AUTOMORPHISM GROUPS OF COMPACT COMPLEX SURFACES: T-JORDAN PROPERTY, TITS ALTERNATIVE AND SOLVABILITY

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Abstract. Let X be a (smooth) compact complex surface. We show that every torsion subgroup of the biholomorphic automorphism group $\mathrm{Aut}(X)$ is virtually nilpotent. Moreover, we study the Tits alternative of $\mathrm{Aut}(X)$ and virtual derived length of virtually solvable subgroups of $\mathrm{Aut}(X)$.

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1. Introduction

1.1. **T-Jordan Property.** The below theorem is our starting point.

Theorem 1.1 ([20]). *Let G be a connected Lie group. Then there is a constant $J = J(G)$ such that every torsion subgroup of G contains a (normal) abelian subgroup of index $\le J$.*

Inspired by the *Jordan property* introduced by Popov [28, Definition 2.1], we propose the following generalisation.

Definition 1.2. A group G is called *T-Jordan* (alternatively, we say that G has the *T-Jordan* property) if there is a constant $J(G)$ such that every torsion subgroup H of G has an abelian subgroup H_1 with the index $[H : H_1] \le J(G)$.

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For a compact complex manifold X, we denote by $\mathrm{Aut}(X)$ the group of biholomorphic automorphisms. As in [28], we propose the following conjecture.

Conjecture 1.3. *Let X be a compact complex manifold. Then $\mathrm{Aut}(X)$ is T-Jordan.*

Using the equivariant Kähler model, Meng and the author gave an affirmative answer to Conjecture 1.3 for compact complex manifolds in Fujiki's class $\mathcal{C}$. A compact complex space is in *Fujiki's class $\mathcal{C}$* if it is the meromorphic image of a compact Kähler manifold (cf. [12, Definition 1.1]).

However, since we only use Theorem 1.4 below in dimension two and the classical result of Fujiki [12, Theorem 4.8] and Lieberman [21, Proposition 2.2] is enough for its proof.

Theorem 1.4 (cf. [17, Corollary 1.5])**.** *Let X be a compact complex space which is in Fujiki's class $\mathcal{C}$. Then $\mathrm{Aut}(X)$ is T-Jordan.*

In this paper, we are focusing on smooth compact complex surfaces, especially the non-Kähler (= non-Fujiki's class $\mathcal{C}$) surfaces. We use the following notation.

- Let Ξ be the set of smooth compact complex surface X in class VII with the algebraic dimension $a(X) = 0$ and the second Betti number $b_2(X) > 0$.
- Let $\Xi_0 \subseteq \Xi$ be those minimal surfaces which have no curve.

Here, surfaces of class VII are those smooth compact complex surfaces with the first Betti number $b_1 = 1$ and Kodaira dimension $\kappa = -\infty$ (cf. [18, § 7]). By a *minimal surface* we mean a smooth compact complex surface that does not contain any (-1)-curve C (i.e., a smooth rational curve C with self-intersection $C^2 = -1$). It is known that a smooth compact complex surface X is minimal if and only if any bimeromorphic holomorphic map $X \longrightarrow X'$ to a smooth compact complex surface X' is an isomorphism.

Proposition 1.5. *Let X be a compact complex surface not in Ξ_0. Then $\mathrm{Aut}(X)$ is T-Jordan.*

Although we are not able to confirm Conjecture 1.3 for surfaces in Ξ_0, we have a slightly weaker result on torsion subgroups of $\mathrm{Aut}(X)$. Recall that a group G is *virtually $\mathcal{P}$*, if there exists a finite-index subgroup of G that has the property $\mathcal{P}$. We say that a group G is *$\mathcal{P}$-by-$\mathcal{Q}$* if G fits into an short exact sequence

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$$

where K has the property $\mathcal{P}$ and H has the property $\mathcal{Q}$.

Proposition 1.6. *Let X be a smooth compact complex surface in Ξ_0. Let $G \leq \mathrm{Aut}(X)$ be a torsion subgroup. Then G is virtually abelian.*

Our main result below is a direct consequence of Propositions 1.5 and 1.6.

Theorem 1.7. *Let X be a smooth compact complex surface. Then any torsion subgroup $G \leq \mathrm{Aut}(X)$ is virtually abelian.*

Note that by the global spherical shell (GSS) conjecture, $\Xi_0 = \emptyset$ (cf. [24, Conjecture 1]).

Remark 1.8. Assume the global spherical shell conjecture. Let X be a smooth compact complex surface. Then $\mathrm{Aut}(X)$ is T-Jordan.

1.2. **Tits Alternative.** In group theory, the *Tits alternative* is an outstanding theorem about the structure of linear groups (cf. [34]). Campana, Wang and Zhang studied the automorphism groups of compact Kähler manifolds and proved a Tits type alternative for such groups (cf. [5, Theorem 1.5]). We generalise their celebrated result to compact complex spaces in Fujiki's class $\mathcal{C}$ and smooth compact complex surfaces.

Theorem 1.9. *Let X be a compact complex space in Fujiki's class $\mathcal{C}$. Then $\mathrm{Aut}(X)$ satisfies the Tits alternative, that is, for any subgroup G of $\mathrm{Aut}(X)$ either it contains a non-abelian free subgroup, or it is virtually solvable, i.e., admits a solvable subgroup of finite index.*

For the definition and properties of Enoki surfaces and Inoue-Hirzebruch surfaces, see [10] and [16], respectively.

Theorem 1.10. *Let X be a smooth compact complex surface. Assume that either $X \notin \Xi$, or $X \in \Xi$ but its minimal model is an Enoki or Inoue-Hirzebruch surface. Then $\mathrm{Aut}(X)$ satisfies the Tits alternative.*

1.3. **Virtual Derived Length.** Once we have established the Tits type alternative for $\mathrm{Aut}(X)$, it is natural to study the properties of those virtually solvable subgroups.

We have the following result, which gives a uniform upper bound of the virtual derived length (cf. Definition 2.1) for virtually solvable subgroups of $\mathrm{Aut}(X)$.

Theorem 1.11. *Let X be a smooth compact complex surface. Assume that either $X \notin \Xi$, or $X \in \Xi$ but its minimal model is an Enoki or Inoue-Hirzebruch surface. Let $G \leq \mathrm{Aut}(X)$ be a virtually solvable subgroup. Then the virtual derived length $\ell_{\mathrm{vir}}(G) \leq 4$.*

Remark 1.12.

(1) Currently, we are not able to prove Theorems 1.10 and 1.11 in full generality for $X \in \Xi$.
(2) The GSS conjecture claims that any minimal surface in Ξ is a Kato surface (cf. [24, Conjecture 1] and [9, Main Theorem]).
(3) Kato surfaces are divided into four classes: Enoki surfaces (including parabolic Inoue surfaces), half Inoue surfaces, Inoue-Hirzebruch surfaces and intermediate surfaces (cf. [33, §3.3.2]).
(4) Fix $b > 0$. The moduli space of framed Enoki surfaces with $b_2 = b$ is an open subspace of the moduli space of framed Kato surfaces with $b_2 = b$ (cf. [7, Remark, pp. 54] and [33, §3.3.4.1]).
(5) When X is a parabolic Inoue surface (which is in Ξ), it has been proved that $\mathrm{Aut}(X)$ is virtually abelian (cf. [11, Theorem 1.1]). In particular, $\mathrm{Aut}(X)$ satisfies the Tits alternative and any subgroup G of $\mathrm{Aut}(X)$ is virtually solvable with virtual derived length $\ell_{\mathrm{vir}}(G) \leq 1$.

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2. Preliminaries

In this section, we gather some general preliminary results.

Definition 2.1. Recall that for a group G, its *n-th derived subgroup* is defined recursively by

$$G^{(0)} := G \quad \text{and} \quad G^{(n)} := [G^{(n-1)}, G^{(n-1)}],\ n \in \mathbb{N}.$$

By definition, the group G is *solvable* if and only if $G^{(n)} = 1$ for some non-negative integer n. The minimum of such n is the *derived length* of G (when G is solvable), denoted by $\ell(G)$.

If G is virtually solvable, we define the *virtual derived length* to be

$$\ell_{\text{vir}}(G) := \min_{G'} \ell(G')$$

where G' runs through all finite-index subgroups of G.

A group G has *bounded torsion subgroups* if there is a constant $T = T(G)$ such that any torsion subgroup $G_1 \le G$ has order $|G_1| \le T$.

Lemma 2.2. *Let*

$$1 \longrightarrow N \longrightarrow G \longrightarrow H$$

be an exact sequence of groups. Then the following assertions hold:

(1) *If N is T-Jordan and H has bounded torsion subgroups, then G is T-Jordan.*
(2) *Assume further that the sequence is also right exact. If N is a torsion group and G is T-Jordan, then H is T-Jordan.*

Proof. (1) is clear. (2) Let $H' \le H$ be a torsion subgroup, and $G' \le G$ be its preimage. Since N is a torsion group, G' is also a torsion group and hence there exists an abelian subgroup $G_1' \le G'$ such that $[G' : G_1'] \le J(G)$ for some constant $J(G)$. Let $H_1' = G_1'/(G_1' \cap N) \le H'$, which is abelian with $[H' : H_1'] \le J(G)$. Therefore, H is T-Jordan. $\square$

The following two lemmas are adapted from [31, Corollaries 4.2, 4.4]. Note that every element of a torsion group has finite order.

