SHEAF STABLE PAIRS, QUOT-SCHEMES, AND BIRATIONAL GEOMETRY

CAUCHER BIRKAR, JIA JIA, AND ARTAN SHESHMANI

Abstract. In this paper we build bridges between moduli theory of sheaf stable pairs on one hand and birational geometry on the other hand. We will in particular treat moduli of sheaf stable pairs on smooth projective curves in detail and present some calculations in low degrees. We will also outline problems in various directions.

Contents

1. Introduction 1
2. Preliminaries 5
3. Higher rank sheaf stable pairs 7
4. Models associated to a sheaf stable pair 13
5. Stable pairs on curves with fixed cokernel divisor 17
6. Stable pairs of rank two over curves 22
7. Degree one stable pairs on curves 26
8. Degree two stable pairs on curves 31
9. Degree three stable pairs on curves 36
References 43

We work over an algebraically closed field k of characteristic 0.

1. Introduction

Given an algebraic variety, the Quot-schemes parameterise flat families of quotient sheaves with fixed numerical characteristics, for instance Hilbert polynomial on that variety. Hilbert and Quot schemes owe their construction to Grothendieck in [Gro61], and later they got further developed following results of Mumford, Altman and Kleiman. Here is a short review:

Let X be a reduced connected projective scheme over k, equipped with a very ample line bundle $\mathcal{O}_X(1)$, and let $P \in \mathbb{Q}[t]$ be a numerical polynomial with rational coefficients. As the family of semi-stable coherent sheaves on X with Hilbert polynomial P is bounded (e.g. [HL10, Theorem 3.3.7, p. 78]), there is $m \in \mathbb{N}$, such that any such sheaf $\mathcal{F}$ is m-regular ([Kle71, Theorem 1.13, p. 623]). From m-regularity of $\mathcal{F}$ it follows that for all $i \geq 0$, the sheaf $\mathcal{F}(i+m)$ is globally generated and

$$H^0(X, \mathcal{O}_X(i)) \otimes H^0(X, \mathcal{F}(m)) \longrightarrow H^0(X, \mathcal{F}(i+m))) \tag{1.1}$$

is surjective ([Mum66, p. 100]). In particular we can find any such $\mathcal{F}$ among quotients of $\mathcal{O}_X^{\oplus n}(-m)$, where $n := P(m)$. So we need to consider the Quot-scheme $\mathrm{Quot}(\mathcal{O}_X^{\oplus n}(-m), P)$

Date: June 3, 2024.
2020 *Mathematics Subject Classification.* 14J10, 14E30, 14H60.
Key words and phrases. sheaf stable pair, moduli space, Quot-scheme, stable minimal model.

([Gro61, Théorém 3.2, p. 260]). The kernels of the quotients $\mathcal{O}_X^{\oplus n}(-m) \twoheadrightarrow \mathcal{F}$ do not have to be globally generated, however, the family of all these kernels is bounded ([Gro61, Proposition 1.2, p. 252]), and hence m'-regular for some $m' \in \mathbb{N}$. Therefore there is $p \in \mathbb{N}$ such that for all $l \geq p$, $\mathcal{O}_X^{\oplus n}(-m)$ is l-regular as well as any $\mathcal{F}$ as above, and the kernel of $\mathcal{O}_X^{\oplus n}(-m) \twoheadrightarrow \mathcal{F}$. Then surjectivity of maps as in (1.1) gives us for each $l \geq p$ a realisation of $\mathrm{Quot}(\mathcal{O}_X^{\oplus n}(-m), P)$ as a closed subscheme

$$\mathrm{Quot}(\mathcal{O}_X^{\oplus n}(-m), P) \hookrightarrow \mathrm{Gr}(N_l - P(l), N_l), \tag{1.2}$$

where $N_l \coloneqq nP_{\mathcal{O}_X}(l-m)$ and $\mathrm{Gr}(N_l - P(l), N_l)$ is the Grassmannian of $N_l - P(l)$-dimensional subspaces in an N_l-dimensional space ([Gro61, Lemmes 3.3 and 3.7]).

The action of $\mathrm{GL}_n(k)$ on $\mathcal{O}_X^{\oplus n}$ induces an action of $\mathrm{GL}_n(k)$ on

$$H^0(X, \mathcal{O}_X^{\oplus n}(l-m))$$

where each $n \times n$ matrix becomes a matrix of $P_{\mathcal{O}_X}(l-m) \times P_{\mathcal{O}_X}(l-m)$ scalar matrices (of matrices entries).[1] Thus we have a right[2] action of $\mathrm{GL}_n(k)$ on $\mathrm{Gr}(N_l - P(l), N_l)$. The Plücker embedding

$$\mathrm{Gr}(N_l - P(l), N_l) \hookrightarrow \mathbb{P}^{M_l}, \quad M_l = \binom{N_l}{P(l)} - 1 \tag{1.3}$$

comes with a $\mathrm{GL}_n(k)$-linearisation of the very ample line bundle, that is induced from the canonical $\mathrm{GL}_{N_l}(k)$-linearisation. As $\mathrm{Quot}(\mathcal{O}_X^{\oplus n}(-m), P) \subseteq \mathrm{Gr}(N_l - P(l), N_l)$ is $\mathrm{GL}_n(k)$-invariant, the induced very ample line bundle on $\mathrm{Quot}(\mathcal{O}_X^{\oplus n}(-m), P)$ is $\mathrm{GL}_n(k)$-linearised (e.g. [HL10, p. 101]). This linearised ample line bundle allows one to describe the moduli space of (semi) stable coherent sheaves, as the GIT quotient of a locus of "*(semi) stable*" quotient sheaves cut out in the Quot-scheme. The moduli space of coherent sheaves became an instrumental tool to study many fundamental problems in modern algebraic geometry. In the 1990s Le Potier studied the moduli space of coherent systems [Le 93a]. These parameterise further, the information of pairs $(V, \mathcal{F})$ composed of coherent sheaf $\mathcal{F}$ with fixed numerical characteristics, together with a subspace V of its space of global sections, that is, morphisms

$$V \otimes \mathcal{O}_X \to \mathcal{F}, \quad V \subset H^0(X, \mathcal{F}),$$

equipped with a suitable notion of stability condition associated to $(V, \mathcal{F})$.

The current article aims at studying a particular instance of coherent systems, known as stable pairs, with support over a fixed algebraic variety and explores the connections between the birational geometry of the underlying variety and the associated moduli space of stable pairs.

Definition 1.1. Let Z be an algebraic variety. In this paper, a *sheaf stable pair* $\mathcal{E}, s$ on Z consists of a torsion-free coherent sheaf $\mathcal{E}$ and a morphism

$$\mathcal{O}_Z^r \xrightarrow{s} \mathcal{E}$$

of sheaves (of $\mathcal{O}_Z$-modules) such that

$$\dim \mathrm{Supp}\,\mathrm{coker}(s) < \dim Z.$$

[1] In particular, scalar $n \times n$-matrices are mapped to scalar matrices.

[2] We regard global sections of $\mathcal{O}_X^{\oplus n}$ as row vectors, i.e. the action of $\mathrm{GL}_n(k)$ is from the right.

It is not difficult to see that $r = \mathrm{rank}(\mathcal{E})$. For simplicity, we will usually drop "sheaf" and just refer to $\mathcal{E}, s$ as a stable pair [PT09].

We say two stable pairs $\mathcal{E}, s$ and $\mathcal{G}, t$ are equivalent if there is a commutative diagram

$$\begin{tikzcd} \mathcal{O}_Z^r \arrow[r, "s"] \arrow[d, equal, "\mathrm{id}"'] & \mathcal{E} \arrow[d, "\wr"', "\text{isomorphism}"] \\ \mathcal{O}_Z^r \arrow[r, "t"'] & \mathcal{G}. \end{tikzcd}$$

The equivalence class of $\mathcal{E}, s$ is denoted by $[\mathcal{E}, s]$.

However, above stable pairs should not be confused with stable pairs studied in birational geometry. In fact, we relate above stable pairs with stable minimal models in birational geometry [Bir22]. It is this connection that inspired this work.

Much of this paper is devoted to understanding the moduli spaces $M_Z(r,n)$ of stable pairs classes $[\mathcal{E}, s]$ on a smooth curve Z where $\mathcal{E}$ is of rank r (mostly $r = 2$) and $\deg \mathcal{E} = n$. Here $\deg \mathcal{E}$ is defined in terms of $\mathrm{coker}(s)$. This case already exhibits a rich geometry. To such $[\mathcal{E}, s]$ one can associate $X = \mathbb{P}(\mathcal{E}) \to Z$ together with divisors D_1, D_2, A. The structure

$$(X, D_1 + D_2), A \longrightarrow Z$$

is a stable minimal model over the generic point of Z but often not over the whole Z. But a birational procedure produces a stable minimal model

$$(X', D_1' + D_2'), A' \longrightarrow Z.$$

It turns out that the moduli space of the initial stable pairs $[\mathcal{E}, s]$ parametrises the **procedure** of going from $(X, D_1 + D_2), A$ to $(X', D_1' + D_2'), A'$. This geometric picture provides a crucial tool to study the above moduli spaces.

We can now state the first result of this paper for stable sheaves on curves.

Theorem 1.2. *Let Z be a smooth projective curve and n be a non-negative integer. Then*

(1) $M_Z(r,n)$ is a smooth projective variety.

(2) Consider the natural morphism $M_Z(r,n) \xrightarrow{\pi} \mathrm{Hilb}_Z^n$ sending $[\mathcal{E}, s]$ to the divisor of $\mathrm{coker}(s)$. The fibre of π over $\sum_1^\ell n_j a_j$ is isomorphic to

$$F_1 \times \cdots \times F_\ell$$

where F_j depends only on n_j (so it is independent of Z and the choice of $\sum_1^\ell n_j a_j$).

(3) F_j in (2) is a normal variety with Cartier canonical divisor.

We have a more precise description of $M_Z(2,n)$ and the fibres of π in low degrees $n \le 3$.

Theorem 1.3. *Let Z be a smooth projective curve. Then $M_Z(2,1)$ is isomorphic to $Z \times \mathbb{P}^1$.*

Theorem 1.4. *Let Z be a smooth projective curve. Then the fibre of*

$$\pi \colon M_Z(2,2) \longrightarrow \mathrm{Hilb}_Z^2$$

over a point $\sum_1^\ell n_j a_j$ is isomorphic to

$$\begin{cases} \text{smooth quadric in } \mathbb{P}^3, & \text{when } \ell = 2, \\ \text{singular quadric in } \mathbb{P}^3, & \text{when } \ell = 1. \end{cases}$$

Theorem 1.5. *Let Z be a smooth projective curve. Then the fibre of*

$$\pi \colon M_Z(2,3) \longrightarrow \mathrm{Hilb}_Z^3$$

over a point $\sum_1^\ell n_j a_j$ *is isomorphic to*

$$\begin{cases} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, & \text{when } \ell = 3, \\ \mathbb{P}^1 \times \text{singular quadric}, & \text{when } \ell = 2, \\ F_3, & \text{when } \ell = 1 \end{cases}$$

where F_3 *is a* $\mathbb{Q}$*-factorial Fano* 3*-fold of Picard number one with canonical singularities along a copy of* $\mathbb{P}^1$*. Moreover,* F_3 *is birational to* $\mathbb{P}^3$*.*

We will give an explicit construction of F_3 from $\mathbb{P}^3$.

The present article is only the beginning of a long term project on sheaf stable pairs and birational geometry. There are various directions to explore. We outline some of these:

- Study $M_Z(2, n)$ over $Z = \mathbb{P}^1$ for degrees $n \geq 4$;
- Study $M_Z(r, n)$ over $Z = \mathbb{P}^1$ for higher ranks $r \geq 3$;
- Study $M_Z(r, n)$ over curves Z of genus $g(Z) \geq 1$;
- Study $M_Z(\mathrm{ch})$ over higher dimensional bases Z with fixed Chern character ch;
- Use techniques of enumerative geometry to get results in birational geometry.

Each direction exhibits its own challenges. Overall, the above program will enrich both birational geometry and enumerative geometry. We believe that deeper connections between the two fields become more apparent when one studies the above moduli spaces over higher dimensional bases. In this article, we mainly apply birational geometry to understand these moduli spaces over curves. But in the higher dimensional case one might be able to go in the opposite direction as well and relate invariants in the two fields.

Finally, for completeness of this discussion we say a few words about our investigation of $M_Z(2, 2)$ over a smooth curve Z. Hope this helps to see how birational geometry comes into the picture. Assume $[\mathcal{E}, s] \in M_Z(2, 2)$. As stated above, we can associate a model

$$X, D_1, D_2, A \longrightarrow Z$$

and from this we can get a stable minimal model

$$(X', B'), A' \longrightarrow Z.$$

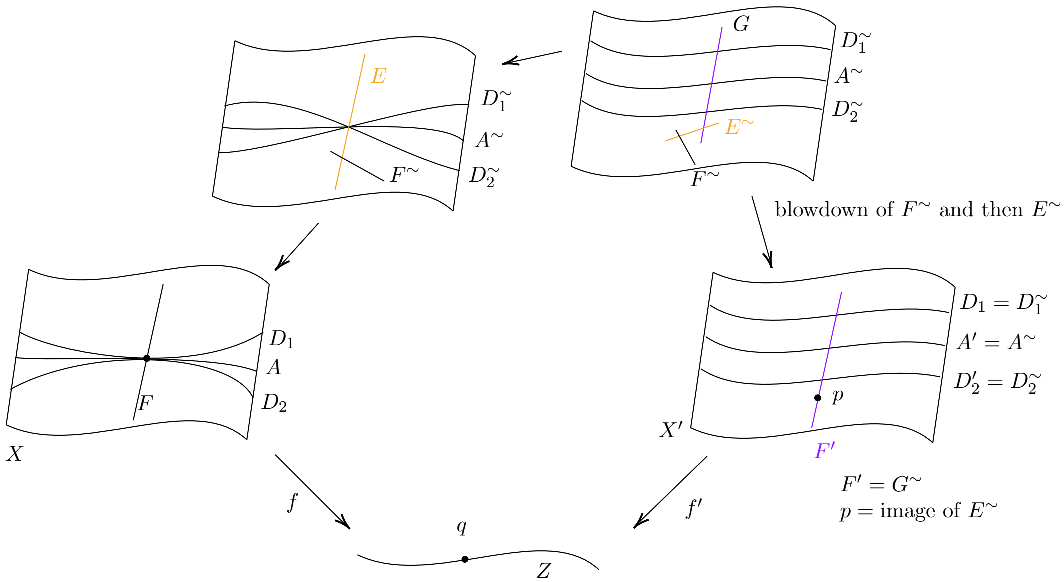
There are two main cases to consider: one is when the cokernel divisor is reduced and the other is when the cokernel divisor is non-reduced.

Assume first that $\mathcal{E}, s$ has reduced cokernel divisor $Q = q_1 + q_2$. The two divisors D_1, D_2 intersect at two distinct points, one over each q_i. To go from X to X' it is enough to blowup these points and then blow down two curves over the q_i. Already from this picture one can guess that the classes $\mathcal{E}, s$ with fixed q_1, q_2 are parametrised by $\mathbb{P}^1 \times \mathbb{P}^1$.

Now assume the cokernel divisor is non-reduced, say $Q = 2q$. This is the more complicated case. In this case, $D_1 \cdot D_2 = 2$ and the intersection points are over q. There are three subcases to be considered.

Case I: The fibre F over q is a component of both D_1, D_2. In this case, $X = X'$ and D_i' is the horizontal part of D_i (similarly for A).

Case II: F is not a component of D_1, D_2, A. Then D_1 and D_2 are tangent to each other at some point over q. Then the stable minimal model is obtained as in the following picture:



Case III: F is a component of one of D_1, D_2, A. Say F is a component of D_1. Then D_2, A are tangent, and the stable minimal model is obtained by a similar but slightly different process. One then considers the case when F is a component of D_2 (resp. A), etc.

The above arguments make it clear that the fibres of $M_Z(2,2) \to \mathrm{Hilb}^2_Z$ are independent of the genus of Z, that is, the fibre only depends on whether the cokernel divisor is reduced or not (and the same arguments apply even if Z is not projective). In fact, it is enough to work in a formal neighbourhood of the cokernel divisor. From this one can reduce the calculation of the fibres to a local problem.

To make the story short, in the non-reduced cokernel case one is reduced to classifying all quotients

$$k[t]/\langle t^2\rangle \oplus k[t]/\langle t^2\rangle \longrightarrow L$$

where L is a $k[t]/\langle t^2\rangle$-module of length 2. Such L is either $k \oplus k$ or $k[t]/\langle t^2\rangle$. Some careful calculations show that such quotients are parametrised by a singular quadric in $\mathbb{P}^3$.

But to investigate fibres in the degree 3 case we need to borrow more sophisticated tools from birational geometry.

Acknowledgement. The first author is supported by a grant from Tsinghua University and a grant of the National Program of Overseas High Level Talent. The second author is supported by Shuimu Tsinghua Scholar Program and China Postdoctoral Science Foundation (2023TQ0172). The third author would like to thank Vladimir Baranovsky for helpful discussions. He is supported by a grant from Beijing Institute of Mathematical Sciences and Applications (BIMSA), and by the National Program of Overseas High Level Talent.

2. Preliminaries

We work over an algebraically closed field k of characteristic zero. All varieties and schemes are defined over k unless stated otherwise. Varieties are assumed to be irreducible.

2.1. **Contractions.** By a *contraction* we mean a projective morphism $f\colon X \to Y$ of schemes such that $f_*\mathcal{O}_X = \mathcal{O}_Y$ In particular, f is surjective and has connected fibres.

We say that a birational map $f\colon X \dashrightarrow Y$ *contracts* a divisor $D \subset X$ if f is defined at the generic point of D and $f(D) \subset Y$ has codimension at least 2. The map f is called a *birational contraction* if f^{-1} does not contract any divisor.

2.2. **Pairs and Singularities.** Let X be a pure dimensional scheme of finite type over k and let M be a $\mathbb{Q}$-divisor on X. We denote the coefficient of a prime divisor D in M by $\mu_D M$.

A *pair* (X, B) consists of a normal quasi-projective variety X and a $\mathbb{Q}$-divisor $B \geq 0$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. We call B the *boundary divisor*.

Let $\phi\colon W \to X$ be a log resolution of a pair (X, B). Let $K_W + B_W$ be the pullback of $K_X + B$. The *log discrepancy* of a prime divisor D on W with respect to (X, B) is defined as

$$a(D, X, B) \coloneqq 1 - \mu_D B_W.$$

A *non-klt* place of (X, B) is a prime divisor D over X, that is, on birational models of X, such that $a(D, X, B) \leq 0$, and a *non-klt centre* is the image of such a D on X. We say (X, B) is lc (resp. klt) if $a(D, X, B)$ is ≥ 0 (resp. > 0) for every D. This means that every coefficient of B_W is ≤ 1 (resp. < 1).

A *log smooth* pair is a pair (X, B) where X is smooth and $\operatorname{Supp} B$ has simple normal crossing singularities.

2.3. **Base locus.** Let $\mathcal{L}$ be an invertible sheaf on a scheme X. The *base locus* of $\mathcal{L}$ is

$$\mathrm{Bs}(\mathcal{L}) = \{x \in X \mid t(x) = 0, \forall t \in H^0(X, \mathcal{L})\}.$$

If $H^0(X, \mathcal{L}) = 0$, by convention, $\mathrm{Bs}(\mathcal{L}) = X$.

2.4. **Types of Models.** Let (X, B) be an lc pair.

Let $X \to Z$ be a contraction to a normal variety and assume that $K_X + B$ is big over Z. We say that the *log canonical model* of (X, B) over Z exists if there is a birational contraction $\varphi\colon X \dashrightarrow Y$ where

- Y is normal and projective over Z;
- $K_Y + B_Y \coloneqq \varphi_*(K_X + B)$ is ample over Z; and
- $\alpha^*(K_X + B) \geq \beta^*(K_Y + B_Y)$ for any common resolution

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "\varphi"'] & & Y. \end{tikzcd}$$

We call (Y, B_Y) the log canonical model of (X, B) over Z.

A *dlt model* of an lc pair (X, B) is a pair (X', B') with a projective birational morphism $\psi\colon X' \to X$ such that

- (X', B') is dlt;
- every exceptional prime divisor of ψ appears in B' with coefficients one;
- $K_{X'} + B' = \psi^*(K_X + B)$.

We say that a pair (Y, B_Y) is a *log minimal model* of an lc pair (X, B) if there exists a birational contraction $\phi\colon X \dashrightarrow Y$ such that

- (Y, B_Y) is $\mathbb{Q}$-factorial dlt;
- $K_Y + B_Y$ is nef; and
- ϕ is $K_X + B$-negative, i.e., for any prime divisor D on X which is exceptional over Y, we have $a(D, X, B) < a(D, Y, B_Y)$.

A log minimal model (Y, B_Y) is *good* if $K_Y + B_Y$ is semi-ample.

2.5. **Stratification.** Let X be a scheme. A *stratification* of X consists of a set of finitely many locally closed subschemes $X_1, \ldots, X_n$ of X, called strata, pairwise disjoint and such that $X = \bigcup_1^n X_i$, i.e., such that we have a surjective morphism $\coprod_1^n X_i \to X$.

3. Higher rank sheaf stable pairs

3.1. **Stability.** Let X be a projective variety of dimension d, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Denote by $P_{\mathcal{F}}$ the *Hilbert polynomial* of a coherent sheaf $\mathcal{F}$ on X.

