

# AUTOMORPHISMS GROUPS OF COMPACT COMPLEX SURFACES: T-JORDAN PROPERTY, TITS ALTERNATIVE AND SOLVABILITY

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ABSTRACT. Let  $X$  be a (smooth) compact complex surface. We show that every torsion subgroup of the biholomorphic automorphisms group  $\text{Aut}(X)$  is virtually nilpotent. Moreover, we study the Tits alternative of  $\text{Aut}(X)$  and virtual derived length of virtually solvable subgroups of  $\text{Aut}(X)$ .

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## 1. INTRODUCTION

1.1. **T-Jordan Property.** The below theorem is our starting point.

**Theorem 1.1** ([Lee76]). *Let  $G$  be a connected Lie group. Then there is a constant  $J = J(G)$  such that every torsion subgroup of  $G$  contains a (normal) abelian subgroup of index  $\leq J$ .*

Inspired by the *Jordan property* introduced by Popov [Pop11, Definition 2.1], we propose the following generalisation. A group  $G$  is called *T-Jordan* (alternatively, we say that  $G$  has the *T-Jordan property*) if there is a constant  $J(G)$  such that every torsion subgroup  $H$  of  $G$  has an abelian subgroup  $H_1$  with the index  $[H : H_1] \leq J(G)$ .

For a compact complex manifold  $X$ , we denote by  $\text{Aut}(X)$  the group of biholomorphic automorphisms. As in [Pop11], we propose the following conjecture.

**Conjecture 1.2.** *Let  $X$  be a compact complex manifold. Then  $\text{Aut}(X)$  is T-Jordan.*

Using the equivariant Kähler model, Meng and the author gave an affirmative answer to Conjecture 1.2 for compact complex manifolds in Fujiki's class  $\mathcal{C}$  (for the definition, see [Fuj78, Definition 1.1]).

**Theorem 1.3** (cf. [JM22, Corollary 1.5]). *Let  $X$  be a compact complex space which is in Fujiki's class  $\mathcal{C}$ . Then  $\text{Aut}(X)$  is T-Jordan.*

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In this paper, we are focusing on smooth compact complex surfaces, especially the non-Kähler (= non-Fujiki's class  $\mathcal{C}$ ) surfaces. We use the following notation.

- Let  $\Xi$  be the set of smooth compact complex surface  $X$  in class VII with the algebraic dimension  $a(X) = 0$  and the second Betti number  $b_2(X) > 0$ .
- Let  $\Xi_0 \subseteq \Xi$  be those minimal surfaces which have no curve.

Here, surfaces of class VII are those smooth compact complex surfaces with the first Betti number  $b_1 = 1$  and Kodaira dimension  $\kappa = -\infty$  (cf. [Kod64, § 7]). By a *minimal surface* we mean a smooth compact complex surface that does not contain any  $(-1)$ -curve  $C$  (i.e., a smooth rational curve  $C$  with self-intersection  $C^2 = -1$ ). It is known that a smooth compact complex surface  $X$  is minimal if and only if any bimeromorphic holomorphic map  $X \rightarrow X'$  to a smooth compact complex surface  $X'$  is an isomorphism.

**Proposition 1.4.** *Let  $X$  be a compact complex surface not in  $\Xi_0$ . Then  $\text{Aut}(X)$  is T-Jordan.*

Although we are not able to confirm Conjecture 1.2 for surfaces in  $\Xi_0$ , we have a slightly weaker result on torsion subgroups of  $\text{Aut}(X)$ . Recall that a group  $G$  is *virtually  $\mathcal{P}$* , if a finite-index subgroup of  $G$  has the property  $\mathcal{P}$ . We say that a group  $G$  is  *$\mathcal{P}$ -by- $\mathcal{Q}$*  if  $G$  fits into an short exact sequence

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$$

where  $K$  has the property  $\mathcal{P}$  and  $H$  has the property  $\mathcal{Q}$ .

**Proposition 1.5.** *Let  $X$  be a smooth compact complex surface in  $\Xi_0$ . Let  $G \leq \text{Aut}(X)$  be a torsion subgroup. Then  $G$  is finite-by-abelian and hence virtually nilpotent and virtually metabelian.*

Since virtually abelian groups are virtually nilpotent, the next result is a direct consequence of Propositions 1.4 and 1.5.

**Theorem 1.6.** *Let  $X$  be a smooth compact complex surface. Then any torsion subgroup  $G \leq \text{Aut}(X)$  is virtually nilpotent.*

Note that by the global spherical shell (GSS) conjecture,  $\Xi_0 = \emptyset$  (cf. [Nak90, Conjecture 1]).

*Remark 1.7.* Assume the global spherical shell conjecture. Let  $X$  be a smooth compact complex surface. Then  $\text{Aut}(X)$  is T-Jordan.

**1.2. Tits Alternative.** In group theory, the *Tits alternative* is an outstanding theorem about the structure of linear groups (cf. [Tit72]). Campana, Wang and Zhang studied the automorphism groups of compact Kähler manifolds and proved a Tits type alternative for such groups (cf. [CWZ13, Theorem 1.5]). We generalise their celebrated result to compact complex spaces in Fujiki's class  $\mathcal{C}$  and smooth compact complex surfaces.

**Theorem 1.8.** *Let  $X$  be a compact complex space in Fujiki's class  $\mathcal{C}$ . Then  $\text{Aut}(X)$  satisfies the Tits alternative, that is, for any subgroup  $G$  of  $\text{Aut}(X)$  either it contains a non-abelian free subgroup, or it is virtually solvable, i.e., admits a solvable subgroup of finite index.*

For the definition and properties of Enoki surfaces and Inoue-Hirzebruch surfaces, see [Eno81] and [Ino77], respectively.

**Theorem 1.9.** *Let  $X$  be a smooth compact complex surface. Assume that either  $X \notin \Xi$ , or  $X \in \Xi$  but its minimal model is an Enoki or Inoue-Hirzebruch surface. Then  $\text{Aut}(X)$  satisfies the Tits alternative.*

**1.3. Virtual Derived Length.** Once we have established the Tits type alternative for  $\text{Aut}(X)$ , it is natural to study the properties of those virtually solvable subgroups.

We have the following result, which gives a uniform upper bound of the virtual derived length (cf. Definition 2.1) for virtually solvable subgroups of  $\text{Aut}(X)$ .

**Theorem 1.10.** *Let  $X$  be a smooth compact complex surface. Assume that either  $X \notin \Xi$ , or  $X \in \Xi$  but its minimal model is an Enoki or Inoue-Hirzebruch surface. Let  $G \leq \text{Aut}(X)$  be a virtually solvable subgroup. Then the virtual derived length  $\ell_{\text{vir}}(G) \leq 4$ .*

*Remark 1.11.*

- (1) Currently, we are not able to prove Theorems 1.9 and 1.10 in full generality for  $X \in \Xi$ .
- (2) The GSS conjecture claims that any minimal surface in  $\Xi$  is a Kato surface (cf. [Nak90, Conjecture 1] and [DOT03, Main Theorem]).
- (3) Kato surfaces are divided into four classes: Enoki surfaces (including parabolic Inoue surfaces), half Inoue surfaces, Inoue-Hirzebruch surfaces and intermediate surfaces (cf. [Tel19, §3.3.2]).
- (4) When  $X$  is a parabolic Inoue surface (which is in  $\Xi$ ), it has been proved that  $\text{Aut}(X)$  is virtually abelian (cf. [Fuj09, Theorem 1.1]). In particular,  $\text{Aut}(X)$  satisfies the Tits alternative and any subgroup  $G$  of  $\text{Aut}(X)$  is virtually solvable with virtual derived length  $\ell_{\text{vir}}(G) \leq 1$ .

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## 2. PRELIMINARIES

In this section, we gather some general preliminary results.

**Definition 2.1.** Recall that for a group  $G$ , its  $n$ -th derived subgroup is defined recursively by

$$G^{(0)} := G \quad \text{and} \quad G^{(n)} := [G^{(n-1)}, G^{(n-1)}], \quad n \in \mathbb{N}.$$

By definition, the group  $G$  is *solvable* if and only if  $G^{(n)} = 1$  for some non-negative integer  $n$ . The minimum of such  $n$  is the *derived length* of  $G$  (when  $G$  is solvable), denoted by  $\ell(G)$ .

If  $G$  is virtually solvable, we define the *virtual derived length* to be

$$\ell_{\text{vir}}(G) := \min_{G'} \ell(G')$$

where  $G'$  runs through all finite-index subgroups of  $G$ .

A group  $G$  has *bounded torsion subgroups* if there is a constant  $T = T(G)$  such that any torsion subgroup  $G_1 \leq G$  has order  $|G_1| \leq T$ .

**Lemma 2.2.** *Let*

$$1 \longrightarrow N \longrightarrow G \longrightarrow H$$

*be an exact sequence of groups. Then the following assertions hold:*

- (1) *If  $N$  is  $T$ -Jordan and  $H$  has bounded torsion subgroups, then  $G$  is  $T$ -Jordan.*
- (2) *Assume further that the sequence is also right exact. If  $N$  is a torsion group and  $G$  is  $T$ -Jordan, then  $H$  is  $T$ -Jordan.*

*Proof.* (1) is clear. (2) Let  $H' \leq H$  be a torsion subgroup, and  $G' \leq G$  be its preimage. Since  $N$  is a torsion group,  $G'$  is also a torsion group and hence there exists an abelian subgroup  $G'_1 \leq G'$  such that  $[G' : G'_1] \leq J(G)$  for some constant  $J(G)$ . Let  $H'_1 = G'_1/(G'_1 \cap N) \leq H'$ , which is abelian with  $[H' : H'_1] \leq J(G)$ . Therefore,  $H$  is T-Jordan.  $\square$

The following two lemmas are adapted from [PS21, Corollaries 4.2, 4.4].