Lemma 2.3. *Let X be an irreducible Hausdorff reduced complex space, and let $\Gamma \le \mathrm{Aut}(X)$ be a torsion subgroup. Suppose that Γ has a fixed point x on X. Then the natural representation*

$$\Gamma \longrightarrow \mathrm{GL}(T_{x,X})$$

is faithful.

Lemma 2.4. *Let X be an irreducible Hausdorff reduced complex space, and let $\Delta \le \mathrm{Aut}(X)$ be a subgroup. Suppose that Δ has a fixed point x on X, and let*

$$\sigma\colon \Delta \longrightarrow \mathrm{GL}(T_{x,X})$$

be the natural representation. Suppose that there is a normal subgroup $\Gamma \le \Delta$ with the quotient group Δ/Γ being torsion, such that the restriction $\sigma|_\Gamma$ is a group monomorphism. Then σ is an embedding as well.

Proof. Let Δ_0 be the kernel of σ. Since $\Delta_0 \cap \Gamma = \{\mathrm{id}\}$ and Δ/Γ is torsion, we see that Δ_0 is also torsion. Thus, Δ_0 is trivial by Lemma 2.3. $\square$

Lemma 2.5. *Let $G \le \mathrm{GL}_n(\mathbb{C})$ be a virtually solvable subgroup. Then $\ell_{\text{vir}}(G) \le n$*

Proof. By passing to a finite-index subgroup, we may assume that G itself is solvable. Then some subgroup G_1 of finite index in G can be put in triangular form (cf. [4, Corollary, pp. 137]). It follows that the derived length of G_1 and hence the virtual derived length of G is at most n. $\square$

Lemma 2.6. *Consider the short exact sequence of groups*

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1.$$

Then

(1) if N is solvable and H is virtually solvable, then G is virtually solvable with $\ell_{\mathrm{vir}}(G) \leq \ell(N) + \ell_{\mathrm{vir}}(H)$.
(2) if N is finite and H is virtually solvable, then G is virtually solvable with $\ell_{\mathrm{vir}}(G) \leq \ell_{\mathrm{vir}}(H) + 1$;
(3) the group G is virtually solvable if and only if both N and H are virtually solvable;
(4) if both N and H satisfy the Tits alternative, then so does G.

Proof. (1) Let $H' \leq H$ be a solvable subgroup of finite index (with $\ell(H') = \ell_{\mathrm{vir}}(H)$), and let $G' \leq G$ be its preimage. Since $[G : G'] = [G/N : G'/N] = [H : H'] < \infty$, we may replace G, H by G', H' and assume that H is solvable. Then it is clear that G is solvable, and $\ell_{\mathrm{vir}}(G) \leq \ell(G) \leq \ell(N) + \ell(H) \leq \ell(N) + \ell_{\mathrm{vir}}(H)$.

(2) As in (1), we may assume that H is solvable. Let $Z_G(N)$ be the centraliser of N in G. Since N is finite, $Z_G(N) \leq G$ is of finite index. Note that $Z_G(N) \cap N$ is abelian (in particular, solvable) and $Z_G(N)/(Z_G(N) \cap N) \leq G/N \simeq H$ is solvable. Hence, $Z_G(N)$ is also solvable with derived length $\ell(Z_G(N)) \leq \ell(H) + 1$. It follows that $\ell_{\mathrm{vir}}(G) \leq \ell_{\mathrm{vir}}(H) + 1$.

(3) The "only if" direction is clear. Suppose that N and H are virtually solvable. As in (2), we may assume that H is solvable. Let $N' \lhd N$ be a finite-index solvable (normal) subgroup of minimal index. Then N' is normal in G; otherwise, if N'' is another conjugate of N', then $N'N'' \lhd N$ is solvable with smaller index, a contradiction to the choice of N'. Consider the following short exact sequence

$$1 \longrightarrow N/N' \longrightarrow G/N' \longrightarrow G/N \simeq H \longrightarrow 1.$$

By assumption, the first term is finite and H is solvable. Thus, G/N' is virtually solvable by (2). Then using the short exact sequence

$$1 \longrightarrow N' \longrightarrow G \longrightarrow G/N' \longrightarrow 1,$$

we obtain that G is virtually solvable by (1).

(4) This follows from (3). Note that if H contains a copy of $\mathbb{Z} * \mathbb{Z}$, an non-abelian,free group with two generators, then G contains a(nother) copy of $\mathbb{Z} * \mathbb{Z}$ as well. $\square$

Using the lemma above, we can give a proof of Theorem 1.9.

Proof of Theorem 1.9. Let $\pi \colon X' \longrightarrow X$ be an $\mathrm{Aut}(X)$-equivariant resolution of singularities (cf. [2, Theorem 13.2(2)(ii)]) with $\mathrm{Aut}(X)$ lifts to a (unique) subgroup of $\mathrm{Aut}(X')$ via π. Note that X' is still in Fujiki's class $\mathcal{C}$. Therefore, we may replace X by X' and assume that X is smooth.

Denote by

$$\mathrm{Aut}_\tau(X) := \{g \in \mathrm{Aut}(X) \mid g^*|_{H^2(X,\mathbb{Q})} = \mathrm{id}\}.$$

Then we have the following short exact sequence

$$1 \longrightarrow G_\tau \longrightarrow G \longrightarrow G|_{H^2(X,\mathbb{Q})} \longrightarrow 1$$

where $G_\tau = G \cap \mathrm{Aut}_\tau(X)$. It suffices to prove that G_τ satisfies the Tits alternative by [34, Theorem 1] and Lemma 2.6 (4). Let $\sigma \colon \widetilde{X} \longrightarrow X$ be a bimeromorphic holomorphic map from a compact Kähler manifold $\widetilde{X}$ such that $\mathrm{Aut}_\tau(X)$ lifts to $\widetilde{X}$ holomorphically via σ (cf. [17, Theorem 1.1]). Viewing $G_\tau \leq \mathrm{Aut}_\tau(X)$ as a subgroup of $\mathrm{Aut}(\widetilde{X})$, we know that G_τ satisfies the Tits alternative (cf. [5, Theorem 1.5]). Therefore, G satisfies the Tits alternative. $\square$

$A = X \setminus C$ biholomorphically and faithfully and π is $\mathrm{Aut}(X)$-equivariant (see the second paragraph of this section).

Consider the short exact sequence

$$1 \longrightarrow K \longrightarrow \mathrm{Aut}(X) \longrightarrow \Gamma \longrightarrow 1,$$

where $K \le \mathrm{Aut}_E(A)$ and $\Gamma \le \mathrm{Aut}(E)$. Since E is an elliptic curve, $\mathrm{Aut}(E)$ and hence Γ are virtually abelian, or equivalently, virtually solvable with virtual derived length ≤ 1. By Proposition 3.2, there exists a finite subset S of E such that the natural map $K \hookrightarrow \prod_{e\in S} \mathrm{Aut}(A_e)$ is an injection. Since $A_e \simeq \mathbb{A}^1$, the group $\prod_{e\in S} \mathrm{Aut}(A_e)$ and hence K are metabelian, or equivalently, solvable with derived length ≤ 2. By Lemma 2.6 (1), $\mathrm{Aut}(X)$ is virtually solvable with virtual derived length ≤ 3. It is clear that any subgroup $G \le \mathrm{Aut}(X)$ is also virtually solvable and its virtual derived length $\ell_{\mathrm{vir}}(G) \le 3$. □

4. Surfaces with Invariant Curves

Lemma 4.1. *Let X be a smooth compact complex surface. Suppose that there is a finite non-empty $\mathrm{Aut}(X)$-invariant set Σ of (irreducible) curves on X. Then $\mathrm{Aut}(X)$ is T-Jordan.*

Proof. Let C be one of the curves from Σ. Then the group $\mathrm{Aut}(X, C)$ of automorphisms of X that preserves the curve C has finite index in $\mathrm{Aut}(X)$. We only need to show that $\mathrm{Aut}(X, C)$ is T-Jordan.

Assume first that C is singular. Then $\mathrm{Aut}(X, C)$, after replacing by a subgroup of finite index, fixes some point on C. Now Lemma 2.3 implies that $\mathrm{Aut}(X, C)$ is embedded into $\mathrm{GL}_2(\mathbb{C})$. Therefore, the group $\mathrm{Aut}(X, C)$ is T-Jordan (cf. Theorem 1.1).