Definition 3.1 ([She16, Definition 2.6])**.** Let $q(m)$ be given by a polynomial with rational coefficients such that its leading coefficient is positive. A pair $\mathcal{O}_X^r \xrightarrow{\phi} \mathcal{F}$, where $\mathcal{F}$ is a pure sheaf, is *τ'-stable* (resp. *τ'-semi-stable*) with respect to (stability parameter) $q(m)$ if

(1) for all proper non-zero subsheaves $\mathcal{G} \subseteq \mathcal{F}$ for which ϕ does not factor through $\mathcal{G}$ we have
$$\frac{P_{\mathcal{G}}}{\mathrm{rk}(\mathcal{G})} < \frac{P_{\mathcal{F}} + q(m)}{\mathrm{rk}(\mathcal{F})}, \quad \text{resp. } (\leq)$$

(2) for all proper subsheaves $\mathcal{G} \subseteq \mathcal{F}$ for which ϕ factors through
$$\frac{P_{\mathcal{G}} + q(m)}{\mathrm{rk}(\mathcal{G})} < \frac{P_{\mathcal{F}} + q(m)}{\mathrm{rk}(\mathcal{F})}, \quad \text{resp. } (\leq).$$

We consider the stability condition when $q(m) \to \infty$.

Definition 3.2. Fix $q(m)$ to be given as a polynomial of degree at least $d+1$ with rational coefficients such that its leading coefficient is positive. A pair $\mathcal{O}_X^r \xrightarrow{\phi} \mathcal{F}$ is called to be *τ'-limit-stable* (resp. τ'-limit-semi-stable) if it is stable (resp. semi-stable) in the sense of Definition 3.1 with respect to this fixed choice of $q(m)$.

Lemma 3.3 ([She16, Lemma 2.7])**.** *Fix $q(m)$ to be given as a polynomial of at least degree $d+1$ with rational coefficients such that its leading coefficient is positive. Then stability and semi-stability coincide. A pair $\mathcal{O}_X^r \xrightarrow{\phi} \mathcal{F}$ is τ'-limit-stable if and only if* $\mathrm{coker}(\phi)$ *is a sheaf with at most $d-1$-dimensional support, i.e.,* $\mathrm{coker}(\phi)$ *is a torsion sheaf.*

Proof. The exact sequence
$$0 \longrightarrow \mathcal{K} \coloneqq \ker(\phi) \longrightarrow \mathcal{O}_X^r \xrightarrow{\phi} \mathcal{F} \longrightarrow \mathcal{Q} \coloneqq \mathrm{coker}(\phi) \longrightarrow 0$$
induces a short exact sequence
$$0 \longrightarrow \mathrm{Im}(\phi) \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0.$$
Hence we obtain a commutative diagram
$$\begin{tikzcd} \mathcal{O}_X^r \arrow[r, "\phi"] \arrow[d, equal] & \mathrm{Im}(\phi) \arrow[d, hook] \\ \mathcal{O}_X^r \arrow[r] & \mathcal{F}. \end{tikzcd}$$
Now we assume that $\mathcal{O}_X^r \to \mathcal{F}$ is τ'-limit-stable:
$$\frac{P_{\mathrm{Im}(\phi)} + q(m)}{\mathrm{rk}(\mathrm{Im}(\phi))} < \frac{P_{\mathcal{F}} + q(m)}{\mathrm{rk}(\mathcal{F})}.$$

By rearrangement we get

$$q(m)\cdot(\mathrm{rk}(\mathcal{F})-\mathrm{rk}(\mathrm{Im}(\phi))) < \mathrm{rk}(\mathrm{Im}(\phi))P_{\mathcal{F}}-\mathrm{rk}(\mathcal{F})P_{\mathrm{Im}(\phi)}. \tag{3.1}$$

Note that the right-hand side of (3.1) is a polynomial in m of degree at most d. However by the choice of $q(m)$, it is a polynomial of degree at least $d+1$ with positive leading coefficients. Hence, as $m\to\infty$, the only way for the inequality (3.1) to hold is $\mathrm{rk}(\mathrm{Im}(\phi))=\mathrm{rk}(\mathcal{F})$ and therefore $\mathcal{Q}$ has rank zero.

For the other direction, we assume that $\mathcal{Q}$ is a torsion sheaf, but $\mathcal{O}_X^r\to\mathcal{F}$ is not τ'-limit-stable. Then there exists a *saturated* subsheaf $\mathcal{G}$ satisfying the destabilising condition:

$$\frac{P_{\mathcal{G}}}{\mathrm{rk}(\mathcal{G})}\geq\frac{P_{\mathcal{F}}+q(m)}{\mathrm{rk}(\mathcal{F})},$$

noting that $\mathrm{Im}(\phi)\nsubseteq\mathcal{G}$. Since $\mathcal{F}$ is pure one has $\mathrm{rk}(\mathcal{G})>0$. But the degree of $q(m)$ is chosen to be sufficiently large, a contradiction. □

Now we study automorphisms of stable pairs.

Lemma 3.4 ([She16, Lemma 3.6])**.** *Given a τ'-limit-stable pair $\mathcal{O}_X^r\xrightarrow{\phi}\mathcal{F}$ and a commutative diagram*

$$\begin{tikzcd} \mathcal{O}_X^r \arrow[r, "\phi"] \arrow[d, equal, "\mathrm{id}"'] & \mathcal{F} \arrow[d, "\rho"] \\ \mathcal{O}_X^r \arrow[r, "\phi"'] & \mathcal{F}. \end{tikzcd} \tag{3.2}$$

The map ρ is given by $\mathrm{id}_{\mathcal{F}}$.

Proof. The diagram (3.2) induces

$$\begin{tikzcd} \mathcal{O}_X^r \arrow[r, two heads, "\phi"] \arrow[d, equal, "\mathrm{id}"'] & \mathrm{Im}(\phi) \arrow[r, hook] \arrow[d, "\rho|_{\mathrm{Im}(\phi)}"] & \mathcal{F} \arrow[d, "\rho"] \\ \mathcal{O}_X^r \arrow[r, two heads, "\phi"] & \mathrm{Im}(\phi) \arrow[r, hook] & \mathcal{F}. \end{tikzcd}$$

By commutativity of (3.2), $\rho\circ\phi=\phi\circ\mathrm{id}=\phi$ then $\rho(\mathrm{Im}(\phi))=\mathrm{Im}(\phi)$. Hence $\rho(\mathrm{Im}(\phi))\subseteq\mathrm{Im}(\phi)$. It follows that $\rho\mid_{\mathrm{Im}(\phi)}=\mathrm{id}_{\mathrm{Im}(\phi)}$. Indeed, if $s\in\mathrm{Im}(\phi)(U)$ where $U\subseteq X$ is affine open with $\widetilde{s}\in\mathcal{O}_X^r(U)$ satisfying $\phi(\widetilde{s})=s$ then

$$\rho(s)=\rho(\phi(\widetilde{s}))=\phi(\mathrm{id}(\widetilde{s}))=\phi(\widetilde{s})=s.$$

Now apply $\mathrm{Hom}(-,\mathcal{F})$ to the short exact sequence

$$0\longrightarrow\mathrm{Im}(\phi)\longrightarrow\mathcal{F}\longrightarrow\mathcal{Q}\longrightarrow 0$$

where $\mathcal{Q}$ is the corresponding cokernel. We obtain

$$0\longrightarrow\mathrm{Hom}(\mathcal{Q},\mathcal{F})\longrightarrow\mathrm{Hom}(\mathcal{F},\mathcal{F})\longrightarrow\mathrm{Hom}(\mathrm{Im}(\phi),\mathcal{F}).$$

Since $\mathcal{O}_X^r\to\mathcal{F}$ is τ'-limit-stable, $\mathcal{Q}$ is a torsion sheaf. Hence by purity of $\mathcal{F}$, $\mathrm{Hom}(\mathcal{Q},\mathcal{F})=0$. We obtain an injection

$$\mathrm{Hom}(\mathcal{F},\mathcal{F})\hookrightarrow\mathrm{Hom}(\mathrm{Im}(\phi),\mathcal{F}).$$

Now $\rho|_{\mathrm{Im}(\phi)}=\mathrm{id}_{\mathrm{Im}(\phi)}=(\mathrm{id}_{\mathcal{F}})|_{\mathrm{Im}(\phi)}$, so $\rho=\mathrm{id}_{\mathcal{F}}$. □

3.2. **Sheaf stable pairs.** In this subsection, we establish some general results regarding sheaf stable pairs on varieties. Inspired by Lemma 3.3, we propose the following definition of such pairs.

Definition 3.5. Assume Z is a variety. A *sheaf stable pairs* is of the form $\mathcal{E}, s$ where

$$\begin{cases} \mathcal{E} \text{ is a torsion-free coherent sheaf of rank } r > 0, \\ \mathcal{O}_Z^r \xrightarrow{s} \mathcal{E} \text{ is a morphism of } \mathcal{O}_Z\text{-modules}, \\ \dim \operatorname{Supp} \operatorname{coker}(s) < \dim Z. \end{cases}$$

To give a morphism $\mathcal{O}_Z^r \xrightarrow{s} \mathcal{E}$ is the same as giving r sections

$$s_1, \ldots, s_r \in H^0(Z, \mathcal{E})$$

where s_i corresponds to the morphism from the i-th summand of $\mathcal{O}_Z^r$ to $\mathcal{E}$ determined by s. We will then sometimes use the notation $\mathcal{E}, s_1, \ldots, s_r$ instead of $\mathcal{E}, s$.

We also often denote $\mathcal{Q} \coloneqq \operatorname{coker}(s)$. So we get an exact sequence

$$\mathcal{O}_Z^r \xrightarrow{s} \mathcal{E} \longrightarrow \mathcal{Q} = \operatorname{coker}(s) \longrightarrow 0.$$

Lemma 3.6. *Assume $\mathcal{E}, s$ is a stable pair. Then the morphism $\mathcal{O}_Z^r \xrightarrow{s} \mathcal{E}$ is injective.*

Proof. Since $\dim \operatorname{Supp} \operatorname{coker}(s) < \dim Z$, s is generically an isomorphism, so $\ker(s)$ is torsion, hence zero. So we get a short exact sequence

$$0 \longrightarrow \mathcal{O}_Z^r \xrightarrow{s} \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0.$$ □

Lemma 3.7. *Assume $\mathcal{E}, s$ is a stable pair on a normal variety Z. Then* $\operatorname{Supp} \operatorname{coker}(s)$ *is empty or of pure codimension one.*

Proof. Let $\mathcal{Q} = \operatorname{coker}(s)$. If $\mathcal{Q} = 0$, then $\operatorname{Supp} \mathcal{Q} = \emptyset$.

Suppose $\mathcal{Q} \neq 0$. Assume $\operatorname{Supp} \mathcal{Q}$ is not of pure codimension one. Then after shrinking we can assume $0 < \dim \operatorname{Supp} \mathcal{Q} \leq \dim Z - 2$. Now we have a diagram

$$\begin{tikzcd} \mathcal{O}_Z^r \arrow[r, "s"] \arrow[d, "\text{isomorphism}"'] & \mathcal{E} \arrow[d] \\ (\mathcal{O}_Z^r)^{\vee\vee} \arrow[r] & \mathcal{E}^{\vee\vee} \end{tikzcd}$$

where ${}^{\vee\vee}$ denotes double dual. All the maps are isomorphisms on $U = Z \setminus \operatorname{Supp} \mathcal{Q}$. But $\mathcal{E}^{\vee\vee}$ is reflexive ([Har80, Corollary 1.2]) and Z is normal, hence $\mathcal{E}^{\vee\vee}$ is determined by $\mathcal{E}^{\vee\vee}|_U \simeq \mathcal{O}_Z^r|_U$ ([Har80, Proposition 1.6]), so $\mathcal{O}_Z^r \to \mathcal{E}^{\vee\vee}$ is an isomorphism. On the other hand, $\mathcal{E}$ is torsion-free by assumption, so the natural morphism $\mathcal{E} \to \mathcal{E}^{\vee\vee}$ is injective. Therefore,

$$\mathcal{O}_Z^r \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}^{\vee\vee}$$

are isomorphisms, contradicting the assumption $\mathcal{Q} \neq 0$. □

Lemma 3.8. *Assume $\mathcal{E}, s$ is a stable pair on a variety Z. Assume $\mathcal{E}$ is locally free, and*

$$X = \mathbb{P}(\mathcal{E}) \xrightarrow{f} Z, \quad \mathcal{O}_X(1)$$

the associated projection and line bundle. Then the base locus $\operatorname{Bs}(\mathcal{O}_X(1))$ *is vertical over Z. More precisely,*

$$f(\operatorname{Bs}(\mathcal{O}_X(1))) \subseteq \operatorname{Supp} \operatorname{coker}(s).$$

Proof. Pulling back $\mathcal{O}_Z^r \xrightarrow{s} \mathcal{E}$ to X we get

$$\mathcal{O}_X^r \longrightarrow f^*\mathcal{E} \longrightarrow \mathcal{O}_X(1) \tag{3.3}$$

where the second morphism is a natural surjective morphism [Har77, Chapter II, Proposition 7.11]. Since $\mathcal{O}_Z^r \xrightarrow{s} \mathcal{E}$ is surjective on

$$U := Z \setminus \operatorname{Supp}\operatorname{coker}(s)$$

the induced morphism $\mathcal{O}_X^r \longrightarrow \mathcal{O}_X(1)$ is surjective on $f^{-1}U$. So $\mathcal{O}_X(1)$ is generated by global sections on $f^{-1}U$, hence $\operatorname{Bs}(\mathcal{O}_X(1)) \subseteq X \setminus f^{-1}U$. □

Corollary 3.9. *Assume $\mathcal{E}, s$ is a stable pair on a smooth projective curve Z, then the associated $\mathcal{O}_X(1)$ is nef (i.e., $\mathcal{E}$ is nef).*

Proof. Recall that an invertible sheaf $\mathcal{L}$ on a projective scheme Y is nef if $\deg \mathcal{L}|_C \geq 0$ for every curve $C \subseteq Y$.

In our case,

$$0 = \dim \operatorname{Supp}\operatorname{coker}(s) < \dim Z = 1,$$

so by Lemma 3.8,

$$\operatorname{Bs}(\mathcal{O}_X(1)) \subseteq \text{union of fibres of } f.$$

Thus if $C \subseteq X$ is a curve with $\deg \mathcal{O}_X(1)|_C < 0$, then $C \subseteq \operatorname{Bs}(\mathcal{O}_X(1)) \subseteq$ union of fibres of f. But $\mathcal{O}_X(1)$ is ample over Z, so there is no such C. This shows that $\mathcal{O}_X(1)$ is nef. □

Example 3.10. If Z is a smooth projective curve and $\mathcal{E} = \mathcal{O}_Z(n_1) \oplus \cdots \oplus \mathcal{O}_Z(n_r), s$ is stable, then $n_i \geq 0$ for every i.

If not, say, $n_1 < 0$, then $\mathcal{O}_X(1)|_S$ is anti-ample where $S \hookrightarrow X$ is the section determined by $\mathcal{O}_Z(n_1)$. But then $\mathcal{O}_X(1)$ is not nef, a contradiction. See also [Laz04, Proposition 6.1.2].

In particular, any stable pair on $\mathbb{P}^1$ is of the form above.

Example 3.11. In general, $\mathcal{O}_X(1)$ may not be base point free. Indeed, assume

$$\begin{aligned} &Z = \text{elliptic curve}, \\ &\mathcal{E} = \mathcal{O}_Z \oplus \mathcal{O}_Z(p), \text{ where } p \text{ is a point on } Z. \end{aligned}$$

The identity morphism $\mathcal{O}_Z \to \mathcal{O}_Z$ and the morphism $\mathcal{O}_Z \to \mathcal{O}_Z(p)$ corresponding to any non-zero section determine a stable pair $\mathcal{E}, s$ with $\operatorname{coker}(s)$ supported at p. Now $X = \mathbb{P}_Z(\mathcal{E})$ is a ruled surface over Z. Consider the section $S \subseteq X$ given by the surjection $\mathcal{E} \to \mathcal{O}_Z(p)$ [Har77, Chapter V, Proposition 2.6]. Then $\mathcal{O}_X(1)|_S$ is isomorphic to $\mathcal{O}_Z(p)$ which is not base point free, hence $\mathcal{O}_X(1)$ is not base point free.

Remark 3.12. Assume $\mathcal{E}, s$ is a stable pair on a variety Z. Then the morphism $\mathcal{O}_Z^r \xrightarrow{s} \mathcal{E}$ determines a rational map

$$\begin{tikzcd} X = \mathbb{P}(\mathcal{E}) \arrow[rr, dashed] \arrow[dr] & & \mathbb{P}_Z^{r-1} \arrow[dl] \\ & Z & \end{tikzcd}$$

Indeed, the summands $\mathcal{O}_Z$ of $\mathcal{O}_Z^r$ determine sections $s_1, \ldots, s_r$ of $\mathcal{E}$ which generate $\mathcal{E}$ outside $\operatorname{Supp}\operatorname{coker}(s)$. We can view $s_1, \ldots, s_r$ as sections of $\mathcal{O}_X(1)$ generating it outside

$$f^{-1} \operatorname{Supp}\operatorname{coker}(s).$$

Therefore, they determine a rational map as in the diagram above which is a morphism outside $f^{-1} \operatorname{Supp}\operatorname{coker}(s)$.

Alternatively, the morphism $\mathcal{O}_Z^r \to \mathcal{E}$ is an isomorphism outside $\operatorname{Supp}\operatorname{coker}(s)$ hence determines the rational map above.

Definition 3.13. Assume Z is a variety, $\mathcal{E}, s$ and $\mathcal{G}, t$ stable pairs. We say the pairs are *equivalent* if there is a commutative diagram

$$\begin{tikzcd} \mathcal{O}_Z^r \arrow[r, "s"] \arrow[d, equal, "\mathrm{id}"'] & \mathcal{E} \arrow[d, "\wr"', "\text{isomorphism}"] \\ \mathcal{O}_Z^r \arrow[r, "t"'] & \mathcal{G} \end{tikzcd}$$

where the vertical arrows are isomorphisms and the left one is the identity. The equivalence class of $\mathcal{E}, s$ is denoted by $[\mathcal{E}, s]$.

If $s_1, \ldots, s_r \in H^0(Z, \mathcal{E})$ and $t_1, \ldots, t_r \in H^0(Z, \mathcal{G})$ are the sections of $\mathcal{E}, \mathcal{G}$ determined by s, t, then the above is equivalent to saying that there exists an isomorphism $\mathcal{E} \to \mathcal{G}$ sending s_i to t_i for every i.

Lemma 3.14. *Assume Z is a variety, $\mathcal{E}, s$ and $\mathcal{G}, t$ are stable pairs. Assume that there is a commutative diagram*

$$\begin{tikzcd} \mathcal{O}_Z^r \arrow[r, "s"] \arrow[d, "\wr"'] & \mathcal{E} \arrow[d, "\wr"', "\textit{isomorphism}"] \\ \mathcal{O}_Z^r \arrow[r, "t"'] & \mathcal{G} \end{tikzcd}$$

where the vertical arrows are isomorphisms and the left one is given by a matrix

$$\begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix}$$

for some $\lambda \in k \setminus \{0\}$. Then

$$[\mathcal{E}, s] = [\mathcal{G}, t].$$

Proof. If $s_1, \ldots, s_r \in H^0(Z, \mathcal{E})$ and $t_1, \ldots, t_r \in H^0(Z, \mathcal{G})$ are the sections of $\mathcal{E}, \mathcal{G}$ determined by s, t respectively, then the above is equivalent to saying that there exists $\lambda \in k \setminus \{0\}$ and an isomorphism $\mathcal{E} \to \mathcal{G}$ sending s_i to λt_i, for each i.

Consider the morphism $\mathcal{G} \to \mathcal{G}$ which sends a section u on an open set to $\frac{1}{\lambda} u$. This is an isomorphism and composing it with $\mathcal{E} \to \mathcal{G}$ gives an isomorphism $\mathcal{E} \to \mathcal{G}$ sending s_i to t_i. Thus $[\mathcal{E}, s] = [\mathcal{G}, t]$. $\square$

Lemma 3.15. *Assume Z is a variety. Then to give a class $[\mathcal{E}, s]$ of rank r with $\mathcal{E}$ reflexive is equivalent to giving a reflexive $\mathcal{K}$ and an inclusion*

$$\mathcal{K} \hookrightarrow \mathcal{O}_Z^r$$

with cokernel of rank zero.

Proof. Given $\mathcal{E}, s$, dualising

$$\mathcal{O}_Z^r \xrightarrow{s} \mathcal{E} \longrightarrow \operatorname{coker}(s) \longrightarrow 0$$

we get

$$0 \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{O}_Z^r$$

with $\mathrm{rank}(\mathcal{O}_Z^r/\mathcal{E}^\vee)=0$. Moreover, if $[\mathcal{E},s]=[\mathcal{G},t]$, then we have a diagram

$$\begin{tikzcd} \mathcal{O}_Z^r \arrow[r, "s"] \arrow[dr, "t"'] & \mathcal{E} \arrow[d] \\ & \mathcal{G} \end{tikzcd}$$

with $\mathcal{E}\to\mathcal{G}$ an isomorphism, and dualising we get

$$\begin{tikzcd} \mathcal{E}^\vee \arrow[r, hook] & \mathcal{O}_Z^r \\ \mathcal{G}^\vee \arrow[u] \arrow[ur, hook] & \end{tikzcd}$$

which means the two inclusions are the same, i.e., their images coincide.