**Lemma 2.3.** *Let  $X$  be an irreducible Hausdorff reduced complex space, and let  $\Gamma \leq \text{Aut}(X)$  be a torsion subgroup. Suppose that  $\Gamma$  has a fixed point  $x$  on  $X$ . Then the natural representation*

$$\Gamma \longrightarrow \text{GL}(T_{x,X})$$

*is faithful.*

**Lemma 2.4.** *Let  $X$  be an irreducible Hausdorff reduced complex space, and let  $\Delta \leq \text{Aut}(X)$  be a subgroup. Suppose that  $\Delta$  has a fixed point  $x$  on  $X$ , and let*

$$\sigma : \Delta \longrightarrow \text{GL}(T_{x,X})$$

*be the natural representation. Suppose that there is a normal subgroup  $\Gamma \leq \Delta$  with the quotient group  $\Delta/\Gamma$  being torsion, such that the restriction  $\sigma|_\Gamma$  is a group monomorphism. Then  $\sigma$  is an embedding as well.*

*Proof.* Let  $\Delta_0$  be the kernel of  $\sigma$ . Since  $\Delta_0 \cap \Gamma = \{\text{id}\}$  and  $\Delta/\Gamma$  is torsion, we see that  $\Delta_0$  is also torsion. Thus,  $\Delta_0$  is trivial by Lemma 2.3.  $\square$

**Lemma 2.5.** *Let  $G \leq \text{GL}_n(\mathbb{C})$  be a virtually solvable subgroup. Then  $\ell_{\text{vir}}(G) \leq n$*

*Proof.* By passing to a finite-index subgroup, we may assume that  $G$  itself is solvable. Then some subgroup  $G_1$  of finite index in  $G$  can be put in triangular form (cf. [Bor12, Corollary, pp. 137]). It follows that the derived length of  $G_1$  and hence the virtual derived length of  $G$  is at most  $n$ .  $\square$

**Lemma 2.6.** *Consider the short exact sequence of groups*

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1.$$

*Then*

- (1) *if  $N$  is solvable and  $H$  is virtually solvable, then  $G$  is virtually solvable with  $\ell_{\text{vir}}(G) \leq \ell(N) + \ell_{\text{vir}}(H)$ .*
- (2) *if  $N$  is finite and  $H$  is virtually solvable, then  $G$  is virtually solvable with  $\ell_{\text{vir}}(G) \leq \ell_{\text{vir}}(H) + 1$ ;*
- (3) *the group  $G$  is virtually solvable if and only if both  $N$  and  $H$  are virtually solvable;*
- (4) *if both  $N$  and  $H$  satisfy the Tits alternative, then so does  $G$ .*

*Proof.* (1) Let  $H' \leq H$  be a solvable subgroup of finite index (with  $\ell(H') = \ell_{\text{vir}}(H)$ ), and let  $G' \leq G$  be its preimage. Since  $[G : G'] = [G/N : G'/N] = [H : H'] < \infty$ , we may replace  $G, H$  by  $G', H'$  and assume that  $H$  is solvable. Then it is clear that  $G$  is solvable, and  $\ell_{\text{vir}}(G) \leq \ell(G) \leq \ell(N) + \ell(H) \leq \ell(N) + \ell_{\text{vir}}(H)$ .

(2) As in (1), we may assume that  $H$  is solvable. Let  $Z_G(N)$  be the centraliser of  $N$  in  $G$ . Since  $N$  is finite,  $Z_G(N) \leq G$  is of finite index. Note that  $Z_G(N) \cap N$  is abelian (in particular, solvable) and  $Z_G(N)/(Z_G(N) \cap N) \leq G/N \simeq H$  is solvable. Hence,  $Z_G(N)$  is also solvable with derived length  $\ell(Z_G(N)) \leq \ell(H) + 1$ . It follows that  $\ell_{\text{vir}}(G) \leq \ell_{\text{vir}}(H) + 1$ .

(3) The “only if” direction is clear. Suppose that  $N$  and  $H$  are virtually solvable. As in (2), we may assume that  $H$  is solvable. Let  $N' \triangleleft N$  be a finite-index solvable (normal) subgroup of minimal index.

Then  $N'$  is normal in  $G$ ; otherwise, if  $N''$  is another conjugate of  $N'$ , then  $N'N'' \triangleleft N$  is solvable with smaller index, a contradiction to the choice of  $N'$ . Consider the following short exact sequence

$$1 \longrightarrow N/N' \longrightarrow G/N' \longrightarrow G/N \simeq H \longrightarrow 1.$$

By assumption, the first term is finite and  $H$  is solvable. Thus,  $G/N'$  is virtually solvable by (2). Then using the short exact sequence

$$1 \longrightarrow N' \longrightarrow G \longrightarrow G/N' \longrightarrow 1,$$

we obtain that  $G$  is virtually solvable by (1).

(4) This follows from (3).  $\square$

Using the lemma above, we can give a proof of Theorem 1.8.

*Proof of Theorem 1.8.* Let  $\pi: X' \longrightarrow X$  be an  $\text{Aut}(X)$ -equivariant resolution of singularities (cf. [BM97, Theorem 13.2]) with  $\text{Aut}(X)$  lifts to a (unique) subgroup of  $\text{Aut}(X')$  via  $\pi$ . Note that  $X'$  is still in Fujiki's class  $\mathcal{C}$ . Therefore, we may replace  $X$  by  $X'$  and assume that  $X$  is smooth.

Denote by

$$\text{Aut}_\tau(X) := \{g \in \text{Aut}(X) \mid g^*|_{H^2(X, \mathbb{Q})} = \text{id}\}.$$

Then we have the following short exact sequence

$$1 \longrightarrow G_\tau \longrightarrow G \longrightarrow G|_{H^2(X, \mathbb{Q})} \longrightarrow 1$$

where  $G_\tau = G \cap \text{Aut}_\tau(X)$ . It suffices to prove that  $G_\tau$  satisfies the Tits alternative by [Tit72, Theorem 1.1] and Lemma 2.6 (4). Let  $\sigma: \tilde{X} \longrightarrow X$  be a bimeromorphic holomorphic map from a compact Kähler manifold  $\tilde{X}$  such that  $\text{Aut}_\tau(X)$  lifts to  $\tilde{X}$  holomorphically via  $\sigma$  (cf. [JM22, Theorem 1.1]). Viewing  $G_\tau \leq \text{Aut}_\tau(X)$  as a subgroup of  $\text{Aut}(\tilde{X})$ , we know that  $G_\tau$  satisfies the Tits alternative (cf. [CWZ13, Theorem 1.5]). Therefore,  $G$  satisfies the Tits alternative.  $\square$

The theorem below is due to [Pin84, §2, Theorem].

**Theorem 2.7.** *Let  $X$  be an Inoue-Hirzebruch surface. Then  $\text{Aut}(X)$  is finite. In particular,  $\text{Aut}(X)$  satisfies the Tits alternative and every subgroup  $G \leq \text{Aut}(X)$  is finite and hence virtually solvable with virtual derived length  $\ell_{\text{vir}}(G) = 0$ .*

The following lemmas will be used in Section 8.

**Lemma 2.8** (cf. [Hal50, §2, 4]). *Let  $G$  be a finitely generated group. Then every finite-index subgroup of  $G$  contains a finite-index subgroup  $G'$  which is characteristic in  $G$ .*

**Lemma 2.9.** *Let  $G$  be a solvable group and  $U \triangleleft G$  a non-trivial unipotent subgroup. Then, after replacing  $G$  by a finite-index subgroup, there is a subgroup  $Z_1 \simeq \mathbb{G}_a \leq U$  which is normal in  $G$ .*

*Proof.* Let  $Z = Z(U)$  be the centre of  $U$  which is nontrivial and normal in  $G$ . Note that the commutative unipotent group  $Z \simeq \mathbb{G}_a^n$ , the additive affine algebraic group of dimension  $n$  for some  $n \geq 1$ . Consider the conjugation action of  $G$  on  $Z$ :

$$c_g: x \mapsto gxg^{-1} \quad \text{for } x \in Z \text{ and } g \in G.$$

One can show that this action is linear and induces a homomorphism  $G \longrightarrow \text{GL}_n(\mathbb{C})$ . Replacing  $G$  by a finite-index subgroup, we may assume that the image  $G'$  of the solvable group  $G$  in  $\text{GL}_n(\mathbb{C})$  is of triangular form (cf. [Bor12, Corollary, pp. 137]). Let  $Z_1 \simeq \mathbb{G}_a$  be an (1-dimensional) eigenspace of  $G'$ . Then  $Z_1$  is stable under the conjugation action of  $G$  and hence is normal in  $G$ .  $\square$

## 3. EQUIVARIANT FIBRATIONS

In this section we will consider smooth compact complex surfaces  $X$  with algebraic dimension  $a(X) = 1$ , or with Kodaira dimension  $\kappa(X) = 1$ . If  $a(X) = 1$ , there is a natural elliptic fibration, the *algebraic reduction*  $\pi: X \rightarrow Y$  (cf. [BHPV04, VI. (5.1) Proposition]), which is a holomorphic map. For  $g \in \text{Aut}(X)$ , the image of a fibre  $F$  of  $\pi$  under  $g$  is another fibre; otherwise the self-intersection number of  $g(F) + F$  is positive and hence  $X$  is projective (cf. [BHPV04, IV, Theorem 6.2]), a contradiction (cf. [BHPV04, IV, Corollary 6.5]). Thus,  $\pi$  is  $\text{Aut}(X)$ -equivariant. When  $\kappa(X) = 1$ , the *pluricanonical fibration* gives a natural elliptic fibration as well, which is also  $\text{Aut}(X)$ -equivariant.

In addition to the above, we will also study affine  $\mathbb{A}^1$ -bundle  $\pi: A \rightarrow E$  over an elliptic curve  $E$ . Note that the fibre of  $\pi: A \rightarrow E$  being an affine line  $\mathbb{A}^1$  cannot dominate the base elliptic curve  $E$ . It follows that  $\pi$  is  $\text{Aut}(A)$ -equivariant.

