathcal{O}_S(-k(f^*K_C + E))) = H^0(S, \mathcal{I}_\Gamma^k \otimes \mathcal{O}_S(k(K_S - f^*K_C - E - K_S))) = 0$$

for all $k \in \mathbb{N}$. Since $C \simeq \mathbb{P}^1$, by the canonical bundle formula, $K_S - f^*K_C - E \sim c_2(S)F/12 - E_0$. Note that $K_S = \delta(f)F$ with $\delta(f) > 0$ (cf. Notation 4.2 (4)). Since f is isotrivial, by Lemma 4.8, the only multiple fibres are of Type ${}_m\mathrm{I}_0$, and hence has Euler number zero. Then

$$\frac{c_2(S)}{12}F - E_0 = \sum_{i \in I} \left(1 - \left(1 - \frac{e(S_{f(E_i)})}{12}\right)\nu_i\right) E_i.$$

Now we obtain

$$K_S - f^*K_C - E \sim_{\mathbb{Q}} \sum_{i \in I} \left(1 - \left(1 - \frac{e(S_{f(E_i)})}{12}\right)\nu_i\right) E_i = \sum_{c \in D} \widetilde{S_c}.$$

Here, the set $D \subseteq C$ consists of the point c such that the reduction of the scheme theoretic fibre S_c is not an elliptic curve. We are left to show that for every $s \in \Gamma$ and every $k \in \mathbb{N}$, the local equation of $k\widetilde{S_{f(s)}}$ does not kill the ideal sheaf $\mathcal{I}_{\Gamma,s}^k$. By Lemma 4.8, the non-multiple fibres are of Kodaira's type II, III, IV, I_0^*, II^*, III^* and IV^*. Now the proof is finished by Remark 4.9 and Table 1. $\square$

Lemma 4.11. *If T_S is pseudo-effective and $Y|_Y$ is not pseudo-effective, then $c_2(S) = 0$.*

Proof. By Proposition 4.7, our f is isotrivial. Suppose to the contrary that $c_2(S) > 0$. Applying Lemmas 4.4 and 4.10, we see that $\xi|_Y$ is not pseudo-effective. By the divisorial Zariski decomposition (cf. [Bou04, Theorem 3.12]), there exists $c > 0$ such that

$$\xi \equiv cY + \sum a_i Y_i + P, \tag{2}$$

where $a_i > 0$, $Y_i\ (\neq Y)$ are prime divisors and P is a modified nef $\mathbb{R}$-divisor (cf. [Bou04, Proposition 2.4]). Suppose first that $c \geq 1$. Then $\xi - Y$ is pseudo-effective. Together with Lemma 4.6, we have $\xi \equiv Y$ and $E_0 \equiv c_2(S)F$ which implies that $c_2(S) = 0$, noting that $E_0^2 < 0$ once it is non-zero (cf. [BHPVdV04, III, (8.2) Lemma]), while F is free. Suppose now that $c < 1$. Then we have

$$(1-c)Y \equiv (\xi - cY) - \pi^* T_{S/C}.$$

Restricting to Y itself, we have

$$(1-c)Y|_Y \equiv (\xi - cY)|_Y - \pi^* T_{S/C}|_Y,$$

where $(\xi - cY)|_Y$ is pseudo-effective by Eq. (2). Since $-T_{S/C}$ is pseudo-effective (cf. Lemma 4.6), it follows that $-\pi^* T_{S/C}|_Y$ is also pseudo-effective, noting that Y is horizontal and irreducible. In particular, $(1-c)Y|_Y$ is pseudo-effective. However, this gives rise to a contradiction to our assumption. □

In the view of Lemma 4.11, we are left to show that $Y|_Y$ is not pseudo-effective. For the convenience of our later use, we formulate the following lemma which is a direct consequence of Zariski's lemma.

Lemma 4.12. *Let $f\colon S \to C$ be a fibration from a smooth projective surface to a smooth curve. Let*

$$S_c := f^{-1}(c) = \sum_{i_c \in I_c} m_{i_c} E_{i_c} + \sum_{j_c \in J_c} n_{j_c} F_{j_c}$$

be the fibre of f over $c \in C$ where E_{i_c} and F_{j_c} are (distinct) irreducible components, and m_{i_c} and n_{j_c} are the corresponding multiplicities in S_c. Then

$$M := \sum_{c \in D} \Big(\sum_{i_c \in I_c} a_{i_c} E_{i_c} - \sum_{j_c \in J_c} b_{j_c} F_{j_c} \Big)$$

is not pseudo-effective for any non-empty finite set $D \subset C$ and any rational numbers $a_{i_c} \geq 0$, $b_{j_c} > 0$ satisfying the following two conditions:

(1) there is at least one $c \in D$ such that $J_c \neq \emptyset$;
(2) for each $c \in D$, if $J_c = \emptyset$, then there exists some $a_{i_c} = 0$ with $i_c \in I_c$.

In particular, $M - \delta F$ is not pseudo-effective for any rational number $\delta > 0$ if M satisfies condition (2), where F is a general fibre of f.