Conversely, given $\mathcal{K}\hookrightarrow\mathcal{O}_Z^r$, dualising we get the class of a stable pair $\mathcal{O}_Z^r \xrightarrow{s} \mathcal{K}^\vee$. $\square$

Definition 3.16. For a smooth projective variety Z, $M_Z(\mathrm{ch})$ denotes the moduli space of stable pair classes $[\mathcal{E},s]$ with Chern character ch on Z. This is a projective scheme; see [Le 93b], [She16, Theorem 3.7] and Lemma 3.4. When Z is a smooth projective curve, we simply write it as $M_Z(r,n)$ where r is the rank and n is the degree of $\mathcal{E}$. In this case the morphism

$$M_Z(r,n)\longrightarrow \mathrm{Hilb}_Z^n$$

sends $[\mathcal{E},s]$ to the divisor determined by $\mathrm{coker}(s)$ (Quot-to-Chow morphism).

Lemma 3.17. *Assume Z is a smooth projective curve. Then $M_Z(r,n)$ is isomorphic to* $\mathrm{Quot}(\mathcal{O}_Z^r,n)$*, hence smooth.*

Proof. Assume $[\mathcal{E},s]\in M_Z(r,n)$. By Lemma 3.15, this class naturally corresponds to the induced inclusion

$$\mathcal{E}^\vee \hookrightarrow \mathcal{O}_Z^r,$$

and if $[\mathcal{E},s]=[\mathcal{G},t]$, then

$$\mathcal{E}^\vee \hookrightarrow \mathcal{O}_Z^r \quad \text{and} \quad \mathcal{G}^\vee \hookrightarrow \mathcal{O}_Z^r$$

have equal images. Therefore the class $[\mathcal{E},s]$ corresponds uniquely to a rank zero quotient of $\mathcal{O}_Z^r$ of degree n, hence to a point of the Quot-scheme $\mathrm{Quot}(\mathcal{O}_Z^r,n)$.

For smoothness of $\mathrm{Quot}(\mathcal{O}_Z^r,n)$, see [HL10, Proposition 2.2.8], noting that for any short exact sequence

$$0\longrightarrow\mathcal{K}\longrightarrow\mathcal{O}_Z^r\longrightarrow\mathcal{Q}\longrightarrow 0$$

with $\mathcal{Q}$ being rank zero one has $\mathrm{Ext}^1(\mathcal{K},\mathcal{Q})\simeq H^1(Z,\mathcal{K}^\vee\otimes\mathcal{Q})=0$ by the Grothendieck vanishing. $\square$

Lemma 3.18. *Assume Z is a smooth projective curve. Assume that for some r,n, the fibre of*

$$M_Z(r,n)\xrightarrow{\pi}\mathrm{Hilb}_Z^n$$

over some point h is of dimension $n(r-1)$ and smooth in codimension one. Then this fibre is an irreducible normal variety.

Proof. First note that Hilb_Z^n is just the n-th symmetric product of Z, hence it is smooth of dimension n (for $Z=\mathbb{P}^1$, $\mathrm{Hilb}_Z^n\simeq\mathbb{P}^n$). Pick general ample divisors $H_1,\dots,H_n$ through h. Then H_i are smooth intersecting transversally at h.

Since $\dim \mathrm{Quot}(\mathcal{O}_Z^r, n) = nr$, general fibres of π are of dimension $n(r-1)$ (in fact, general fibres of π are $(\mathbb{P}^{r-1})^n$). By upper semi-continuity of fibre dimension, every fibre of π over some neighbourhood of h is of dimension $n(r-1)$. Thus we can see that

$$\text{fibre over } h = \pi^* H_1 \cap \cdots \cap \pi^* H_n$$

is a locally complete intersection and hence Cohen-Macaulay [Har77, Chapter II, Proposition 8.23]. And it is smooth in codimension one by assumption. Therefore, it is normal by Serre criterion [Har77, Chapter II, Proposition 8.23]. Here we are using smoothness of $M_Z(r,n)$ (see Lemma 3.17). Then the fibre is irreducible as π has connected fibres. □

4. Models associated to a sheaf stable pair

Definition 4.1. Assume $[\mathcal{E}, s]$ is a stable pair of rank r on a variety Z, with $\mathcal{E}$ *locally free*. Let $s_1, \dots, s_r$ be the sections of $\mathcal{E}$ determined by s. We introduce the notation

$$\begin{aligned} &X = \mathbb{P}(\mathcal{E}) \xrightarrow{f} Z, \\ &\mathcal{O}_X(1) = \text{the associated invertible sheaf}, \\ &D_i = \text{divisor of } s_i, \\ &A = \text{divisor of } s_1 + \cdots + s_r. \end{aligned}$$

Here we view s_i as sections of $\mathcal{O}_X(1)$ via the morphism (3.3), so D_i is the divisor of this section (similarly for $s_1 + \cdots + s_r$). We call

$$X, D_1, \dots, D_r, A \xrightarrow{f} Z$$

the *model associated to* $[\mathcal{E}, s]$.

Now assume $\mathcal{G}, t$ is another stable pair on Z with $\mathcal{G}$ locally free. Let

$$Y, E_1, \dots, E_r, C \longrightarrow Z$$

be its associated model. We say the above two associated models are *isomorphic* if there is an isomorphism over Z

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

mapping $\mathcal{O}_X(1)$ to $\mathcal{O}_Y(1)$ and mapping D_i to E_i and A to C.

Lemma 4.2. *Assume $\mathcal{E}, s$ and $\mathcal{G}, t$ are stable pairs on a normal variety Z, with $\mathcal{E}, \mathcal{G}$ locally free. Then $[\mathcal{E}, s] = [\mathcal{G}, t]$ if and only if the associated models of $\mathcal{E}, s$ and $\mathcal{G}, t$ are isomorphic.*

Proof. Assume $[\mathcal{E}, s] = [\mathcal{G}, t]$. Then there exists an isomorphism $\mathcal{E} \to \mathcal{G}$ sending the corresponding sections $s_1, \dots, s_r$ to $t_1, \dots, t_r$. This induces an isomorphism

$$\begin{tikzcd} \mathbb{P}(\mathcal{G}) \arrow[rr, "\sim"] \arrow[dr] & & \mathbb{P}(\mathcal{E}) \arrow[dl] \\ & Z & \end{tikzcd} \tag{4.1}$$

mapping $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$ to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and mapping the divisor of t_i to the divisor of s_i. It also maps the divisor of $t_1 + \cdots + t_r$ to the divisor of $s_1 + \cdots + s_r$.

Conversely, assume the associated models are isomorphic. So we have an isomorphism as in (4.1) mapping $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$ onto $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and mapping the divisor of t_i to the divisor of s_i and divisor of $t_1 + \cdots + t_r$ to divisor of $s_1 + \cdots + s_r$. This gives an isomorphism $\mathcal{E} \to \mathcal{G}$.

However, divisor of a section does not determine the section, it determines it only up to scaling [Har77, Chapter II, Proposition 7.7] (as $\mathbb{P}(\mathcal{E})$, $\mathbb{P}(\mathcal{G})$ are normal), i.e., there exist $\lambda_i, \lambda \in k \setminus \{0\}$ such that s_i is mapped to $\lambda_i t_i$ and $s_1 + \cdots + s_r$ is mapped to $\lambda(t_1 + \cdots + t_r)$. But then

$$\sum \lambda_i t_i = \lambda(t_1 + \cdots + t_r).$$

However, the sections $t_1, \dots, t_r$ are linearly independent over k as $\mathcal{G}, t$ is a stable pair so $\mathcal{O}_Z^r \xrightarrow{t} \mathcal{G}$ is injective. Therefore, $\lambda_i = \lambda$ and s_i is mapped to λt_i for all i. This shows $[\mathcal{E}, s] = [\mathcal{G}, t]$ by Lemma 3.14. $\square$

Next we discuss connection with the theory of stable minimal models.

Definition 4.3. Assume Z is a variety. In this paper a (lc) *stable minimal model* over Z is of the form

$$(X, B), A \xrightarrow{f} Z$$

where

$$\begin{cases} (X,B) \text{ is a log canonical pair equipped with a projective morphism } X \xrightarrow{f} Z, \\ K_X + B \text{ is semi-ample over } Z, \\ A \geq 0 \text{ is an integral divisor on } X, \\ K_X + B + uA \text{ is ample over } Z \text{ for } 0 < u \ll 1, \\ (X, B + uA) \text{ is log canonical for } 0 < u \ll 1. \end{cases}$$

For more details see [Bir22]. Note however that the above definition is in the relative situation while the setting in [Bir22] is global. Also one should not confuse the above notation with the one in [Bir22]; in the setting above, Z is a fixed base and $K_X + B$ is semi-ample over Z defining a contraction $X \longrightarrow S/Z$; we have suppressed S in the notation because in this paper we usually deal with the situation where $S = Z$.

Example 4.4. Assume $Z = \operatorname{Spec} k$ is a point, and $\mathcal{E}, s$ is stable of rank r. Then $\mathcal{E}$ is a k-vector space of dimension r. And $\mathcal{O}_Z^r \xrightarrow{s} \mathcal{E}$ is an isomorphism. Let

$$X, D_1, \dots, D_r, A \longrightarrow Z$$

be the associated model. Put $B = \sum D_i$.

We want to argue that $(X, B), A$ is a stable minimal model. Identifying $\mathcal{E}$ with k^r via $\mathcal{O}_Z^r \xrightarrow{s} \mathcal{E}$, we can assume

$$X = \mathbb{P}^{r-1} = \operatorname{Proj} k[s_1, \dots, s_r].$$

Then $D_1, \dots, D_r$ are the standard coordinate hyperplanes, hence (X, B) is lc (this is the standard toric structure on $\mathbb{P}^{r-1}$). To show $(X, B), A$ is a stable minimal model, it is enough to show $s_1 + \cdots + s_r$ does not identically vanish on $\bigcap_{i \in I} D_i$ for any subset $I \subseteq \{1, \dots, r\}$. Assume not. Then extending I, we can assume $|I| = r - 1$ in which case $\bigcap_{i \in I} D_i$ is one point. We may assume $I = \{1, \dots, r-1\}$. Then $s_1 + \cdots + s_r$ vanishes on $\bigcap_{i \in I} D_i = (0 : \cdots : 0 : 1)$ which is not the case, a contradiction.

Remark 4.5. Given a stable pair $\mathcal{E}, s$ on a variety Z with $\mathcal{E}$ locally free, we can define a model $(X, B), A \to Z$ as in the previous example. The example shows that this model is a stable minimal model over some open subset of Z. But we will see that it is often not a stable minimal model over the whole Z even when Z is a smooth curve and $r = 2$. It is however possible to modify the above model birationally and get a stable minimal model over the whole Z. This is our next aim.

Construction 4.6. Assume Z is a normal variety and $\mathcal{E}, s$ is a stable pair on Z. We will associate a stable minimal model $(X', B'), A' \xrightarrow{f'} Z$. Let $Q = \operatorname{Supp}\operatorname{coker}(s)$, and $U = Z \setminus Q$. Since $\mathcal{O}_Z^r \xrightarrow{s} \mathcal{E}$ is an isomorphism on U, $\mathcal{E}$ is locally free on U. So by Definition 4.1, we have an associated model

$$X^U, D_1^U, \dots, D_r^U, A^U \longrightarrow U$$

over U. If F is the fibre over any closed point $z \in U$,

$$(F, \sum D_i^U|_F), A^U|_F$$

is a stable minimal model, by Example 4.4. Letting $B^U = \sum D_i^U$, we see that $(X^U, B^U), A^U$ is a stable minimal model over the *smooth locus* of U. But it may not be a stable minimal model over the whole Z (or even U).

Let X be a *compactification* of X^U over Z. Denote $X \to Z$ by f. Take a log resolution $X'' \xrightarrow{\varphi} X$. We can take this resolution so that it is an isomorphism over the smooth locus of U and so that letting

$$\begin{aligned} B'' &= \operatorname{Supp}((B^U)^\sim + \operatorname{Exc}(\varphi) + \varphi^{-1} f^{-1} Q), \\ A'' &= (A^U)^\sim, \end{aligned} \qquad (\sim \text{ denotes birational transform})$$

the pair $(X'', B'' + A'')$ is log smooth.

Run an MMP on $K_{X''} + B''$ over Z. Since $(X^U, B^U), A^U \to U$ is a stable minimal model over the smooth locus of U, and since φ is an isomorphism over this locus, the MMP does not modify X'' over this locus.

Assume the MMP terminates with a good minimal model X'''. Then $K_{X'''} + B'''$ is semi-ample over Z defining a contraction $X''' \to T \to Z$. The MMP is also an MMP on $K_{X''} + B'' + uA''$ for any $0 < u \ll 1$. Now run another MMP on $K_{X'''} + B''' + uA'''$ over T. Assume this terminates with a good minimal model X' over T. Then A' is semi-ample over T defining a birational contraction, since A' is big over Z and hence big over T. Replacing X' with the base of this contraction we get

$$(X', B'), A' \longrightarrow Z$$

which is a stable minimal model, noting that A' does not pass through any non-klt centre of (X', B'). Over the smooth locus of U, $X \dashrightarrow X'$ and $X'' \dashrightarrow X'$ are isomorphisms.

Proposition 4.7. *Let Z be a smooth variety and $\mathcal{E}, s$ a stable pair of rank r with $\mathcal{E}$ locally free and Q the divisor of* $\operatorname{coker}(s)$ *being simple normal crossing. Then the associated stable minimal model*

$$(X', B'), A' \longrightarrow Z$$

exists and it depends only on Z, r and $\operatorname{Supp} Q$.

Proof. Consider the identity morphism $\mathcal{O}_Z^r \to \mathcal{O}_Z^r$ and the associated model

$$X', D_1', \dots, D_r', A' \xrightarrow{f'} Z$$

as in Definition 4.1. Let $B' = \sum D_i' + f'^* \operatorname{Supp} Q$. Let

$$X, D_1, \dots, D_r, A \xrightarrow{f} Z$$

be the associated model of $\mathcal{E}, s$ as in Definition 4.1. Set $B = \operatorname{Supp}(\sum D_i + f^{-1} Q)$.

By Remark 3.12, there is a rational map

$$\begin{tikzcd} X \arrow[rr, dashed, "\alpha"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Z & \end{tikzcd}$$

induced by the sections $s_1, \dots, s_r$ determined by s.

Let $t_1, \dots, t_r$ be the sections of $\mathcal{O}_Z^r$ determined by the summand $\mathcal{O}_Z$. The morphism

$$\mathcal{O}_Z^r \xrightarrow{s} \mathcal{E}$$

maps t_i to s_i. Viewing t_i as sections of $\mathcal{O}_{X'}(1)$ and s_i as sections of $\mathcal{O}_X(1)$, the above rational map α pulls back t_i to s_i. This shows $D_i = \alpha^* D_i'$, $A = \alpha^* A'$. Moreover, α is an isomorphism over U where $U = Z \setminus \operatorname{Supp} Q$.

Let $X'' \xrightarrow{\varphi} X$ be a log resolution of $(X, B + A)$ so that it is an isomorphism over U. Let

$$B'' = \operatorname{Supp}(B^\sim + \operatorname{Exc}(\varphi) + \varphi^{-1} f^{-1} Q)$$

and let A'' be the horizontal part of $A^\sim$. We can assume $X'' \xrightarrow{\psi} X'$ is a morphism.

Then $\psi_* B'' = B'$. In fact, $B'' = \operatorname{Supp}(B'^\sim + \operatorname{Exc}(\psi))$. Thus, we can run an MMP on $K_{X''} + B''$ over X' ending with a dlt model of (X', B'), say (Y, B_Y). Moreover, since A' does not contain any non-klt centre of (X', B'), the pullback of A' to Y, say A_Y, is the birational transform of A'. Therefore, $(X', B' + uA')$ is the lc model of both $(Y, B_Y + uA_Y)$ and $(X'', B'' + uA'')$ for any $0 < u \ll 1$. So $(X', B'), A' \to Z$ is the stable minimal model associated to $\mathcal{E}, s$. $\square$

Remark 4.8. Under the assumption of Construction 4.6, assume there exists an effective Cartier divisor L on Z with $\operatorname{Supp} L = Q$ and U is smooth. For example this holds when X is smooth. Then one can show that we can run MMPs as in Construction 4.6 terminating with good minimal models as required.

Lemma 4.9. *Let $\mathcal{E}, s$ be a stable pair on a smooth projective curve Z. Let*

$$X, D_1, D_2, A \xrightarrow{f} Z$$

be the model of $\mathcal{E}, s$ as in Definition 4.1. Then $D_1 \cdot D_2 = \deg \mathcal{E}$.

Proof. We have a formula

$$K_X + D_1 + D_2 \sim f^*(K_Z + \det \mathcal{E}).$$

Let T_1 be the horizontal part of D_1. We can write $D_1 = T + P$ where $P \geq 0$ is vertical. Then

$$\begin{aligned} (K_X + D_1 + D_2) &= f^*(K_Z + \det \mathcal{E}) \cdot D_1 = f^*(K_Z + \det \mathcal{E}) \cdot T_1 \\ &= 2g - 2 + \deg \mathcal{E} \end{aligned}$$

where g is the genus of Z. Moreover,

$$\begin{aligned} (K_X + D_1 + D_2) \cdot D_1 &= (K_X + D_1) \cdot D_1 + D_1 \cdot D_2 \\ &= (K_X + T_1 + P) \cdot (T_1 + P) + D_1 \cdot D_2 \\ &= (K_X + T_1) \cdot T_1 + (K_X + 2T_1) \cdot P + D_1 \cdot D_2 \\ &= 2g - 2 + D_1 \cdot D_2 \end{aligned}$$

where we use the fact $(K_X + 2T_1) \equiv 0/Z$. Therefore, $D_1 \cdot D_2 = \deg \mathcal{E}$. $\square$

5. Stable pairs on curves with fixed cokernel divisor

In this section we study stable pairs on a curve with cokernel given by a fixed divisor on the curve. This is motivated by the construction of stable minimal models explained in the previous section.

Given a stable pair $\mathcal{E}, s$ on a smooth curve, assume $Q = \sum_1^\ell n_j q_j$ is the divisor determined by $\operatorname{coker}(s)$. Going from the associated model

$$X, D_1, \dots, D_r, A \longrightarrow Z$$

to the stable minimal model

$$(X', B'), A' \longrightarrow Z$$

resolves the singularities of sections $s_1, \dots, s_r$ determined by s, and untwists $\mathcal{E}$ to turn it into $\mathcal{O}_Z^r$. This procedure makes modifications only over the points q_j and in an independent way over different q_j. This leads to the following statement which will be proved in a more direct fashion.

Theorem 5.1. *Let Z be a smooth curve and $Q = \sum_1^\ell n_j q_j$ an effective divisor. Let*

$$M_Q = \{[\mathcal{E}, s] \mid \mathcal{E}, s \textit{ is a stable pair of rank } r \textit{ with cokernel divisor } Q\},$$
$$M_j = \{[\mathcal{E}, s] \mid \mathcal{E}, s \textit{ is a stable pair of rank } r \textit{ with cokernel divisor } n_j q_j\}.$$

Then M_Q and M_j carry natural projective scheme structures and

$$M_Q \simeq M_1 \times \cdots \times M_\ell.$$

We will make some preparations before giving the proof. We will apply the theorem to understand the fibres of

$$M_Z(r, n) \longrightarrow \operatorname{Hilb}_Z^n$$

when Z is a smooth projective curve. In fact this connection already appears in the proof of the theorem.

Lemma 5.2. *Assume Z is a scheme and $V, W \subseteq Z$ are open subsets such that $Z = V \cup W$. Also assume that*

$$\mathcal{F}_V \longrightarrow \mathcal{E}_V \quad \textit{and} \quad \mathcal{F}_W \longrightarrow \mathcal{E}_W$$

are morphisms of $\mathcal{O}_V$-modules and $\mathcal{O}_W$-modules respectively such that after restriction to $V \cap W$ we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}_V|_{V\cap W} & \xrightarrow{\text{isomorphism}} & \mathcal{F}_W|_{V\cap W} \\ \downarrow & & \downarrow \\ \mathcal{E}_V|_{V\cap W} & \xrightarrow[\text{isomorphism}]{} & \mathcal{E}_W|_{V\cap W}. \end{array}$$

Then these uniquely determine a morphism of $\mathcal{O}_X$-modules

$$\mathcal{F} \longrightarrow \mathcal{E}$$

whose restriction to V, W coincide with $\mathcal{F}_V \to \mathcal{E}_V$ and $\mathcal{F}_W \to \mathcal{E}_W$ respectively.