Now we assume that C is smooth and let $G \le \mathrm{Aut}(X, C)$ be a torsion subgroup. Let $\mathcal{N}_{C/X}$ be the normal bundle of C in X. Consider the natural group homomorphism

$$\mathcal{N}\colon\ \mathrm{Aut}(X, C) \longrightarrow \mathrm{Aut}(\mathcal{N}_{C/X} \to C)$$

via $\sigma \mapsto \mathcal{N}_\sigma$ (cf. [23, Notation 2.2]). By [23, Lemma 2.3], $\ker \mathcal{N} \cap G = \{\mathrm{id}\}$. Moreover, by the proof of [23, Theorem 3.1], there is a monomorphism

$$\mathrm{Aut}(\mathcal{N}_{C/X} \to C) \longrightarrow \mathrm{Aut}(\mathbb{P}_C(\mathcal{N}_{C/X} \oplus \mathcal{O})).$$

So we may view G as a subgroup of $\mathrm{Aut}(\mathbb{P}_C(\mathcal{N}_{C/X} \oplus \mathcal{O}))$. Note that C is projective. So $\mathbb{P}_C(\mathcal{N}_{C/X} \oplus \mathcal{O})$ is also a projective manifold. Then the theorem follows from Theorem 1.4. □

Corollary 4.2. *Let X be a compact complex surface with $a(X) = 0$. Assume that X contains at least one curve. Then $\mathrm{Aut}(X)$ is T-Jordan.*

Proof. It follows from Lemma 4.1, since X contains at most a finite number of (irreducible) curves (cf. [1, IV. Theorem 8.2]). □

Proposition 4.3. *Let X be a compact complex surface with $e(X) \ne 0$ and $a(X) = 1$. Then $\mathrm{Aut}(X)$ is T-Jordan.*

Proof. Let $\pi\colon X \longrightarrow Y$ be the algebraic reduction (cf. Section 3), so that Y is a smooth curve and π is an $\mathrm{Aut}(X)$-equivariant elliptic fibration. Since $e(X) \ne 0$, the fibration π has at least one fibre X_y such that $F = (X_y)_{\mathrm{red}}$ is not a smooth elliptic curve by Suzuki's formula (cf. [1, III, Proposition 11.4]). Note that π has only finitely many singular fibres, whose union is $\mathrm{Aut}(X)$-invariant. Then the assertion follows from Lemma 4.1. □

The theorem below is due to [27, §2, Theorem].

Theorem 2.7. *Let X be an Inoue-Hirzebruch surface. Then* $\mathrm{Aut}(X)$ *is finite. In particular,* $\mathrm{Aut}(X)$ *satisfies the Tits alternative and every subgroup $G \leq \mathrm{Aut}(X)$ is finite and hence virtually solvable with virtual derived length $\ell_{\mathrm{vir}}(G) = 0$.*

The following lemmas will be used in Section 8.

Lemma 2.8 (cf. [13, §2, 4])**.** *Let G be a finitely generated group. Then every finite-index subgroup of G contains a finite-index subgroup G' which is characteristic in G.*

Lemma 2.9. *Let G be a solvable group and $U \lhd G$ a non-trivial unipotent (normal) subgroup. Then, after replacing G by a finite-index subgroup, there is a subgroup $Z_1 \simeq \mathbb{G}_a \leq U$ which is normal in G.*

Proof. Let $Z = Z(U)$ be the centre of U which is nontrivial and normal in G. Note that the commutative unipotent group Z is isomorphic to $\mathbb{G}_a^n$, the additive affine algebraic group of dimension n for some $n \geq 1$. Consider the conjugation action of G on Z:

$$c_g \colon x \mapsto gxg^{-1} \quad \text{for } x \in Z \text{ and } g \in G.$$

This conjugation action c_g is linear: for a fixed $x \in Z \simeq \mathbb{G}_a^n$, the morphism $z \mapsto f(z) \coloneqq c_g(z.x) - z.c_g(x)$ is trivial for all integer points $z \in \mathbb{G}_a$, so $f(z) = 0$ for all points $z \in \mathbb{G}_a$.

Therefore, one has an induces a homomorphism $G \longrightarrow \mathrm{GL}_n(\mathbb{C})$. Replacing G by a finite-index subgroup, we may assume that the image G' of the solvable group G in $GL_n(\mathbb{C})$ is of triangular form (cf. [4, Corollary, pp. 137]). Let $Z_1 \simeq \mathbb{G}_a$ be an (1-dimensional) eigenspace of G'. Then Z_1 is stable under the conjugation action of G and hence is normal in G. □

3. Equivariant Fibrations

In this section we will consider smooth compact complex surfaces X with algebraic dimension $a(X) = 1$, or with Kodaira dimension $\kappa(X) = 1$. If $a(X) = 1$, there is a natural elliptic fibration, the *algebraic reduction* $\pi \colon X \longrightarrow Y$ (cf. [1, VI. (5.1) Proposition]), which is a holomorphic map. Here, by an *elliptic fibration*, we mean a surjective morphism between two compact complex spaces with connected fibre, such that a general fibre is isomorphic to a smooth elliptic curve. For $g \in \mathrm{Aut}(X)$, the image of a fibre F of π under g is another fibre; otherwise the self-intersection number of $g(F) + F$ is positive and hence X is projective (cf. [1, IV, Theorem 6.2]), a contradiction (cf. [1, IV, Corollary 6.5]). Thus, π is $\mathrm{Aut}(X)$-equivariant. When $\kappa(X) = 1$, the *pluricanonical fibration* gives a natural elliptic fibration as well, which is also $\mathrm{Aut}(X)$-equivariant.

In addition to the above, we will also study affine $\mathbb{A}^1$-bundle $\pi \colon A \longrightarrow E$ over an elliptic curve E. Note that the fibre of $\pi \colon A \longrightarrow E$ being an affine line $\mathbb{A}^1$ cannot dominate the base elliptic curve E. It follows that π is $\mathrm{Aut}(A)$-equivariant.

Let $\pi \colon X \longrightarrow Y$ be a morphism between complex spaces. Let G be a complex Lie group contained in $\mathrm{Aut}(X)$ such that $\pi \circ g = \pi$ for all $g \in G$, i.e., $G \leq \mathrm{Aut}_Y(X)$. For any finite subset $S \subseteq Y$, denote by $G_S \coloneqq \{g \in G \mid g|_{\pi^{-1}(S)} = \mathrm{id}\}$, i.e., G_S is the kernel of the natural group homomorphism $G \longrightarrow \prod_{y \in S} \mathrm{Aut}(X_y)$. Note that any Lie group is second countable and hence has only countably many connected components.

The ideas of Lemma 3.1 and Proposition 3.2 come from Sheng Meng.

Lemma 3.1. *Suppose that* $\dim G = 0$. *Then* G_y *is trivial for very general* $y \in Y$.

Proof. Let $Y_g := \{y \in Y \mid g|_{X_y} = \mathrm{id}\}$. Then Y_g is a proper closed subspace of Y for $g \neq \mathrm{id}$. Let $Z := \bigcup_{\mathrm{id} \neq g \in G} Y_g$. Then any $y \in Y \setminus Z$ is a very general choice. Here we use the condition that G is countable since $\dim G = 0$. □

A very general finite subset $S \subseteq Y$ means that every element of S is very generally chosen in Y.

Proposition 3.2. *For very general finite subset* $S \subseteq Y$, G_S *is trivial.*

Proof. First, we show by induction on $\dim G$ that $\dim G_S = 0$ for very general finite subset $S \subseteq Y$. When $\dim G = 0$, the claim follows from Lemma 3.1. Now assume that $\dim G \geq 1$. Denote by $T := \{y \in Y \mid G_0|_{X_y} = \mathrm{id}\}$, where G_0 is the neutral component of G. Note that $T \subsetneq Y$ is a closed subspace. Then for $y \notin T$, we have $\dim G_y < \dim G$. By induction, $\dim(G_y)_S = 0$ for any $y \notin T$ and very general finite subset $S \subseteq Y$ (depending on y). Note that $(G_y)_S = G_{\{y\} \cup S}$. So the claim is proved.

Then by Lemma 3.1, we see that $G_{S \cup \{y\}} = (G_S)_y$ is trivial for very general $y \in Y$. □

Lemma 3.3. *Let* X *be either a (primary or secondary) Kodaira surface, or a minimal smooth compact complex surface of Kodaira dimension* 1. *Then there is an* $\mathrm{Aut}(X)$*-equivariant elliptic fibration* $\pi \colon X \longrightarrow Y$ *to a smooth curve* Y. *Also, the natural image of* $\mathrm{Aut}(X)$ *in* $\mathrm{Aut}(Y)$ *is finite.*

Proof. We may take π to be either the algebraic reduction or the pluricanonical fibration (see the first paragraph of this section). Then our claim follows from [32, Corollary 3.4, Lemma 3.5] and [30, Proposition 1.2]. □

Theorem 3.4. *Let* X *be either a (primary or secondary) Kodaira surface or a minimal smooth compact complex surface of Kodaira dimension* 1. *Then* $\mathrm{Aut}(X)$ *is virtually abelian. In particular,* $\mathrm{Aut}(X)$ *satisfies T-Jordan property, the Tits alternative and every subgroup* $G \leq \mathrm{Aut}(X)$ *is virtually abelian and hence virtually solvable with virtual derived length* $\ell_{\mathrm{vir}}(G) \leq 1$.

Proof. By Lemma 3.3, there is a short exact sequence

$$1 \longrightarrow \mathrm{Aut}_Y(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow \Gamma \longrightarrow 1,$$

where $\pi \colon X \longrightarrow Y$ is an elliptic fibration and Γ is a finite subgroup of $\mathrm{Aut}(Y)$. By Proposition 3.2, there exists a finite subset S of Y such that the natural map $\mathrm{Aut}_Y(X) \hookrightarrow \prod_{y \in S} \mathrm{Aut}(X_y)$ is an injection. Since X_y is an elliptic curve, $\prod_{c \in S} \mathrm{Aut}(X_y)$ and hence $\mathrm{Aut}_Y(X)$ are virtually abelian. Then it is clear that $\mathrm{Aut}(X)$ is virtually abelian. □

Theorem 3.5. *Let* $X \in \Xi$ *be an Enoki surface. Then* $\mathrm{Aut}(X)$ *is virtually solvable (and hence* $\mathrm{Aut}(X)$ *satisfies the Tits alternative) and every subgroup* $G \leq \mathrm{Aut}(X)$ *is virtually solvable with* $\ell_{\mathrm{vir}}(G) \leq 3$.