Let  $\pi: X \rightarrow Y$  be a morphism between complex spaces. Let  $G$  be a complex Lie group contained in  $\text{Aut}(X)$  such that  $\pi \circ g = \pi$  for all  $g \in G$ , i.e.,  $G \leq \text{Aut}_Y(X)$ . For any finite subset  $S \subseteq Y$ , denote by  $G_S := \{g \in G \mid g|_{\pi^{-1}(S)} = \text{id}\}$ , i.e.,  $G_S$  is the kernel of the natural group homomorphism  $G \rightarrow \prod_{y \in S} \text{Aut}(X_y)$ . Note that any Lie group is second countable and hence has only countably many connected components.

The ideas of Lemma 3.1 and Proposition 3.2 come from Doctor Sheng Meng.

**Lemma 3.1.** *Suppose that  $\dim G = 0$ . Then  $G_y$  is trivial for very general  $y \in Y$ .*

*Proof.* Let  $Y_g := \{y \in Y \mid g|_{X_y} = \text{id}\}$ . Then  $Y_g$  is a proper closed subspace of  $Y$  for  $g \neq \text{id}$ . Let  $Z := \bigcup_{\text{id} \neq g \in G} Y_g$ . Then any  $y \in Y \setminus Z$  is a very general choice. Here we use the condition that  $G$  is countable since  $\dim G = 0$ .  $\square$

A very general finite subset  $S \subseteq Y$  means that every element of  $S$  is very generally chosen in  $Y$ .

**Proposition 3.2.** *For very general finite subset  $S \subseteq Y$ ,  $G_S$  is trivial.*

*Proof.* First, we show by induction on  $\dim G$  that  $\dim G_S = 0$  for very general finite subset  $S \subseteq Y$ . When  $\dim G = 0$ , the claim follows from Lemma 3.1. Now assume that  $\dim G \geq 1$ . Denote by  $T := \{y \in Y \mid G_0|_{X_y} = \text{id}\}$ , where  $G_0$  is the neutral component of  $G$ . Note that  $T \subsetneq Y$  is a closed subspace. Then for  $y \notin T$ , we have  $\dim G_y < \dim G$ . By induction,  $\dim(G_y)_S = 0$  for any  $y \notin T$  and very general finite subset  $S \subseteq Y$  (depending on  $y$ ). Note that  $(G_y)_S = G_{\{y\} \cup S}$ . So the claim is proved.

Then by Lemma 3.1, we see that  $G_{S \cup \{y\}} = (G_S)_y$  is trivial for very general  $y \in Y$ .  $\square$

**Lemma 3.3.** *Let  $X$  be either a (primary or secondary) Kodaira surface, or a minimal smooth compact complex surface of Kodaira dimension 1. Then there is an  $\text{Aut}(X)$ -equivariant elliptic fibration  $\pi: X \rightarrow Y$  to a smooth curve  $Y$ . Also, the natural image of  $\text{Aut}(X)$  in  $\text{Aut}(Y)$  is finite.*

*Proof.* We may take  $\pi$  to be either the algebraic reduction or the pluricanonical fibration (see the first paragraph of this section). Then our claim follows from [Shr20, Corollary 3.4, Lemma 3.5] and [PS20b, Proposition 1.2].  $\square$

**Theorem 3.4.** *Let  $X$  be either a (primary or secondary) Kodaira surface or a minimal smooth compact complex surface of Kodaira dimension 1. Then  $\text{Aut}(X)$  is virtual abelian. In particular,  $\text{Aut}(X)$  satisfies  $T$ -Jordan property, the Tits alternative and every subgroup  $G \leq \text{Aut}(X)$  is virtually abelian and hence virtually solvable with virtual derived length  $\ell_{\text{vir}}(G) \leq 1$ .*

*Proof.* By Lemma 3.3, there is a short exact sequence

$$1 \rightarrow \text{Aut}_Y(X) \rightarrow \text{Aut}(X) \rightarrow \Gamma \rightarrow 1,$$

where  $\pi: X \rightarrow Y$  is an elliptic fibration and  $\Gamma$  is a finite subgroup of  $\text{Aut}(Y)$ . By Proposition 3.2, there exists a finite subset  $S$  of  $Y$  such that the natural map  $\text{Aut}_Y(X) \hookrightarrow \prod_{y \in S} \text{Aut}(X_y)$  is an injection. Since  $X_y$  is an elliptic curve,  $\prod_{y \in S} \text{Aut}(X_y)$  and hence  $\text{Aut}_Y(X)$  are virtually abelian. Then it is clear that  $\text{Aut}(X)$  is virtually abelian.  $\square$

**Theorem 3.5.** *Let  $X \in \Xi$  be an Enoki surface. Then  $\text{Aut}(X)$  is virtually solvable (and hence  $\text{Aut}(X)$  satisfies the Tits alternative) and every subgroup  $G \leq \text{Aut}(X)$  is virtually solvable with  $\ell_{\text{vir}}(G) \leq 3$ .*

*Proof.* By [Eno81, Main Theorem],  $X$  has a cycle of rational curves  $C$  such that  $A := X \setminus C$ , the complement of  $C$  in  $X$ , is an affine  $\mathbb{A}^1$ -bundle  $\pi: A \rightarrow E$  over an elliptic curve  $E$ . Since the algebraic dimension  $a(X) = 0$ , the surface  $X$  contains finitely many (irreducible) curves (cf. [BHPV04, IV. Theorem 8.2]). Replacing  $\text{Aut}(X)$  by a finite-index subgroup, we may assume that  $C$  is  $\text{Aut}(X)$ -invariant. Then  $\text{Aut}(X)$  acts on  $A = X \setminus C$  biholomorphically and faithfully and  $\pi$  is  $\text{Aut}(X)$ -equivariant (see the second paragraph of this section).

Consider the short exact sequence

$$1 \rightarrow K \rightarrow \text{Aut}(X) \rightarrow \Gamma \rightarrow 1,$$

where  $K \leq \text{Aut}_E(A)$  and  $\Gamma \leq \text{Aut}(E)$ . Since  $E$  is an elliptic curve,  $\text{Aut}(E)$  and hence  $\Gamma$  are virtually abelian, or equivalently, virtually solvable with virtual derived length  $\leq 1$ . By Proposition 3.2, there exists a finite subset  $S$  of  $E$  such that the natural map  $K \hookrightarrow \prod_{e \in S} \text{Aut}(A_e)$  is an injection. Since  $A_e \simeq \mathbb{A}^1$ , the group  $\prod_{e \in S} \text{Aut}(A_e)$  and hence  $K$  are metabelian, or equivalently, solvable with derived length  $\leq 2$ . By Lemma 2.6 (1),  $\text{Aut}(X)$  is virtually solvable with virtual derived length  $\leq 3$ . It is clear that any subgroup  $G \leq \text{Aut}(X)$  is also virtually solvable and its virtual derived length  $\ell_{\text{vir}}(G) \leq 3$ .  $\square$

#### 4. SURFACES WITH INVARIANT CURVES

**Lemma 4.1.** *Let  $X$  be a smooth compact complex surface. Suppose that there is a finite non-empty  $\text{Aut}(X)$ -invariant set  $\Sigma$  of (irreducible) curves on  $X$ . Then  $\text{Aut}(X)$  is T-Jordan.*

*Proof.* Let  $C$  be one of the curves from  $\Sigma$ . Then the group  $\text{Aut}(X, C)$  of automorphisms of  $X$  that preserves the curve  $C$  has finite index in  $\text{Aut}(X)$ . We only need to show that  $\text{Aut}(X, C)$  is T-Jordan.

Assume first that  $C$  is singular. Then  $\text{Aut}(X, C)$ , after replacing by a subgroup of finite index, fixes some point on  $C$ . Now Lemma 2.3 implies that  $\text{Aut}(X, C)$  is embedded into  $\text{GL}_2(\mathbb{C})$ . Therefore, the group  $\text{Aut}(X, C)$  is T-Jordan (cf. Theorem 1.1).

Now we assume that  $C$  is smooth and let  $G \leq \text{Aut}(X, C)$  be a torsion subgroup. Let  $\mathcal{N}_{C/X}$  be the normal bundle of  $C$  in  $X$ . Consider the natural group homomorphism

$$\mathcal{N}: \text{Aut}(X, C) \rightarrow \text{Aut}(\mathcal{N}_{C/X} \rightarrow C)$$

via  $\sigma \mapsto \mathcal{N}_\sigma$  (cf. [MPZ22, Notation 2.2]). By [MPZ22, Lemma 2.3],  $\ker \mathcal{N} \cap G = \{\text{id}\}$ . Moreover, by the proof of [MPZ22, Theorem 3.1], there is a monomorphism

$$\text{Aut}(\mathcal{N}_{C/X} \rightarrow C) \rightarrow \text{Aut}(\mathbb{P}_C(\mathcal{N}_{C/X} \oplus \mathcal{O})).$$

So we may view  $G$  as a subgroup of  $\text{Aut}(\mathbb{P}_C(\mathcal{N}_{C/X} \oplus \mathcal{O}))$ . Note that  $C$  is projective. So  $\mathbb{P}_C(\mathcal{N}_{C/X} \oplus \mathcal{O})$  is also a projective manifold. Then the theorem follows from Theorem 1.3.  $\square$

**Corollary 4.2.** *Let  $X$  be a compact complex surface with  $a(X) = 0$ . Assume that  $X$  contains at least one curve. Then  $\text{Aut}(X)$  is T-Jordan.*

*Proof.* It follows from Lemma 4.1, since  $X$  contains at most a finite number of (irreducible) curves (cf. [BHPV04, IV. Theorem 8.2]).  $\square$



**Proposition 4.3.** *Let  $X$  be a compact complex surface with  $e(X) \neq 0$  and  $a(X) = 1$ . Then  $\text{Aut}(X)$  is  $T$ -Jordan.*