Proof. The sheaves $\mathcal{F}_V$ and $\mathcal{F}_W$ glue via the isomorphism $\mathcal{F}_V|_{V\cap W} \to \mathcal{F}_W|_{V\cap W}$ to make an $\mathcal{O}_X$-module $\mathcal{F}$. Similarly, we get an $\mathcal{O}_X$-module $\mathcal{E}$ from $\mathcal{E}_V$ and $\mathcal{E}_W$. For similar reason, the morphisms also glue to give $\mathcal{F} \to \mathcal{E}$ which is uniquely determined. $\square$

Proof of Theorem 5.1. First we reduce to the case when Z is projective. Let $\overline{Z}$ be the compactification of Z. Let $V = Z$ and $W = \overline{Z} \setminus \operatorname{Supp} Q$. Given any stable pair $\mathcal{E}, s$ on Z with cokernel divisor Q, considering

$$\mathcal{O}_V^r \xrightarrow{s} \mathcal{E}_V := \mathcal{E} \quad \text{and} \quad \mathcal{O}_W^r \xrightarrow{\mathrm{id}} \mathcal{O}_W^r$$

and the diagram

$$\begin{tikzcd}
\mathcal{O}_{V\cap W}^r \arrow[r, "\mathrm{id}"] \arrow[d, "s|_{V\cap W}"'] & \mathcal{O}_{V\cap W}^r \arrow[d, "\mathrm{id}"] \\
\mathcal{E}_V|_{V\cap W} \arrow[r, "\alpha"'] & \mathcal{O}_{V\cap W}^r
\end{tikzcd}$$

where α is determined by the other three arrows (which are all isomorphisms), and applying Lemma 5.2, we can extend $\mathcal{E}, s$ to $\overline{Z}$ preserving the cokernel divisor. Note that α is simply the inverse of $s|_{V\cap W}$. Similar remarks apply to stable pairs with cokernel divisor $n_j q_j$. Thus from now on we assume Z is projective. Then we can identify

$$\begin{aligned}
&M_Q = \text{fibre of } M_Z(r,n) \longrightarrow \operatorname{Hilb}_Z^n \text{ over } Q \qquad (n = \textstyle\sum n_j) \\
&M_j = \text{fibre of } M_Z(r,n) \longrightarrow \operatorname{Hilb}_Z^{n_j} \text{ over } n_j q_j
\end{aligned}$$

so we get projective scheme structures on M_Q and M_j.

Next we will define a map on k-rational points

$$M_Q \longrightarrow M_1 \times \cdots \times M_\ell.$$

Pick $[\mathcal{E}, s] \in M_Q$. Let $V = Z \setminus \{q_2, \dots, q_\ell\}$ and $W = Z \setminus \{q_1\}$. Then the morphisms

$$\mathcal{O}_V^r \xrightarrow{s|_V} \mathcal{E}|_V \quad \text{and} \quad \mathcal{O}_W^r \xrightarrow{\mathrm{id}} \mathcal{O}_W^r$$

glue together via

$$\begin{tikzcd}
\mathcal{O}_{V\cap W}^r \arrow[r, "\mathrm{id}"] \arrow[d, "s|_{V\cap W}"'] & \mathcal{O}_{V\cap W}^r \arrow[d, "\mathrm{id}"] \\
\mathcal{E}|_{V\cap W} \arrow[r, "\beta"'] & \mathcal{O}_{V\cap W}^r
\end{tikzcd}$$

where β is determined by other arrows. This gives a stable pair $\mathcal{E}^1, s^1$ with cokernel divisor $n_1 q_1$. A similar construction produces $\mathcal{E}^j, s^j$ with cokernel divisor $n_j q_j$.

Assume $[\mathcal{E}, s] = [\mathcal{G}, t] \in M_Q$. Then there is a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_Z^r \arrow[r, "s"] \arrow[d, equal] & \mathcal{E} \arrow[d, "\text{isomorphism}"] \\
\mathcal{O}_Z^r \arrow[r, "t"'] & \mathcal{G}
\end{tikzcd}$$

Let again $V = Z \setminus \{q_2, \dots, q_\ell\}$ and $W = Z \setminus \{q_1\}$, and consider

$$\begin{tikzcd}
\mathcal{O}_V^r \arrow[r, "s|_V"] \arrow[d, "\mathrm{id}"'] & \mathcal{E}|_V \arrow[d] \\
\mathcal{O}_V^r \arrow[r, "t|_V"'] & \mathcal{G}|_V
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{O}_W^r \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \mathcal{O}_W^r \arrow[d, "\mathrm{id}"] \\
\mathcal{O}_W^r \arrow[r, "\mathrm{id}"'] & \mathcal{O}_W^r.
\end{tikzcd} \tag{5.1}$$

Restricting to $V \cap W$ we get a commutative diagram

$$\begin{tikzcd}
\mathcal{O}^r_{V\cap W} \arrow[rr, "\mathrm{id}"] \arrow[dr, "s|_{V\cap W}"] \arrow[dd, "\mathrm{id}"'] & & \mathcal{O}^r_{V\cap W} \arrow[dr, "\mathrm{id}"] \arrow[dd, dotted, "\mathrm{id}" near end] & \\
& \mathcal{E}|_{V\cap W} \arrow[rr] \arrow[dd] & & \mathcal{O}^r_{V\cap W} \arrow[dd, "\mathrm{id}"] \\
\mathcal{O}^r_{V\cap W} \arrow[rr, dotted, "\mathrm{id}" near end] \arrow[dr, "t|_{V\cap W}"'] & & \mathcal{O}^r_{V\cap W} \arrow[dr, dotted, "\mathrm{id}"] & \\
& \mathcal{G}|_{V\cap W} \arrow[rr] & & \mathcal{O}^r_{V\cap W}
\end{tikzcd}$$

where $\mathcal{E}|_{V\cap W} \to \mathcal{O}^r_{V\cap W}$ and $\mathcal{G}|_{V\cap W} \to \mathcal{O}^r_{V\cap W}$ are inverses of $s|_{V\cap W}$ and $t|_{V\cap W}$. Here we are making use of the fact that $V \cap W = Z \setminus \operatorname{Supp} Q$, so $s|_{V\cap W}$ and $t|_{V\cap W}$ are both isomorphisms. Therefore, the diagrams in (5.1) glue together giving a commutative diagram

$$\begin{tikzcd}
\mathcal{O}^r_Z \arrow[r, "s^1"] \arrow[d, equal] & \mathcal{E}^1 \arrow[d, "\text{isomorphism}"] \\
\mathcal{O}^r_Z \arrow[r, "t^1"'] & \mathcal{G}^1
\end{tikzcd}$$

showing $[\mathcal{E}^1, s^1] = [\mathcal{G}^1, t^1]$. The same holds for every j. Thus we have a well-defined map

$$M_Q \longrightarrow M_1 \times \cdots \times M_\ell.$$

We claim that this map is bijective. Pick $[\mathcal{E}_j, s_j] \in M_j$. Inductively, we can assume that there exists a unique stable pair $\widetilde{\mathcal{E}}, \widetilde{s}$ on Z such that

$$\operatorname{coker}(\widetilde{s}) = \sum_{j=2}^{\ell} n_j q_j \quad \text{and} \quad \widetilde{\mathcal{E}}^j, \widetilde{s}^j = \mathcal{E}_j, s_j \quad \forall 2 \leq j \leq \ell.$$

Using the notation V, W from above, the two morphisms

$$\mathcal{O}^r_V \xrightarrow{s_1|_V} \mathcal{E}_1 \quad \text{and} \quad \mathcal{O}^r_W \xrightarrow{\widetilde{s}|_W} \widetilde{\mathcal{E}}|_W$$

restricted to $V \cap W$ give a diagram

$$\begin{tikzcd}
\mathcal{O}^r_{V\cap W} \arrow[r, "\mathrm{id}"] \arrow[d, "s_1|_{V\cap W}"'] & \mathcal{O}^r_{V\cap W} \arrow[d, "\widetilde{s}|_{V\cap W}"] \\
\mathcal{E}_1|_{V\cap W} \arrow[r, "\text{isomorphism}"'] & \widetilde{\mathcal{E}}|_{V\cap W}
\end{tikzcd}$$

where the lower isomorphism is determined by the other three isomorphisms. So the above two morphisms glue to give a morphism $\mathcal{O}^r_Z \xrightarrow{s} \mathcal{E}$. From the construction we see that $\mathcal{E}^j, s^j = \mathcal{E}_j, s_j$ for all j.

One can also show that the class $[\mathcal{E}, s]$ is also uniquely determined by the classes $[\mathcal{E}_j, s_j]$. We leave this to the reader.

It remains to show $M_Q \to M_1 \times \cdots \times M_\ell$ is a morphism of schemes. Note that M_Q and M_j are (fine) moduli spaces with universal families $\mathcal{M}_j \to M_j$ giving $\prod \mathcal{M}_j \to \prod M_j$. The latter is a family for moduli functor of M_Q, hence there is a moduli map $\prod M_j \to M$. One may check that this coincides with inverse of $M_Q \to \prod M_j$ constructed above. $\square$

Remark 5.3. Assume Z is a smooth curve, and $Q = nq \geq 0$ a divisor supported at one point q. We outline an algebraic method to calculate

$$M_Q = \{[\mathcal{E}, s] \mid \mathcal{E}, s \text{ is a stable pair of rank } r \text{ with cokernel divisor } Q\}.$$

We can compactify Z hence assume it is projective. Then by Lemma 3.17 it is enough to parametrises quotients

$$\mathcal{O}_Z^r \longrightarrow \mathcal{L}$$

with $\mathcal{L}$ of rank zero whose first Chern class is Q [Ful98, Example 15.2.16(b)]. Two such quotients are considered the same if their kernels coincide.

Since $\mathcal{L}$ is supported at q, the quotient is determined by the induced quotient

$$\mathcal{O}_q^r \longrightarrow \mathcal{L}_q$$

where $\mathcal{O}_q$ is the local ring of Z at q; this is because $\mathcal{L}_q = H^0(Z, \mathcal{L})$ and giving a surjection $\mathcal{O}_Z^r \to \mathcal{L}$ (resp. $\mathcal{O}_q^r \to \mathcal{L}_q$) is the same as giving r sections of $\mathcal{L}$ generating $\mathcal{L}$ (resp. r elements of $\mathcal{L}_q$ generating $\mathcal{L}_q$).

Now $\mathcal{O}_q$ is a PID, so $\mathcal{L}_q$ is an $\mathcal{O}_q$-module of the form

$$\mathcal{O}_q/I_1 \oplus \cdots \oplus \mathcal{O}_q/I_d$$

where $d \leq r$ and

$$\sum_1^d \operatorname{length}(\mathcal{O}_q/I_j) = n.$$

If t is a local parametre at q, then $I_j = \langle t^{\ell_j} \rangle$ for some $\ell_j \leq n$, hence

$$\mathcal{O}_q/I_j \simeq k[t]/\langle t^{\ell_j} \rangle \quad \text{and} \quad \sum_1^d \ell_j = n.$$

So M_Q parametrises quotients

$$\bigoplus_1^r k[t] \longrightarrow \bigoplus_1^d k[t]/\langle t^{\ell_j} \rangle \quad \text{with} \quad \sum_1^d \ell_j = n.$$

In turn this corresponds to quotients

$$k[t]^{\oplus r} \longrightarrow \bigoplus_1^r k[t]/\langle t^n \rangle \longrightarrow \bigoplus_1^d k[t]/\langle t^{\ell_j} \rangle \quad \text{with} \quad \sum_1^d \ell_j = n. \tag{5.2}$$

In particular, this shows that M_Q depends only on n and thus to calculate M_Q we could assume $Z = \mathbb{P}^1$.

Remark 5.4. This is a continuation of Remark 5.3. If $n = 1$, the quotients of (5.2) are of the form

$$k^{\oplus r} \longrightarrow k,$$

which in turn implies $M_q \simeq \mathbb{P}^{r-1}$.

In the following we assume $r, n \geq 2$. There are $p_r(n) \geq 2$ possibilities for $L = \oplus k[t]/\langle t^{\ell_j} \rangle$ satisfies (5.2), where $p_r(n)$ is the number of partitions of n into r non-zero parts.

(1) First, we take $L = k[t]/\langle t^n \rangle$. A quotient as in (5.2) is determined by

$$e_i = a_{i,0} + a_{i,1}t + \cdots + a_{i,n-1}t^{n-1}, \quad 1 \leq i \leq r$$

such that $a_{i,0} \neq 0$ for at least one i. Note that e_i is invertible in L if and only if $a_{i,0} \neq 0$.

The quotients with e_i invertible are parametrised by $\mathbb{A}^{n(r-1)}$ via

$$\begin{pmatrix} a_{1,0} & a_{1,1} & \cdots & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,0} & a_{i-1,1} & \cdots & a_{i-1,n-1} \\ 1 & 0 & \cdots & 0 \\ a_{i+1,0} & a_{i+1,1} & \cdots & a_{i+1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r,0} & a_{r,1} & \cdots & a_{r,n-1} \end{pmatrix}$$

whose entries are the coefficients of $e_i^{-1}e_j$ in L for $1 \leq j \leq r$. All of these glue together to form a smooth open subset S_1 of M_Q of dimension $n(r-1)$.

(2) Next we take $L = k[t]/\langle t^{n-1}\rangle \oplus k$. Then a quotient as in (5.2) is determined

$$e_i = a_{i,0} + a_{i,1}t + \cdots + a_{i,n-2}t^{n-2} \text{ and } b_i, \quad 1 \leq i \leq r$$

such that

$$\mathrm{rk}\begin{pmatrix} a_{1,0} & b_1 \\ a_{2,0} & b_2 \\ \vdots & \vdots \\ a_{r,0} & b_r \end{pmatrix} = 2.$$

The quotients with

$$\begin{pmatrix} a_{i,0} & b_i \\ a_{j,0} & b_j \end{pmatrix}$$

invertible are parametrised by $\mathbb{A}^{n(r-1)-2}$ via

$$\begin{pmatrix} a_{1,0} & a_{1,1} & \cdots & a_{1,n-2} & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{i-1,0} & a_{i-1,1} & \cdots & a_{i-1,n-2} & b_{i-1} \\ 1 & 0 & \cdots & 0 & 0 \\ a_{i+1,0} & a_{i+1,1} & \cdots & a_{i+1,n-2} & b_{i+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{j-1,0} & a_{j-1,1} & \cdots & a_{j-1,n-2} & b_{j-1} \\ 0 & a_{j,1} & \cdots & a_{j,n-2} & 1 \\ a_{j+1,0} & a_{j+1,1} & \cdots & a_{j+1,n-2} & b_{j+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{r,0} & a_{r,1} & \cdots & a_{r,n-2} & b_r \end{pmatrix}.$$

All of these glue together to form smooth locally closed subset S_2 of M_Q of dimension $n(r-1)-2$.

We conclude that M_Q has a stratification with locally closed smooth subsets $\{S_i\}_1^{p_r(n)}$ where $\dim S_1 = n(r-1)$ and $\dim S_j \leq n(r-1)-2$ for $1 \leq i \leq p_r(n)$ and $1 < j \leq p_r(n)$. In particular, M_Q is of dimension $n(r-1)$ and smooth in codimension one. Therefore, M_Q is normal by Lemma 3.18.

Proof of Theorem 1.2. (1) follows from Lemma 3.17. (2) is an immediate consequence of Theorem 5.1 and Remark 5.3.

For (3), as in Remark 5.3, F_j depends only on n_j and hence we may take $Z = \mathbb{P}^1$. If $n_j = 1$, then $F_j \simeq \mathbb{P}^{r-1}$ and everything is clear. So we assume $n_j \geq 2$ and then the normality of F_j follows from Remark 5.4. Note that every fibre of π has dimension equal $n(r-1)$. So π is flat [Eis95, Theorem 18.16], and then K_{F_j} is Cartier since K_X is Cartier [Mat87, Theorem 23.4]. □

6. Stable pairs of rank two over curves

In this section we continue our study from the previous section but for rank two case with irreducible cokernel support.

Theorem 6.1. *Assume Z is a smooth curve, $Q = nq$ with $n \geq 0$ and q a point. Then*

$$M_Q = \{[\mathcal{E}, s] \mid \mathcal{E}, s \text{ is a stable pair of rank } 2 \text{ with cokernel divisor } Q\}$$

has a natural stratification by locally closed subsets

$$M_Q = \bigcup_{\substack{m\geq 0 \text{ such that} \\ m+2\ell=n \text{ for some } \ell\geq 0}} G_m$$

where

$$G_m \subseteq M_{mq} = \{[\mathcal{G}, t] \text{ stable of rank } 2 \text{ with cokernel divisor } mq\}$$

is an open subset.

Proof. As in the proof of Theorem 5.1 we can compactify Z hence assume it is projective. If $Q = 0$, then $\mathcal{O}_Z^2 \xrightarrow{s} \mathcal{E}$ is an isomorphism, so there exists only one equivalent class $[\mathcal{E}, s]$. In this case M_Q is one point corresponding to $m = 0 = n$. So we assume $Q \neq 0$.

Let $X, D_1, D_2, A \to Z$ be the model associated to $\mathcal{E}, s$. Since $\mathcal{E}, s$ is stable, D_1, D_2, A have no common horizontal component over Z. But they may have common components mapping to q.

Let $X_0 = X$, $D_{i,0} = D_i$. Also let R_0 be the largest divisor such that $R_0 \leq D_{i,0}$ for $i = 1, 2$. Let $M_{i,0} = D_i - R_0$. If s_i is the section corresponding to $D_{i,0}$, then R_0 is the fixed part of the linear system generated by s_1, s_2.

Now $M_{1,0}$ and $M_{2,0}$ have no common component and $M_{i,0} = D_i$ over $Z \setminus \{q\}$. Moreover, since $D_{1,0} \sim D_{2,0}$, $M_{1,0} \sim M_{2,0}$. Assume $M_{1,0} \cap M_{2,0} = \emptyset$. In this case, $R_0 = \frac{n}{2} F$ where F is the fibre of $X \to Z$ over q. Indeed, this follows from Lemma 4.9 and

$$\begin{aligned} n = D_{1,0} \cdot D_{2,0} &= (M_{1,0} + R_0) \cdot (M_{2,0} + R_0) = M_{1,0} \cdot R_0 + M_{2,0} \cdot R_0 \\ &= 2M_{1,0} \cdot R_0 \end{aligned}$$

and $M_{1,0} \cdot F = 1$. Moreover, since $M_{1,0} \sim M_{2,0}$ and $M_{1,0} \cap M_{2,0} = \emptyset$, the linear system $|M_{1,0}| = |M_{2,0}|$ gives an isomorphism $X \simeq Z \times \mathbb{P}^1$ and $\mathcal{E} \otimes \mathcal{O}_Z(-\frac{n}{2}) \simeq \mathcal{O}_Z^2$. Thus the case $M_{1,0} \cap M_{2,0} = \emptyset$ corresponds to $m = 0$, $2\ell = n$ and there is only one class $[\mathcal{E}, s]$ with this property, and this gives $G_0 = \mathrm{pt}$ when n is even. This case cannot happen when n is odd.

Assume that $M_{1,0} \cap M_{2,0} \neq \emptyset$. Note that at least one of $M_{1,0}$ and $M_{2,0}$ is purely horizontal, i.e., have no vertical component. So $M_{1,0} \cap F$ or $M_{2,0} \cap F$ is one point (with multiplicity one). Moreover, $M_{1,0}, M_{2,0}$ cannot intersect outside F. This implies $M_{1,0} \cap M_{2,0}$ is one point (but maybe with multiplicity > 1). And if $M_{i,0}$ is purely horizontal, then $M_{i,0} \cap F = M_{1,0} \cap M_{2,0}$ (as sets).

Let $X_1 \xrightarrow{\varphi_1} X_0$ be the blowup of the intersection point $M_{1,0} \cap M_{2,0}$, say r_0, and E_1 the exceptional divisor. Let

$$M_{i,1} = \varphi_1^* M_{i,0} - R_1$$

where R_1 is the largest divisor with $R_1 \leq \varphi_1^* M_{i,0}$, $i = 1, 2$. By the previous paragraph, $F \not\subseteq M_{i,0}$ for some i and $M_{i,0} \cdot F = 1$, so $M_{i,0}$ is smooth at r_0 for this i, so $R_1 = E_1$ and

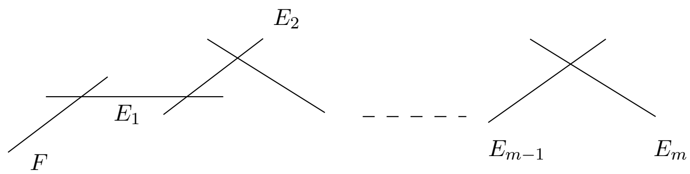
$$M_{i,1} \cdot E_1 = 1 \text{ for both } i = 1, 2.$$

Also $M_{i,1}$ is purely horizontal for some i, intersecting each fibre of $X_1 \to Z$ at one point transversally. So $M_{1,1} \cap M_{2,1}$ is empty or consists of one point only (maybe with multiplicity > 1).

If $M_{1,1} \cap M_{2,1} = \emptyset$, we stop. If not, we blowup $M_{1,1} \cap M_{2,1}$ and repeat the above. This gives a sequence

$$X_m \xrightarrow{\varphi_m} X_{m-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\varphi_1} X_0 = X \tag{6.1}$$

such that $X_{j+1} \to X_j$ blows up a point on $E_j = \operatorname{Exc}(\varphi_j)$ not belonging to any other component of the fibre of $X_j \to Z$ over q, and such that $M_{1,m} \cap M_{2,m} = \emptyset$. So the fibre of $X_m \to Z$ over q is a reduced chain of $\mathbb{P}^1$:



with self-intersections $-1, -2, \ldots, -2, -1$, respectively. Also $E_m \not\subseteq M_{i,m}$ for $i = 1, 2$, and $M_{i,m}$ is purely horizontal for some i, hence this $M_{i,m}$ intersects E_m but no other E_i or F.

