EQUIVARIANT KÄHLER MODEL FOR FUJIKI'S CLASS

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Abstract. Let X be a compact complex manifold in Fujiki's class $\mathcal{C}$, i.e., admitting a big $(1,1)$-class $[\alpha]$. Consider $\mathrm{Aut}(X)$ the group of biholomorphic automorphisms and $\mathrm{Aut}_{[\alpha]}(X)$ the subgroup of automorphisms preserving the class $[\alpha]$ via pullback. We show that X admits an $\mathrm{Aut}_{[\alpha]}(X)$-equivariant Kähler model: there is a bimeromorphic holomorphic map $\sigma\colon \widetilde{X} \to X$ from a Kähler manifold $\widetilde{X}$ such that $\mathrm{Aut}_{[\alpha]}(X)$ lifts holomorphically via σ.

There are several applications. We show that $\mathrm{Aut}_{[\alpha]}(X)$ is a Lie group with only finitely many components. This generalizes an early result of Fujiki and Lieberman on the Kähler case. We also show that every torsion subgroup of $\mathrm{Aut}(X)$ is almost abelian, and $\mathrm{Aut}(X)$ is finite if it is a torsion group.

Contents

1. Introduction

Let X be a compact complex manifold in *Fujiki's class* $\mathcal{C}$, i.e., one of the following three equivalent assumptions is satisfied:

(1) X is the meromorphic image of a compact Kähler manifold;
(2) X is bimeromorphic to a compact Kähler manifold;
(3) X admits a big $(1,1)$-class $[\alpha]$.

We refer to [Fuj78, Definition 1.1 and Lemma 1.1], [Var89, Chapter IV, Theorem 5] and [DP04, Theorem 0.7] for the equivalence and some properties of Fujiki's class $\mathcal{C}$.

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It is a natural question that can we study the group of biholomorphic automorphisms $\mathrm{Aut}(X)$ via some well-chosen Kähler model of X? So this requires us to trace back to the construction of the Kähler model. Indeed, Demailly and Paun showed in the proof of [DP04, Theorem 3.4] that if a compact complex manifold X admits a big $(1,1)$-class $[\alpha]$, then there is a bimeromorphic holomorphic map $\sigma\colon X' \to X$ from a Kähler manifold X' obtained by a sequence of blowups along smooth centres determined by the ideal sheaf $\mathcal{J}$ corresponding to some Kähler current T with analytic singularities (cf. Definition 2.2) in $[\alpha]$. In general, $\mathrm{Aut}(X)$ does not lift via σ, for the first very simple reason that the class $[\alpha]$ may not be preserved by $\mathrm{Aut}(X)$. Then we focus ourselves on the subgroup

$$\mathrm{Aut}_{[\alpha]}(X) := \{g \in \mathrm{Aut}(X) \mid g^*[\alpha] = [\alpha]\}.$$

However, one still cannot expect the lifting of $\mathrm{Aut}_{[\alpha]}(X)$, for the second reason that the blown-up ideal sheaf $\mathcal{J}$ is not $\mathrm{Aut}_{[\alpha]}(X)$-invariant. Therefore, we need to find another Kähler model in a more natural way. Our idea is to consider the ideal sheaf generated by $g^*\mathcal{J}$ for all $g \in \mathrm{Aut}_{[\alpha]}(X)$.

The following is our main result. Note that the lift action of $\mathrm{Aut}_{[\alpha]}(X)$ on $\widetilde{X}$ is given by $g(-) = (\sigma^{-1} \circ g \circ \sigma)(-)$. By a holomorphic lifting we mean that the lift action is also holomorphic, i.e., the induced map $\mathrm{Aut}_{[\alpha]}(X) \times \widetilde{X} \to \widetilde{X}$ is holomorphic.

Theorem 1.1. *Let X be a compact complex manifold in Fujiki's class $\mathcal{C}$. For any big $(1,1)$-class $[\alpha]$ on X, there exists a bimeromorphic holomorphic map $\sigma\colon \widetilde{X} \to X$ from a Kähler manifold $\widetilde{X}$ such that $\mathrm{Aut}_{[\alpha]}(X)$ lifts holomorphically via σ.*

Remark 1.2. The bimeromorphic holomorphic map σ in Theorem 1.1 is indeed obtained by a sequence of $\mathrm{Aut}_{[\alpha]}(X)$-equivariant blowups along smooth centres.