*Proof.* Let  $\pi: X \rightarrow Y$  be the algebraic reduction (cf. Section 3), so that  $Y$  is a smooth curve and  $\pi$  is an  $\text{Aut}(X)$ -equivariant elliptic fibration. Since  $e(X) \neq 0$ , the fibration  $\pi$  has at least one fibre  $X_y$  such that  $F = (X_y)_{\text{red}}$  is not a smooth elliptic curve by Suzuki's formula (cf. [BHPV04, III, Proposition 11.4]). Note that  $\pi$  has only finitely many singular fibres, whose union is  $\text{Aut}(X)$ -invariant. Then the assertion follows from Lemma 4.1.  $\square$

## 5. HOPF SURFACES

In this section, we study the automorphism groups of Hopf surfaces. By definition, a *Hopf surface*  $X$  is a smooth compact complex surface with universal covering being isomorphic to  $\mathbb{C}^2 \setminus \{0\}$ . Then,  $X$  can be obtained from  $\mathbb{C}^2 \setminus \{0\}$  as a quotient by a free action of a discrete group  $\Gamma \simeq \pi_1(X)$ . If  $\pi_1(X) \simeq \mathbb{Z}$ , we call such  $X$  a *primary* Hopf surface. In this case, after an appropriate choice of coordinates of  $\mathbb{C}^2$ , the generator of  $\Gamma$  has the form

$$(5.1) \quad (z, w) \mapsto (az + \lambda w^m, bw),$$

where  $m \in \mathbb{Z}_{>0}$  and  $a, b, \lambda$  are complex constants subject to the restrictions

$$(a - b^m)\lambda = 0, \quad 0 < |a| \leq |b| < 1.$$

Any Hopf surface is either primary or a finite étale quotient of a primary Hopf surface, which are called the Hopf surface (cf. [Kod66, Theorem 30])

By above,  $\Gamma$  contains  $\Lambda \simeq \mathbb{Z}$  as a subgroup of finite index. In particular,  $\Gamma$  is virtually solvable. After replacing  $\Lambda$  by a suitable subgroup  $\Lambda_0 \simeq \mathbb{Z}$ , we may assume that  $\Lambda$  is a characteristic subgroup of  $\Gamma$  (cf. [PS21, Lemma 2.10]), with the generator of  $\Lambda$  having the form Eq. (5.1). Now there is a short exact sequence

$$(5.2) \quad 1 \rightarrow \Gamma \rightarrow \widetilde{\text{Aut}}(X) \rightarrow \text{Aut}(X) \rightarrow 1$$

where  $\widetilde{\text{Aut}}(X)$  acts on  $\mathbb{C}^2 \setminus \{0\}$  biholomorphically. It follows from Hartogs extension theorem,  $\widetilde{\text{Aut}}(X)$  can be extended to biholomorphic actions on  $\mathbb{C}^2$ , which fixes the origin  $0 \in \mathbb{C}$ . By assumption,  $\Lambda \triangleleft \widetilde{\text{Aut}}(X)$  is a normal subgroup.

**Theorem 5.1.** *Let  $X$  be a Hopf surface. Then the group  $\text{Aut}(X)$  satisfies the Tits alternative.*

*Proof.* Assume first that  $X$  is a primary Hopf surface. Then we may identify  $\widetilde{\text{Aut}}(X)$  with a subgroup of  $\text{GL}_2(\mathbb{C}) \times \mathbb{C}$  by [Nam74, § 2] and [Weh81, pp. 24]. Since  $\text{GL}_2(\mathbb{C}) \times \mathbb{C}$  satisfies the Tits alternative, so do  $\widetilde{\text{Aut}}(X)$  and  $\text{Aut}(X)$  (cf. Lemma 2.6 (4)).

Now let  $X$  be a secondary Hopf surface. Then either  $\widetilde{\text{Aut}}(X) \leq \text{GL}_2(\mathbb{C})$ , or  $\widetilde{\text{Aut}}(X) \simeq \mathbb{C} \rtimes \mathbb{C}^*$  or  $\widetilde{\text{Aut}}(X) \simeq \mathbb{C} \rtimes (\mathbb{C}^*)^2$  (cf. [MN00, Theorem 1]). In the first case,  $\widetilde{\text{Aut}}(X)$  and hence  $\text{Aut}(X)$  satisfies the Tits alternative as  $\text{GL}_2(\mathbb{C})$  does; in the latter two cases,  $\widetilde{\text{Aut}}(X)$  and hence  $\text{Aut}(X)$  are already solvable. In a word,  $\text{Aut}(X)$  satisfies the Tits alternative.  $\square$

**Theorem 5.2.** *Let  $X$  be a Hopf surface and  $G \leq \text{Aut}(X)$  a virtually solvable subgroup. Then  $\ell_{\text{vir}}(G) \leq 2$ .*

*Proof.* Let  $\tilde{G} \leq \widetilde{\text{Aut}}(X)$  be the preimage of  $G$ . Then  $\tilde{G}$  is virtually solvable by Lemma 2.6 (3) as  $\Gamma$  is virtually solvable. Therefore,  $\ell_{\text{vir}}(G) \leq \ell_{\text{vir}}(\tilde{G}) \leq \ell_{\text{vir}}(\widetilde{\text{Aut}}(X)) \leq 2$  by Lemma 2.5 and proof of Theorem 5.1.  $\square$

The lemma below is a simple linear algebra, which is taken from [PS21, Lemma 6.3].



**Lemma 5.3.** *Let*

$$M = \begin{pmatrix} a & \lambda \\ 0 & b \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

*be an upper triangular matrix, and  $Z \leq \mathrm{GL}_2(\mathbb{C})$  the centraliser of  $M$ . Then the following assertions hold:*

- (i) *If  $a = b$  and  $\lambda = 0$ , then  $Z = \mathrm{GL}_2(\mathbb{C})$ .*
- (ii) *If  $a \neq b$  and  $\lambda = 0$ , then  $Z \simeq (\mathbb{C}^*)^2$ .*
- (iii) *If  $a = b$  and  $\lambda \neq 0$ , then  $Z \simeq \mathbb{C}^* \times \mathbb{C}^+$ .*

**Theorem 5.4.** *Let  $X$  be a Hopf surface. Then  $\mathrm{Aut}(X)$  is T-Jordan.*

*Proof.* The proof is adapted from [PS21, Lemma 6.4].

Consider the short exact sequence Eq. (5.2). The image of the generator of  $\Lambda$  is mapped by the natural homomorphism

$$\sigma: \widetilde{\mathrm{Aut}}(X) \longrightarrow \mathrm{GL}(T_{0,\mathbb{C}^2}) \simeq \mathrm{GL}_2(\mathbb{C})$$

to the matrix

$$M = \begin{pmatrix} a & \lambda \delta_1^m \\ 0 & b \end{pmatrix}$$

where  $\delta$  is the Kronecker symbol.

Let  $G \leq \mathrm{Aut}(X)$  be a torsion subgroup, and  $\tilde{G}$  its preimage in  $\widetilde{\mathrm{Aut}}(X)$ . Thus, one has  $G \simeq \tilde{G}/\Gamma$ . By Lemma 2.4,  $\sigma|_{\tilde{G}}$  is an embedding since  $\sigma_\Lambda$  is a group monomorphism and  $\tilde{G}/\Lambda$  is torsion. Let  $\Omega$  be the normaliser of  $\sigma(\Lambda)$  in  $\mathrm{GL}_2(\mathbb{C})$ . By construction  $\sigma(\tilde{G})$  is contained in the normaliser of  $\sigma(\Gamma)$  in  $\mathrm{GL}_2(\mathbb{C})$ , which in turn is contained in  $\Omega$  because  $\Lambda$  is a characteristic subgroup of  $\Gamma$ . Hence, every torsion subgroup of  $\mathrm{Aut}(X)$  is contained in the group  $\Omega/\sigma(\Gamma)$ . On the other hand,  $\Omega/\sigma(\Gamma)$  is a quotient of  $\Omega/\sigma(\Lambda)$  by a finite subgroup isomorphic to  $\sigma(\Gamma)/\sigma(\Lambda)$ . Thus, by Lemma 2.2, it is sufficient to show that the group  $\Omega/\sigma(\Lambda)$  is T-Jordan.

Since  $\sigma(\Lambda) \simeq \mathbb{Z}$ , the group  $\Omega$  has a (normal) subgroup  $\Omega'$  of index at most 2 that coincides with the centraliser of the matrix  $M$ . It remains to check that the group  $\Omega'/\sigma(\Lambda)$  is T-Jordan. If  $\lambda = 0$  and  $a = b$ , it follows from Lemma 5.3(i) that  $\Omega'/\sigma(\Lambda)$  is a connected Lie group and hence T-Jordan by Theorem 1.1. If either  $\lambda = 0$  and  $a \neq b$ , or  $\lambda \neq 0$  and  $m \geq 2$ , then this follows from Lemma 5.3(ii) that  $\Omega'/\sigma(\Lambda)$  is abelian. If  $\lambda \neq 0$  and  $m = 1$ , then this follows from Lemma 5.3(iii) that  $\Omega'/\sigma(\Lambda)$  is abelian.  $\square$

## 6. INOUE SURFACES

An *Inoue surface*  $X$  is a compact complex surface obtained from  $W := \mathbb{H} \times \mathbb{C}$  as a quotient by an infinite discrete group, where  $\mathbb{H}$  is the upper half complex plane. Inoue surfaces are minimal surfaces in class VII, contain no curve, and have the following numerical invariants:

$$a(X) = 0, \quad b_1(X) = 1, \quad b_2(X) = 0.$$

There are three families of Inoue surfaces:  $S_M$ ,  $S^{(+)}$ , and  $S^{(-)}$  (cf. [Ino74]), and we will study their automorphisms separately.