Now we can run an MMP on X_m over Z. First we contract F, then $E_1, \ldots$, then E_{m-1}. We arrive at a surface X' such that E_m is the fibre of $X' \to Z$ over q. Since one of $M_{i,m}$ does not intersect any of the curved contracted by the MMP, $M_{1,m}$ and $M_{2,m}$ remains disjoint in the process. Therefore,

$$X' \simeq Z \times \mathbb{P}^1.$$

Assume $R_0 = \ell F$. Then, by Lemma 4.9,

$$\begin{aligned} n = D_{1,0} \cdot D_{2,0} &= (M_{1,0} + R_0) \cdot (M_{2,0} + R_0) \\ &= M_{1,0} \cdot M_{2,0} + M_{1,0} \cdot R_0 + M_{2,0} \cdot R_0 \\ &= M_{1,0} \cdot M_{2,0} + 2\ell \end{aligned}$$

where we use the facts

$$M_{1,0} \sim M_{2,0} \quad \text{and} \quad M_{i,0} \cdot F = 1.$$

Also we can see from the construction of sequence (6.1) that $m = M_{1,0} \cdot M_{2,0}$. Thus $n = m + 2\ell$. If s_1, s_2 are the sections determined by s, then we get sections t_1, t_2 of $\mathcal{E} \otimes \mathcal{O}_Z(-\ell q)$ whose zero divisors on X are $M_{1,0}$ and $M_{2,0}$. In fact,

$$\mathcal{O}_Z^2 \xrightarrow{t=(t_1,t_2)} \mathcal{E} \otimes \mathcal{O}_Z(-\ell q)$$

gives a stable pair $\mathcal{E} \otimes \mathcal{O}_Z(-\ell q), t$ with cokernel divisor mq.

Conversely, from any stable pair $\mathcal{G}, t$ with cokernel divisor mq we get a stable pair $\mathcal{E}, s$ with cokernel divisor $Q = nq$ as above. This identifies

$$G_m = \{[\mathcal{E}, s] \in M_Q \text{ with } M_{1,0} \cdot M_{2,0} = m\}$$

with the open subset

$$\{[\mathcal{G}, t] \in M_{mq} \text{ with } M_1 \text{ and } M_2 \text{ have no common component}\}$$

where M_1, M_2 are the divisors determined by t. Note that the case $m = n$, $\ell = 0$ gives an open subset $G_n \subseteq M_Q$. $\square$

Remark 6.2. We use notation in the proof above. Let D_i' be the birational transform of $M_{i,0}$, and A' birational transform of $A_0 = A - R_0$. Also let F' be the fibre of $X' \to Z$ over q. We will see below that

$$(X', B' = D_1' + D_2' + F'), A' \longrightarrow Z$$

is nothing but the stable minimal model associated to $\mathcal{E}, s$.

It is clear from the construction above that we can recover

$$X, M_{1,0}, M_{2,0}, A_0 \longrightarrow Z$$

from

$$X', D_1', D_2', A' \longrightarrow Z$$

by reversing $X_m \to X'$ and then reversing $X_m \to X$.

In fact, G_m is parametrising all such precesses. This amounts to choosing and blowing up a point on F', then blowup a point on the exceptional curve not belonging to the birational transform of F', and so on.

We argue that $M_{1,0}, M_{2,0}, A_0$ are the pullbacks of D_1', D_2', A' under the birational map

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & X' \arrow[dl] \\ & Z. & \end{tikzcd}$$

One way to see this is to note that $X \dashrightarrow X' \simeq Z \times \mathbb{P}^1$ is the map defined by the sections t_1, t_2 of $\mathcal{E} \otimes \mathcal{O}_Z(-\ell q)$ corresponding to $M_{1,0}, M_{2,0}$.

Here is another argument. If $M_{i,0}$ is purely horizontal, then

$$M_{i,m} = \psi^*\psi_* M_{i,m} = \psi^* D_i'$$

where ψ denotes $X_m \to X'$, hence the same holds for both $i = 1, 2$ noting that $\psi_* M_{i,m} = D_i'$. This shows $M_{i,0}$ is the pullback of D_i'.

On the other hand, let $A_m \geq 0$ be the divisor on X_m such that $A_m \sim M_{i,m}$ for $i = 1, 2$ and such that pushdown of A_m to X is A_0. Then $E_m \not\subseteq A_m$ and A' is the pushdown of A_m. Thus $D_1' \sim D_2' \sim A'$, so if $\overline{A}$ is the pullback of A' to X, then

$$\overline{A} \sim M_{1,0} \sim M_{2,0} \sim A_0 \quad \text{and} \quad \overline{A} = A_0 \text{ over } Z \setminus \{q\},$$

so $\overline{A} = A_0$.

Remark 6.3. We explain how the construction in the proof of Theorem 6.1 relates to stable minimal models. We use notation introduced in the proof.

By construction, (X_m, B_m) is log smooth where

$$B_m = \operatorname{Supp}(D_1^\sim + D_2^\sim + \operatorname{Exc}(\varphi) + \varphi^{-1} f^{-1} Q)$$

and where $\sim$ denotes birational transform and φ denotes $X_m \to X$. Also recall $A_m \sim M_{i,m}$. Let B_j, A_j be the pushdowns of B_m, A_m to X_j via $X_m \to X_j$. We claim that

$$(X_m, B_m + uA_m^h)$$

is lc but

$$(X_j, B_j + uA_j^h) \quad 0 \leq j < m$$

is not lc for any $0 < u \ll 1$ where A_j^h denotes the horizontal part of A_j. The pushdown of B_m to X' is $B' = D_1' + D_2' + F'$, and the pushdown of A_m^h is A'. Moreover, D_1', D_2', A' are disjoint, and $(X', B' + uA')$ is lc. Since $X_m \to X'$ is a sequence of blowups, $(X_m, B_m + uA_m^h)$ is lc.

Now assume both $M_{1,m}$ and $M_{2,m}$ are purely horizontal. Then $M_{1,j}$ and $M_{2,j}$ intersects at some point on the fibre of $X_j \to Z$ over q, so (X_j, B_j) is not lc. Then we can assume one of $M_{1,m}$ and $M_{2,m}$ is purely horizontal but the other is not. The same holds for $M_{1,j}$ and $M_{2,j}$. This implies A_0 is also purely horizontal hence A_j is purely horizontal, so $A_j^h = A_j$.

Then again $M_{1,j}$ and $M_{2,j}$ intersect at some point and A_j also passes through this point. Thus

$$(X_j, B_j + uA_j^h)$$

is not lc. This proves the claim.

Moreover, A' does not contain any non-klt centre of (X', B'), so

$$(X', B'), A' \longrightarrow Z$$

is a stable minimal model.

Since B_m equals $B'^{\sim}$ plus the exceptional divisor of $X_m \to X'$, running MMPs as in Construction 4.6 ends with (X', B'), so the above is the stable minimal model of $\mathcal{E}, s$.

In summary: each $X_{j+1} \to X_j$ is the blowup of a point where

$$(X_j, B_j + uA_j^h)$$

fails to be lc. When we arrive at X_m,

$$(X_m, B_m + uA_m^h)$$

is lc, and using MMP we modify it to get a stable minimal model. So the whole process is dictated by transforming $(X_0, B_0 + uA_0^h)$ into a stable minimal model.

6.4. **The variety of invariant subspaces.** Here is another point of view from embeddings.

Lemma 6.5. *Let Z be a smooth projective curve, and let $\alpha\colon \mathcal{K} \hookrightarrow \mathcal{O}_Z^2$ be a subsheaf of co-length n. Suppose that $\wedge^2\alpha$ given by a section $s \in H^0(Z, \mathcal{O}_Z(n))$. Then there is a diagram*

$$\begin{tikzcd} \mathcal{O}_Z(-n)^{\oplus 2} \arrow[r, hook] \arrow[rr, bend right, "s\oplus s"'] & \mathcal{K} \arrow[r, hook, "\alpha"] & \mathcal{O}_Z^2. \end{tikzcd}$$

Proof. Let $\mathcal{E} = \mathcal{K}^\vee$. Note that $\mathcal{E} \simeq \mathcal{K} \otimes \mathcal{O}_Z(n)$. The adjoint $\alpha^\vee\colon \mathcal{O}_Z^2 \longrightarrow \mathcal{E}$, which is defined via the adjunct matrix of α, can be twisted by $\mathcal{O}_Z(-n)$. The fact that composition is $s \oplus s$ is essentially Cramer's rule. $\square$

Therefore $\mathcal{K}$ is uniquely determined by a subsheaf of the Artinian sheaf

$$\mathcal{A}_s \coloneqq (\mathcal{O}_Z/s\mathcal{O}_Z(-n))^{\oplus 2}$$

of length n, and

$$\mathcal{O}_Z^2/\alpha\mathcal{K} \longrightarrow (\mathcal{O}_Z/s\mathcal{O}_Z(-n))^{\oplus 2} = \mathcal{A}_s$$

while $\mathrm{length}(\mathcal{A}_s) = 2n$. Suppose that s vanishes at points $q_1, \dots, q_r$ with multiplicities $n_1, \dots, n_\ell$, then $s = s_1 \cdots s_\ell$ where s_j vanishes exactly at q_j with multiplicity n_j.

Denote by $\mathrm{Quot}(s)$ the space of subsheaves $\mathcal{K}$ that correspond to a fixed $s \in H^0(Z, \mathcal{O}_Z(n))$. Then we have the factorisation property

$$\mathcal{A}_s = \mathcal{A}_{s_1} \oplus \cdots \oplus \mathcal{A}_{s_\ell},$$
$$\mathrm{Quot}(s) = \mathrm{Quot}(s_1) \times \cdots \times \mathrm{Quot}(s_\ell).$$

So it suffices to assume s vanishes at a single point q with multiplicity n. In other words $s = t^n$ where $t \in H^0(\mathcal{O}(1))$ vanishes at q. Then $\mathrm{Quot}(t^n)$ is the variety of n-dimensional subspaces $V \subseteq (k[t]/\langle t^n \rangle)^{\oplus 2}$ satisfying $t \cdot V \subseteq V$.

Write

$$(k[t]/\langle t^n\rangle)^{\oplus 2} = W \oplus tW \oplus \cdots \oplus t^{n-1}W$$

with $\dim W = 2$. Then V is a module over $k[t]$ and by classification of modules over PID,

$$V \simeq k[t]/\langle t^m\rangle \oplus k[t]/\langle t^{n-m}\rangle, \quad m = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor.$$

There are at most 2 terms since the null space $\mathrm{Null}(V \xrightarrow{t\cdot} V) \hookrightarrow t^{n-1}W$ and $\dim W = 2$.

Lemma 6.6. *For fixed* $m = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor =: b$,

$$V = k[t] \cdot v + (t^{n-m}W \oplus \cdots \oplus t^{n-1}W)$$

and $v \in t^m W \oplus t^{m+1}W \oplus \cdots \oplus t^{n-1}W$ *has a non-zero projection* $v \pmod{t^{m+1}} \in t^m W$.

Note that we only care about the components $v_m + v_{m+1} + \cdots + v_{n-m-1}$ of v in $t^m W \oplus t^{m+1}W \oplus \cdots \oplus t^{n-m-1}W$. Since we only consider V as a subspace of $(k[t]/\langle t^n\rangle)^{\oplus 2}$, the component v_m is well-defined only as a point in $\mathbb{P}(t^m W) = \mathbb{P}^1$. Other components $v_{m+1} + \cdots + v_{n-m-1}$ are determined up to an action of the multiplication group $1 + kt + \cdots + kt^{n-2m-1} \simeq \mathbb{A}^{n-2m-1}$. So $\dim \mathrm{Quot}(t^n) = n$ and there is a stratification

$$\mathrm{Quot}(t^n) = \mathrm{Quot}(t^n, 0) \cup \mathrm{Quot}(t^n, 1) \cup \cdots \cup \mathrm{Quot}(t^n, b)$$

according to the value of $m = 0, 1, \ldots, b$ and each stratum $\mathrm{Quot}(t^n, m)$ is an $\mathbb{A}^{n-2m-1}$-bundle over $\mathbb{P}^1$, except that for $n = 2b$ and $m = b$ one has $V = t^b W \oplus \cdots \oplus t^{2b-1}W$. So the stratum $\mathrm{Quot}(t^{2b}, b)$ is a single point.

For example, in the special case $n = 3$ one has $b = 1$ and $m = 0, 1$. The 3-dimensional space $\mathrm{Quot}(t^3) \simeq \mathrm{Quot}(t^3, 0) \cup \mathrm{Quot}(t^3, 1)$. Since $\mathrm{Quot}(t^3, 0)$ is an $\mathbb{A}^2$-bundle over $\mathbb{P}^1$ and $\mathrm{Quot}(t^3, 1)$ is isomorphic to $\mathbb{P}^1$, the dimension of stratum drops by 2.

7. Degree one stable pairs on curves

In this and following sections we give an in depth analysis of the moduli spaces $M_Z(2, n)$ and the fibre of

$$M_Z(2, n) \longrightarrow \mathrm{Hilb}^n_Z$$

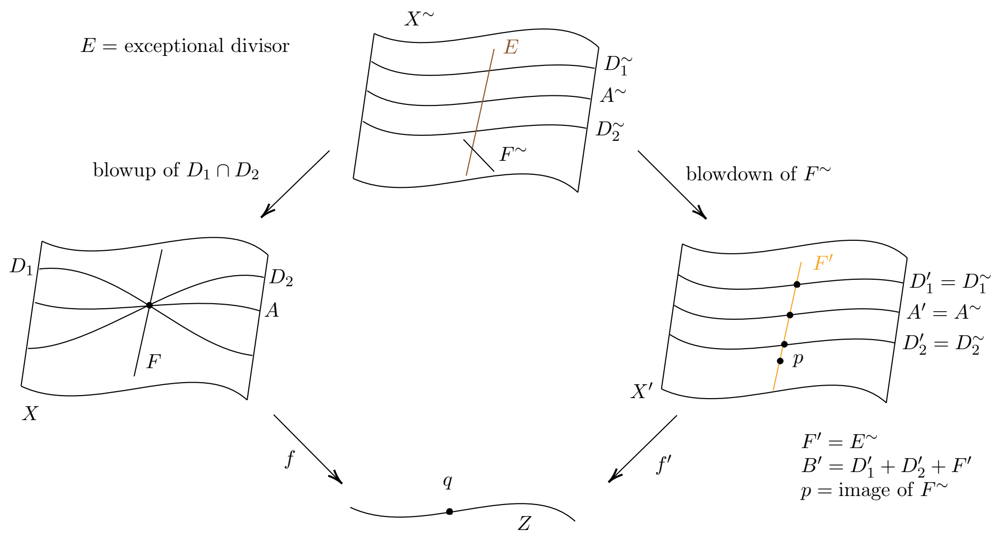
for rank two stable pairs on curves and of degree $n \leq 3$.

Example 7.1 (Degree one)**.** Assume Z is a smooth projective curve and assume $\mathcal{E}, s$ is a stable pair of rank 2 and degree 1. Let

$$X, D_1, D_2, A \longrightarrow Z$$

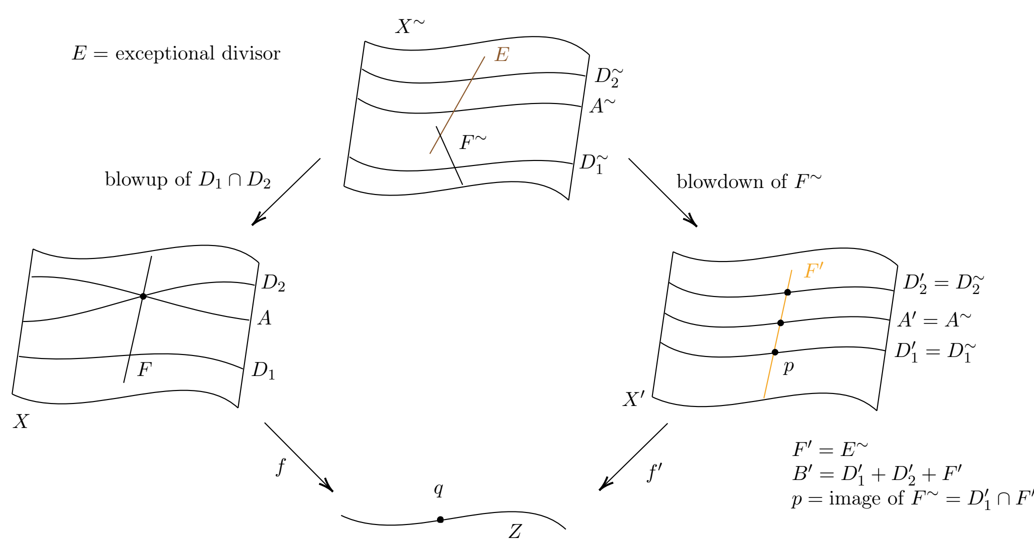
be the associated model. By Lemma 4.9, $D_1 \cdot D_2 = 1$, so $D_1 \cap D_2 \subseteq$ one fibre of f, and divisor $Q = \mathrm{coker}(s)$ is one point q. Let F be the fibre over q.

Case 1: D_1, D_2, A have no vertical components. Then the associated stable minimal model is obtained as in this picture:



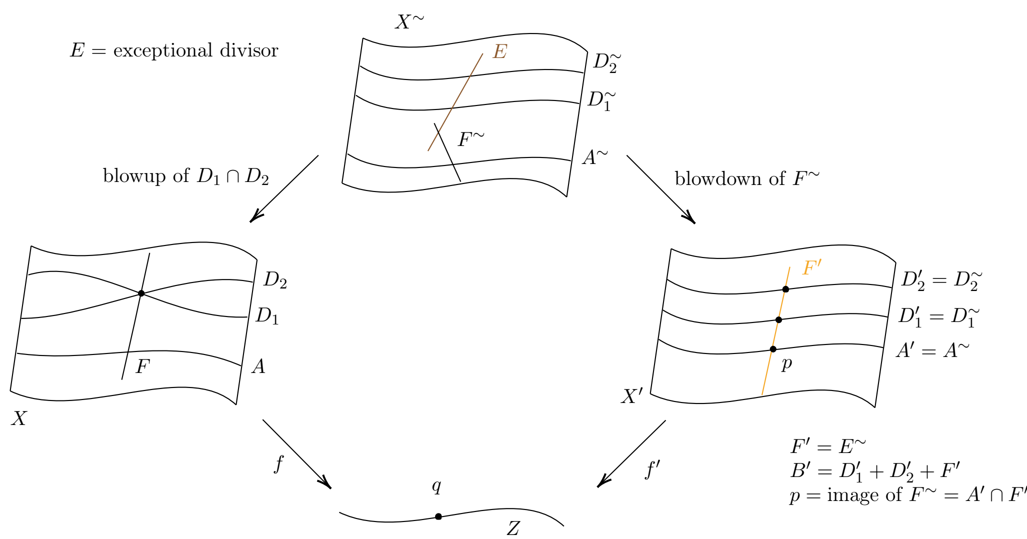
Here $X' \simeq Z \times \mathbb{P}^1$ and D_1', D_2', A' are fibres on the projection $X' \to \mathbb{P}^1$.

Case 2: D_1 or D_2 has a vertical component. Since $D_1 \cdot D_2 = 1$ and since D_1, D_2 have no common horizontal component, only one of D_1, D_2 can have a vertical component, say D_1. For similar reason, A cannot have a vertical component. Then the associated stable minimal model is obtained as in this picture:



Again $X' \simeq Z \times \mathbb{P}^1$ and D_1', D_2', A' are fibres of the projection $X' \to \mathbb{P}^1$.

Case 3: A has a vertical component. In this case D_1, D_2 cannot have a vertical component. Then the associated stable minimal model is obtained as in this picture:



Again $X' \simeq Z \times \mathbb{P}^1$ and D_1', D_2', A' are fibres of the projection $X' \to \mathbb{P}^1$.

In summary, the class $[\mathcal{E}, s]$ is determined by the model $X, D_1, D_2, A \to Z$ which is in turn determined by the fixed model $X', D_1', D_2', A' \to Z$ and the point p on the fibre F'. Therefore, the fibre of

$$M_Z(2,1) \longrightarrow \mathrm{Hilb}^1_Z = Z$$

over each point q is $\mathbb{P}^1$ and $M_Z(2,1) \simeq Z \times \mathbb{P}^1$.

This also completes the proof of Theorem 1.3.

Example 7.2 (Degree one on $\mathbb{P}^1$)**.** Assume $Z = \mathbb{P}^1$. Then for any $[\mathcal{E}, s] \in M_Z(2,1)$, $\mathcal{E} \simeq \mathcal{O}_Z \oplus \mathcal{O}_Z(1)$ as $\mathcal{E}$ is nef (Corollary 3.9). Moreover, by Lemma 3.15, $M_Z(2,1) \simeq \mathrm{Quot}(\mathcal{O}_Z^2, 1)$, more precisely, $M_Z(2,1)$ is parametrising embeddings

$$\mathcal{O}_Z \oplus \mathcal{O}_Z(-1) \subseteq \mathcal{O}_Z \oplus \mathcal{O}_Z.$$

Tensoring with $\mathcal{O}_Z(1)$ we get

$$\mathcal{O}_Z(1) \oplus \mathcal{O}_Z \subseteq \mathcal{O}_Z(1) \oplus \mathcal{O}_Z(1).$$

Since both sheaves are generated by global sections, the above inclusion can be recovered from

$$H^0(Z, \mathcal{O}_Z(1) \oplus \mathcal{O}_Z) \subseteq H^0(Z, \mathcal{O}_Z(1) \oplus \mathcal{O}_Z(1)).$$

Thus we get an embedding into a Grassmannian:

$$M_Z(2,1) \hookrightarrow \mathrm{Grass}(3,4) \simeq \mathbb{P}^3.$$

Since $M_Z(2,1)$ has a $\mathbb{P}^1$-fibration onto Z, which is of Picard number two, $M_Z(2,1)$ is a hypersurface in $\mathbb{P}^3$ of degree 2. Therefore,

$$M_Z(2,1) \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

This is of course consistent with Example 7.1 where giving a class $[\mathcal{E}, s] \in M_Z(2,1)$ is the same as a picking point $q \in Z = \mathbb{P}^1$ and then picking a point p on the fibre $F' = \mathbb{P}^1$ of $X' \to Z$.