In the proof below, by *a cycle of rational curves*, we mean a reduced (singular) complex curve with only nodes singularities such that its dual graph is a cycle and each component of its normalisation is a smooth rational curve.

Proof. By [10, Main Theorem], X has a cycle of rational curves C such that $A := X \setminus C$, the complement of C in X, is an affine $\mathbb{A}^1$-bundle $\pi \colon A \longrightarrow E$ over an elliptic curve E. Since the algebraic dimension $a(X)$ is 0, the surface X contains finitely many (irreducible) curves (cf. [1, IV. Theorem 8.2]). Replacing $\mathrm{Aut}(X)$ by a finite-index subgroup, we may assume that C is $\mathrm{Aut}(X)$-invariant. Then $\mathrm{Aut}(X)$ acts on

5. Hopf Surfaces

In this section, we study the automorphism groups of Hopf surfaces. By definition, a *Hopf surface* X is a smooth compact complex surface with universal covering being isomorphic to $\mathbb{C}^2 \setminus \{0\}$. Then, X can be obtained from $\mathbb{C}^2 \setminus \{0\}$ as a quotient by a free action of a discrete group $\Gamma \simeq \pi_1(X)$. If $\pi_1(X) \simeq \mathbb{Z}$, we call such X a *primary* Hopf surface. In this case, after an appropriate choice of coordinates of $\mathbb{C}^2$, the generator of Γ has the form

$$(z, w) \longmapsto (az + \lambda w^m, bw), \tag{5.1}$$

where $m \in \mathbb{Z}_{>0}$ and a, b, λ are complex constants subject to the restrictions

$$(a - b^m)\lambda = 0, \quad 0 < |a| \leq |b| < 1.$$

Any Hopf surface is either primary or a finite étale quotient of a primary Hopf surface, where the latter is called the *secondary* Hopf surface (cf. [19, Theorem 30]).

By above, Γ contains $\Lambda \simeq \mathbb{Z}$ as a subgroup of finite index. In particular, Γ is virtually solvable. After replacing Λ by a suitable subgroup $\Lambda_0 \simeq \mathbb{Z}$, we may assume that Λ is a characteristic subgroup of Γ (cf. [31, Lemma 2.10]), with the generator of Λ having the form Eq. (5.1). Now there is a short exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \widetilde{\mathrm{Aut}}(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow 1 \tag{5.2}$$

where $\widetilde{\mathrm{Aut}}(X)$ acts on $\mathbb{C}^2 \setminus \{0\}$ biholomorphically. It follows from the Hartogs extension theorem that $\widetilde{\mathrm{Aut}}(X)$ can be extended to biholomorphic actions on $\mathbb{C}^2$ fixing the origin $0 \in \mathbb{C}$. By assumption, $\Lambda \lhd \widetilde{\mathrm{Aut}}(X)$ is a normal subgroup.

Theorem 5.1. *Let X be a Hopf surface. Then the group $\mathrm{Aut}(X)$ satisfies the Tits alternative.*

Proof. Assume first that X is a primary Hopf surface. Then we may identify $\widetilde{\mathrm{Aut}}(X)$ with a subgroup of $\mathrm{GL}_2(\mathbb{C}) \times \mathbb{C}$ by [25, § 2] and [35, pp. 24]. Since $\mathrm{GL}_2(\mathbb{C}) \times \mathbb{C}$ satisfies the Tits alternative, so do $\widetilde{\mathrm{Aut}}(X)$ and $\mathrm{Aut}(X)$ (cf. Lemma 2.6 (4)).

Now let X be a secondary Hopf surface. Then either $\widetilde{\mathrm{Aut}}(X) \leq \mathrm{GL}_2(\mathbb{C})$, or $\widetilde{\mathrm{Aut}}(X) \simeq \mathbb{C} \rtimes \mathbb{C}^*$ or $\widetilde{\mathrm{Aut}}(X) \simeq \mathbb{C} \rtimes (\mathbb{C}^*)^2$ (cf. [22, Theorem 1]). In the first case, $\widetilde{\mathrm{Aut}}(X)$ and hence $\mathrm{Aut}(X)$ satisfies the Tits alternative as $\mathrm{GL}_2(\mathbb{C})$ does; in the latter two cases, $\widetilde{\mathrm{Aut}}(X)$ and hence $\mathrm{Aut}(X)$ are already solvable. In a word, $\mathrm{Aut}(X)$ satisfies the Tits alternative. □

Theorem 5.2. *Let X be a Hopf surface and $G \leq \mathrm{Aut}(X)$ a virtually solvable subgroup. Then $\ell_{\mathrm{vir}}(G) \leq 2$.*

Proof. Let $\widetilde{G} \leq \widetilde{\mathrm{Aut}}(X)$ be the preimage of G. Then $\widetilde{G}$ is virtually solvable by Lemma 2.6 (3) as Γ is virtually solvable. Therefore, $\ell_{\mathrm{vir}}(G) \leq \ell_{\mathrm{vir}}(\widetilde{G}) \leq \ell_{\mathrm{vir}}(\widetilde{\mathrm{Aut}}(X)) \leq 2$ by Lemma 2.5 and proof of Theorem 5.1. □

The lemma below is a simple linear algebra, which is taken from [31, Lemma 6.3].

Lemma 5.3. *Let*

$$M = \begin{pmatrix} a & \lambda \\ 0 & b \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

be an upper triangular matrix, and $Z \leq \mathrm{GL}_2(\mathbb{C})$ the centraliser of M. Then the following assertions hold:

(i) *If $a = b$ and $\lambda = 0$, then $Z = \mathrm{GL}_2(\mathbb{C})$.*

(ii) If $a \neq b$ and $\lambda = 0$, then $Z \simeq (\mathbb{C}^*)^2$.

(iii) If $a = b$ and $\lambda \neq 0$, then $Z \simeq \mathbb{C}^* \times \mathbb{C}^+$.

Theorem 5.4. *Let X be a Hopf surface. Then $\mathrm{Aut}(X)$ is T-Jordan.*

Proof. The proof is adapted from [31, Lemma 6.4].

Consider the short exact sequence Eq. (5.2). The image of the generator of Λ is mapped by the natural homomorphism

$$\sigma\colon \widetilde{\mathrm{Aut}}(X) \longrightarrow \mathrm{GL}(T_{0,\mathbb{C}^2}) \simeq \mathrm{GL}_2(\mathbb{C})$$

to the matrix

$$M = \begin{pmatrix} a & \lambda\delta_1^m \\ 0 & b \end{pmatrix}$$

where δ is the Kronecker symbol.

Let $G \leq \mathrm{Aut}(X)$ be a torsion subgroup, and $\widetilde{G}$ its preimage in $\widetilde{\mathrm{Aut}}(X)$. Thus, one has $G \simeq \widetilde{G}/\Gamma$. By Lemma 2.4, $\sigma|_{\widetilde{G}}$ is an embedding since σ_Λ is a group monomorphism and $\widetilde{G}/\Lambda$ is torsion. Let Ω be the normaliser of $\sigma(\Lambda)$ in $\mathrm{GL}_2(\mathbb{C})$. By construction $\sigma(\widetilde{G})$ is contained in the normaliser of $\sigma(\Gamma)$ in $\mathrm{GL}_2(\mathbb{C})$, which in turn is contained in Ω because Λ is a characteristic subgroup of Γ. Hence, every torsion subgroup of $\mathrm{Aut}(X)$ is contained in the group $\Omega/\sigma(\Gamma)$. On the other hand, $\Omega/\sigma(\Gamma)$ is a quotient of $\Omega/\sigma(\Lambda)$ by a finite subgroup isomorphic to $\sigma(\Gamma)/\sigma(\Lambda)$. Thus, by Lemma 2.2, it is sufficient to show that the group $\Omega/\sigma(\Lambda)$ is T-Jordan.

Since $\sigma(\Lambda) \simeq \mathbb{Z}$, the group Ω has a (normal) subgroup Ω' of index at most 2 that coincides with the centraliser of the matrix M. It remains to check that the group $\Omega'/\sigma(\Lambda)$ is T-Jordan. If $\lambda = 0$ and $a = b$, it follows from Lemma 5.3(i) that $\Omega'/\sigma(\Lambda)$ is a connected Lie group and hence T-Jordan by Theorem 1.1. If either $\lambda = 0$ and $a \neq b$, or $\lambda \neq 0$ and $m \geq 2$, then this follows from Lemma 5.3(ii) that $\Omega'/\sigma(\Lambda)$ is abelian. If $\lambda \neq 0$ and $m = 1$, then this follows from Lemma 5.3(iii) that $\Omega'/\sigma(\Lambda)$ is abelian. □

6. Inoue Surfaces

An *Inoue surface* X is a compact complex surface obtained from $W := \mathbb{H} \times \mathbb{C}$ as a quotient by an infinite discrete group, where $\mathbb{H}$ is the upper half complex plane. Inoue surfaces are minimal surfaces in class VII, contain no curve, and have the following numerical invariants:

$$a(X) = 0, \quad b_1(X) = 1, \quad b_2(X) = 0.$$

There are three families of Inoue surfaces: S_M, $S^{(+)}$, and $S^{(-)}$ (cf. [15]), and we will study their automorphisms separately.