Since every holomorphic map from  $\mathbb{C}$  to  $\mathbb{H}$  is constant, any automorphism  $u$  of  $W$  has the form

$$(6.1) \quad u(w, z) = (s(w), t(w, z))$$

where

$$s(w) = \frac{aw + b}{cw + d}, \quad a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0.$$

We may expand  $t$  in the power series of  $w$  and  $z$  at  $(\sqrt{-1}, 0)$ :

$$t(w, z) = \sum_{p \geq 0} C_{p,0}(w - \sqrt{-1})^p + C_{p,1}(w - \sqrt{-1})^p z.$$

**6.1. Type  $S_M$ .** Let  $M = (m_{i,j}) \in \mathrm{SL}_3(\mathbb{Z})$  be a matrix with eigenvalues  $\alpha, \beta, \bar{\beta}$  such that  $\alpha > 1$  and  $\beta \neq \bar{\beta}$ . Take  $(a_1, a_2, a_3)^T$  to be a real eigenvector of  $M$  corresponding to  $\alpha$ , and  $(b_1, b_2, b_3)^T$  an eigenvector corresponding to  $\beta$ . Let  $G_M$  be the group of automorphisms of  $W$  generated by

$$\begin{aligned} g_0(w, z) &= (\alpha w, \beta z), \\ g_i(w, z) &= (w + a_i, z + b_i), \quad i = 1, 2, 3, \end{aligned}$$

which satisfy these conditions

$$\begin{aligned} g_0 g_i g_0^{-1} &= g_1^{m_{i,1}} g_2^{m_{i,2}} g_3^{m_{i,3}}, \\ g_i g_j &= g_j g_i, \quad i, j = 1, 2, 3. \end{aligned}$$

It can be shown that the action of  $G_M$  on  $W$  is free and properly discontinuous. The quotient  $X := W/G_M$  is an Inoue surface of type  $S_M$ . Thus, there is a short exact sequence of groups

$$1 \longrightarrow G_M \longrightarrow \widetilde{\mathrm{Aut}}(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow 1$$

where  $\widetilde{\mathrm{Aut}}(X)$  acts biholomorphically on  $W$ , which is the normaliser of  $G_M$  in  $\mathrm{Aut}(W)$ .

We will study  $\widetilde{\mathrm{Aut}}(X)$  in details. Assume that  $u \in \widetilde{\mathrm{Aut}}(X)$ . Since  $G_M \triangleleft \widetilde{\mathrm{Aut}}(X)$ , we have  $ugu^{-1} \in G_M$  for all  $g \in G_M$ . Indeed, we only need to verify this for the generators of  $G_M$ .

By the commutative relations of  $G_M$ , we may assume that

$$(6.2) \quad ug_i u^{-1} = g_1^{n_{i,1}} g_2^{n_{i,2}} g_3^{n_{i,3}} g_0^{k_i}, \quad 0 \leq i \leq 3$$

for some  $k_i \in \mathbb{Z}$ ,  $\mathbf{n} = (n_{0,1}, n_{0,2}, n_{0,3}) \in \mathbb{Z}^3$  and  $N := (n_{i,j})_{1 \leq i,j \leq 3} \in \mathrm{Mat}_{3 \times 3}(\mathbb{Z})$ . Then

$$g_1^{n_{i,1}} g_2^{n_{i,2}} g_3^{n_{i,3}} g_0^{k_i} u(w, z) = (\alpha^{k_i} s(w) + \sum_{j=1}^3 n_{i,j} a_j, \beta^{k_i} t(w, z) + \sum_{j=1}^3 n_{i,j} b_j).$$

Note that

$$\begin{aligned} ug_0(w, z) &= u(\alpha w, \beta z) = (s(\alpha w), t(\alpha w, \beta z)), \\ ug_i(w, z) &= u(w + a_i, z + b_i) = (s(w + a_i), t(z + b_i)). \end{aligned}$$

Then, Eq. (6.2) implies

$$(6.3) \quad s(\alpha w) = \alpha^{k_0} s(w) + \sum_j n_{0,j} a_j,$$

$$(6.4) \quad t(\alpha w, \beta z) = \beta^{k_0} t(w, z) + \sum_j n_{0,j} b_j,$$

$$(6.5) \quad s(w + a_i) = \alpha^{k_i} s(w) + \sum_j n_{i,j} a_j,$$

$$(6.6) \quad t(z + b_i) = \beta^{k_i} t(w, z) + \sum_j n_{i,j} b_j.$$

By Eq. (6.3), we have

$$(6.7) \quad \begin{cases} ac(1 - \alpha^{k_0}) = c^2 \sum_j n_{0,j} a_j, \\ ad(\alpha - \alpha^{k_0}) + bc(1 - \alpha^{k_0+1}) = cd(1 + \alpha) \sum_j n_{0,j} a_j, \\ bd(1 - \alpha^{k_0}) = d^2 \sum_j n_{0,j} a_j. \end{cases}$$

If  $cd \neq 0$ , then  $a(1 - \alpha^{k_0}) = c \sum_j n_{0,j} a_j$  and  $b(1 - \alpha^{k_0}) = d \sum_j n_{0,j} a_j$  by the first and the third equalities of Eq. (6.7). Using the fact that  $ad - bc > 0$ , the second equality of Eq. (6.7) implies  $k_0 = 0$  and  $\sum_j n_{0,j} a_j = 0$ , which contradicts the middle equality above. Therefore, either  $c = 0$  or  $d = 0$ .

From Eq. (6.5) one deduces that

$$(6.8) \quad \begin{cases} ac(1 - \alpha^{k_i}) = c^2 \sum_j n_{i,j} a_j, \\ (aca_i + ad + bc)(1 - \alpha^{k_i}) = (c^2 a_i + 2cd) \sum_j n_{i,j} a_j, \\ (ad - bc\alpha^{k_i})a_i + bd(1 - \alpha^{k_i}) = (cda_i + d^2) \sum_j n_{i,j} a_j, \end{cases} \quad \text{for all } i.$$

If  $d = 0$ , then  $bc\alpha^{k_i} a_i = 0$  for all  $i$  by the last equality of Eq. (6.8). Since  $bc \neq 0$  now and  $\alpha > 1$ , the third equality of Eq. (6.8) implies  $a_i = 0$  for all  $i$ , which contradicts the assumption that  $(a_1, a_2, a_3)$  is an eigenvector of  $M$ . Hence,  $c = 0$  and we may rewrite  $s$  as  $s(w) = aw + b$  with  $a > 0$  and  $b \in \mathbb{R}$  (i.e.,  $d = 1$ ). Then, by Eq. (6.7),  $k_0 = 1$  and  $(1 - \alpha)b = \sum_j n_{0,j} a_j$  and Eq. (6.8) gives  $k_i = 0$  and  $aa_i = \sum_j n_{i,j} a_j$ .

Comparing the coefficients of  $z$  in Eq. (6.4), we have

$$(6.9) \quad \begin{aligned} \sum_p C_{p,0}(\alpha w - \sqrt{-1})^p &= \beta \sum_p C_{p,0}(w - \sqrt{-1})^p + \sum_j n_{0,j} b_j, \\ \beta \sum_p C_{p,1}(\alpha w - \sqrt{-1})^p &= \beta \sum_p C_{p,1}(w - \sqrt{-1})^p. \end{aligned}$$

It follows that  $C_{p,0} = C_{p,1} = 0$  for all  $p \geq 1$ . Thus, we may rewrite  $t$  as  $t(w, z) = Az + B$  for some  $A \neq 0, B \in \mathbb{C}$ . Then Eq. (6.9) becomes  $(1 - \beta)B = \sum_j n_{0,j} b_j$ , and Eq. (6.6) gives  $Ab_i = \sum_j n_{i,j} b_j$ .

We conclude that  $u(w, z) = (aw + b, Az + B)$  where  $a, b, A, B$  satisfy

$$\mathbf{n} \cdot \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} = \begin{pmatrix} (1 - \alpha)b & (1 - \beta)B \end{pmatrix}$$

and

$$(6.10) \quad N \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad N \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = A \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

It follows that  $M$  and  $N$  are simultaneously diagonalisable. So  $M$  and  $N$  commute. Moreover, the matrix associated to  $u^{-1}$  is  $N^{-1} \in \text{Mat}_{3 \times 3}(\mathbb{Z})$ , so  $\det N = \pm 1$  and  $N \in \text{SL}_3(\mathbb{Z})$ .

Consider the following subgroups of  $\widetilde{\text{Aut}}(X)$ :

$$K = \left\{ (w, z) \mapsto \left( w + \frac{1}{1 - \alpha} \sum_{j=1}^3 n_j a_j, z + \frac{1}{1 - \beta} \sum_{j=1}^3 n_j b_j \right) \mid n_i \in \mathbb{Z} \right\} \simeq \mathbb{Z}^3$$

and

$$\Gamma := \{ N \in \text{SL}_3(\mathbb{Z}) \mid N \text{ and } M \text{ are simultaneously diagonalisable} \},$$

which are abelian groups. By the construction,  $\widetilde{\text{Aut}}(X) = K \rtimes \Gamma$ . Note also that  $G_M = G_1 \rtimes G_0$  where  $G_1 = \{g_1^{n_1} g_2^{n_2} g_3^{n_3} \mid n_i \in \mathbb{Z}\} \simeq \mathbb{Z}^3$  and  $G_0 = \langle g_0 \rangle \simeq \mathbb{Z}$ . Now we have the following commutative diagram

(by the snake lemma)

(6.11)

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & G_1 & \xrightarrow{\phi} & K & \longrightarrow & F \longrightarrow 1 \\
 & & \downarrow & & \downarrow \iota & & \downarrow \\
 1 & \longrightarrow & G_M & \longrightarrow & \widetilde{\text{Aut}}(X) & \longrightarrow & \text{Aut}(X) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & G_0 & \longrightarrow & \Gamma & \longrightarrow & \Gamma/G_0 \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array}$$

where  $\phi$  is given by

$$\phi(\mathbf{n}) = \mathbf{n} \cdot (I - M) \quad \text{with } \mathbf{n} = (n_1, n_2, n_3).$$

Since  $I - M$  is invertible, the group  $F$  being the cokernel of  $\phi$ , is finite (and abelian). Then, by the last column of the commutative diagram,  $\text{Aut}(X)$  is finite-by-abelian and metabelian. By Eq. (6.11), we have:

**Theorem 6.1.** *Let  $X$  be an Inoue surface of type  $S_M$ . Then  $\text{Aut}(X)$  is solvable. Moreover, for any  $G \leq \text{Aut}(X)$ , one has  $\ell_{\text{vir}}(G) \leq 2$ .*

Next, we claim that any torsion subgroup of  $\Gamma/\mathbb{Z}$  is finite. Let  $n_0$  be the largest integer such that, for some  $n_0$ -th root of  $M$ ,  $M^{\frac{1}{n_0}} \in \text{SL}_3(\mathbb{Z})$ . Let  $N \in \Gamma$  be an element such that its image in  $\Gamma/\mathbb{Z}$  is torsion. Then there is a pair of coprime integers  $(m, n)$  such that  $N^n = M^m$  and hence  $M^{\frac{m}{n}} \in \text{SL}_3(\mathbb{Z})$ . Since  $m$  and  $n$  are coprime, there are integers  $a$  and  $b$  such that  $am + bn = 1$ . Then

$$\frac{1}{n} = a \frac{m}{n} + b.$$

It follows that  $M^{\frac{1}{n}} \in \text{SL}_3(\mathbb{Z})$  and hence  $n \leq n_0$ . Therefore,

$$\{(m, n) \mid M^{\frac{m}{n}} \in \text{SL}_3(\mathbb{Z}), m, n \text{ coprime}, m < n\}$$

is a finite set. It follows that the set of torsion elements of  $\Gamma/\mathbb{Z}$  is finite (with the bound depending on  $X$ ), which proves the claim. This claim and Eq. (6.11) imply:

**Theorem 6.2.** *Let  $X$  be an Inoue surface of type  $S_M$ . Then any torsion subgroup  $G \leq \text{Aut}(X)$  is finite. In particular,  $\text{Aut}(X)$  is  $T$ -Jordan.*

**6.2. Types  $S^{(+)}$  and  $S^{(-)}$ .** Since the constructions of Inoue surfaces of type  $S^{(+)}$  and  $S^{(-)}$  are almost parallel, we will only focus on type  $S^{(+)}$  in this subsection. The same argument works for Inoue surfaces of type  $S^{(-)}$ .

Let  $M \in \text{SL}_2(\mathbb{Z})$  be a matrix with two real eigenvalues  $\alpha$  and  $1/\alpha$  with  $\alpha > 1$ . Let  $(a_1, a_2)^T$  and  $(b_1, b_2)^T$  be real eigenvectors of  $M$  corresponding to  $\alpha$  and  $1/\alpha$ , respectively, and fix integers  $p_1, p_2, r$  ( $r \neq 0$ ) and a complex number  $\tau$ . Define  $(c_1, c_2)^T$  to be the solution of the following equation

$$(I - M) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \frac{b_1 a_2 - b_2 a_1}{r} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

where

$$e_i = \frac{1}{2} m_{i,1} (m_{i,1} - 1) a_1 b_1 + \frac{1}{2} m_{i,2} (m_{i,2} - 1) a_2 b_2 + m_{i,1} m_{i,2} b_1 a_2, \quad i = 1, 2.$$

Let  $G_M^{(+)}$  be the group of analytic automorphisms of  $W = \mathbb{H} \times \mathbb{C}$  generated by

$$\begin{aligned} g_0 &: (w, z) \mapsto (\alpha w, z + \tau), \\ g_i &: (w, z) \mapsto (w + a_i, z + b_i w + c_i), \quad i = 1, 2, \\ g_3 &: (w, z) \mapsto \left(w, z + \frac{b_1 a_2 - b_2 a_1}{r}\right). \end{aligned}$$

We have the following relations between these generators

$$\begin{aligned} g_3 g_i &= g_i g_3 \quad \text{for } i = 0, 1, 2, \quad g_1^{-1} g_2^{-1} g_1 g_2 = g_3^r, \\ g_0 g_j g_0^{-1} &= g_1^{m_{j,1}} g_2^{m_{j,2}} g_3^{p_j} \quad \text{for } j = 1, 2. \end{aligned}$$

The action of  $G_M^{(+)}$  is free and properly discontinuous. The quotient space  $X := W/G_M^{(+)}$  is an Inoue surface of type  $S^{(+)}$ . Note that  $G_M^{(+)} \simeq H(r) \rtimes \mathbb{Z}$  as an abstract group, where  $H(r) = \langle g_1, g_2, g_3 \mid g_3 g_i = g_i g_3, g_1^{-1} g_2^{-1} g_1 g_2 = g_3^r \rangle$  and  $\mathbb{Z}$  is generated by  $g_0$ . In fact, the centre  $Z(H(r)) \simeq \mathbb{Z}$  is generated by  $g_3$  and  $H(r)/Z(H(r)) \simeq \mathbb{Z}^2$ .

Similarly, there is a short exact sequence of groups

$$1 \longrightarrow G_M^{(+)} \longrightarrow \widetilde{\text{Aut}}(X) \longrightarrow \text{Aut}(X) \longrightarrow 1,$$

where  $\widetilde{\text{Aut}}(X) \leq \text{Aut}(W)$  is the normaliser of  $G_M^{(+)}$ . Now suppose that  $u \in \widetilde{\text{Aut}}(X)$  as in Eq. (6.1) and

$$(6.12) \quad u g_i u^{-1} = g_1^{n_{i,1}} g_2^{n_{i,2}} g_3^{l_i} g_0^{k_i}, \quad 0 \leq i \leq 3, \quad j = 1, 2$$

for some  $l_i, k_i \in \mathbb{Z}$ ,  $\mathbf{n}_0 = (n_{0,1}, n_{0,2})$ ,  $\mathbf{n}_3 = (n_{3,1}, n_{3,2}) \in \mathbb{Z}^2$  and  $N = (n_{i,j})_{i,j=1,2} \in \text{Mat}_{2 \times 2}(\mathbb{Z})$ . Then,

$$\begin{aligned} &g_1^{n_{i,1}} g_2^{n_{i,2}} g_3^{l_i} g_0^{k_i} u(w, z) \\ &= \left( \alpha^{k_i} s(w) + \sum_j n_{i,j} a_j, t(w, z) + k_i \tau + l_i \frac{b_1 a_2 - b_2 a_1}{r} + \left( \sum_j n_{i,j} b_j \right) \alpha^{k_i} s(w) + \sum_j n_{i,j} c_j + e_i(n) \right), \end{aligned}$$

where

$$e_i(n) = \frac{1}{2} n_{i,1} (n_{i,1} - 1) a_1 b_1 + \frac{1}{2} n_{i,2} (n_{i,2} - 1) a_2 b_2 + n_{i,1} n_{i,2} a_2 b_1.$$

As in the previous subsection, Eq. (6.12) implies

$$(6.13) \quad s(\alpha w) = \alpha^{k_0} s(w) + n_{0,1} a_1 + n_{0,2} a_2,$$

$$(6.14) \quad t(\alpha w, z + t) = t(w, z) + k_0 \tau + l_0 \frac{b_1 a_2 - b_2 a_1}{r} + \left( \sum_j n_{0,j} b_j \right) \alpha^{k_0} s(w) + \sum_j n_{0,j} c_j + e_0(n),$$

$$(6.15) \quad s(w + a_i) = \alpha^{k_i} s(w) + n_{i,1} a_1 + n_{i,2} a_2,$$

$$(6.16) \quad t(w + a_i, z + b_i w + c_i) = t(w, z) + k_i \tau + l_i \frac{b_1 a_2 - b_2 a_1}{r} + \left( \sum_j n_{i,j} b_j \right) \alpha^{k_i} s(w) + \sum_j n_{i,j} c_j + e_i(n),$$

$$(6.17) \quad s(w) = \alpha^{k_3} s(w) + n_{3,1} a_1 + n_{3,2} a_2,$$

$$(6.18) \quad t\left(w, z + \frac{b_1 a_2 - b_2 a_1}{r}\right) = t(w, z) + k_3 \tau + l_3 \frac{b_1 a_2 - b_2 a_1}{r} + \left( \sum_j n_{3,j} b_j \right) \alpha^{k_3} s(w) + \sum_j n_{3,j} c_j + e_3(n).$$

By Eq. (6.13), using the assumption that  $\alpha > 1$ , we have

$$(6.19) \quad \begin{cases} ac(1 - \alpha^{k_0}) = c^2(n_{0,1} a_1 + n_{0,2} a_2), \\ ad(\alpha - \alpha^{k_0}) + bc(1 - \alpha^{k_0+1}) = cd(1 + \alpha)(n_{0,1} a_1 + n_{0,2} a_2), \\ bd(1 - \alpha^{k_0}) = d^2(n_{0,1} a_1 + n_{0,2} a_2). \end{cases}$$

If  $c \neq 0$ , then  $k_0 = -1$  by the first two equalities of Eq. (6.19) since  $ad - bc > 0$ . Similarly, if  $d \neq 0$ , then one has  $k_0 = 1$  by the last two equalities of Eq. (6.19). Therefore, either  $c = 0$  and  $k_0 = 1$ , or  $d = 0$  and  $k_0 = -1$ .