Remark 7.3. Here is another point of view.

Assume $Z = \mathbb{P}^1$. Let $\mathcal{G}_1 = \mathcal{O}_Z^{r-1}$, $\mathcal{G}_2 = \mathcal{O}_Z(n)$ and $\mathcal{E} \coloneqq \mathcal{G}_1 \oplus \mathcal{G}_2$. We consider the moduli space $\mathcal{M}$ of maps $\mathcal{O}_Z^r \xrightarrow{s} \mathcal{E}$ with the condition that its cokernel has 0-dimensional support and the following equivalence holds

$$\begin{tikzcd} \mathcal{O}_Z^r \arrow[r, "s"] \arrow[d, equal, "\mathrm{id}"'] & \mathcal{E} \arrow[d, "\wr"] \\ \mathcal{O}_Z^r \arrow[r, "s'"'] & \mathcal{E}. \end{tikzcd}$$

This $\mathcal{M}$ can be viewed as a subvariety of the Quot-scheme $\mathrm{Quot}(\mathcal{O}_Z^r, n)$, while the latter is smooth of dimension nr.

Such a map $s\colon \mathcal{O}_Z^r \to \mathcal{E}$ is given by

$$M = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_r \\ \beta_1 & \beta_2 & \cdots & \beta_r \end{pmatrix} \in \overline{\mathcal{M}}$$

where $\alpha_i \in H^0(\mathcal{G}_2) =: H$ and $\beta_i \in H^0(\mathcal{G}_1)$ such that its determinant is a non-zero vector in H. In particular, those β_i's determine an $(r-1)$-dimensional subspace in $V \coloneqq H^0(\mathcal{O}_Z^r)$, denoted by β_M. Then there is a morphism

$$\overline{\pi}\colon \overline{\mathcal{M}} \longrightarrow H \times V, \quad M = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_r \\ \beta_1 & \beta_2 & \cdots & \beta_r \end{pmatrix} \longmapsto (\det M, \beta_M).$$

Note that

$$\mathrm{Aut}(\mathcal{E}) = (\mathrm{Aut}(\mathcal{G}_1) \times \mathrm{Aut}(\mathcal{G}_2)) \ltimes \mathrm{Hom}(\mathcal{G}_1, \mathcal{G}_2) = (k^* \times \mathrm{GL}(r-1)) \ltimes \mathrm{Hom}(\mathcal{O}_Z^{r-1}, \mathcal{O}_Z(n))$$

acts on $\overline{\mathcal{M}}$ by matrix multiplication

$$\begin{pmatrix} \gamma_1 & \varphi \\ 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_r \\ \beta_1 & \beta_2 & \cdots & \beta_r \end{pmatrix} = \begin{pmatrix} \gamma_1\alpha_1 + \varphi\beta_1 & \gamma_1\alpha_2 + \varphi\beta_2 & \cdots & \gamma_1\alpha_r + \varphi\beta_r \\ \gamma_2\beta_1 & \gamma_2\beta_2 & \cdots & \gamma_2\beta_r \end{pmatrix}$$

where $\gamma_1 \in k^*$, $\gamma_2 \in \mathrm{GL}(r-1)$ and $\varphi \in \mathrm{Hom}(\mathcal{O}_Z^{r-1}, \mathcal{O}_Z(n))$. It is not hard to see that $\det M$ is invariant up to scaling of $\gamma_1 {\cdot} \det \gamma_2$ under the action of $\mathrm{Aut}(\mathcal{E})$, while β_M is invariant as a linear subspace of V. Hence there is an induced morphism $\pi\colon \mathcal{M} = \overline{\mathcal{M}}/\mathrm{Aut}(\mathcal{E}) \to \mathbb{P}(H) \times \mathbb{P}(V)$. The injectivity and surjectivity can be derived from the definition. The surjectivity of π is clear. Assume that $\pi(M) = \pi(M')$ for some $M, M' \in \overline{\mathcal{M}}$ with

$$M = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_r \\ \beta_1 & \beta_2 & \ldots & \beta_r \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} \alpha'_1 & \alpha'_2 & \ldots & \alpha'_r \\ \beta'_1 & \beta'_2 & \ldots & \beta'_r \end{pmatrix}.$$

After multiplying by matrices from $\mathrm{Aut}(\mathcal{E})$ we may assume that both $(\beta_2, \beta_3, \ldots, \beta_r)$ and $(\beta'_2, \beta'_3, \ldots, \beta'_r)$ are the identity matrix I_{r-1}, and $\det M = \det M'$. Then $\beta_1 = \beta'_1$ and we may take $\gamma_1 = 1$ and $\gamma_2 = I_{r-1}$. It is not hard to calculate that $\det M = \alpha_1 - \sum_{j=1}^{r-1} \beta_{1j}\alpha_{j+1}$. Now we define $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{r-1}) \in \mathrm{Hom}(\mathcal{O}_Z^{r-1}, \mathcal{O}_Z(n))$ by letting $\varphi_j = \alpha'_{j+1} - \alpha_{j+1}$. Then

$$\begin{pmatrix} 1 & \varphi \\ 0 & I_{r-1} \end{pmatrix} M = M'$$

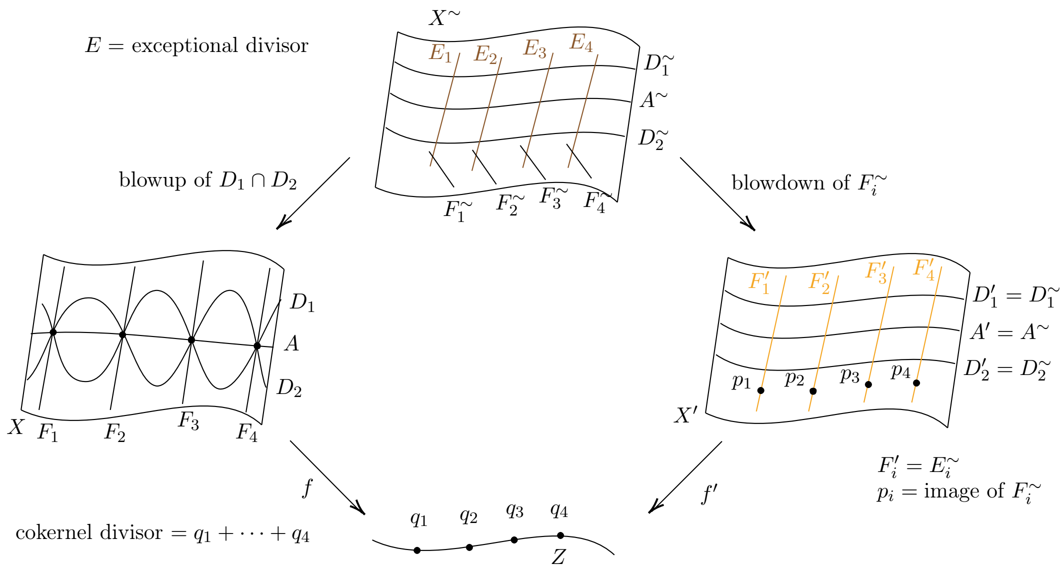
and hence π is injective. Since $\mathcal{M}$ is connected and $\mathbb{P}(H) \times \mathbb{P}(V)$ is smooth, the bijective morphism π is an isomorphism.

In the case $n = 1$, any quotient of $\mathcal{O}_Z^r$ of degree 1 has kernel $\mathcal{O}_Z^{r-1} \oplus \mathcal{O}_Z(-1)$. Those quotients are corresponding to stable pairs $\mathcal{O}_Z^r \to \mathcal{O}_Z^{r-1} \oplus \mathcal{O}_Z(1)$. In particular, we obtain $\mathrm{Quot}(\mathcal{O}_Z^r, 1) \simeq \mathbb{P}^{r-1} \times \mathbb{P}^1$.

Example 7.4 (Reduced cokernel divisor)**.** Assume Z is a smooth projective curve, $\mathcal{E}, s$ a stable pair with associated model

$$X, D_1, D_2, A \xrightarrow{f} Z$$

such that the cokernel divisor Q is reduced, say $Q = q_1 + \cdots + q_n$. Assume $F_1, \ldots, F_n$ are the fibres passing through the points in $D_1 \cap D_2$, i.e., fibres over the q_i. Then the stable minimal model $(X', B' = D_1' + D_2'), A' \to Z$ is obtained by blowing up $D_1 \cap D_2$ followed by blowing down $F_1^{\sim}, \ldots, F_n^{\sim}$. This picture illustrates the case $n = 4$ when D_1, D_2, A have no vertical component.



The stable minimal model $(X', B'), A' \to Z$ satisfies

$$X' = Z \times \mathbb{P}^1, \quad B' = D_1' + D_2' + \sum_1^n F_i'$$

and this is equipped with the marked points $p_i \in F_i'$. When D_1, D_2, A have no vertical component we have

$$p_i \in F_i' \setminus (D_1' \cup D_2' \cup A') \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

The marked stable minimal model

$$(X', B'), A', p_1, \ldots, p_n$$

determines the equivalence class $[\mathcal{E}, s]$. Moreover, starting with n distinct points $q_1, \ldots, q_n \in Z$ and considering a marked stable minimal model as above, the same process produces an equivalence class $[\mathcal{E}, s]$. In other words, all such marked stable minimal models are parametrised by

$$\mathrm{Hilb}^n_{Z,g} \times \mathbb{P}^1$$

where $\mathrm{Hilb}_{Z,g} \subseteq \mathrm{Hilb}^n_Z$ is the locus corresponding to reduced divisors (all coefficients equal 1) on Z of degree n.

The classes $[\mathcal{E}, s]$ with the properties above form an open subset of $M_Z(2, n)$. Thus $M_Z(2, n)$ is birational to the product above.

Note also that $p_i \in D_1'$ if and only if D_1 has a vertical component over q_i. A similar remark applies to D_2' and A'.

8. Degree two stable pairs on curves

Example 8.1 (Degree two on $\mathbb{P}^1$, geometric treatment)**.** Assume $Z = \mathbb{P}^1$ and $[\mathcal{E}, s] \in M_Z(2,2)$. Then $\mathcal{E} = \mathcal{O}_Z(1) \oplus \mathcal{O}_Z(1)$ or $\mathcal{E} = \mathcal{O}_Z \oplus \mathcal{O}_Z(2)$ as $\mathcal{E}$ is nef.

(1) Assume $\mathcal{E}, s$ has reduced cokernel divisor $Q = q_1 + q_2$ as in Example 7.4. Let

$$X, D_1, D_2, A \longrightarrow Z$$

be the associated model and

$$(X', B'), A', p_1, p_2 \longrightarrow Z$$

be the associated marked stable minima model. We claim that

$$\mathcal{E} = \mathcal{O}_Z \oplus \mathcal{O}_Z(2) \iff p_1, p_2 \in \text{the same fibre of the second projection } X' \longrightarrow \mathbb{P}^1.$$

Proof. ($\Longrightarrow$) There exists a (-2)-curve $S \subseteq X$ corresponding to the summand $\mathcal{O}_Z$ of $\mathcal{E}$, i.e., corresponding to the surjection $\mathcal{E} \twoheadrightarrow \mathcal{O}_Z$. Let F_1, F_2 be the fibre of $X \to Z$ over the points q_1, q_2 respectively. There is a section $u \in H^0(\mathcal{E}) \simeq H^0(\mathcal{O}_X(1))$ whose divisor is $S + F_1 + F_2$. None of D_1, D_2 can pass through $S \cap F_1$ or $S \cap F_2$ otherwise $2 = \deg \mathcal{E} = D_i \cdot (S + F_1 + F_2) > 2$, a contradiction. So if $S' \subseteq X'$ is the birational transform of S, then $S'^2 = 0$ and $p_1, p_2 \in S'$. But we can check $S' \sim D_i'$, so $S' \cap D_i' = \emptyset$, hence S' is a fibre of the second projection $X' = Z \times \mathbb{P}^1 \to \mathbb{P}^1$.

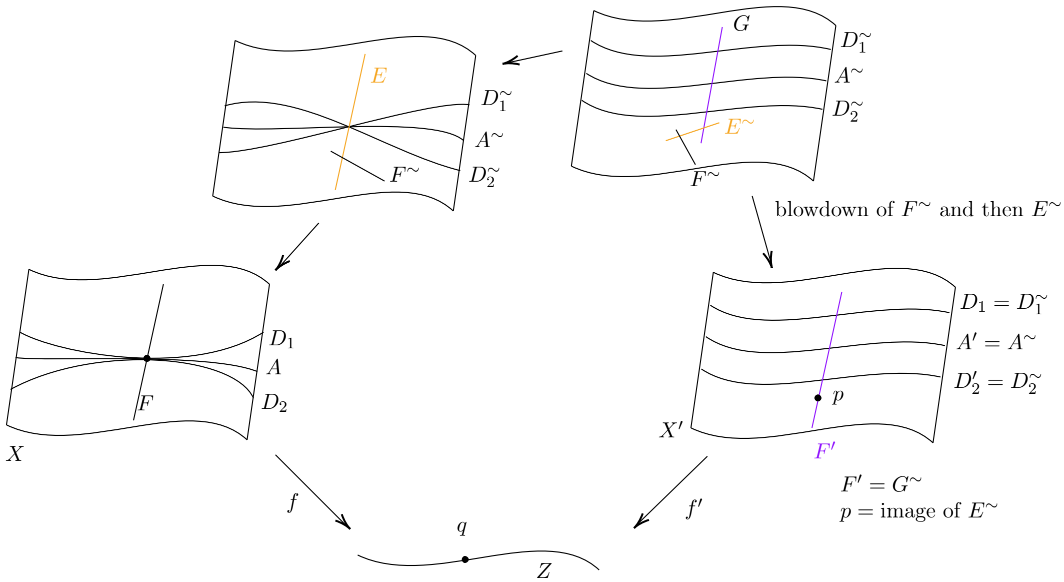
($\Longleftarrow$) Let S' be the fibre containing p_1, p_2. Then we can check that $S^2 = -2$ where $S \subseteq X$ is the birational transform of S'. So this is possible only if $\mathcal{E} = \mathcal{O}_Z \oplus \mathcal{O}_Z(2)$. □

The claim shows that the points $[\mathcal{E}, s] \in M_Z(2,2)$ with $\mathcal{E} = \mathcal{O}_Z \oplus \mathcal{O}_Z(2)$ and cokernel divisor $q_1 + q_2$ are parametrised by a copy of $\mathbb{P}^1$.

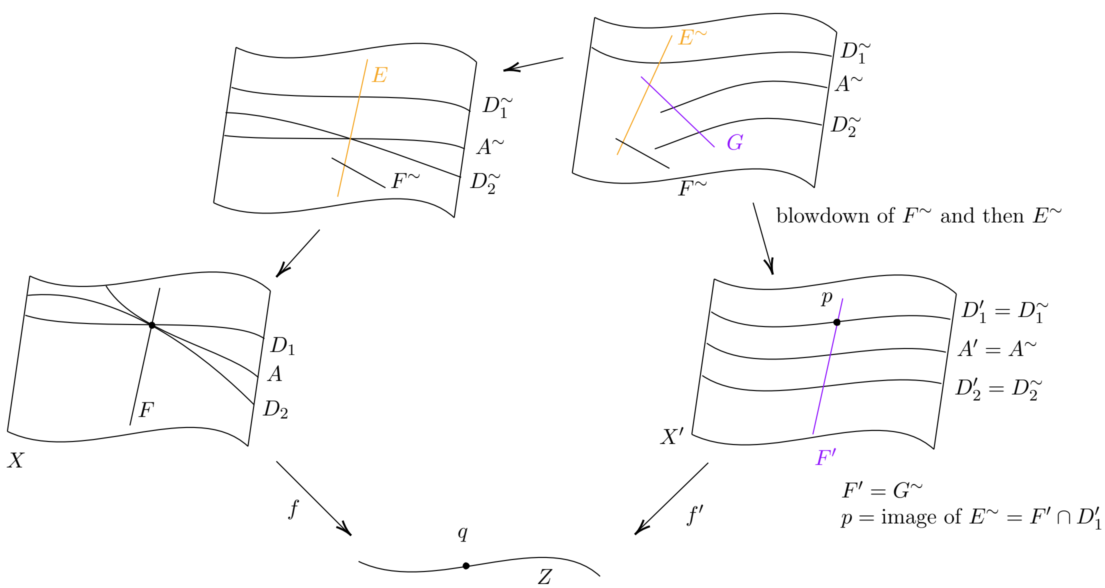
(2) Now assume $\operatorname{coker}(s)$ divisor is non-reduced, say $Q = 2q$.

Case I: The fibre F over q is a component of both D_1, D_2. Then $X = X'$ and D_i' is the horizontal part of D_i (similarly for A).

Case II: F is not a component of D_1, D_2, A. Then D_1 and D_2 are tangent to each other at some point: indeed if A intersects D_1 at any point, then it also intersects D_2 at the same point because A is the divisor of $s_1 + s_2$. Then the stable minimal model is obtained as follows:



Case III: F is a component of one of D_1, D_2, A. (Now if F is a component of two of D_1, D_2, A, then we are done in Case I). Say F is a component of D_1. Then D_2, A are tangent. Then the stable minimal model is obtained as follows:



If F is a component of D_2 (resp. A), then the picture is similar with $p = D_2' \cap F'$ (resp. $p' = A' \cap F'$).

(3) We consider the fibre of $M_Z(2,2) \to \mathrm{Hilb}^2_Z$. The fibres over points corresponding to reduced divisors, are $\mathbb{P}^1 \times \mathbb{P}^1$ by Example 7.4. The fibre over a point corresponding to $2q$ is

$$M_{2q} = \{[\mathcal{E}, s] \in M_Z(2,2) \mid \mathrm{coker}(s)\,\mathrm{div} = 2q\}$$

which has a stratification as in Theorem 6.1,

$$M_{2q} = G_0 \cup G_2.$$

Here G_0 is one point corresponding to the case when the fibre F over q is a component of both D_1 and D_2. On the other hand G_2 parametrising the case when D_1, D_2 have no common component. By the proof of Theorem 6.1, G_2 admits a morphism

$$G_2 \longrightarrow \mathbb{P}^1$$

whose fibres are $\mathbb{A}^1$. This is seen from the process of going from X to X'.

Example 8.2 (Degree two on $\mathbb{P}^1$, algebraic treatment)**.**

(1) Then $M_Z(2,2) \simeq \mathrm{Quot}(\mathcal{O}_Z^2, 2)$ and the latter parametrising embeddings

$$\mathcal{K} \subseteq \mathcal{O}_Z \oplus \mathcal{O}_Z$$

where either

$$\mathcal{K} = \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z(-1) \quad \text{or} \quad \mathcal{K} = \mathcal{O}_Z \oplus \mathcal{O}_Z(-2).$$

Tensoring with $\mathcal{O}_Z(1)$,

$$\mathcal{K}(1) \subseteq \mathcal{O}_Z(1) \oplus \mathcal{O}_Z(1).$$

In the first case, $\mathcal{K}$ can be recovered from

$$H^0(Z, \mathcal{K}(1)) \subseteq H^0(Z, \mathcal{O}_Z(1) \oplus \mathcal{O}_Z(1)) \tag{8.1}$$

but not in the second case. In any case, the later inclusion gives a birational morphism

$$g\colon\ \mathrm{Quot}(\mathcal{O}_Z^2, 2) \longrightarrow \mathrm{Gr}(2,4)$$

and $\mathrm{Gr}(2,4) \subseteq \mathbb{P}^5$ is a quadric hypersurface. So we get a diagram

$$\begin{tikzcd} & M_Z(2,2) \arrow[dl, "g"'] \arrow[dr, "\pi"] & \\ \mathrm{Gr}(2,4) \arrow[rr, dashed, "\text{rational map}"'] & & \mathrm{Hilb}^2_Z \simeq \mathbb{P}^2. \end{tikzcd} \tag{8.2}$$

Think of points in $\mathrm{Gr}(2,4)$ as (projective) lines in $\mathbb{P}(V \otimes S^1)$ where $V = H^0(\mathcal{O}^{\oplus 2})$ and $S^\ell = H^0(\mathbb{P}^1, \mathcal{O}(\ell))$. There are three types of lines in $\mathbb{P}(V \otimes S^1)$.

(i) $\mathbb{P}(v \otimes S^1)$ that generates $v \otimes S^\ell \subseteq V \otimes S^\ell$ of codimension $\ell + 1$.

(ii) $\mathbb{P}(V \otimes f(x,y))$ for some linear function $f \neq 0$. By change of coordinates, we may assume $f(x,y) = x$. In $V \otimes S^\ell$, it generates (a subspace isomorphic to) $V \otimes S^{\ell-1}$ which has codimension 2, where its complement space is generated by $V \otimes y^\ell$.