Let X be a compact complex manifold. It is known that $\mathrm{Aut}(X)$ has the natural structure of a complex Lie group acting biholomorphically on X (cf. [Dou66]). Denote by $\mathrm{Aut}_0(X)$ the connected component of $\mathrm{Aut}(X)$ containing the identity. Since it is connected, the pullback action of $\mathrm{Aut}_0(X)$ on the (discrete) lattice $H^2(X,\mathbb{Z})$ is trivial. When the $\partial\overline{\partial}$-lemma holds on X, we have the Hodge decomposition that $H^{1,1}(X,\mathbb{R})$ is a subspace of $H^2(X,\mathbb{R})$ and then $\mathrm{Aut}_0(X)$ acts also trivially on $H^{1,1}(X,\mathbb{R})$. Note that $\partial\overline{\partial}$-lemma holds when X admits a big $(1,1)$-class. So in this paper, our $\mathrm{Aut}_0(X)$ is always a subgroup of $\mathrm{Aut}_{[\alpha]}(X)$ for any big $(1,1)$-class $[\alpha] \in H^{1,1}(X,\mathbb{R})$. We refer to [DGMS75, Lemma (5.15) and Proposition (5.17)] and [Fuj78, Proposition 1.6 and Corollary 1.7] for the details.

When X is a Kähler manifold with a Kähler form α, Fujiki (cf. [Fuj78, Theorem 4.8]) and Lieberman (cf. [Lie78, Proposition 2.2]), separately, proved that

$$[\mathrm{Aut}_{[\alpha]}(X) : \mathrm{Aut}_0(X)] < \infty.$$

Their proof heavily relies on the Kähler form α (or at least the existence of a Kähler form).

Nevertheless, with the help of our Theorem 1.1, we can show the following result.

Corollary 1.3. *Let X be a compact complex manifold (in Fujiki's class $\mathcal{C}$). Then*

$$[\operatorname{Aut}_{[\alpha]}(X) : \operatorname{Aut}_0(X)] < \infty$$

for any big $(1,1)$-class $[\alpha]$ on X.

We also give applications on torsion group actions. When X is a projective variety defined over any field k of characteristic 0, Javanpeykar [Jav21, Theorem 1.4] showed that the group of k-automorphisms $\operatorname{Aut}_k(X)$ is finite if it is torsion. We show that the same result holds true for normal compact complex spaces in Fujiki's class $\mathcal{C}$.

Corollary 1.4. *Let X be a normal compact complex space in Fujiki's class $\mathcal{C}$. Then $\operatorname{Aut}(X)/\operatorname{Aut}_0(X)$ has bounded torsion subgroups, i.e., there is a constant C such that $|G| \le C$ for any torsion subgroup $G \le \operatorname{Aut}(X)/\operatorname{Aut}_0(X)$. In particular, $\operatorname{Aut}(X)$ is a torsion group if and only if it is finite.*

In the previous work [MPZ20], Perroni, Zhang and the second author showed the Jordan property for $\operatorname{Aut}(X)$ when X is a compact complex space in Fujiki's class $\mathcal{C}$. The proof there developed a trick by finding some invariant Kähler submanifold Z (with $\operatorname{Aut}(X)$ shrunk a bit) and transferring the attention to a new compact Kähler manifold X': the compactified normal bundle $\mathbb{P}_Z(\mathcal{N}_{Z/X} \oplus \mathcal{O}_Z)$. Note however $\operatorname{Aut}(X')$ only keeps tracking of finite subgroups (more generally reductive subgroups or torsion subgroups) of $\operatorname{Aut}(X)$.

Now with the help of Corollary 1.3, we can provide an alternative proof.