By Eq. (6.15), we have

$$(6.20) \quad \begin{cases} ac(1 - \alpha^{k_i}) = c^2(n_{i,1}a_1 + n_{i,2}a_2), \\ (aca_i + bc + ad)(1 - \alpha^{k_i}) = (c^2a_i + 2cd)(n_{i,1}a_1 + n_{i,2}a_2), \\ ada_i + bd(1 - \alpha^{k_i}) = \alpha^{k_i}bca_i + (cda_i + d^2)(n_{i,1}a_1 + n_{i,2}a_2). \end{cases}$$

If  $d = 0$ , we have  $\alpha^{k_i}bca_i = 0$  for  $i = 1, 2$  by the last equality of Eq. (6.20). This implies that  $a_i = 0$  ( $i = 1, 2$ ) since  $\alpha > 1$  and  $ad - bc > 0$ , which contradicts the fact that  $(a_1, a_2)^T$  is an eigenvector of  $M$ . So,  $c = 0$  and  $k_0 = 1$ . Also, the second equality of Eq. (6.20), which becomes  $ad = \alpha^{k_i}ad$ , implies that  $k_i = 0$  for  $i = 1, 2$ . Now we may rewrite  $s$  as  $s(w) = aw + b$  with  $a > 0$  and  $b \in \mathbb{R}$ . Then Eqs. (6.19) and (6.20) imply that

$$(6.21) \quad \begin{cases} (1 - \alpha)b = n_{0,1}a_1 + n_{0,2}a_2, \\ aa_i = n_{i,1}a_1 + n_{i,2}a_2, \quad i = 1, 2. \end{cases}$$

Comparing the coefficients of  $z$  of Eq. (6.14), we have

$$\sum_p C_{p,1}(\alpha w - \sqrt{-1})^p = \sum_p C_{p,1}(w - \sqrt{-1})^p,$$

which implies that  $C_{p,1} = 0$  for all  $p \geq 1$ . For the  $z$ -constant part of Eq. (6.14), one has the following equality

$$\begin{aligned} & \sum_p C_{p,0}(\alpha w - \sqrt{-1})^p + C_{0,1}\tau \\ &= \sum_p C_{p,0}(w - \sqrt{-1})^p + (n_{0,1}b_1 + n_{0,2}b_2)\alpha aw + \tau + l_0 \frac{b_1a_2 - b_2a_1}{r} + n_{0,1}c_1 + n_{0,1}c_2 + e_0(n). \end{aligned}$$

Similarly, we conclude that  $C_{p,0} = 0$  for  $p \geq 2$ . Hence, we may rewrite  $t$  as  $t(w, z) = Aw + B + Cz$  for some  $A, B, C \in \mathbb{C}$  with  $C \neq 0$ . Now comparing the coefficients of  $w$  of Eqs. (6.14) and (6.16), we get

$$(6.22) \quad \begin{cases} \left(1 - \frac{1}{\alpha}\right) \frac{A}{a} = n_{0,1}b_1 + n_{0,2}b_2, \\ (C - 1)\tau = l_0 \frac{b_1a_2 - b_2a_1}{r} + (n_{0,1}b_1 + n_{0,2}b_2)\alpha b + n_{0,1}c_1 + n_{0,2}c_2 + e_0(n), \\ Cb_i = a(n_{i,1}b_1 + n_{i,2}b_2), \\ Aa_i + Cc_i = l_i \frac{b_1a_2 - b_2a_1}{r} + (n_{i,1}b_1 + n_{i,2}b_2)b + n_{i,1}c_1 + n_{i,2}c_2 + e_i(n). \end{cases}$$

Finally, Eqs. (6.17) and (6.18) simply give that  $k_3 = 0$ ,  $l_3 = C$  and  $n_{3,1} = n_{3,2} = 0$ .

Therefore,  $N$  has two eigenvalues  $a$  and  $C/a$  with eigenvectors  $(a_1, a_2)^T$  and  $(b_1, b_2)^T$ , respectively. It is not hard to see that the matrix associated to  $u^{-1}$  is  $N^{-1}$ . Then  $\det N = \pm 1$  and hence  $N \in \text{SL}_2(\mathbb{Z})$ . Moreover, by Eqs. (6.21) and (6.22),  $u$  (if exists) is determined by  $\mathbf{n}_0$ ,  $N$  and  $B$ .

Let

$$\Gamma := \{N \in \text{SL}_2(\mathbb{Z}) \mid N \text{ and } M \text{ are simultaneously diagonalisable}\},$$

which is an abelian group, and let  $\tau: \widetilde{\text{Aut}}(X) \rightarrow \Gamma$  be the homomorphism (not necessarily surjective) defined by  $u \mapsto N$ , where  $N$  is the matrix associated with  $u$  as constructed above. Let  $K$  be the kernel of this homomorphism, with image  $\Gamma' \leq \Gamma$ . It is clear that any automorphism in  $K$  has the form

$$(w, z) \mapsto (w + b, z + Aw + B)$$

for some  $b \in \mathbb{R}$  and  $A, B \in \mathbb{C}$  satisfying Eqs. (6.21) and (6.22). An explicit calculation gives that the centre  $Z(K) \simeq \mathbb{C}$ , which is generated by

$$(w, z) \mapsto (w, z + B).$$

Further,  $K/Z(K) \simeq \mathbb{Z} \rtimes \mathbb{Z}$  is generated by the images of

$$(w, z) \mapsto (w + b, z) \quad \text{and} \quad (w, z) \mapsto (w, z + Aw).$$

Combining these, the snake lemma shows the diagrams below are commutative with exact rows and columns.

$$\begin{array}{ccccccc} & 1 & & 1 & & 1 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \longrightarrow & H(r) & \longrightarrow & K & \longrightarrow & F & \longrightarrow 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \longrightarrow & G_M^{(+)} & \longrightarrow & \widetilde{\text{Aut}}(X) & \longrightarrow & \text{Aut}(X) & \longrightarrow 1 \\ & \downarrow & & \downarrow \tau & & \downarrow & \\ 1 \longrightarrow & \mathbb{Z} & \longrightarrow & \Gamma' & \longrightarrow & \Gamma'/\mathbb{Z} & \longrightarrow 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 1 & & 1 & & 1 & \end{array} \qquad \begin{array}{ccccccc} & 1 & & 1 & & 1 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \longrightarrow & Z(H(r)) & \longrightarrow & H(r) & \longrightarrow & H(r)/Z(H(r)) & \longrightarrow 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \longrightarrow & Z(K) & \longrightarrow & K & \longrightarrow & K/Z(K) & \longrightarrow 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 1 & & 1 & & 1 & \end{array}$$

Note that  $F'$  is abelian and  $F''$  is finite and metabelian. Consequently,  $F$ , as the quotient of  $K$  by  $H(r)$ , is abelian-by-finite and abelian-by-metabelian.

By the above diagrams:

**Theorem 6.3.** *Let  $X$  be an Inoue surface of type  $S^{(+)}$  or  $S^{(-)}$ . Then  $\text{Aut}(X)$  is solvable. Moreover, for any subgroup  $G \leq \text{Aut}(X)$ , one has  $\ell_{\text{vir}}(G) \leq 4$ .*

Similar to the argument before Theorem 6.2, any torsion subgroup of  $\Gamma'/\mathbb{Z}$  is finite. It follows from the diagram above that each torsion subgroup of  $\text{Aut}(X)$  has an abelian subgroup with bounded index depending on  $X$ .

**Theorem 6.4.** *Let  $X$  be an Inoue surface of type  $S^{(+)}$  or  $S^{(-)}$ . Then  $\text{Aut}(X)$  is  $T$ -Jordan.*

## 7. CLASS VII SURFACES WITH $b_2 > 0$

We will prove Proposition 1.5 in the section.

*Proof of Proposition 1.5.* We may assume that  $G$  is an infinite group. By Corollary 4.2, we only need to consider such  $X$  that does not have any curves. Consider the action of  $\text{Aut}(X)$  on  $H^*(X, \mathbb{Q})$ , and let  $\text{Aut}^*(X)$  be the kernel of  $\text{Aut}(X) \rightarrow \text{GL}(H^*(X, \mathbb{Q}))$ . Note that the image of  $G$  in  $\text{GL}(H^*(X, \mathbb{Q}))$  is finite (cf. [JM22, Lemma 5.2]). By passing to a finite index subgroup, we may assume that  $G \leq \text{Aut}^*(X)$ .

Pick  $\text{id} \neq g \in G$ , and let  $G'$  be the centraliser of  $\langle g \rangle$  in  $G$ . Since  $g$  has finite order,  $[G : G']$  is finite. Replacing  $G$  by the finite-index subgroup  $G'$ , we may assume that  $g \in Z(G)$ , the centre of  $G$ . Note that  $g$  has  $b_2(X)$  fixed points (cf. [PS20a, Lemma 5.4]). Let  $Y = X/\langle g \rangle$ , which is singular, and let  $\tilde{Y} \rightarrow Y$  be the minimal  $\text{Aut}(Y)$ -equivariant resolution. Then  $\tilde{Y} \notin \Xi_0$  since it contains some curves.

By the construction,  $G$  acts on  $Y$  and hence on  $\tilde{Y}$  (but may not be faithful):

$$1 \longrightarrow G'' \longrightarrow G \longrightarrow G|_Y \longrightarrow 1.$$



It follows from Proposition 3.2 that the natural map  $G'' \hookrightarrow \prod_{y \in S} \text{Aut}(X_y)$  is an injection for some very general finite subset  $S$  of  $Y$ . Since  $X \rightarrow Y$  is a finite morphism,  $G''$  is a finite group. Note that  $G|_Y \simeq G|_{\tilde{Y}}$  is a torsion group. By Proposition 1.4,  $G|_{\tilde{Y}}$  and hence  $G|_Y$  are virtually abelian. Then  $G$  is virtually nilpotent (cf. Proof of Lemma 2.6 (2)) and hence virtually solvable with  $\ell_{\text{vir}}(G) \leq 2$ .  $\square$

## 8. COMPACT KÄHLER SURFACES

In this section, we deal with smooth compact surfaces which are Kähler. The idea of this section comes from Professor De-Qi Zhang.

Let  $X$  be a compact Kähler manifold. For a subgroup  $G$  of  $\text{Aut}(X)$ , define  $G^0 := G \cap \text{Aut}_0(X)$  and denote by  $N(G)$  the set of all elements in  $G$  with zero entropy.