(iii) line intersects Segre quadric in 2 points. We may assume it is represented by the subspace spanned by $v_1 \otimes x$ and $v_2 \otimes y$ (as 2×2 matrices). In $V \otimes S^\ell$, it generates a subspace spanned by $v_1 \otimes xg(x,y)$, $v_2 \otimes xg(x,y)$, which also has codimension 2, where its complement space is spanned by $v_1 \otimes y^\ell$ and $v_2 \otimes x^\ell$.

Now the non-injective case is (i), i.e., when $H^0(\mathcal{K}(1)) = \langle v \rangle \times S^1 \hookrightarrow \Gamma(V \otimes S^1) \simeq V \otimes S^1$. This happens when $H^0(\mathcal{K}|_{k\langle v \rangle}) \subseteq H^0(V \otimes \mathcal{O}_Z) = V$. That is, $\mathcal{K} = \mathcal{O}_Z \oplus \mathcal{O}_Z(-2)$. The point in $\mathrm{Gr}(2,4)$ has no information about the embeddings $\mathcal{O}_Z(-2) \hookrightarrow \mathcal{O}_Z^2$ and that can be added in after tensoring $\mathcal{K} \hookrightarrow \mathcal{O}_Z^2$ by $\mathcal{O}_Z(2)$ and taking global sections. That will be a subspace

$$\Big\langle \langle v \rangle \otimes S^2 + \langle w \rangle \Big\rangle \subseteq V \otimes S^2,$$

where $\langle w \rangle$ is the image of $H^0(\mathcal{O}_Z(-2) \otimes \mathcal{O}_Z(2))$. Note that $\langle w \rangle$ is a well-defined line in $(V/\langle v \rangle) \otimes S^2$, i.e., a point of $\mathbb{P}^2 \simeq \mathbb{P}(S^2)$. So the special points of $\mathrm{Quot}(\mathcal{O}_Z^2, 2)$ form a $\mathbb{P}(S^2)$-bundle over $\mathbb{P}(V)$. Indeed, it is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^1$; see the calculation below and also Remark 7.3. So the birational morphism g is blowup of a copy of $\mathbb{P}^1 = \mathbb{P}(V)$ [AW93, Corollary 4.11]. Thus the diagram (8.2) is the resolution of indeterminacies of the lower rational map. It is possible to write down this map explicitly in coordinate.

To compute the normal bundle of $\mathbb{P}(V) \simeq \mathbb{P}^1 \hookrightarrow G := \mathrm{Gr}(2, V \otimes S^1)$ explicitly, use the Plüker map

$$G \longrightarrow \mathbb{P}^5 = \mathbb{P}(\wedge^2(V \otimes S^1))$$

and $\wedge^2(V \otimes S^1) = (\wedge^2 V \otimes S^2) \oplus (\mathrm{Sym}^2 V \otimes \wedge^2 S^1)$. On one hand, the Plüker coordinates of $\mathbb{P}(V) \subseteq G$ is $[0 : x_1^2 : x_1x_2 : x_1x_2 : x_2^2 : 0]$, which is not contained in any plane in G, where x_0, x_1 are the homogeneous coordinates of $\mathbb{P}(V)$. On the other hand, a subspace $W \subseteq V \otimes S^1$ has form $v \otimes S^1$ if and only if the projection from $\wedge^2 W$ to the first factor $\wedge^2 V \otimes S^2$ is zero. The space $\wedge^2 w$ for $[w] \in G \subseteq \mathbb{P}^5$ is the fibre of determinant of the universal subbundle which is $\mathcal{O}_{\mathbb{P}^5}(-1)|_G =: \mathcal{O}_G(-1)$. So $\mathbb{P}(V) \subseteq G$ is the zero locus of a canonical bundle homomorphism

$$\mathcal{O}_G(-1) \longrightarrow \mathcal{O}_G \otimes \wedge^2 V \otimes S^2.$$

Alternatively it is the zero locus for a section of $\mathcal{O}_G(1) \otimes \wedge^2 V \otimes S^2$ and the normal bundle is isomorphic to

$$(\mathcal{O}_G(1) \otimes \wedge^2 V \otimes S^2)|_{\mathbb{P}(V)}$$

which is $\mathcal{O}_{\mathbb{P}(V)}(2)^{\oplus 3}$, noting that $\mathbb{P}(V)$ is a conic in $\mathbb{P}^5$.

Since g is a blowup, the canonical divisor of $M_Z(2,2)$ is linearly equivalent to $g^*K_G - 2E \sim 4g^*H|_G - 2E$, where E is the g-exceptional divisor, $H \subseteq \mathbb{P}^5$ is a hyperplane. Let $\ell \subseteq G$ be a line that intersects $\mathbb{P}(V)$ at a point q (with multiplicity one). By the blowup formula [Ful98, Theorem 6.7], $g^*[\ell] = [\widetilde{\ell}] + [L]$ in the Chow group $A_1(M_Z(2,2))$, where $\widetilde{\ell}$ is the proper transform of ℓ and $L \subseteq \mathbb{P}^2 \subseteq M_Z(2,2)$ is a line in the fibre of g over q. Since $\overline{\mathrm{NE}}(M_Z(2,2))$ is generated by the classes of $\widetilde{\ell}$ and L, Kleiman's criterion implies that $4g^*H|_G - 2E$ is ample and hence $M_Z(2,2)$ is Fano; see also [Muk89, Theorem 7].

(2) We calculate the fibres of π. We need to parametrise quotients

$$\mathcal{O}_Z^2 \longrightarrow \mathcal{L}$$

with $\mathcal{L}$ of rank zero whose divisor if of degree 2.

First consider the case when $\mathcal{L}$ is supported at distinct points q_1, q_2. By Theorem 5.1 and Remark 5.3, such quotients are parametrised by $\mathbb{P}^1 \times \mathbb{P}^1$. This agrees with the geometric picture above.

Now consider fibre over $2q$, that is, when $\mathcal{L}$ is supported at one point q. By Remark 5.3, it is enough to parametrise quotients

$$k[t]/\langle t^2 \rangle \oplus k[t]/\langle t^2 \rangle \longrightarrow L$$

where L is a $k[t]/\langle t^2 \rangle$-module of length 2. Such L is either $k \oplus k$ or $k[t]/\langle t^2 \rangle$.

First consider $L = k[t]/\langle t^2 \rangle$. A quotient is determined by the choice of

$$e = a_1 + b_1 t, \quad h = a_2 + b_2 t \in k[t]/\langle t^2 \rangle$$

such that either $a_1 \neq 0$ or $a_2 \neq 0$. Then kernel of the quotient is

$$\{(f,g) \in k[t]/\langle t^2 \rangle \oplus k[t]/\langle t^2 \rangle \mid fe + gh = 0\}.$$

If $a_1 \neq 0$, then e is invertible in $k[t]/\langle t^2 \rangle$, so if (f,g) is in the kernel, then

$$f = -e^{-1}gh,$$

so

$$(f,g) = (-e^{-1}gh, g) = e^{-1}g(-h, e).$$

If $a_2 \neq 0$, we can see

$$(f,g) = h^{-1}f(h,-e).$$

So in any case the kernel is $\langle(-h,e)\rangle$, i.e., the submodule generated by $(-h,e)$.

When e is invertible

$$\langle(-h,e)\rangle = \langle(-e^{-1}h,1)\rangle$$

where

$$-e^{-1}h =: m_1 + m_2 t$$

is uniquely determined. Thus such kernels are parametrised by the points of $\mathbb{A}^1 \times \mathbb{A}^1$ with coordinates m_1, m_2. Similarly, when h is invertible, the corresponding kernel is $\langle(1,-h^{-1}e)\rangle$ where

$$-h^{-1}e =: l_1 + l_2 t.$$

Such kernels are parametrised by $\mathbb{A}^1 \times \mathbb{A}^1$ with coordinates l_1, l_2.

When both e, h are invertible,

$$\langle(m_1 + m_2 t, 1)\rangle = \langle(1, l_1 + l_2 t)\rangle$$

so

$$(m_1 + m_2 t)(l_1 + l_2 t) = 1 \text{ in } k[t]/\langle t^2\rangle.$$

This means the two $\mathbb{A}^1 \times \mathbb{A}^1$ are glued along $\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1$ by the isomorphism

$$\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1$$
$$(m_1, m_2) \longmapsto \Big(\frac{1}{m_1}, -\frac{m_2}{m_1^2}\Big).$$

The union of the two copies $\mathbb{A}^1 \times \mathbb{A}^1$ under the above isomorphism is a variety mapping to $\mathbb{P}^1$ and with fibres $\mathbb{A}^1$. This corresponds to the morphism $G_2 \to \mathbb{P}^1$ determined in the geometric discussion above.

Consider

$$V(xz + y^2) \subseteq \mathbb{P}^3$$

where we consider the coordinates x, y, z, u on $\mathbb{P}^3$. The points with $x \neq 0$ are

$$(1 : y : -y^2, u)$$

which are parametrised $\mathbb{A}^1 \times \mathbb{A}^1$ with coordinates y, u. And the points with $z \neq 0$ are

$$(-y^2 : y : 1 : u)$$

which are parametrised $\mathbb{A}^1 \times \mathbb{A}^1$ with coordinates $-y, u$. The intersection of the two sets of points is the set of points with $x \neq 0$, $z \neq 0$, $y \neq 0$ which consists of points

$$(1 : y : -y^2 : u) = \Big(-\frac{1}{y^2} : -\frac{1}{y} : 1 : -\frac{u}{y^2}\Big)$$

corresponding to an isomorphism from $\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1$ given by

$$(y,u) \longmapsto \Big(\frac{1}{y} : -\frac{u}{y^2}\Big)$$

with respect to the coordinates above. So $V(xz + y^2)$ consists of the union of two copies of $\mathbb{A}^1 \times \mathbb{A}^1$ glued by the above isomorphism together with the singular point $(0 : 0 : 0 : 1)$.

Now consider all the quotients

$$k[t]/\langle t^2\rangle \oplus k[t]/\langle t^2\rangle \longrightarrow L$$

with $L = k \oplus k$. Such a quotient is given by sending $(1,0),(0,1)$ to $(a_1,b_1),(a_2,b_2) \in k^2$ such that

$$\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \neq 0.$$

The quotient send t to zero, so it factors through a surjection $k \oplus k \to L$ which should be an isomorphism. So the kernel is $\langle t \rangle \oplus \langle t \rangle$ and this is unique.

Finally by Lemma 3.18, all the quotients for $L = k[t]/\langle t^2 \rangle$ and $L = k \oplus k$ are parametrised by a normal surface. And the above discussion shows that this normal surface is isomorphic to $V(xz + y^2)$ when we remove one point from each side. But since both are normal surfaces, they are isomorphic.

This completes the proof of Theorem 1.4.

9. Degree three stable pairs on curves

9.1. **Algebraic treatment.** Assume $Z = \mathbb{P}^1$. We consider sheaf stable pairs $\mathcal{O}_Z^2 \to \mathcal{E}$ of rank 2 and degree 3. Since $\mathcal{E}$ is nef, either $\mathcal{E} \simeq \mathcal{O}_Z(1) \oplus \mathcal{O}_Z(2)$, or $\mathcal{E} \simeq \mathcal{O}_Z \oplus \mathcal{O}_Z(3)$.

We first focus on the former, which is the generic case. Let $\mathcal{M}$ be the moduli space of stable pairs with $\mathcal{E} = \mathcal{O}_Z(1) \oplus \mathcal{O}_Z(2)$.

Denote $V \coloneqq H^0(\mathcal{O}_Z(1) \oplus \mathcal{O}_Z(1))$, $H \coloneqq H^0(\det \mathcal{E}) \simeq H^0(\mathcal{O}_{\mathbb{P}^1}(3))$, which are both vector spaces of dimension 4. We claim that there is an injective morphism $\mathcal{M} \to \mathbb{P}(H) \times \mathbb{P}(V)$.

As in Remark 7.3, a stable map $s\colon \mathcal{O}_Z^2 \to \mathcal{E} = \mathcal{O}_Z(1) \oplus \mathcal{O}_Z(2)$ is given by

$$M = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \in \overline{\mathcal{M}}$$

where $\alpha_i \in H^0(\mathcal{O}_{\mathbb{P}^1}(2))$ and $\beta_i \in H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ such that its determinant is a non-zero vector in $H = H^0(\det \mathcal{E})$. Note that

$$\mathrm{Aut}(\mathcal{E}) = (\mathrm{Aut}(\mathcal{G}_1) \times \mathrm{Aut}(\mathcal{G}_2)) \ltimes \mathrm{Hom}(\mathcal{G}_1, \mathcal{G}_2) = (k^* \times k^*) \ltimes \mathrm{Hom}(\mathcal{O}_Z(1), \mathcal{O}_Z(2))$$

acts on $\overline{\mathcal{M}}$ by matrix multiplication

$$\begin{pmatrix} \gamma_1 & \varphi \\ 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = \begin{pmatrix} \gamma_1\alpha_1 + \varphi\beta_1 & \gamma_2\alpha_2 + \varphi\beta_2 \\ \gamma_2\beta_1 & \gamma_2\beta_2 \end{pmatrix}$$

where $\gamma_i \in k^*$ and $\varphi \in \mathrm{Hom}(\mathcal{O}_Z(1), \mathcal{O}_Z(2))$. We still see that the determinant $\det M = \alpha_1\beta_2 - \alpha_2\beta_1$ is invariant up to scaling of $\gamma_1\gamma_2$. View $[\beta_1 : \beta_2]$ as an element of $V \simeq \mathbb{A}^4$, which is invariant up to scaling of γ_2. So there is an induced morphism

$$\mathcal{M} \longrightarrow \mathbb{P}(H) \times \mathbb{P}(V), \quad M \longmapsto (\det M, [\beta_1 : \beta_2]).$$

Assume that $\pi(M) = \pi(M')$ for some $M, M' \in \overline{\mathcal{M}}$ with

$$M = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} \alpha_1' & \alpha_2' \\ \beta_1' & \beta_2' \end{pmatrix}.$$

After multiplying by matrices from $\mathrm{Aut}(\mathcal{E})$ we may assume that $[\beta_1 : \beta_2] = [\beta_1' : \beta_2']$ and $\det M = \det M'$. Then we may take $\gamma_1 = \gamma_2 = 1$. In the case when β_1 and β_2 are proportional, let $\varphi \in \mathrm{Hom}(\mathcal{O}_Z(1), \mathcal{O}_Z(2))$ be any homomorphism satisfying $\varphi\beta_1 = \alpha_1' - \alpha_1$; in the case when β_1 and β_2 are not proportional, let $\varphi = 0$ be given by $\varphi\beta_j = \alpha_j' - \alpha_j$ for $j = 1, 2$. Then

$$\begin{pmatrix} 1 & \varphi \\ 0 & 1 \end{pmatrix} M = M'$$

and hence π is injective.

Now we turn to the general setting. Note that $M_Z(2,3) \simeq \mathrm{Quot}(\mathcal{O}_Z^2, 3)$ and the latter parametrising embeddings

$$\mathcal{K} \longhookrightarrow \mathcal{O}_Z \oplus \mathcal{O}_Z$$

where either $\mathcal{K} = \mathcal{O}_Z(-1) \oplus \mathcal{O}_Z(-2)$ or $\mathcal{K} = \mathcal{O}_Z \oplus \mathcal{O}_Z(-3)$. In both cases,

$$H^0(\mathcal{K}(2)) \longhookrightarrow H^0(\mathcal{O}_Z(2) \oplus \mathcal{O}_Z(2)) = V \otimes S^2$$

is a 3-dimensional subspace in 6-dimensional vector space, where $V = H^0(\mathcal{O}_Z \oplus \mathcal{O}_Z)$ and $S^\ell = H^0(\mathcal{O}_Z(\ell))$. But $\mathcal{K}$ is globally generated only in the first (generic) case.

Assume that $\oplus_{\ell \geq 0} S^\ell \simeq k[x,y]$ as graded modules. For a 3-dimensional subspace $W \hookrightarrow V \otimes S^2$ there is a linear map

$$W \oplus W \longrightarrow V \otimes S^3, \quad (w_1, w_2) \longmapsto w_1 x + w_2 y.$$

The image is expected to be $H^0(\mathcal{K}(3))$ and hence of codimension 3 in the 8-dimensional vector space $V \otimes S^3$. On the Grassmannian $G = \mathrm{Gr}(3,6)$ we have a universal subbundle $\mathcal{S} \hookrightarrow \mathcal{O}_G \otimes V \otimes S^2$ of rank 3, with a morphism $\sigma \colon \mathcal{S} \oplus \mathcal{S} \to \mathcal{O}_G \otimes V \otimes S^3$ for which we like the image of σ to have rank 5. By general theory in [Ful98, Chapter 14], the *expected codimension* of the *degeneracy locus*

$$D_5(\sigma) = \{x \in G \mid \operatorname{rk} \sigma(x) \leq 5\}$$

in G is 3. By [Băn91, § 4.1], the degeneracy locus $D_5(\sigma)$ has singularities along $D_4(\sigma)$. This corresponds to the case when $\mathcal{K} = \mathcal{O}_Z \oplus \mathcal{O}_Z(-3)$. In this case $H^0(\mathcal{K}(2)) = v \otimes S^2$ for some vector $v \in V$, which spans $H^0(\mathcal{K}) \hookrightarrow H^0(\mathcal{O}_Z \otimes V)$. The image of $W \oplus W \to V \otimes S^3$ for $W = v \otimes S^2$ is $v \otimes S^3$ which has dimension 4. To get all of 5-dimensional $H^0(\mathcal{K}(3)) \hookrightarrow H^0(V \otimes \mathcal{O}(3)) = V \otimes S^3$ we need to specify an additional line in

$$V \otimes S^3 / v \otimes S^3 \subseteq (V/\langle v \rangle) \otimes S^3.$$

This line is $H^0(\mathcal{O}_Z(-3) \otimes \mathcal{O}_Z(3)) \hookrightarrow H^0(\mathcal{K}(3))$, so if $\mathrm{Quot}(\mathcal{O}_Z^2, 3) = \mathrm{Quot}_{1,2} \cup \mathrm{Quot}_{0,3}$ then the map $\pi \colon \mathrm{Quot}(\mathcal{O}_Z^2, 3) \to G$ is injective on $\mathrm{Quot}_{1,2}$ and collapses $\mathrm{Quot}_{0,3} \simeq \mathbb{P}^2 \times \mathbb{P}^1$ onto $\mathbb{P}^1 = \mathbb{P}(V)$ which is the singular locus of $\pi(\mathrm{Quot}(\mathcal{O}_Z^2, 3)) \subseteq \mathrm{Gr}(3,6)$.

If we consider $W \subseteq V \otimes S^2$ and want $\dim xW + yW = 5$ for $xW + yW \subseteq V \otimes S^3$, then this means that there exist $w_1, w_2 \in W$ such that $w_1 x = w_2 y$. As $W \subseteq V \otimes S^2$, there exists $v \in V \otimes S^1$ such that $w_1 = vy$ and $w_2 = vx$. This $v \in V \otimes S^1$ is unique up to rescaling since $\dim xW \cap yW = 1$ (as subspaces in $V \otimes S^3$). Hence, on an open subset of $\mathrm{Quot}(\mathcal{O}_Z^2, 3)$, we take $v \in V \otimes S^1$ then construct $\langle vx, vy \rangle$ in $V \otimes S^2$, which are always independent vectors, and complete this to a 3-dimensional subspace W by choosing a line

$$\langle w \rangle \subseteq V \otimes S^2 / \langle vx, vy \rangle.$$

This is an open subset in a $\mathbb{P}^3$-bundle over $\mathbb{P}^3$: although each $\langle v \rangle$, $\langle w \rangle$ corresponds a point of $\mathbb{P}^3$, occasionally we will end up with $\dim xW \cap yW \geq 2$ in $V \otimes S^3$.

Suppose $\dim xW \cap yW \geq 2$. Then there exist $v_1, v_2 \in V \otimes S^1$ such that $v_1 x, v_1 y, v_2 x, v_2 y \in W$. But W is 3-dimensional, so $\mathrm{Span}\langle v_1 x, v_1 y \rangle \cap \mathrm{Span}\langle v_2 x, v_2 y \rangle \neq 0$. This means that there exists $u \in V$ such that $uS^2 \subseteq W$ and thus $uS^2 = W$. Such subspaces W are parametrised by locus $\langle u \rangle \in \mathbb{P}(V) = \mathbb{P}^1$ and their π-preimage in $\mathrm{Quot}(\mathcal{O}_Z^2, 3)$ is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^1$. Since $\pi^{-1}(\mathbb{P}(V))$ is not a divisor on $\mathrm{Quot}(\mathcal{O}_Z^2, 3)$, this $\mathbb{P}(V) = \mathbb{P}^1 \subseteq G = \mathrm{Gr}(3,6)$ is singular in the π-image of $\mathrm{Quot}(\mathcal{O}_Z^2, 3)$.

9.2. **Description the fibre of Hilbert-Chow morphism via equations.** Let $Z = \mathbb{P}^1$. Denote by F be the fibre of $\pi\colon M_Z(2,3) \to \mathrm{Hilb}^3_Z$ over $3q$ for some closed point $q \in Z$. We may assume that q is the origin.