Corollary 1.5. *Let X be a compact complex space in Fujiki's class $\mathcal{C}$. Then there exists a constant J such that any torsion subgroup G of $\operatorname{Aut}(X)$ has an abelian subgroup $H \le G$ with $[G : H] \le J$. In particular, $\operatorname{Aut}(X)$ has Jordan property.*

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2. Preliminaries

Let X be a compact complex manifold with a fixed positive definite Hermitian form ω. Let α be a closed $(1,1)$-form. We use $[\alpha]$ to represent its class in $H^{1,1}(X,\mathbb{R})$. We define the following positivity notions (independent of the choice of ω):

- $[\alpha]$ is *Kähler* if it contains a Kähler form, i.e., if there is a smooth function φ such that $\alpha + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi \geq \epsilon\omega$ on X for some $\epsilon > 0$.
- $[\alpha]$ is *big* if it contains a Kähler current T, i.e., if there is a quasi-plurisubharmonic function $\varphi\colon X \to \mathbb{R}\cup\{-\infty\}$ such that $T := \alpha + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi \geq \epsilon\omega$ holds weakly as currents on X for some $\epsilon > 0$.

Recall that a *quasi-plurisubharmonic* function means that locally it is given by the sum of a plurisubharmonic function plus a smooth function.

We recall the definition of the non-Kähler locus of a big class $[\alpha]$.

Definition 2.1. Let X be a compact complex manifold and $[\alpha]$ a big $(1,1)$-class. Then the *non-Kähler locus* of $[\alpha]$ (or α) is defined and denoted by

$$E_{nK}(\alpha) := E_{nK}([\alpha]) := \bigcap_{T\in[\alpha]} \operatorname{Sing}(T),$$

where the intersection ranges over all Kähler currents $T = \alpha + \sqrt{-1}\partial\bar{\partial}\varphi$ in the class $[\alpha]$, and $\operatorname{Sing}(T)$ is the complement of the set of points $x \in X$ such that φ is smooth near x.

We recall the basic definition of analytic singularities (cf. [Bou02, Section 2.1]). Note that the data $(\mathcal{J}, c)$ below is not uniquely determined by the function φ with analytic singularities.

Definition 2.2. Let X be a compact complex manifold and $[\alpha]$ a closed $(1,1)$-class.

(1) Given a coherent ideal sheaf $\mathcal{J}$ and a constant $c > 0$, we say that a function φ has *singularities of type* $(\mathcal{J}, c)$ if locally it can be written as

$$\varphi = c\log\left(\sum_{j=1}^{n}|f_j|^2\right) + h$$

for some local generators (f_j) of $\mathcal{J}$ and some smooth function h.

(2) We say that φ has *analytic singularities* if it has singularities of type $(\mathcal{J}, c)$ for some coherent ideal sheaf $\mathcal{J}$ and some constant $c > 0$.

(3) We also say that a closed current $T \in [\alpha]$ has *analytic singularities* if it can be written as $T = \alpha + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi$ such that the potential function φ has analytic singularities.

The following is a direct application of the regularization theorem by Demailly [Dem92]; see also [DP04, Theorem 3.2].

Theorem 2.3. *Let X be a compact complex manifold with a big $(1,1)$-class $[\alpha]$. Then there exists a Kähler current $T \in [\alpha]$ with analytic singularities.*

Boucksom [Bou04, Theorem 3.17] observed further that indeed one can find a Kähler current with analytic singularities and also with minimal singular locus.

Theorem 2.4. *Let X be a compact complex manifold with a big $(1,1)$-class $[\alpha]$. Then there exists a Kähler current $T \in [\alpha]$ with analytic singularities such that $\mathrm{Sing}(T) = E_{nK}(\alpha)$. In particular, $E_{nK}(\alpha)$ is a closed analytic subspace of X and $E_{nK}(\alpha) = \emptyset$ if and only if $[\alpha]$ is Kähler.*

3. Equivariant Kähler model

In this section, we prove Theorem 1.1.

First, we recall the following theorem on the equivariant log resolution of an ideal sheaf by Bierstone and Milman [BM97, Theorem 1.10].

Theorem 3.1. *Let X be a compact complex manifold with a coherent ideal sheaf $\mathcal{J}$. Let $G \leq \mathrm{Aut}(X)$ such that $g^*\mathcal{J} = \mathcal{J}$ for any $g \in G$. Then there is a finite sequence*

$$X_k \xrightarrow{\sigma_k} \cdots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 = X$$

of G-equivariant blowups σ_j, $j = 1, \ldots, k$, along smooth centres, such that $\sigma^{-1}\mathcal{J} \cdot \mathcal{O}_{X_k}$ is a normal-crossings divisor.