Proof. Suppose to the contrary that M is pseudo-effective. Let

$$M \equiv P + N = P + N_0 + \sum_{c \in D} \sum_{i_c \in I_c} d_{i_c} E_{i_c}$$

be the Zariski decomposition, where P is nef, N is effective ($d_{i_c} \geq 0$) such that N_0 and E_{i_c}'s do not have any common components. Then we have

$$\sum_{c \in D} \sum_{i_c \in I_{c,1}} a'_{i_c} E_{i_c} = P + N_0 + \sum_{c \in D} \Big(\sum_{i_c \in I_{c,2}} d'_{i_c} E_{i_c} + \sum_{j_c \in J_c} b_{j_c} F_{j_c} \Big)$$

with $a_{i_c} \geq a'_{i_c} > 0$, $d'_{i_c} > 0$, $I_{c,1} \cap I_{c,2} = \emptyset$ and $I_{c,1} \cup I_{c,2} \subseteq I_c$ for each $c \in D$. Note that there exists at least one, say $c' \in D$, such that $I_{c',1} \neq \emptyset$, since the RHS is not trivial by (1). Take the intersection with

$$\sum_{c \in D} \sum_{i_c \in I_{c,1}} a'_{i_c} E_{i_c}$$

on both sides. Then we get the contradiction by Zariski's lemma (cf. [BHPVdV04, III, (8.2) Lemma]):

$$\Big(\sum_{c \in D} \sum_{i_c \in I_{c,1}} a'_{i_c} E_{i,c} \Big)^2 < 0,$$

$$\Big(P + N_0 + \sum_{c \in D} \Big(\sum_{i_c \in I_{c,2}} d'_{i_c} E_{i_c} + \sum_{j_c \in J_c} b_{j_c} F_{j_c} \Big) \Big) \Big(\sum_{c \in D} \sum_{i_c \in I_{c,1}} a'_{i_c} E_{i,c} \Big) \geq 0,$$

noting that $\sum_{i_c \in I_{c,1}} a'_{i_c} E_{i_c}$ is never a (positive) rational multiple of S_c by condition (2).

For the last assertion, note that F is numerically equivalent to any fibre S_c. It is not hard to see that $M - \delta F$ satisfies both (1) and (2) whenever M satisfies (2). □

Now we come to the last part of the proof of the isotrivial case for Theorem 4.1.

Lemma 4.13. *Suppose that f is isotrivial. If $c_2(S) > 0$, then $Y|_Y$ is not pseudo-effective.*

Proof. Since $Y \equiv \xi + \pi^*(c_2(S)F/12 - E_0)$ (cf. Lemma 4.5), we need to calculate

$$\Big(\xi + \pi^*\Big(\frac{c_2(S)}{12}F - E_0\Big)\Big)|_Y = \xi|_Y + (\pi|_Y)^*\Big(\frac{c_2(S)}{12}F - E_0\Big)$$

and show that it is not pseudo-effective on Y. Recall that Y is isomorphic to the blow-up of the ideal sheaf $\mathcal{I}_\Gamma$. By Notation 2.2 (9), $\mathcal{O}_W(1)|_Y \simeq \mathcal{O}_Y(1) \simeq \mathcal{O}_Y(-E_Y) \otimes (\pi|_Y)^*\mathcal{O}_S(-f^*K_C - E)$, where E_Y is the exceptional divisor of $\pi|_Y \colon Y \to S$. In other words, $\xi|_Y \sim -E_Y - (\pi|_Y)^*(f^*K_C + E)$. Therefore,

$$\begin{aligned}
Y|_Y &\equiv \Big(\xi + \pi^*\Big(\frac{c_2(S)}{12}F - E_0\Big)\Big)|_Y \\
&\sim -E_Y - (\pi|_Y)^*(f^*K_C + E) + (\pi|_Y)^*\Big(\frac{c_2(S)}{12}F - E_0\Big) \\
&= -E_Y + (\pi|_Y)^*\Big(\Big(\frac{c_2(S)}{12} - (2g(C) - 2) - \sum_{j\in J}(1 - \frac{1}{m_j})\Big)F - 2E_0\Big) \\
&= -E_Y + (\pi|_Y)^*\Big(2\sum_{c\in C}\widetilde{S_c} - \delta(f)F\Big)
\end{aligned}$$

where the last equality is due to Lemma 4.6 and $\delta(f) > 0$ is defined in Notation 4.2 (4). We use Remark 4.9 and the notations therein. The pullback $(\pi|_Y)^*\widetilde{S_c}$ via the blow-up $\pi|_Y$ equals