Since every holomorphic map from $\mathbb{C}$ to $\mathbb{H}$ is constant, any automorphism u of W has the form

$$u(w, z) = (s(w), t(w, z)) \tag{6.1}$$

where

$$s(w) = \frac{aw + b}{cw + d}, \quad a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0.$$

We may expand t in the power series of w and z at $(\sqrt{-1}, 0)$:

$$t(w, z) = \sum_{p \geq 0} C_{p,0}(w - \sqrt{-1})^p + C_{p,1}(w - \sqrt{-1})^p z.$$

6.1. **Type S_M.** Let $M = (m_{i,j}) \in \mathrm{SL}_3(\mathbb{Z})$ be a matrix with eigenvalues $\alpha, \beta, \overline{\beta}$ such that $\alpha > 1$ and $\beta \neq \overline{\beta}$. Take $(a_1, a_2, a_3)^T$ to be a real eigenvector of M corresponding to α, and $(b_1, b_2, b_3)^T$ an eigenvector corresponding to β. Let G_M be the group of automorphisms of W generated by

$$\begin{aligned} g_0(w,z) &= (\alpha w, \beta z), \\ g_i(w,z) &= (w + a_i, z + b_i), \quad i = 1,2,3, \end{aligned}$$

which satisfy these conditions

$$\begin{aligned} g_0 g_i g_0^{-1} &= g_1^{m_{i,1}} g_2^{m_{i,2}} g_3^{m_{i,3}}, \\ g_i g_j &= g_j g_i, \quad i,j = 1,2,3. \end{aligned}$$

It can be shown that the action of G_M on W is free and properly discontinuous. The quotient $X \coloneqq W/G_M$ is an Inoue surface of type S_M. Thus, there is a short exact sequence of groups

$$1 \longrightarrow G_M \longrightarrow \widetilde{\mathrm{Aut}}(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow 1$$

where $\widetilde{\mathrm{Aut}}(X)$ acts biholomorphically on W, which is the normaliser of G_M in $\mathrm{Aut}(W)$.

We will study $\widetilde{\mathrm{Aut}}(X)$ in detail. Assume that $u \in \widetilde{\mathrm{Aut}}(X)$. Since $G_M \lhd \widetilde{\mathrm{Aut}}(X)$, we have $ugu^{-1} \in G_M$ for all $g \in G_M$. Indeed, we only need to verify this for the generators of G_M.

By the commutative relations of G_M, we may assume that

$$ug_iu^{-1} = g_1^{n_{i,1}} g_2^{n_{i,2}} g_3^{n_{i,3}} g_0^{k_i}, \quad 0 \leq i \leq 3 \tag{6.2}$$

for some $k_i \in \mathbb{Z}$, $\mathbf{n} = (n_{0,1}, n_{0,2}, n_{0,3}) \in \mathbb{Z}^3$ and $N \coloneqq (n_{i,j})_{1 \leq i,j \leq 3} \in \mathrm{Mat}_{3\times 3}(\mathbb{Z})$. Then

$$g_1^{n_{i,1}} g_2^{n_{i,2}} g_3^{n_{i,3}} g_0^{k_i} u(w,z) = (\alpha^{k_i} s(w) + \sum_{j=1}^{3} n_{i,j} a_j, \beta^{k_i} t(w,z) + \sum_{j=1}^{3} n_{i,j} b_j).$$

Note that

$$\begin{aligned} ug_0(w,z) &= u(\alpha w, \beta z) = (s(\alpha w), t(\alpha w, \beta z)), \\ ug_i(w,z) &= u(w + a_i, z + b_i) = (s(w + a_i), t(z + b_i)). \end{aligned}$$

Then, Eq. (6.2) implies

$$s(\alpha w) = \alpha^{k_0} s(w) + \sum_j n_{0,j} a_j, \tag{6.3}$$

$$t(\alpha w, \beta z) = \beta^{k_0} t(w,z) + \sum_j n_{0,j} b_j, \tag{6.4}$$

$$s(w + a_i) = \alpha^{k_i} s(w) + \sum_j n_{i,j} a_j, \tag{6.5}$$

$$t(z + b_i) = \beta^{k_i} t(w,z) + \sum_j n_{i,j} b_j. \tag{6.6}$$

By Eq. (6.3), we have

$$\begin{cases} ac(1 - \alpha^{k_0}) = c^2 \sum_j n_{0,j} a_j, \\ ad(\alpha - \alpha^{k_0}) + bc(1 - \alpha^{k_0+1}) = cd(1+\alpha) \sum_j n_{0,j} a_j, \\ bd(1 - \alpha^{k_0}) = d^2 \sum_j n_{0,j} a_j. \end{cases} \tag{6.7}$$

If $cd \neq 0$, then $a(1-\alpha^{k_0}) = c\sum_j n_{0,j}a_j$ and $b(1-\alpha^{k_0}) = d\sum_j n_{0,j}a_j$ by the first and the third equalities of Eq. (6.7). Using the fact that $ad - bc > 0$, the second equality of Eq. (6.7) implies $k_0 = 0$ and $\sum_j n_{0,j}a_j = 0$, which contradicts the middle equality above. Therefore, either $c = 0$ or $d = 0$.

From Eq. (6.5) one deduces that

$$\begin{cases} ac(1-\alpha^{k_i}) = c^2\sum_j n_{i,j}a_j, \\ (aca_i + ad + bc)(1-\alpha^{k_i}) = (c^2a_i + 2cd)\sum_j n_{i,j}a_j, \\ (ad - bc\alpha^{k_i})a_i + bd(1-\alpha^{k_i}) = (cda_i + d^2)\sum_j n_{i,j}a_j, \end{cases} \quad \text{for all } i. \tag{6.8}$$

If $d = 0$, then $bc\alpha^{k_i}a_i = 0$ for all i by the last equality of Eq. (6.8). Since $bc \neq 0$ now and $\alpha > 1$, the third equality of Eq. (6.8) implies $a_i = 0$ for all i, which contradicts the assumption that (a_1, a_2, a_3) is an eigenvector of M. Hence, $c = 0$ and we may rewrite s as $s(w) = aw + b$ with $a > 0$ and $b \in \mathbb{R}$ (i.e., $d = 1$). Then, by Eq. (6.7), $k_0 = 1$ and $(1-\alpha)b = \sum_j n_{0,j}a_j$ and Eq. (6.8) gives $k_i = 0$ and $aa_i = \sum_j n_{i,j}a_j$.

Comparing the coefficients of z in Eq. (6.4), we have

$$\begin{aligned} \sum_p C_{p,0}(\alpha w - \sqrt{-1})^p &= \beta\sum_p C_{p,0}(w - \sqrt{-1})^p + \sum_j n_{0,j}b_j, \\ \beta\sum_p C_{p,1}(\alpha w - \sqrt{-1})^p &= \beta\sum_p C_{p,1}(w - \sqrt{-1})^p. \end{aligned} \tag{6.9}$$

It follows that $C_{p,0} = C_{p,1} = 0$ for all $p \geq 1$. Thus, we may rewrite t as $t(w,z) = Az + B$ for some $A \neq 0, B \in \mathbb{C}$. Then Eq. (6.9) becomes $(1-\beta)B = \sum_j n_{0,j}b_j$, and Eq. (6.6) gives $Ab_i = \sum_j n_{i,j}b_j$.

We conclude that $u(w,z) = (aw + b, Az + B)$ where a, b, A, B satisfy

$$\mathbf{n} \cdot \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} = \begin{pmatrix} (1-\alpha)b & (1-\beta)B \end{pmatrix}$$

and

$$N\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad N\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = A\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \tag{6.10}$$

It follows that M and N are simultaneously diagonalisable. So M and N commute. Moreover, the matrix associated to u^{-1} is $N^{-1} \in \mathrm{Mat}_{3\times 3}(\mathbb{Z})$, so $\det N = \pm 1$ and $N \in \mathrm{GL}_3(\mathbb{Z})$.