Let t be the local coordinate of the origin in $\mathbb{A}^1 \subseteq \mathbb{P}^1$. Then all possible embeddings $\mathcal{K} \hookrightarrow \mathcal{O}_Z^2$ with $\det \mathcal{K}^\vee = \mathcal{O}_Z(3q)$ are in bijective correspondence with 3-dimensional subspaces in

$$(k[t]/\langle t^3 \rangle)^{\oplus 2} := M$$

which are t-invariant; see Section 6.4. There are two types of such submodules $N \subseteq M$ with $\dim N = 3$:

(1) $N = k[t]/\langle t^3 \rangle$ as a $k[t]$ module. This is an open subset of F (of dimension 3) corresponds to the case when N projects onto a line in M/tM.
(2) $N \simeq k[t]/\langle t^2 \rangle \oplus k[t]/\langle t \rangle$ corresponds to the case when $t^2 M \subsetneq N \subsetneq M$. This is the singular set, isomorphic to $\mathbb{P}^1$, which is the set of lines in tM/t^2M.

Local equations can be written in local coordinates on the Grassmannian $\mathrm{Gr}(3, M)$. If $W \subseteq M$ is a 3-dimensional subspace and one chooses a splitting $M \simeq W \oplus M/W$, then these subspaces in the neighbourhood of W correspond to graphs of linear maps $W \to M/W$. For example, W itself corresponds to zero map.

Write $M = \mathrm{Span}(u, ut, ut^2, v, vt, vt^2)$ and choose $W = \mathrm{Span}(ut, ut^2, vt^2)$. This corresponds to the Case (2) above and they are all isomorphic. Then $M/W \simeq \mathrm{Span}(u, v, vt)$. If we order the basis as $(ut, ut^2, vt^2, u, v, vt)$ then subspaces near W are spanned by rows of

$$\begin{pmatrix} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & i \end{pmatrix}$$

where the right half of the matrix, i.e.,

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

is the linear map $W \to M/W$. By assumption, this subspace is t-invariant. Denote the j-th row by R_j, $1 \leq j \leq 4$. Since $t(ut + au + bv + cvt) = ut^2 + aut + bvt + cvt^2$, we have $tR_1 = (a, 1, c, 0, 0, b)$ and hence $tR_1 = aR_1 + R_2 + cR_3$. This gives equations

$$\begin{cases} -a^2 - d - gc = 0, \\ -ab - e - hc = 0, \\ b - ac - f - ci = 0. \end{cases}$$

Repeat this process, we have the full set of equations on 9 variables:

$$\begin{cases} -a^2-d-gc=0, & (9.1)\\ -ab-e-ch=0, \\ b-ac-f-ci=0, & (9.2)\\ -ad-fg=0, \\ -bd-fh=0, \\ e-cd-fi=0, \\ -ag-gi=0, & (9.3)\\ -bg-hi=0, & (9.4)\\ h-cg-i^2=0. & (9.5)\end{cases}$$

In Eq. (9.3) we have $g(a+i)=0$. If $g=0$ then Eq. (9.4) gives $hi=0$ and from Eq. (9.5) we obtain $h=i^2$. So both h,i are also zero. Remaining equations then give:

$$\begin{cases} -a^2-d=0, & (9.6)\\ -ab-e=0, \\ b-ac-f=0, \\ -ad=0, & (9.7)\\ -bd=0, \\ e-cd=0.\end{cases}$$

From Eqs. (9.6) and (9.7) we have $a=d=0$. Then $e=0$, $b=f$ and c is a free variable. This gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 & b & c\\ 0 & 1 & 0 & 0 & 0 & b\\ 0 & 0 & 1 & 0 & 0 & 0\end{pmatrix} \tag{9.8}$$

which is 2-dimensional.

If $g\neq 0$ in Eq. (9.3), then $i=-a$ from Eq. (9.3), $b=f$ from Eq. (9.2) and $h=-d$ from Eqs. (9.1) and (9.5). So we are left with variables a,b,c,d,e,g and equations

$$\begin{cases} -a^2-d-gc=0,\\ -ab-e+dc=0,\\ -ad-bg=0.\end{cases}$$

Note that d,e can be eliminated from the first two equations while the last one gives

$$-a(-a^2-gc)-bg=0 \quad\Rightarrow\quad a^3=g(b-ac)$$

in $\mathbb{A}^4$ with coordinates a,b,c,g. Finally collect together

$$\begin{cases} b=f, h=-d, i=-a,\\ d=-a^2-gc,\\ e=dc-ab,\end{cases}$$

which gives matrix

$$\begin{pmatrix} 1 & 0 & 0 & a & b & c\\ 0 & 1 & 0 & -a^2-gc & dc-ab & b\\ 0 & 0 & 1 & g & -d & -a\end{pmatrix}$$

with relation $a^3 + agc = bg$. In particular, we can take $a = g = d = 0$ and that recovers Eq. (9.8) which we considered above.

Therefore, locally around a single point, the equation of F can be written as $a^3 = g(b - ac)$ in a, b, c, g coordinates which is a cA_2-singularity. In particular, F has canonical singularities.

The calculation above can be generalised to higher dimension.

Proposition 9.1. *Let Q_n be the variety of n-dimensional submodules in $M_n = (k[t]/\langle t^n \rangle)^{\oplus 2}$.*

(1) $\dim Q_n = n$;
(2) the singular locus $Q_n^{\mathrm{Sing}} \simeq Q_{n-2}$ when $n \geq 2$;
(3) the smooth locus $Q_n^\circ \coloneqq Q_n \setminus Q_n^{\mathrm{Sing}}$ is an $\mathbb{A}^{n-1}$-bundle over $\mathbb{P}^1$;
(4) if $x \in Q_{n-2}^\circ \subseteq Q_{n-2} \subseteq Q_n$, then in some local coordinates around x, Q_n is isomorphic to $\mathbb{A}^{n-2} \times$ (singular surface $xy = z^n$), which is a higher dimensional cA_{n-1}-singularity.

Now we can give a proof of Theorem 1.5.

Proof of Theorem 1.5. By Theorem 5.1 we can assume $Z = \mathbb{P}^1$. And by Remark 5.3 it is enough to consider the fibre F of π over $3q$.

As in the previous section, it is enough to parametrise all quotients

$$k[t]/\langle t^3 \rangle \oplus k[t]/\langle t^3 \rangle \longrightarrow L$$

where L is a $k[t]/\langle t^3 \rangle$-module with length 3. The only possibilities for L are

$$\begin{cases} k[t]/\langle t^3 \rangle \\ k[t]/\langle t^2 \rangle \oplus k. \end{cases}$$

It is enough to consider the case

$$L = k[t]/\langle t^3 \rangle$$

as this is the generic case by Remark 5.4. This corresponds to the case when the divisors of s_1, s_2 have no common component, that is, $3 = n = m$ as in Theorem 6.1. Each quotient is determined by

$$\begin{aligned} (1,0) &\longmapsto e \\ (0,1) &\longmapsto h \end{aligned} \qquad (e \text{ or } h \text{ is invertible})$$

and then the kernel of the quotient is the submodule $\langle (-h, e) \rangle$.

The quotients with e invertible are parametrised by $\mathbb{A}^3$ via

$$\begin{aligned} \mathbb{A}^3 &\longrightarrow U \subseteq F \\ (m_1, m_2, m_3) &\longmapsto \langle (m_1 + m_2 t + m_3 t^2, 1) \rangle. \end{aligned}$$

Similarly, the quotients with h invertible are parametrised by $\mathbb{A}^3$ via

$$\begin{aligned} \mathbb{A}^3 &\longrightarrow V \subseteq F \\ (l_1, l_2, l_3) &\longmapsto \langle (1, l_1 + l_2 t + l_3 t^2) \rangle. \end{aligned}$$

When both e, h are invertible we have

$$\begin{aligned} &\langle (m_1 + m_2 t + m_3 t^2, 1) \rangle = \langle (1, l_1 + l_2 t + l_3 t^2) \rangle \\ = &\left\langle \left(1, \frac{1}{m_1} - \frac{m_2}{m_1^2} t + \frac{m_2^2 - m_1 m_3}{m_1^3} t^2 \right) \right\rangle \end{aligned}$$

inducing a birational map

$$U \dashrightarrow V$$
$$(m_1, m_2, m_3) \longmapsto \Big(\frac{1}{m_1}, \frac{m_2}{m_1^2}, \frac{m_2^2 - m_1m_3}{m_1^3}\Big)$$

which is an isomorphism on points with non-zero first coordinates.

The above birational map induces a birational map

$$\varphi\colon X = \mathbb{P}^3 \dashrightarrow \mathbb{P}^3 = Y$$
$$(m_1 : m_2 : m_3 : m_4) \longmapsto (m_1^2m_4 : -m_1m_2m_4 : m_2^2m_4 - m_1m_3m_4 : m_1^3) = (l_1 : l_2 : l_3 : l_4)$$

where we identify U with points with $m_4 \neq 0$ and V with $l_4 \neq 0$.

Define the hyperplanes

$$A_i : \quad m_i = 0 \qquad \text{and} \qquad H_i : \quad l_i = 0.$$

By construction, φ gives an isomorphism

$$X \setminus (A_1 \cup A_4) \longrightarrow Y \setminus (H_1 \cup H_4)$$

and one has $\varphi^2 = \mathrm{id}$. Also we can see that

$$\varphi \text{ contracts } A_1, A_4 \quad \text{and} \quad \varphi^{-1} \text{ contracts } H_1, H_4$$

and no other divisors are contracted. Pick the canonical divisor K_Y such that

$$K_Y + H_1 + H_2 + H_3 + H_4 = 0 \quad (= 0 \text{ not just} \sim 0).$$

We want to compute

$$\varphi^*(K_Y + \sum_1^4 H_i)$$

in terms of K_X and A_i. First note that

$$\begin{aligned} \varphi^*H_1 &= 2A_1 + A_4, \\ \varphi^*H_2 &= A_1 + A_2 + A_4, \\ \varphi^*H_3 &= A_4 + G, \\ \varphi^*H_4 &= 3A_1, \end{aligned}$$

where $G \subseteq X$ is given by $m_2^2 - m_1m_3 = 0$. Moreover,

$$\varphi^*K_Y = K_X + aA_1 + bA_4$$

where K_X is determined as a Weil divisor by K_Y, φ, and $a, b \in \mathbb{Z}$. So

$$\begin{aligned} 0 = \varphi^*(K_Y + \sum_1^4 H_i) &= K_X + aA_1 + bA_4 + 2A_1 + A_4 + A_1 + A_2 + A_4 + A_4 + G + 3A_1 \\ &= K_X + (a+6)A_1 + A_2 + (b+3)A_4 + G. \end{aligned}$$

Since $\deg G = 2$ and $\deg K_X = -4$,

$$-4 + (a+6) + 1 + (b+3) + 2 = 0$$

hence

$$(a+6) + (b+3) = 1.$$

On the other hand, $(Y, \sum_1^4 H_i)$ has lc singularities, so

$$a + 6 \leq 1 \quad \text{and} \quad b + 3 \leq 1.$$

Thus one of $a+6$ and $b+3$ is zero and the other is one.

We claim that $a+6=0$. Assume not. Then

$$(X, A_1 + A_2 + G)$$

is lc. But then

$$(A_1, A_2|_{A_1} + G|_{A_1})$$

is lc [KM98, Theorem 5.50] which is not the case, so we have $a+6=0$ and $b+3=1$ as claimed (note $G|_{A_1}$ is a double line). Summarising, we have

$$\varphi^*(K_Y + \sum_1^4 H_i) = K_X + A_2 + A_4 + G. \tag{9.9}$$

Since the construction are symmetric for X, Y, we also have

$$(\varphi^{-1})^*(K_X + \sum_1^4 A_i) = K_Y + H_2 + H_4 + P$$

for some appropriate choices of K_X, K_Y where P is a hypersurface of degree two. Let $p\colon W' \to X$ and $q\colon W' \to$ be a common resolution of X and Y. Then we have

$$p^*(K_X + \frac{1}{2}A_1 + A_2 + A_3 + \frac{1}{2}A_4 + \frac{1}{2}G) = q^*(K_Y + \frac{1}{2}H_1 + H_2 + H_3 + \frac{1}{2}H_4 + \frac{1}{2}P).$$

Therefore, the log discrepancy $a(A_1, Y, \frac{1}{2}H_1 + H_2 + H_3 + \frac{1}{2}H_4 + \frac{1}{2}P) = \frac{1}{2}$ and hence we can extract A_1 from Y by an extremal birational contraction $\rho\colon W \to Y$ where W is of Fano type [BZ16, Lemma 4.6]. Then by (9.9) we have $a(A_1, Y, \sum_1^4 H_i) = 1$ and thus

$$\rho^*(K_Y + \sum_1^4 H_i) = K_W + \sum_1^4 H_i^\sim$$

where $\sim$ denotes birational transform. Moreover,

$$\begin{aligned} \rho^* H_1 &= H_1^\sim + 2A_1^\sim, \\ \rho^* H_2 &= H_2^\sim + A_1^\sim, \\ \rho^* H_3 &= H_3^\sim, \\ \rho^* H_4 &= H_4^\sim + 3A_1^\sim. \end{aligned} \qquad (A_2^\sim = H_2^\sim \text{ and } G^\sim = H_3^\sim)$$

Claim 9.2. The cone of effective divisors of W is generated by $H_4^\sim$ and $A_1^\sim$.

Proof of the Claim 9.2. Pick a Weil divisor $D \geq 0$ on W. We can write

$$\rho^*\rho_* D = D + \alpha A_1^\sim$$

for some $\alpha \in \mathbb{Z}$. Letting $\beta = \deg \rho_* D$ and noting $\rho_* D \sim \beta H_4$, we get

$$\beta H_4^\sim + 3\beta A_1^\sim \sim D + \alpha A_1^\sim.$$

Then the left side of

$$\begin{aligned} (\rho^{-1}\varphi)^*(\beta H_4^\sim + (3\beta - \alpha)A_1^\sim) &= \varphi^* H_4 - (\rho^{-1}\varphi)^* \alpha A_1^\sim \\ &\leq 3A_1 - \alpha A_1 \end{aligned}$$

is pseudo-effective on X which shows

$$\alpha \leq 3.$$

Thus $3\beta - \alpha \geq 0$ implying the claim. Note that if $\beta = 0$, then $D = -\alpha A_1^\sim$ with $-\alpha \geq 0$. $\square$

Claim 9.3. The Kodaira dimension $\kappa(H_4^\sim) = 0$.

Proof of the Claim 9.3. Assume not. Then $H_4^\sim$ is movable and there exist $e \in \mathbb{N}$ and $M \geq 0$ such that $eH_4^\sim \sim M$ where $H_4^\sim \not\subseteq \operatorname{Supp} M$. Then from

$$\rho^*\rho_* H_4^\sim = \rho^* H_4 = eH_4^\sim + 3A_1^\sim$$

we get

$$\rho^*\rho_* M = M + 3eA_1^\sim.$$

So

$$\varphi^*(\rho_* M) = R + 3eA_1 \quad \text{for some } R \geq 0.$$

Moreover, $e = \deg \rho_* M$, so $\rho_* M$ is a hypersurface defined by a polynomial Π of degree e. Thus $\varphi^*(\rho_* M)$ is given by the polynomial

$$\Pi(m_1^2 m_4, -m_1 m_2 m_4, m_2^2 m_4 - m_1 m_3 m_4, m_1^3)$$

of degree $3e$. But then $R = 0$ and $\varphi^*(\rho_* M) = 3eA_1$.

Since H_1, H_4 are the only exceptional divisors of φ^{-1} and since $H_4 \not\subseteq \operatorname{Supp} \rho_* M$, we have $\rho_* M = eH_1$. Thus $\varphi^*\rho_* M = 2eA_1 + eA_4$, a contradiction. So we have proved the claim. □

Now run the $H_4^\sim$-MMP and let T be the resulting model. We can run this MMP as W is of Fano type. Since W is $\mathbb{Q}$-factorial of Picard number two [BZ16, Lemma 4.6], T is $\mathbb{Q}$-factorial of Picard number one as $H_4^\sim$ is contracted, given $\kappa(H_4^\sim) = 0$. In particular, T is a (klt) Fano variety.

By construction, W contains *big open subsets* of both U and V, i.e., open subset with complement of codimension ≥ 2:

- for U we use the fact that $U \dashrightarrow W$ contracts no divisor; recall that $U \setminus A_1 \xrightarrow{\sim} V \setminus H_1$ and $A_1^\sim \subseteq W$;
- for V it is clear as $W \to Y$ is a birational morphism.

In fact the complement in W of the union the two big open sets contains only one prime divisor, $H_4^\sim$. Now $W \dashrightarrow T$ is an isomorphism outside $H_4^\sim$ and $H_4^\sim$ does not intersect the mentioned big open sets. Thus T also contains big open subsets of U and V. Moreover, the complement in T of the union of two big open sets, is of codimension ≥ 2.

Therefore T is isomorphic to the fibre F of π because both T, F are normal varieties with isomorphic open subsets with codimension ≥ 2 complements. Now since F has klt singularities and K_F is Cartier by Theorem 1.2 (3), it has canonical singularities, along a copy of $\mathbb{P}^1$ as in Section 9.1. □

References

[AW93] Marco Andreatta and Jarosław A Wiśniewski. "A Note on Nonvanishing and Applications". In: *Duke Mathematical Journal* **72** (3) (1993), pp. 739–755 (cit. on p. 33).

[Băn91] Constantin Bănică. "Smooth Reflexive Sheaves". In: *Proceedings of the Colloquium on Complex Analysis and the Sixth Romanian-Finnish Seminar*. Vol. 36. 1991, pp. 571–593 (cit. on p. 37).

[Bir22] Caucher Birkar. *Moduli of Algebraic Varieties*. 2022. arXiv: `2211.11237 [math.AG]`. preprint (cit. on pp. 3, 14).

[BZ16] Caucher Birkar and De-Qi Zhang. "Effectivity of Iitaka Fibrations and Pluricanonical Systems of Polarized Pairs". In: *Publications mathématiques de l'IHÉS* **123** (1) (2016), pp. 283–331 (cit. on pp. 42, 43).

[Eis95] David Eisenbud. *Commutative Algebra*. Vol. 150. Graduate Texts in Mathematics. New York: Springer, 1995 (cit. on p. 22).

[Ful98] William Fulton. *Intersection Theory*. 2nd ed. New York: Springer, 1998 (cit. on pp. 20, 34, 37).

[Gro61] A. Grothendieck. "Techniques de construction et théorèmes d'existence en géometrie algébrique IV: les schémas de Hilbert." In: *Séminaire Bourbaki* **221, Astérisque tome 6** (1961), pp. 249–276 (cit. on pp. 1, 2).

[Har77] Robin Hartshorne. *Algebraic Geometry*. Vol. 52. Graduate Texts in Mathematics. New York: Springer-Verlag, 1977 (cit. on pp. 10, 13, 14).

[Har80] Robin Hartshorne. "Stable Reflexive Sheaves". In: *Mathematische Annalen* **254** (2) (1980), pp. 121–176 (cit. on p. 9).

[HL10] Daniel Huybrechts and Manfred Lehn. *The Geometry of Moduli Spaces of Sheaves*. 2nd ed. Cambridge University Press, 2010 (cit. on pp. 1, 2, 12).

[Kle71] Steven L. Kleiman. *Théorèmes de finitude pour le foncteur de Picard*. Vol. Exp XIII. Springer, 1971, pp. 616–666 (cit. on p. 1).

[KM98] János Kollár and Shigefumi Mori. *Birational Geometry of Algebraic Varieties*. 1st ed. Cambridge Tracts in Math 134. Cambridge University Press, 1998 (cit. on p. 42).

[Laz04] Robert Lazarsfeld. *Positivity in Algebraic Geometry II*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2004 (cit. on p. 10).

[Le 93a] Joseph Le Potier. *Faisceaux semi-stables et systémes cohérents*. Vol. London Math. Soc. Lecture Note Ser, vol 288. Press, Cambridge, 1993 (cit. on p. 2).

[Le 93b] Joseph Le Potier. *Systèmes cohérents et structures de niveau*. Astérisque 214. Société mathématique de France, 1993 (cit. on p. 12).

[Mat87] Hideyuki Matsumura. *Commutative Ring Theory*. Trans. by Miles Reid. 1st ed. Cambridge University Press, 1987 (cit. on p. 22).

[Muk89] Shigeru Mukai. "Biregular Classification of Fano 3-Folds and Fano Manifolds of Coindex 3". In: *Proceedings of the National Academy of Sciences* **86** (9) (1989), pp. 3000–3002 (cit. on p. 34).

[Mum66] D. Mumford. *Lectures on curves on an algebraic surface*. Princeton University Press, 1966 (cit. on p. 1).

[PT09] R. Pandharipande and R. P. Thomas. "Curve Counting via Stable Pairs in the Derived Category". In: *Inventiones mathematicae* **178** (2) (2009), pp. 407–447 (cit. on p. 3).

[She16] Artan Sheshmani. "Higher Rank Stable Pairs and Virtual Localization". In: *Communications in Analysis and Geometry* **24** (1) (2016), pp. 139–193 (cit. on pp. 7, 8, 12).

Yau Mathematical Sciences Center, Jingzhai, Tsinghua University, Haidian District, Beijing, China, 100084
Email address: birkar@tsinghua.edu.cn

Yau Mathematical Sciences Center, Jingzhai, Tsinghua University, Haidian District, Beijing, China, 100084
Email address: jia_jia@u.nus.edu,mathjiajia@tsinghua.edu.cn

Beijing Institute of Mathematical Sciences and Applications, No. 544, Hefangkou Village, Huaibei Town, Huairou District, Beijing, China, 101408
Email address: artan@bimsa.cn

Massachusetts Institute of Technology (MIT), IAiFi Institute, 182 Memorial Drive, Cambridge, MA 02139, USA
Email address: artan@mit.edu