$$(\pi|_Y)_*^{-1}(\widetilde{S_c}) + \begin{cases}
\frac{1}{3}Y_1 & \mathrm{II} \\
\frac{1}{2}Y_{1,2} & \mathrm{III} \\
\frac{1}{2}Y_{1,2,3} & \mathrm{IV} \\
\frac{1}{2}(Y_{1,2} + Y_{1,3} + Y_{1,4} + Y_{1,5}) & \mathrm{I}_0^* \\
\frac{3}{2}Y_{1,2} + \frac{7}{6}Y_{2,3} + \frac{5}{6}Y_{3,4} + \frac{1}{2}(Y_{4,5} + Y_{6,7}) + \frac{1}{3}Y_{6,8} + \frac{1}{6}Y_{5,6} + Y_{8,9} & \mathrm{II}^* \\
\frac{5}{4}(Y_{1,2} + Y_{7,8}) + \frac{3}{4}(Y_{2,3} + Y_{6,7}) + \frac{1}{2}Y_{4,5} + \frac{1}{4}(Y_{3,4} + Y_{4,6}) & \mathrm{III}^* \\
Y_{1,2} + Y_{3,4} + Y_{5,6} + \frac{1}{3}(Y_{2,7} + Y_{4,7} + Y_{6,7}) & \mathrm{IV}^*
\end{cases}$$

where $(\pi|_Y)_*^{-1}(\widetilde{S_c})$ is the proper transform and $Y_{i,j}$ is the exceptional divisor over $e_i \cap e_j$ (scheme-theoretically).

Remark 4.14. We remind readers that, when the blown-up point s lies in the fibre of Types I_0^*, II^*, III^* or IV^*, the ideal sheaf $\mathcal{I}_{\Gamma,s}$ is reduced and the corresponding blow-up is the usual one; hence the coefficient of $(\pi|_Y)^*\widetilde{S_c}$ along $Y_{i,j}$ is simply the sum of the coefficients of $\widetilde{S_c}$ along e_i and e_j. However, the blown-up point in the fibre of Types II, III or IV is non-reduced (cf. Table 1), in which cases, the coefficient of $(\pi|_Y)^*\widetilde{S_c}$ along $Y_{i,j}$ is a bit more involved. We refer to Example 4.15 for the explicit calculation on the case of Type IV for the convenience of the reader but skip other cases for the organisation of our paper.

Now, we come back to the proof of our lemma. By the above list, within each singular fibre S_c, there exists a pair of indices $i_1 \neq i_2 \in I$ such that $e_{i_1} \cap e_{i_2} \neq \emptyset$ and the coefficient

$$\operatorname{coeff}_{(\pi|_Y)^*(\widetilde{S_c})} Y_{i_1,i_2} \leq \frac{1}{2}.$$

Therefore,

$$2(\pi|_Y)^*\widetilde{S_c} - \sum_{s\in\Gamma\cap S_c} Y_s$$

is a linear combination of some components of the fibre of $Y \to C$ over $c \in C$ satisfying (2) of Lemma 4.12. Noting that $E_Y = \sum_{s\in\Gamma} Y_s$ and $\delta(f) > 0$, we see that the divisor

$$Y|_Y \equiv \xi|_Y + (\pi|_Y)^*\Big(\frac{c_2(S)}{12}F - E_0\Big) \sim_{\mathbb{Q}} \sum_{c\in C}\Big(-\sum_{s\in\Gamma\cap S_c} Y_s + 2(\pi|_Y)^*\widetilde{S_c}\Big) - \delta(f)(\pi|_Y)^*F$$

is not pseudo-effective by Lemma 4.12. Our lemma is thus proved. $\square$

Example 4.15. In this example, we calculate the blow-up of the non-reduced point in the fibre of Type IV (cf. Table 1). Let $f(x,y)=(y-x)(y-\zeta x)(y-\zeta^2 x)=y^3-x^3$ be the union of three lines on $\mathbb{A}^2$, where $\zeta=\exp(2i\pi/3)$ is the primitive root of unity. Then the partial derivatives $f_x=-3x^2$ and $f_y=3y^2$. So the blown-up ideal is $\mathcal{I}=(x^2,y^2)$ (cf. [Ser96, Proof of Proposition 3.1]). Let $[a:b]$ be the homogeneous coordinates of $\mathbb{P}^1$. Then the blow-up π of $\mathcal{I}$ is defined by $y^2a-x^2b=0$ in $\mathbb{A}^2\times\mathbb{P}^1$. Consider the pullback of L, L_1 and $L+L_1$, where $L\coloneqq\{y-x=0\}$ is a component of $f(x,y)=0$, and $L_1\coloneqq\{y+x=0)\}$. These pullbacks are defined by

$$(\pi|_Y)^*L:\begin{cases}y^2a-x^2b=0,\\ y-x=0,\end{cases}\qquad(\pi|_Y)^*L_1:\begin{cases}y^2a-x^2b=0,\\ y+x=0,\end{cases}\quad\text{and}\quad(\pi|_Y)^*(L+L_1):\begin{cases}y^2a-x^2b=0,\\ y^2-x^2=0.\end{cases}$$

Their proper transforms are

$$\widetilde{L}:\begin{cases}y^2a-x^2b=0,\\ a-b=0,\\ y-x=0,\end{cases}\qquad\widetilde{L_1}:\begin{cases}y^2a-x^2b=0,\\ a-b=0,\\ y+x=0,\end{cases}\quad\text{and}\quad\widetilde{L}+\widetilde{L_1}:\begin{cases}y^2a-x^2b=0,\\ a-b=0,\end{cases}$$