Consider the following subgroups of $\widetilde{\mathrm{Aut}}(X)$:

$$K = \left\{(w,z) \longmapsto \left(w + \frac{1}{1-\alpha}\sum_{j=1}^{3} n_i a_i, z + \frac{1}{1-\beta}\sum_{j=1}^{3} n_i b_i\right) \mid n_i \in \mathbb{Z}\right\} \simeq \mathbb{Z}^3$$

and

$$\Gamma \coloneqq \{N \in \mathrm{GL}_3(\mathbb{Z}) \mid N \text{ and } M \text{ are simultaneously diagonalisable}\},$$

which are abelian groups. By the construction, $\widetilde{\mathrm{Aut}}(X) = K \rtimes \Gamma$. Note also that $G_M = G_1 \rtimes G_0$ where $G_1 = \{g_1^{n_1}g_2^{n_2}g_3^{n_3} \mid n_i \in \mathbb{Z}\} \simeq \mathbb{Z}^3$ and $G_0 = \langle g_0 \rangle \simeq \mathbb{Z}$. Now we have the following commutative diagram

(by the snake lemma)

(6.11)
$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & G_1 \arrow[r, "\phi"] \arrow[d] & K \arrow[r] \arrow[d, "\iota"] & F \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & G_M \arrow[r] \arrow[d] & \widetilde{\mathrm{Aut}}(X) \arrow[r] \arrow[d] & \mathrm{Aut}(X) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & G_0 \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & \Gamma/G_0 \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

where ϕ is given by

$$\phi(\mathbf{n}) = \mathbf{n} \cdot (I - M) \quad \text{with } \mathbf{n} = (n_1, n_2, n_3).$$

Since $I - M$ is invertible, the group F being the cokernel of ϕ, is finite (and abelian). Then, by the last column of the commutative diagram, $\mathrm{Aut}(X)$ is metabelian. By Eq. (6.11) and Lemma 2.6 (1), we have:

Theorem 6.1. *Let X be an Inoue surface of type S_M. Then $\mathrm{Aut}(X)$ is solvable. Moreover, for any $G \le \mathrm{Aut}(X)$, one has $\ell_{\mathrm{vir}}(G) \le 2$.*

Next, we claim that any torsion subgroup of $\Gamma/\mathbb{Z}$ is finite. Let n_0 be the largest integer such that, for some n_0-th root of M, $M^{\frac{1}{n_0}} \in \mathrm{GL}_3(\mathbb{Z})$. Let $N \in \Gamma$ be an element such that its image in $\Gamma/\mathbb{Z}$ is torsion. Then there is a pair of coprime integers (m, n) such that $N^n = M^m$ and hence $M^{\frac{m}{n}} \in \mathrm{GL}_3(\mathbb{Z})$. Since m and n are coprime, there are integers a and b such that $am + bn = 1$. Then

$$\frac{1}{n} = a\frac{m}{n} + b.$$

It follows that $M^{\frac{1}{n}} \in \mathrm{GL}_3(\mathbb{Z})$ and hence $n \le n_0$. Therefore,

$$\{(m, n) \mid M^{\frac{m}{n}} \in \mathrm{GL}_3(\mathbb{Z}), m, n \text{ coprime}, m < n\}$$

is a finite set. It follows that the set of torsion elements of $\Gamma/\mathbb{Z}$ is finite (with the bound depending on X), which proves the claim. This claim and Eq. (6.11) imply:

Theorem 6.2. *Let X be an Inoue surface of type S_M. Then any torsion subgroup $G \le \mathrm{Aut}(X)$ is finite. In particular, $\mathrm{Aut}(X)$ is T-Jordan.*

6.2. **Types $S^{(+)}$ and $S^{(-)}$.** Since the constructions of Inoue surfaces of type $S^{(+)}$ and $S^{(-)}$ are almost parallel, we will only focus on type $S^{(+)}$ in this subsection. The same argument works for Inoue surfaces of type $S^{(-)}$.

Let $M \in \mathrm{SL}_2(\mathbb{Z})$ be a matrix with two real eigenvalues α and $1/\alpha$ with $\alpha > 1$. Let $(a_1, a_2)^T$ and $(b_1, b_2)^T$ be real eigenvectors of M corresponding to α and $1/\alpha$, respectively, and fix integers p_1, p_2, r $(r \ne 0)$ and a complex number τ. Define $(c_1, c_2)^T$ to be the solution of the following equation

$$(I - M)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \frac{b_1 a_2 - b_2 a_1}{r}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

where

$$e_i = \frac{1}{2}m_{i,1}(m_{i,1}-1)a_1b_1 + \frac{1}{2}m_{i,2}(m_{i,2}-1)a_2b_2 + m_{i,1}m_{i,2}b_1a_2, \quad i=1,2.$$

Let $G_M^{(+)}$ be the group of analytic automorphisms of $W = \mathbb{H}\times\mathbb{C}$ generated by

$$\begin{aligned} g_0 &\colon (w,z) \mapsto (\alpha w, z+\tau), \\ g_i &\colon (w,z) \mapsto (w+a_i, z+b_iw+c_i), \quad i=1,2, \\ g_3 &\colon (w,z) \mapsto \Big(w, z+\frac{b_1a_2-b_2a_1}{r}\Big). \end{aligned}$$

We have the following relations between these generators

$$g_3g_i = g_ig_3 \quad \text{for } i=0,1,2, \quad g_1^{-1}g_2^{-1}g_1g_2 = g_3^r,$$
$$g_0g_jg_0^{-1} = g_1^{m_{j,1}}g_2^{m_{j,2}}g_3^{p_j} \quad \text{for } j=1,2.$$

The action of $G_M^{(+)}$ is free and properly discontinuous. The quotient space $X \coloneqq W/G_M^{(+)}$ is an Inoue surface of type $S^{(+)}$. Note that $G_M^{(+)} \simeq H(r)\rtimes\mathbb{Z}$ as an abstract group, where $H(r) = \langle g_1,g_2,g_3 \mid g_3g_i = g_ig_3, g_1^{-1}g_2^{-1}g_1g_2 = g_3^r\rangle$ and $\mathbb{Z}$ is generated by g_0. In fact, the centre $Z(H(r))\simeq\mathbb{Z}$ is generated by g_3 and $H(r)/Z(H(r))\simeq\mathbb{Z}^2$.

Similarly, there is a short exact sequence of groups

$$1 \longrightarrow G_M^{(+)} \longrightarrow \widetilde{\mathrm{Aut}}(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow 1,$$

where $\widetilde{\mathrm{Aut}}(X) \le \mathrm{Aut}(W)$ is the normaliser of $G_M^{(+)}$. Now suppose that $u\in\widetilde{\mathrm{Aut}}(X)$ as in Eq. (6.1) and

$$ug_iu^{-1} = g_1^{n_{i,1}}g_2^{n_{i,2}}g_3^{l_i}g_0^{k_i}, \quad 0\le i\le 3,\ j=1,2 \tag{6.12}$$

for some $l_i,k_i\in\mathbb{Z}$, $\mathbf{n}_0=(n_{0,1},n_{0,2}), \mathbf{n}_3=(n_{3,1},n_{3,2})\in\mathbb{Z}^2$ and $N=(n_{i,j})_{i,j=1,2}\in\mathrm{Mat}_{2\times2}(\mathbb{Z})$. Then,

$$\begin{aligned} &g_1^{n_{i,1}}g_2^{n_{i,2}}g_3^{l_i}g_0^{k_i}u(w,z) \\ &= \Big(\alpha^{k_i}s(w)+\sum_j n_{i,j}a_j, t(w,z)+k_i\tau+l_i\frac{b_1a_2-b_2a_1}{r}+\Big(\sum_j n_{i,j}b_j\Big)\alpha^{k_i}s(w)+\sum_j n_{i,j}c_j+e_i(n)\Big), \end{aligned}$$

where

$$e_i(n) = \frac{1}{2}n_{i,1}(n_{i,1}-1)a_1b_1+\frac{1}{2}n_{i,2}(n_{i,2}-1)a_2b_2+n_{i,1}n_{i,2}a_2b_1.$$

As in the previous subsection, Eq. (6.12) implies

$$s(\alpha w) = \alpha^{k_0}s(w)+n_{0,1}a_1+n_{0,2}a_2, \tag{6.13}$$

$$t(\alpha w, z+t) = t(w,z)+k_0\tau+l_0\frac{b_1a_2-b_2a_1}{r}+\Big(\sum_j n_{0,j}b_j\Big)\alpha^{k_0}s(w)+\sum_j n_{0,j}c_j+e_0(n), \tag{6.14}$$

$$s(w+a_i) = \alpha^{k_i}s(w)+n_{i,1}a_1+n_{i,2}a_2, \tag{6.15}$$

$$t(w+a_i, z+b_iw+c_i) = t(w,z)+k_i\tau+l_i\frac{b_1a_2-b_2a_1}{r}+\Big(\sum_j n_{i,j}b_j\Big)\alpha^{k_i}s(w)+\sum_j n_{i,j}c_j+e_i(n), \tag{6.16}$$

$$s(w) = \alpha^{k_3}s(w)+n_{3,1}a_1+n_{3,2}a_2, \tag{6.17}$$

$$t\Big(w, z+\frac{b_1a_2-b_2a_1}{r}\Big) = t(w,z)+k_3\tau+l_3\frac{b_1a_2-b_2a_1}{r}+\Big(\sum_j n_{3,j}b_j\Big)\alpha^{k_3}s(w)+\sum_j n_{3,j}c_j+e_3(n). \tag{6.18}$$

By Eq. (6.13), using the assumption that $\alpha > 1$, we have

$$\begin{cases} ac(1-\alpha^{k_0}) = c^2(n_{0,1}a_1 + n_{0,2}a_2), \\ ad(\alpha-\alpha^{k_0}) + bc(1-\alpha^{k_0+1}) = cd(1+\alpha)(n_{0,1}a_1+n_{0,2}a_2), \\ bd(1-\alpha^{k_0}) = d^2(n_{0,1}a_1+n_{0,2}a_2). \end{cases} \tag{6.19}$$

If $c \neq 0$, then $k_0 = -1$ by the first two equalities of Eq. (6.19) since $ad - bc > 0$. Similarly, if $d \neq 0$, then one has $k_0 = 1$ by the last two equalities of Eq. (6.19). Therefore, either $c = 0$ and $k_0 = 1$, or $d = 0$ and $k_0 = -1$.