**Lemma 8.1.** *Let  $X$  be a smooth compact Kähler surface, and  $G \leq \text{Aut}(X)$  a virtually solvable subgroup. Then  $\ell_{\text{vir}}(G) \leq 1 + \ell_{\text{vir}}(N(G))$ .*

*Proof.* By taking a finite-index subgroup, we may assume that  $G$  is a solvable,  $N(G)$  is a normal subgroup of  $G$  and  $G/N(G) \simeq \mathbb{Z}^r$  for some  $r = 0, 1$  (cf. [CWZ13, Theorem 1.5]). Take a finite-index subgroup  $N' \leq N(G)$  such that  $\ell(N') = \ell_{\text{vir}}(N(G))$  and  $\ell(N'/N'^0) \leq 1$  as in [DLOZ22, Theorem 1.2 and Proposition 2.6].

Note that  $N(G)|_{H^2(X, \mathbb{C})}$  is finitely generated (cf. [CWZ13, Theorem 2.2(1)]). Since  $N'|_{H^2(X, \mathbb{C})}$  has finite index in  $N(G)|_{H^2(X, \mathbb{C})}$ , after replacing  $N'$  by a finite-index subgroup, we may assume that  $N'|_{H^2(X, \mathbb{C})}$  is characteristic in  $N(G)|_{H^2(X, \mathbb{C})}$  by Lemma 2.8. In particular,  $N'|_{H^2(X, \mathbb{C})} \triangleleft G|_{H^2(X, \mathbb{C})}$ .

Let  $N \triangleleft N(G)$  be the preimage of  $N'|_{H^2(X, \mathbb{C})}$ . Then  $N \triangleleft G$ ,  $N(G)/N$  is finite,  $\ell(N) = \ell_{\text{vir}}(N(G))$  and  $\ell(N/N^0) \leq 1$ . Since  $(G/N)/(N(G)/N) \simeq \mathbb{Z}^r$ , replacing  $G$  by a finite-index subgroup, we may assume  $G/N \simeq \mathbb{Z}^r$  (cf. [Zha13, Lemma 2.4]). It follows that

$$\ell_{\text{vir}}(G) \leq \ell(G) \leq \ell(G/N) + \ell(N) \leq 1 + \ell_{\text{vir}}(N(G)). \quad \square$$

Indeed, it follows from the above proof that

$$\ell_{\text{vir}}(G) \leq 1 + \ell(N) \leq 1 + \ell(N/N^0) + \ell(N^0) \leq 2 + \ell(N^0).$$

When  $\kappa(X) \geq 0$ ,  $\text{Aut}_0(X)$  is a complex torus (cf. [Fuj78, Corollary 5.11]). In particular,  $N^0 = N \cap \text{Aut}_0(X)$  is abelian and hence of derived length (at most) 1. Thus, we have

**Proposition 8.2.** *Let  $X$  be a smooth compact Kähler surface with Kodaira dimension  $\kappa(X) \geq 0$ . Let  $G \leq \text{Aut}(X)$  be virtually solvable. Then  $\ell_{\text{vir}}(G) \leq 3$ .*

Suppose that now  $X$  is a compact Kähler surface with  $\kappa(X) = -\infty$ . Then  $X$  is projective.

**Proposition 8.3.** *Let  $X$  be a smooth projective surface. Suppose that  $G \leq \text{Aut}(X)$  is virtually solvable, and  $U \leq \text{Aut}_0(X)$  is non-trivial unipotent with  $U \triangleleft G$ . Then the derived length  $\ell(U) \leq 2$  and after replacing by a finite-index subgroup, we have  $G = G^0$ .*

*Proof.* Replacing by closure or finite-index subgroup, we may assume that  $G \leq \text{Aut}(X)$  is closed and solvable,  $G|_{\text{NS}(X)_{\mathbb{C}}}$  is connected and closed, and  $U \leq \text{Aut}_0(X)$  is closed and unipotent (cf. [Ogu06, Proof of Lemma 2.1(2)]).

By Lemma 2.9, after replacing  $G$  by a finite-index subgroup, there is a subgroup  $Z_1 \simeq \mathbb{G}_a \leq U$  which is normal in  $G$ . Let  $X \dashrightarrow Y = X/Z_1$  be the quotient map to the cycle space (cf. [Fuj78, § 4]). Then  $G$  acts biregularly on  $Y$  and the map is  $G$ -equivariant. Replacing the map by the graph and taking equivariant resolutions, we may assume that both  $X$  and  $Y$  are smooth, and the map  $X \rightarrow Y$  is a

morphism, a  $\mathbb{P}^1$ -fibration, whose general fibre is a curve with dense orbit  $Z_1 \cdot x \simeq \mathbb{A}^1$  and whose section at infinity is fixed by  $G$ . Consider the exact sequence

$$1 \longrightarrow K \longrightarrow U \longrightarrow U|_Y \longrightarrow 1.$$

Then both  $K$  and  $U|_Y$  are unipotent. Now the restriction to generic fibre  $K \longrightarrow K|_{X_{\overline{k(Y)}}$  is injective with image contained in a closed unipotent group acting faithfully on  $X_{\overline{k(Y)}} = \mathbb{P}_{\overline{k(Y)}}^1$ , and hence is abelian. It is clear that either  $U|_Y = \{1\}$ , or  $U|_Y \simeq \mathbb{G}_a$  and  $Y = \mathbb{P}^1$ . Hence  $\ell(U) \leq \ell(U|_Y) + 1 \leq 2$ .

Note that  $G$  fixes a big class  $H_Y$  on the curve  $Y$ . Take the class  $H$  to be the sum of the pullback of  $H_Y$  and the section at infinity of the  $\mathbb{P}^1$ -fibration  $X \longrightarrow Y$ . Then  $G$  fixes the big class  $H$  on  $X$  and hence replaced by a finite-index subgroup,  $G$  is contained in  $\text{Aut}_0(X)$  (cf. [Fuj78, Theorem 4.8], [Lie78, Proposition 2.2] and [DHZ15, Corollary 2.2]). Thus  $G = G^0$ .  $\square$

**Proposition 8.4.** *Let  $G$  be a smooth projective surface. Suppose that  $G \leq \text{Aut}(X)$  is virtually solvable. Then  $\ell_{\text{vir}}(G) \leq 4$ .*

*Proof.* Replacing by finite-index subgroup, we may assume that  $G$  is solvable.

First assume that  $G = G^0$  (and connected). Let  $U$  be the unipotent radical of (the linear part of)  $G = G^0$  so that  $G/U$  is a semi-torus and hence commutative (cf. [Lit76, Lemma 4]). Thus,  $\ell_{\text{vir}}(G) \leq 1 + \ell(U) \leq 3$  by Proposition 8.3.

Now we may assume that  $G \neq G^0$  even after replacing  $G$  by a finite-index subgroup. Let  $G_0^0 \triangleleft G^0$  be the identity component, Then by Proposition 8.3, the unipotent radical of (the linear part of)  $G_0^0$  is trivial and hence  $G_0^0$  itself is a semi-torus and hence commutative.

If  $G$  is of zero entropy, by [DLOZ22, Theorem 1.2] one has  $\ell(G/G^0) \leq 1$  after replacing  $G$  by a finite-index subgroup. Consider the short exact sequence

$$1 \longrightarrow G^0/G_0^0 \longrightarrow G/G_0^0 \longrightarrow G/G^0 \longrightarrow 1.$$

Since the first term is a finite group, its centraliser in  $G/G_0^0$  is of finite index (say of index 1, after replacing  $G$  by a finite index subgroup). Thus,

$$\begin{aligned} \ell(G/G_0^0) &\leq 1 + \ell(G/G^0) = 2, \\ \ell_{\text{vir}}(G) &\leq \ell(G) \leq \ell(G/G_0^0) + \ell(G_0^0) \leq 3. \end{aligned}$$

If  $G$  is not of zero entropy, using Lemma 8.1 and the argument above,  $\ell_{\text{vir}}(G) \leq 1 + \ell_{\text{vir}}(N(G)) \leq 1 + 3 = 4$ .  $\square$

## 9. SUMMARY

In the Kähler setting, Proposition 1.4 follows from Theorem 1.3; Theorem 1.9 has been proved in [CWZ13, Theorem 1.5]; Theorem 1.10 follows from Propositions 8.2 and 8.4.

Let  $X$  be a smooth compact complex surface, which is not Kähler. In particular,  $X$  is not rational or ruled. Thus, there is a unique minimal surface  $X'$  bimeromorphic to  $X$  such that

$$\text{Aut}(X) \subseteq \text{Bim}(X) \simeq \text{Bim}(X') = \text{Aut}(X').$$

See [PS21, Proposition 3.5]. It suffices for us to prove Proposition 1.4 and Theorems 1.9 and 1.10 for minimal surfaces (in the non-Kähler setting). Note that non-Kähler minimal surfaces are surfaces of class VII, (primary or secondary) Kodaira surfaces and some properly elliptic surfaces (cf. [BHPV04, IV. Theorem 3.1 and VI. Theorem 1.1]). Since every minimal surface of class VII with vanishing  $b_2$  is either a Hopf surface or an Inoue surface (cf. [Bog76]) and minimal surface of class VII with algebraic

dimension 1 is a Hopf surface (cf. [BHPV04, V. Theorem 18.6]), minimal surfaces of class VII not in  $\Xi$  are exactly Hopf surfaces or Inoue surfaces.

TABLE 1. non-Kähler minimal smooth compact complex surfaces

class of the surface $X$	$\kappa(X)$	$a(X)$	$b_1(X)$	$b_2(X)$	$e(X)$
surfaces of class VII	$-\infty$	$0, 1$	$1$	$\geq 0$	$\geq 0$
primary Kodaira surfaces	$0$	$0$	$3$	$4$	$0$
secondary Kodaira surfaces	$0$	$0$	$1$	$0$	$0$
properly elliptic surfaces	$1$	$1$			$\geq 0$

Then Proposition 1.4 follows from Theorems 3.4, 5.4, 6.2 and 6.4 and Corollary 4.2; Theorem 1.9 follows from Theorems 2.7, 3.4, 3.5, 5.1, 6.1 and 6.3; and Theorem 1.10 follows from Theorems 2.7, 3.4, 3.5, 5.2, 6.1 and 6.3. Finally, Proposition 1.5 has been proved in Section 7.

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