Note that $\widetilde{L}+\widetilde{L_1}$ is a Cartier divisor on the blow-up of $\mathcal{I}$. The π-exceptional divisor E is defined by

$$\begin{cases}y^2a-x^2b=0,\\ x^2=0,\\ y^2=0,\end{cases}\qquad\text{or just}\quad\begin{cases}y^2a-x^2b=0,\\ x^2=0,\end{cases}\quad\text{on the affine chart } a\neq 0.$$

On the affine chart $a\neq 0$ (equivalently, $a=1$), we may calculate the lengths:

$$\mathbb{C}[x,y,b]/(y^2-x^2,1-b,y-x,x^2)\simeq\mathbb{C}[x]/(x^2)\quad\text{has length 2;}$$
$$\mathbb{C}[x,y,b]/(y^2-x^2,1-b,y+x,x^2)\simeq\mathbb{C}[x]/(x^2)\quad\text{has length 2;}$$
$$\mathbb{C}[x,y,b]/(y^2-x^2,1-b,x^2)\simeq\mathbb{C}[x,y]/(x^2,y^2)\quad\text{has length 4.}$$

By [Ful98, Proposition 7.1], we have

$$\widetilde{L}.E\leq 2,\quad\widetilde{L_1}.E\leq 2,\quad(\widetilde{L}+\widetilde{L_1}).E=4.$$

Therefore, $\widetilde{L}.E=\widetilde{L_1}.E=2$. Assume the pullback of L is $\pi^*L=\widetilde{L}+tE$. Then it follows from the projection formula that

$$0=(\widetilde{L}+tE)\cdot E=2+tE^2.$$

By [Ful98, Page 79, the paragraph before Example 4.3.1], the anti-self-intersection $-E^2$ is the (Samuel) multiplicity of the blown-up point, which is 4. Therefore, we have $t=1/2$ and hence $\pi^*L=L+1/2E$.

Corollary 4.16. *Suppose that f is isotrivial. If T_S is pseudo-effective, then $c_2(S)=0$.*

Proof. Assume that $c_2(S)>0$. Then $Y|_Y$ is not pseudo-effective by Lemma 4.13. On the other hand, by Lemma 4.11, we have $c_2(S)=0$, a contradiction. □

Proof of Theorem 4.1. The equivalence of (2) and (3) follows from Lemma 4.4. The implication (2) ⇒ (1) is proved in Lemma 4.5. The implication (1) ⇒ (2) follows from Proposition 4.7 and Corollary 4.16. □

In the last part of this section, we study the Kodaira dimension $\kappa(\mathbb{P}(T_S),\mathcal{O}(1))$.

Lemma 4.17. *If T_S is pseudo-effective, then $\mathcal{O}_{\mathbb{P}(T_S)}(1)$ is $\mathbb{Q}$-linearly equivalent to an effective divisor. In particular, $\kappa(\mathbb{P}(T_S),\mathcal{O}(1))\geq 0$.*

Proof. By Lemma 4.5, we know that $\xi\sim Y+\pi^*T_{S/C}$ and $T_{S/C}\equiv E_0-\frac{1}{12}c_2(S)F$. Since T_S is pseudo-effective, applying Theorem 4.1, we have $c_2(S)=0$ and hence $E_0=0$; in particular, $T_{S/C}\equiv 0$. We shall show that $T_{S/C}\sim_{\mathbb{Q}}0$. Indeed, in the view of Notation 2.2 (7), Notation 4.2 (3) and Serre duality, we have

$$T_{S/C}=-K_S+f^*K_C+\sum(m_j-1)F_j+E_0=-f^*((R^1f_*\mathcal{O}_S)^\vee)=-f^*(f_*\omega_{S/C}).$$

Since $\deg(f_*\omega_{S/C}) = \chi(S, \mathcal{O}_S) = (c_1(S)^2 + c_2(S))/12 = 0$, by [BHPVdV04, III, (18.3) Proposition], $f_*\omega_{S/C}$ and hence $T_{S/C}$ are $\mathbb{Q}$-trivial. Consequently, $\xi \sim_{\mathbb{Q}} Y$ and thus $\kappa(\mathbb{P}(T_S), \mathcal{O}(1)) \geq 0$. □

Lemma 4.18. *Let $f\colon S \to C$ be a locally trivial elliptic fibration over a smooth projective curve C of genus $g = g(C) \geq 1$. Then $\kappa(\mathbb{P}(T_S), \mathcal{O}(1)) = 1 - \kappa(C)$.*