By Eq. (6.15), we have

$$\begin{cases} ac(1-\alpha^{k_i}) = c^2(n_{i,1}a_1 + n_{i,2}a_2), \\ (aca_i + bc + ad)(1-\alpha^{k_i}) = (c^2 a_i + 2cd)(n_{i,1}a_1 + n_{i,2}a_2), \\ ada_i + bd(1-\alpha^{k_i}) = \alpha^{k_i} bca_i + (cda_i + d^2)(n_{i,1}a_1+n_{i,2}a_2). \end{cases} \tag{6.20}$$

If $d = 0$, we have $\alpha^{k_i} bca_i = 0$ for $i = 1,2$ by the last equality of Eq. (6.20). This implies that $a_i = 0$ $(i = 1,2)$ since $\alpha > 1$ and $ad - bc > 0$, which contradicts the fact that $(a_1, a_2)^T$ is an eigenvector of M. So, $c = 0$ and $k_0 = 1$. Also, the second equality of Eq. (6.20), which becomes $ad = \alpha^{k_i} ad$, implies that $k_i = 0$ for $i = 1,2$. Now we may rewrite s as $s(w) = aw + b$ with $a > 0$ and $b \in \mathbb{R}$. Then Eqs. (6.19) and (6.20) imply that

$$\begin{cases} (1-\alpha)b = n_{0,1}a_1 + n_{0,2}a_2, \\ aa_i = n_{i,1}a_1 + n_{i,2}a_2, \quad i = 1,2. \end{cases} \tag{6.21}$$

Comparing the coefficients of z of Eq. (6.14), we have

$$\sum_p C_{p,1}(\alpha w - \sqrt{-1})^p = \sum_p C_{p,1}(w-\sqrt{-1})^p,$$

which implies that $C_{p,1} = 0$ for all $p \geq 1$. For the z-constant part of Eq. (6.14), one has the following equality

$$\begin{aligned} &\sum_p C_{p,0}(\alpha w - \sqrt{-1})^p + C_{0,1}\tau \\ &= \sum_p C_{p,0}(w-\sqrt{-1})^p + (n_{0,1}b_1 + n_{0,2}b_2)\alpha a w + \tau + l_0 \frac{b_1 a_2 - b_2 a_1}{r} + n_{0,1}c_1 + n_{0,1}c_2 + e_0(n). \end{aligned}$$

Similarly, we conclude that $C_{p,0} = 0$ for $p \geq 2$. Hence, we may rewrite t as $t(w,z) = Aw + B + Cz$ for some $A, B, C \in \mathbb{C}$ with $C \neq 0$. Now comparing the coefficients of w of Eqs. (6.14) and (6.16), we get

$$\begin{cases} \left(1-\frac{1}{\alpha}\right)\frac{A}{a} = n_{0,1}b_1 + n_{0,2}b_2, \\ (C-1)\tau = l_0\frac{b_1a_2-b_2a_1}{r} + (n_{0,1}b_1+n_{0,2}b_2)\alpha b + n_{0,1}c_1 + n_{0,2}c_2 + e_0(n), \\ Cb_i = a(n_{i,1}b_1 + n_{i,2}b_2), \\ Aa_i + Cc_i = l_i\frac{b_1a_2-b_2a_1}{r} + (n_{i,1}b_1+n_{i,2}b_2)b + n_{i,1}c_1 + n_{i,2}c_2 + e_i(n). \end{cases} \tag{6.22}$$

Finally, Eqs. (6.17) and (6.18) simply give that $k_3 = 0$, $l_3 = C$ and $n_{3,1} = n_{3,2} = 0$.

Therefore, N has two eigenvalues a and C/a with eigenvectors $(a_1, a_2)^T$ and $(b_1, b_2)^T$, respectively. It is not hard to see that the matrix associated to u^{-1} is N^{-1}. Then $\det N = \pm 1$ and hence $N \in \mathrm{GL}_2(\mathbb{Z})$. Moreover, by Eqs. (6.21) and (6.22), u (if exists) is determined by $\mathbf{n}_0$, N and B.

Let

$$\Gamma \coloneqq \{N \in \mathrm{GL}_2(\mathbb{Z}) \mid N \text{ and } M \text{ are simultaneously diagonalisable}\},$$

which is an abelian group, and let $\tau\colon \widetilde{\mathrm{Aut}}(X) \longrightarrow \Gamma$ be the homomorphism (not necessarily surjective) defined by $u \mapsto N$, where N is the matrix associated with u as constructed above. Let K be the kernel of this homomorphism, with image $\Gamma' \leq \Gamma$. It is clear that any automorphism in K has the form

$$(w, z) \longmapsto (w + b, z + Aw + B)$$

for some $b \in \mathbb{R}$ and $A, B \in \mathbb{C}$ satisfying Eqs. (6.21) and (6.22). An explicit calculation gives that the centre $Z(K) \simeq \mathbb{C}$, which is generated by

$$(w, z) \longmapsto (w, z + B).$$

Further, $K/Z(K) \simeq \mathbb{Z} \rtimes \mathbb{Z}$ is generated by the images of

$$(w, z) \longmapsto (w + b, z) \quad \text{and} \quad (w, z) \longmapsto (w, z + Aw).$$

Combining these, the snake lemma shows the diagrams below are commutative with exact rows and columns.

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & H(r) \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & G_M^{(+)} \arrow[r] \arrow[d] & \widetilde{\mathrm{Aut}}(X) \arrow[r] \arrow[d, "\tau"] & \mathrm{Aut}(X) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \Gamma' \arrow[r] \arrow[d] & \Gamma'/\mathbb{Z} \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & Z(H(r)) \arrow[r] \arrow[d] & H(r) \arrow[r] \arrow[d] & H(r)/Z(H(r)) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & Z(K) \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K/Z(K) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & F' \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & F'' \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

Note that F' is abelian and F'' is finite and metabelian. Consequently, F, as the quotient of K by $H(r)$, is virtually abelian and abelian-by-metabelian.

By the above diagrams and Lemma 2.6 (1):

Theorem 6.3. *Let X be an Inoue surface of type $S^{(+)}$ or $S^{(-)}$. Then $\mathrm{Aut}(X)$ is solvable. Moreover, for any subgroup $G \leq \mathrm{Aut}(X)$, one has $\ell_{\mathrm{vir}}(G) \leq 4$.*

Similar to the argument before Theorem 6.2, any torsion subgroup of $\Gamma'/\mathbb{Z}$ is finite. It follows from the diagram above that each torsion subgroup of $\mathrm{Aut}(X)$ has an abelian subgroup with bounded index depending on X.

Theorem 6.4. *Let X be an Inoue surface of type $S^{(+)}$ or $S^{(-)}$. Then $\mathrm{Aut}(X)$ is T-Jordan.*

7. Class VII Surfaces with $b_2 > 0$

We will prove Proposition 1.6 in this section.

Proof of Proposition 1.6. We may assume that G is an infinite group. By Corollary 4.2, we only need to consider such an X that does not have any curves. Consider the action of $\mathrm{Aut}(X)$ on $H^*(X, \mathbb{Q})$, and let $\mathrm{Aut}^*(X)$ be the kernel of $\mathrm{Aut}(X) \longrightarrow \mathrm{GL}(H^*(X, \mathbb{Q}))$. Note that the image of G in $\mathrm{GL}(H^*(X, \mathbb{Q}))$ is finite (cf. [17, Lemma 5.2]). By passing to a finite index subgroup, we may assume that $G \leq \mathrm{Aut}^*(X)$.

Pick $\mathrm{id} \neq g \in G$, and let G' be the centraliser of $\langle g \rangle$ in G. Since g has finite order, $[G : G']$ is finite. Replacing G by the finite-index subgroup G', we may assume that $g \in Z(G)$, the centre of G. Note that the fixed point set $\mathrm{Fix}(g)$ of g is finite with cardinality $|\mathrm{Fix}(g)| = b_2(X)$ (cf. [29, Lemma 5.4]). Consider the action of G on the finite set $\mathrm{Fix}(g)$; noting that g is in the centre of G. Replacing G by a subgroup of index $\leq b_2(X)!$, we may assume that G fixes some point $x \in \mathrm{Fix}(g)$. Applying Lemma 2.3, the torsion group G is embedded into $\mathrm{GL}_2(\mathbb{C})$. By Theorem 1.1, we have G is virtually abelian. □

8. Compact Kähler Surfaces

In this section, we deal with smooth compact surfaces which are Kähler. The idea of this section comes from De-Qi Zhang.

Let X be a compact Kähler manifold. For a subgroup G of $\mathrm{Aut}(X)$, define $G^0 := G \cap \mathrm{Aut}_0(X)$ and denote by $N(G)$ the set of all elements in G with zero entropy.

Lemma 8.1. *Let X be a smooth compact Kähler surface, and $G \leq \mathrm{Aut}(X)$ a virtually solvable subgroup. Then $\ell_{\mathrm{vir}}(G) \leq 1 + \ell_{\mathrm{vir}}(N(G))$.*

Proof. By taking a finite-index subgroup, we may assume that G is a solvable, $N(G)$ is a normal subgroup of G and $G/N(G) \simeq \mathbb{Z}^r$ for some $r = 0, 1$ (cf. [5, Theorem 1.5]). Take a finite-index subgroup $N' \leq N(G)$ such that $\ell(N') = \ell_{\mathrm{vir}}(N(G))$ and $\ell(N'/N'^0) \leq 1$ as in [8, Theorem 1.2 and Proposition 2.6].

Note that $N(G)|_{H^2(X,\mathbb{C})}$ is finitely generated (cf. [5, Theorem 2.2(1)]). Since $N'|_{H^2(X,\mathbb{C})}$ has finite index in $N(G)|_{H^2(X,\mathbb{C})}$, after replacing N' by a finite-index subgroup, we may assume that $N'|_{H^2(X,\mathbb{C})}$ is characteristic in $N(G)|_{H^2(X,\mathbb{C})}$ by Lemma 2.8. In particular, $N'|_{H^2(X,\mathbb{C})} \lhd G|_{H^2(X,\mathbb{C})}$.

Let $N \lhd N(G)$ be the preimage of $N'|_{H^2(X,\mathbb{C})}$. Then $N \lhd G$, $N(G)/N$ is finite, $\ell(N) = \ell_{\mathrm{vir}}(N(G))$ and $\ell(N/N^0) \leq 1$. Since $(G/N)/(N(G)/N) \simeq \mathbb{Z}^r$, replacing G by a finite-index subgroup, we may assume $G/N \simeq \mathbb{Z}^r$ (cf. [36, Lemma 2.4]). It follows that

$$\ell_{\mathrm{vir}}(G) \leq \ell(G) \leq \ell(G/N) + \ell(N) \leq 1 + \ell_{\mathrm{vir}}(N(G)).$$

□

Indeed, it follows from the above proof that

$$\ell_{\mathrm{vir}}(G) \leq 1 + \ell(N) \leq 1 + \ell(N/N^0) + \ell(N^0) \leq 2 + \ell(N^0).$$

When $\kappa(X) \geq 0$, $\mathrm{Aut}_0(X)$ is a complex torus (cf. [12, Corollary 5.11]). In particular, $N^0 = N \cap \mathrm{Aut}_0(X)$ is abelian and hence of derived length (at most) 1. Thus, we have

Proposition 8.2. *Let X be a smooth compact Kähler surface with Kodaira dimension $\kappa(X) \geq 0$. Let $G \leq \mathrm{Aut}(X)$ be virtually solvable. Then $\ell_{\mathrm{vir}}(G) \leq 3$.*

Suppose that now X is a compact Kähler surface with $\kappa(X) = -\infty$. Then X is projective.

Proposition 8.3. *Let X be a smooth projective surface. Suppose that $G \le \mathrm{Aut}(X)$ is virtually solvable, and $U \le \mathrm{Aut}_0(X)$ is non-trivial unipotent with $U \lhd G$. Then the derived length $\ell(U) \le 2$ and after replacing by a finite-index subgroup, we have $G = G^0$.*

Proof. Replacing by closure or finite-index subgroup, we may assume that $G \le \mathrm{Aut}(X)$ is closed and solvable, $G|_{\mathrm{NS}(X)_\mathbb{C}}$ is connected and closed, and $U \le \mathrm{Aut}_0(X)$ is closed and unipotent (cf. [26, Proof of Lemma 2.1(2)]).

By Lemma 2.9, after replacing G by a finite-index subgroup, there is a subgroup $Z_1 \simeq \mathbb{G}_a \le U$ which is normal in G. Let $X \dashrightarrow Y = X/Z_1$ be the quotient map to the cycle space (cf. [12, § 4]). Then G acts biregularly on Y and the map is G-equivariant. Replacing the map by the graph and taking equivariant resolutions, we may assume that both X and Y are smooth, and the map $X \longrightarrow Y$ is a morphism, a $\mathbb{P}^1$-fibration, whose general fibre is a curve with dense orbit $Z_1 \cdot x \simeq \mathbb{A}^1$ and whose section at infinity is fixed by G. Consider the exact sequence

$$1 \longrightarrow K \longrightarrow U \longrightarrow U|_Y \longrightarrow 1.$$

Then both K and $U|_Y$ are unipotent. Now the restriction to generic fibre $K \longrightarrow K|_{X_{\overline{k(Y)}}}$ is injective with image contained in a closed unipotent group acting faithfully on $X_{\overline{k(Y)}} = \mathbb{P}^1_{\overline{k(Y)}}$, and hence is abelian. It is clear that either $U|_Y = \{1\}$, or $U|_Y \simeq \mathbb{G}_a$ and $Y = \mathbb{P}^1$. Hence $\ell(U) \le \ell(U|_Y) + 1 \le 2$.

Note that G fixes a big class H_Y on the curve Y. Take the class H to be the sum of the pullback of H_Y and the section at infinity of the $\mathbb{P}^1$-fibration $X \longrightarrow Y$. Then G fixes the big class H on X and hence replaced by a finite-index subgroup, G is contained in $\mathrm{Aut}_0(X)$ (cf. [12, Theorem 4.8], [21, Proposition 2.2] and [6, Corollary 2.2]). Thus $G = G^0$. □

Proposition 8.4. *Let G be a smooth projective surface. Suppose that $G \le \mathrm{Aut}(X)$ is virtually solvable. Then $\ell_{\mathrm{vir}}(G) \le 4$.*

Proof. Replacing by finite-index subgroup, we may assume that G is solvable.

First assume that $G = G^0$ (and connected). Let U be the unipotent radical of (the linear part of) $G = G^0$ so that G/U is a semi-torus and hence commutative (cf. [14, Lemma 4]). Thus, $\ell_{\mathrm{vir}}(G) \le 1 + \ell(U) \le 3$ by Proposition 8.3.

Now we may assume that $G \ne G^0$ even after replacing G by a finite-index subgroup. Let $G_0^0 \lhd G^0$ be the identity component, Then by Proposition 8.3, the unipotent radical of (the linear part of) G_0^0 is trivial and hence G_0^0 itself is a semi-torus and hence commutative.

If G is of zero entropy, by [8, Theorem 1.2] one has $\ell(G/G^0) \le 1$ after replacing G by a finite-index subgroup. Consider the short exact sequence

$$1 \longrightarrow G^0/G_0^0 \longrightarrow G/G_0^0 \longrightarrow G/G^0 \longrightarrow 1.$$

Since the first term is a finite group, its centraliser in G/G_0^0 is of finite index (say of index 1, after replacing G by a finite index subgroup). Thus,

$$\ell(G/G_0^0) \le 1 + \ell(G/G^0) = 2,$$
$$\ell_{\mathrm{vir}}(G) \le \ell(G) \le \ell(G/G_0^0) + \ell(G_0^0) \le 3.$$

If G is not of zero entropy, using Lemma 8.1 and the argument above, $\ell_{\mathrm{vir}}(G) \le 1 + \ell_{\mathrm{vir}}(N(G)) \le 1 + 3 = 4$. □

9. Summary

In the Kähler setting, Proposition 1.5 follows from Theorem 1.4; Theorem 1.10 has been proved in [5, Theorem 1.5]; Theorem 1.11 follows from Propositions 8.2 and 8.4.

Let X be a smooth compact complex surface, which is not Kähler. In particular, X is not rational or ruled. Thus, there is a unique minimal surface X' bimeromorphic to X such that

$$\mathrm{Aut}(X) \subseteq \mathrm{Bim}(X) \simeq \mathrm{Bim}(X') = \mathrm{Aut}(X').$$

See [31, Proposition 3.5]. It suffices for us to prove Proposition 1.5 and Theorems 1.10 and 1.11 for minimal surfaces (in the non-Kähler setting). Note that non-Kähler minimal surfaces are surfaces of class VII, (primary or secondary) Kodaira surfaces and some properly elliptic surfaces (cf. [1, IV. Theorem 3.1 and VI. Theorem 1.1]). Since every minimal surface of class VII with vanishing b_2 is either a Hopf surface or an Inoue surface (cf. [3]) and minimal surface of class VII with algebraic dimension 1 is a Hopf surface (cf. [1, V. Theorem 18.6]), minimal surfaces of class VII not in Ξ are exactly Hopf surfaces or Inoue surfaces.

Table 1. non-Kähler minimal smooth compact complex surfaces

class of the surface X	$\kappa(X)$	$a(X)$	$b_1(X)$	$b_2(X)$	$e(X)$
surfaces of class VII	$-\infty$	$0, 1$	1	≥ 0	≥ 0
primary Kodaira surfaces	0	1	3	4	0
secondary Kodaira surfaces	0	1	1	0	0
properly elliptic surfaces	1	1			≥ 0

Then Proposition 1.5 follows from Theorems 3.4, 5.4, 6.2 and 6.4 and Corollary 4.2; Theorem 1.10 follows from Theorems 2.7, 3.4, 3.5, 5.1, 6.1 and 6.3; and Theorem 1.11 follows from Theorems 2.7, 3.4, 3.5, 5.2, 6.1 and 6.3. Finally, Proposition 1.6 has been proved in Section 7.

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