Proof. Since $\chi(S, \mathcal{O}_S) = 0$, after replacing S by a further étale cover, we may assume that the relative tangent bundle $T_{S/C}$ is trivial (cf. [BHPVdV04, III, (18.3) Proposition] and Lemma 2.8). We consider the symmetric power of the following short exact sequence

$$0 \to \mathcal{O}_S \to T_S \to f^*T_C \to 0,$$

noting that $T_{S/C} = \mathcal{O}_S$ by our reduction. For each positive integer m, we have

$$0 \to \operatorname{Sym}^{m-1} T_S \to \operatorname{Sym}^m T_S \to f^*(-mK_C) \to 0.$$

Hence, we have

$$h^0(S, \operatorname{Sym}^m T_S) \leq 1 + \sum_{i=1}^{m} h^0(S, f^*(-iK_C)).$$

In particular, if $g(C) \geq 2$, then $h^0(S, \operatorname{Sym}^m T_S) \leq 1$ for any m. On the other hand, since f is smooth, it follows from Theorem 4.1 that T_S is pseudo-effective. Together with Lemma 4.17, we have $h^0(S, \operatorname{Sym}^m T_S) = 1$ for any m whenever $g(C) \geq 2$. As a result, $\kappa(\mathbb{P}(T_S), \mathcal{O}(1)) = 0$ when $g(C) \geq 2$.

If $g(C) = 1$, then by the vanishing of $\delta(f)$ (cf. Notation 4.2 (4)), we have $\kappa(S) = 0$. Since S is also minimal, our lemma follows from Lemma 3.1. □

As a corollary, we show our main theorem when S is assumed to be minimal.

Corollary 4.19. *Let S be a smooth minimal projective surface, i.e., K_S is nef. Then the tangent bundle T_S is pseudo-effective if and only if the second Chern class $c_2(S) = 0$. Moreover, after a Galois étale cover, S admits a locally trivial elliptic fibration onto a smooth projective curve C with $\kappa(S) = \kappa(C)$. In particular, after a further étale cover, S is either an abelian surface or isomorphic to a product of an elliptic curve and a smooth curve of genus ≥ 2.*

Proof. The first statement follows from Lemma 3.1, Proposition 3.3, and Theorem 4.1. Let $f\colon S \to C$ be an elliptic fibration. Note that $\kappa(S) = 0$ or 1 and $c_2(S) = 0$ if and only if the only singular fibres of f are multiple of elliptic curves. If $C \simeq \mathbb{P}^1$, then f has at least three multiple fibres, since $\delta(f) \geq 0$ by [BHPVdV04, V, (12.5) Proposition]. Applying [GMM21, Lemma 1.1.9], we get a finite Galois étale map $g_S\colon S' \to S$ which is induced by a ramified base change $g\colon C' \to C$ and $f'\colon S' \to C'$ has no multiple fibre. That is, f' is a locally trivial elliptic fibration. Note that when $\kappa(S) = 1$, one has $\delta(f') > 0$ and hence $g(C') \geq 2$; when $\kappa(S) = 0$, one has $0 \leq \kappa(C') \leq \kappa(S') = 0$ and hence $\kappa(C') = 0$. Finally, the last part of our corollary follows from [Har10, Corollary 26.5]. □

5. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. In the view of Propositions 3.2 and 3.3 and Theorem 4.1, our main task left is to exclude the non-minimal surface which is of Kodaira dimension one and has pseudo-effective tangent bundle. The following theorem is our main result of this section.

Theorem 5.1. *Let S be a smooth projective surface of $\kappa(S) = 1$. If T_S is pseudo-effective, then K_S is nef, i.e., S is minimal.*

Proof. In the view of Lemmas 2.7 and 4.3, we only need to exclude the case when $S \coloneqq S_2$ is a blow-up of a smooth minimal surface S_1. Suppose to the contrary that the tangent bundle T_{S_2} is pseudo-effective.

Let us consider the following commutative diagram as in Notation 2.3

$$\begin{tikzcd}
\mathbb{P}(T_{S_1}) \arrow[d, "\pi_1"] & \mathbb{P}(h^*T_{S_1}) \arrow[l, "\widetilde{h}"'] \arrow[d, "\widetilde{\pi_1}"] \arrow[r, dashed] & \mathbb{P}(T_{S_2}) \arrow[d, "\pi_2"] \\
S_1 \arrow[d, "f_1"] & S_2 \arrow[l, "h"'] \arrow[dl, "f_2"] \arrow[r, equal] & S_2 \\
C & &
\end{tikzcd}$$

where $h\colon S_2 \to S_1$ is a single blow-up between smooth projective surfaces with an exceptional (-1)-curve D, $f_1\colon S_1 \to C$ is an elliptic fibration onto a smooth curve C, $f_2 \coloneqq f_1 \circ h$, and $\pi_i\colon \mathbb{P}(T_{S_i}) \to S_i$ is the natural projection. Let ξ_i be the tautological divisor of the projective bundle $\mathbb{P}(T_{S_i})$ and let Y_i be the divisor in $\mathbb{P}(T_{S_i})$ defined in Notation 2.2 (8). Then we have $\xi_i \sim Y_i + \pi_i^* T_{S_i/C}$ (cf. Notation 2.2 (9)).

We apply Corollary 4.19 to get an étale morphism $S_1' \to S_1$, which is induced by a (possibly ramified) base change $g\colon C' \to C$, such that S_1' admits a locally trivial elliptic fibration over a smooth curve C' of genus ≥ 2. Then we have the following commutative diagram induced by the base change

$$\begin{tikzcd}
S_2' \arrow[r, "g_2"] \arrow[d, "h'"'] & S_2 \arrow[d, "h"] \\
S_1' \arrow[r, "g_1"] \arrow[d, "f_1'"'] & S_1 \arrow[d, "f_1"] \\
C' \arrow[r, "g"] & C
\end{tikzcd}$$

where h' is the blow-up along smooth (reduced) points and g_2 is étale. Then $T_{S_2'} = g_2^* T_{S_2}$ is pseudo-effective (cf. Lemma 2.8). Replacing S_i with S_i', we may assume that f_1 is locally trivial with $g(C) \geq 2$.

Since the divisor ξ_2 is pseudo-effective, it follows from the divisorial Zariski decomposition (cf. [Bou04, Theorem 3.12]) that

$$\xi_2 \equiv \sum a_i P_i + N,$$

where $a_i > 0$, P_i are prime divisors and N is a modified nef $\mathbb{R}$-divisor. Since T_{S_2} is pseudo-effective, applying Lemma 2.7 and Corollary 4.19, we know that $c_2(S_1) = 0$ (indeed, we can even make S_1 a product variety); hence, $T_{S_1/C} \equiv 0$ (cf. Lemmas 4.5 and 4.6). We calculate the difference (cf. Notation 2.2 (7))

$$h^* T_{S_1/C} - T_{S_2/C} = -h^* K_{S_1} + K_{S_2} + h^* E_{S_1} - E_{S_2} = D + h^* E_{S_1} - E_{S_2} = D,$$

where E_{S_i} is defined in Notation 2.2 (5). Therefore, by Notation 2.2 (9), we have

$$Y_2 - \pi_2^* D \sim \xi_2 \equiv \sum a_i P_i + N.$$

Without loss of generality, we may assume that $P_1 = Y_2$ with a_1 possibly being zero. Since $0 \neq -\pi_2^* D$ is anti-pseudo-effective, we see that $0 \leq a_1 < 1$. From the numerical equivalence

$$(1 - a_1) Y_2|_{Y_2} \equiv \sum_{i \geq 2} a_i P_i|_{Y_2} + N|_{Y_2} + (\pi_2|_{Y_2})^* D,$$

the pseudo-effectiveness of $Y_2|_{Y_2}$ follows, noting that $\pi_2^* D|_{Y_2}$ is effective. On the other hand, we have the following equivalence (cf. Notation 2.2 (9))

$$Y_2|_{Y_2} \equiv (\xi_2 + \pi_2^* D)|_{Y_2} \sim -\operatorname{Exc}(\pi_2|_{Y_2}) + (\pi_2|_{Y_2})^*(-f_2^* K_C + D).$$

Since $g(C) \geq 2$, after moving one free fibre from $f^* K_C$ to $f_2^{-1}(f_2(D))$, our $Y_2|_{Y_2}$ is anti-pseudo-effective and non-zero, which contradicts $Y_2|_{Y_2}$ being pseudo-effective. So we finish the proof of our theorem. □

Proof of Theorem 1.3. In the view of Propositions 3.2 and 3.3, we may assume that $\kappa(S) = 1$. Then, the first half follows from Theorems 4.1 and 5.1. The second half follows from Lemmas 3.1 and 4.18 and Corollary 4.19. □

6. Ruled surface over non-rational base, Proof of Proposition 1.7

In the last section, we study non-rational uniruled surfaces with pseudo-effective tangent bundles. First, we study the pseudo-effectiveness of the tangent bundle of a projective bundle over a smooth curve, and reduce the classification problem to Question 1.6. We begin with the ruled surfaces.

Lemma 6.1 (cf. [Kim22])**.** *The tangent bundle T_X of any smooth projective ruled surface $X = \mathbb{P}_C(\mathcal{E})$ is pseudo-effective. In particular, T_X is pseudo-effective but non-big if and only if $\mathcal{E}$ is semi-stable and $C \not\cong \mathbb{P}^1$.*

Proof. In the view of [Kim22], we only need to treat the case when $g(C) \geq 1$ and $\mathcal{E}$ is semi-stable, and show that T_X is pseudo-effective. We consider the following exact sequence

$$0 \longrightarrow T_{X/C} \longrightarrow T_X \longrightarrow f^*T_C \longrightarrow 0,$$

where the relative tangent bundle $T_{X/C} := \Omega_{X/C}^{\vee} = -K_X + f^*K_C$. By [Miy87a, Theorem 3.1], $T_{X/C}$ is nef and hence pseudo-effective. So our lemma follows from Lemma 2.5. □

With a similar strategy, the following lemma is a further extension of Lemma 6.1 and [Kim22] which has its own independent interest. As a result, to classify projective bundles over curves with pseudo-effective tangent bundles, we only need to consider the base curve having genus ≥ 2, which is Question 1.6.

Lemma 6.2. *The tangent bundle of any projective bundle $X = \mathbb{P}_C(\mathcal{E})$ over an elliptic curve C is pseudo-effective. In particular, T_X is pseudo-effective but non-big if and only if $\mathcal{E}$ is semi-stable.*

Proof. Similar to the proof of Lemma 6.1, we only need to consider the case when $\mathcal{E}$ is semi-stable, and show that T_X is pseudo-effective; indeed, it is nef. In this case, it follows from [Ati57] that, after an étale base change and a suitable twist, $\mathcal{E}$ is of the following form

$$\mathcal{E} = \bigoplus_{i=1}^{n} \mathcal{F}_{k_i} \otimes \mathcal{L}_{k_i}$$

where $\mathcal{F}_{k_i}$ is the unique non-trivial extension of $\mathcal{F}_{k_i-1}$ and $\mathcal{O}$ with $\mathcal{F}_1 = \mathcal{O}$ and $\mathcal{L}_{k_i}$ is a degree 0 line bundle. In particular, both $\mathcal{E}$ and $\mathcal{E}^{\vee}$ are nef and hence numerically flat, noting that each $\mathcal{F}_{k_i}$ is self-dual and nef (cf. [Ati57, Theorem 5]).

Consider the following short exact sequence

$$0 \longrightarrow T_{X/C} \longrightarrow T_X \longrightarrow f^*T_C \longrightarrow 0,$$

where $f : X \to C$ is the natural projection. We shall show that $T_{X/C}$ is nef. Then we conclude our result by Lemma 2.5. Indeed, by the relative Euler sequence, we have

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1) \otimes f^*\mathcal{E}^{\vee} \longrightarrow T_{X/C} \longrightarrow 0.$$

Taking the m-th symmetric power, we have

$$0 \longrightarrow \operatorname{Sym}^{m-1} f^*\mathcal{E}^{\vee} \otimes \mathcal{O}_X(m-1) \longrightarrow \operatorname{Sym}^m f^*\mathcal{E}^{\vee} \otimes \mathcal{O}_X(m) \longrightarrow \operatorname{Sym}^m T_{X/C} \longrightarrow 0$$

Since $\mathcal{E}$ is numerically flat, we see that $\operatorname{Sym}^m f^*\mathcal{E}^{\vee} \otimes \mathcal{O}_X(m)$ is nef (as the tensor product of two nef vector bundles); in particular, its quotient bundle $\operatorname{Sym}^m T_{X/C}$ and hence $T_{X/C}$ are nef. □

Now we are in the position of proving Proposition 1.7.

Proof of Proposition 1.7. Let $h\colon S_2 \to S_1$ be the blow-up of $S_1 = S$ along p. We may assume that $\mathcal{E}$ is normalised (cf. [Har77, Chapter V, Notation 2.8.1]). Since T_{S_1} is pseudo-effective but not big, by Lemma 6.1, $\mathcal{E}$ is semi-stable. Together with $\mathcal{E}$ being normalised, we have $\deg\det\mathcal{E} \geq 0$. Denote by C_0 the tautological divisor on $S_1 = \mathbb{P}_C(\mathcal{E})$, which is not necessarily effective. Then $\overline{\mathrm{NE}}(S_1) = \mathrm{Nef}(S_1) = \mathbb{R}_+(2C_0 - f_1^*\det\mathcal{E}) + \mathbb{R}_+F$, where F is a fibre of $f_1 = f\colon S_1 = S \to C$ (cf. [Miy87a, Proposition 3.1]).

Consider the following commutative diagram as in Notation 2.3.

$$\begin{tikzcd}
\mathbb{P}(T_{S_1}) \arrow[d, "\pi_1"] & \mathbb{P}(h^*T_{S_1}) \arrow[l, "\widetilde{h}"'] \arrow[r, dashed] \arrow[d, "\widetilde{\pi_1}"] & \mathbb{P}(T_{S_2}) \arrow[d, "\pi_2"] \\
S_1 \arrow[d, "f_1"] & S_2 \arrow[l, "h"'] \arrow[r, equal] \arrow[dl, "f_2"] & S_2 \\
C & &
\end{tikzcd}$$

Let D be the h-exceptional (-1)-curve, $f_2 \coloneqq f_1 \circ h$ the induced composite map, ξ_i the tautological divisor of the projective bundle $\mathbb{P}(T_{S_i})$, and Y_i the divisor in $\mathbb{P}(T_{S_i})$ defined in Notation 2.2 (8). Then we have (cf. Notation 2.2 (9))

$$\xi_i \sim Y_i + \pi_i^* T_{S_i/C}.$$

Let us calculate the difference

$$h^*T_{S_1/C} - T_{S_2/C} = -h^*K_{S_1} + K_{S_2} + h^*E_{S_1} - E_{S_2} = D + h^*E_{S_1} - E_{S_2} = D,$$

where E_{S_i} is defined in Notation 2.2 (5). Here, since f_1 is smooth and the only singular fibre of f_2 is reduced, we have $h^*E_{S_1} = E_{S_2} = 0$ and

$$T_{S_1/C} = -K_{S_1} + f_1^*K_C \sim 2C_0 - f_1^* \det \mathcal{E}.$$

Claim 6.3. *The divisor $T_{S_2/C} = h^*T_{S_1/C} - D$ is pseudo-effective if and only if there exist some positive integer m and some line bundle $\mathcal{L} \equiv T_{S_1/C}$ such that $H^0(S_1, \mathfrak{m}_p^m \otimes \mathcal{L}^{\otimes m}) \neq 0$, where $\mathfrak{m}_p$ is the maximal ideal of $\mathcal{O}_{S_1,p}$.*

Proof of Claim 6.3. One direction is clear. Let us assume that $h^*T_{S_1/C} - D$ is pseudo-effective. Then we have the Zariski decomposition $h^*T_{S_1/C} - D \equiv \sum_i a_i P_i + N$, where N is nef and $\sum_i a_i P_i$ has negative definite intersection matrix. Since $(h^*T_{S_1/C} - \mu D)^2 < 0$ for any $\mu > 0$, there exists at least one, say P_1, which is not h-exceptional such that $P_1^2 < 0$. Pushing this forward to S_1, we obtain $T_{S_1/C} \equiv \sum_i a_i h_* P_i + h_* N$. Since $T_{S_1/C}$ is extremal, $T_{S_1/C} \equiv t h_* P_1$ for some $t > 0$. Suppose to the contrary that for any positive integer m and any line bundle $\mathcal{L} \equiv T_{S_1/C}$, one has $h^0(S_1, \mathfrak{m}_p^m \otimes \mathcal{L}^{\otimes m}) = 0$. By our assumption, $h^*(h_*P_1) = P_1 + sD$ with $st < 1$; cf. [Har77, Chapter V, Proposition 3.6]. Hence, $h^*T_{S_1/C} - D \equiv tP_1 - (1 - st)D$ which is not pseudo-effective by noting that P_1 is extremal and not parallel to D. This leads to a contradiction to our assumption. $\square$

We come back to the proof of Proposition 1.7. If $H^0(S_1, \mathfrak{m}_p^m \otimes \mathcal{L}^{\otimes m}) \neq 0$ for some positive integer m and some line bundle $\mathcal{L} \equiv T_{S_1/C}$, then $T_{S_2/C} = h^*T_{S_1/C} - D$ is pseudo-effective and thus T_{S_2} is pseudo-effective, noting that $\xi_2 \sim Y_2 + \pi_2^* T_{S_2/C}$ (cf. Lemma 2.5).

Now we assume that $H^0(S_1, \mathfrak{m}_p^m \otimes \mathcal{L}^{\otimes m}) = 0$ for any positive integer m and any line bundle $\mathcal{L} \equiv T_{S_1/C}$. Suppose that ξ_2 is pseudo-effective. It follows from the divisorial Zariski decomposition (cf. [Bou04, Theorem 3.12]) that

$$Y_2 + \pi_2^* T_{S_2/C} \sim \xi_2 \equiv \sum a_i P_i + N,$$

where $a_i > 0$, P_i are prime divisors and N is a modified nef $\mathbb{R}$-divisor. By Notation 2.2 (9), $\xi_2|_{Y_2} = -\operatorname{Exc}(\pi_2|_{Y_2}) - (\pi_2|_{Y_2})^* f_2^* K_C$, which is anti-pseudo-effective and non-zero. Without loss of generality, we may assume that $P_1 = Y_2$ with $a_1 > 0$. Note that now $\pi_2^* T_{S_2/C} \sim \xi_2 - Y_2$ is not pseudo-effective by Claim 6.3 (cf. [Nak04, Chapter II, Lemma 5.6]). Consequently, we deduce that $0 < a_1 < 1$. Consider the following restriction

$$\begin{aligned}
(\xi_2 - a_1 Y_2)|_{Y_2} &= ((1 - a_1)\xi_2 + a_1 \pi_2^* T_{S_2/C})|_{Y_2} \\
&= (1 - a_1)(-\operatorname{Exc}(\pi_2|_{Y_2}) - (\pi_2|_{Y_2})^* f_2^* K_C) + a_1 (\pi_2|_{Y_2})^* T_{S_2/C}
\end{aligned}$$

which is a pseudo-effective divisor on Y_2. Pushing this forward to S_2 along the birational morphism $\pi_2|_{Y_2}$, we have the pseudo-effectiveness of

$$(\pi_2|_{Y_2})_*((\xi_2 - a_1 Y_2)|_{Y_2}) = -(1-a_1)f_2^* K_C + a_1 T_{S_2/C},$$

while the RHS is never pseudo-effective whenever $0 < a_1 < 1$ and $g(C) \geq 1$. In particular, our assumption is absurd and ξ_2 is thus not pseudo-effective. So we finish the proof of our proposition. □

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