Proof. In [BM97, Theorem 1.10], the sequence is taken as blowups along $\mathrm{inv}_{\mathcal{J}}$-admissible centres. Note that an $\mathrm{inv}_{\mathcal{J}}$-admissible centre is determined by $\mathcal{J}$ itself and hence G-invariant. So the blowups are G-equivariant. We refer to [Wlo09] for the details. □

The following lemma is well-known, and we give a proof for the convenience of the readers. The result holds true in the algebraic setting with the same proof.

Lemma 3.2. *Let $\sigma\colon X' \to X$ be a bimeromorphic holomorphic map of compact complex spaces. Let G be a complex Lie group acting holomorphically on X such that G lifts via σ. Then the induced action of G on X' is also holomorphic.*

Proof. Consider the holomorphic map

$$\phi\colon G \times X' \times X' \to G \times X \times X$$

via $(g, x', y') \mapsto (g, \sigma(x'), \sigma(y'))$.

Consider the graph of G on X:

$$\Gamma := \{(g, x, y) \in G \times X \times X \mid y = g(x)\}$$

which is a closed analytic subspace of $G \times X \times X$ since the G-action on X is holomorphic.

For each $g \in G$, let $\Gamma_{g|_X}$ be the graph of $g|_X$ viewed as a fibre of $\Gamma \to G$ over g. Define $\Gamma_{g|_{X'}}$ similarly. Then $\Gamma_{g|_{X'}}$ is the proper transform of $\Gamma_{g|_X}$ in $G \times X' \times X'$ via ϕ. Let Γ' be the proper transform of Γ in $G \times X' \times X'$. Then Γ' is a closed analytic subspace of $G \times X' \times X'$. By taking the first projection, the fibre of $\Gamma' \to G$ over g is just $\Gamma_{g|_{X'}} \cong X'$ since $g \colon X' \to X'$ is biholomorphic. Therefore, the projection to the first two factors $\Gamma' \to G \times X'$ is biholomorphic. Note that Γ' is the graph of G on X'. So the G-action on X' is holomorphic. $\square$

Proof of Theorem 1.1. By Theorem 2.3, there exists a Kähler current

$$T = \alpha + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi \in [\alpha]$$

with analytic singularities of type $(\mathcal{J}_\varphi, c)$ for some coherent ideal sheaf $\mathcal{J}_\varphi$ and $c > 0$. This means that we may write (locally) that

$$\varphi = c\log\left(\sum_{j=1}^{n}|f_j|^2\right) + h$$

where (f_j) are local generators of $\mathcal{J}_\varphi$ and h is smooth. Consider the $\mathrm{Aut}_{[\alpha]}(X)$-invariant ideal sheaf

$$\mathcal{J} := \sum_{g \in \mathrm{Aut}_{[\alpha]}(X)} g^*\mathcal{J}_\varphi.$$

Since X is compact, $\mathcal{J}$ is also coherent (cf. [Dem97, Chapter II, Property (3.22)]). Then we may write

$$\mathcal{J} = \sum_{i=1}^{m}\mathcal{J}_i$$

where $\mathcal{J}_i := g_i^*\mathcal{J}_\varphi$ for some $g_i \in \mathrm{Aut}_{[\alpha]}(X)$.

By Theorem 3.1, there is an $\mathrm{Aut}_{[\alpha]}(X)$-equivariant biholomorphic holomorphic map

$$\sigma \colon \widetilde{X} \to X$$

which is obtained by a sequence of blowups along smooth centres such that $\sigma^{-1}\mathcal{J} \cdot \mathcal{O}_{\widetilde{X}}$ is invertible.

Let $\mathcal{J}_i' := \sigma^{-1}\mathcal{J}_i \cdot \mathcal{O}_{\widetilde{X}}$. Then

$$\sum_{i=1}^{m}\mathcal{J}_i' = \sigma^{-1}\mathcal{J} \cdot \mathcal{O}_{\widetilde{X}}$$

is invertible. Denote by

$$\widetilde{\mathcal{J}}_i := \mathcal{J}_i' \cdot \left(\sigma^{-1}\mathcal{J} \cdot \mathcal{O}_{\widetilde{X}}\right)^{-1}$$

which is still an ideal sheaf. Then

$$\sum_{i=1}^{m} \widetilde{\mathcal{J}}_i = \mathcal{O}_{\widetilde{X}}.$$

We give several more notations.

- Let s be the local generator of invertible sheaf $\sigma^{-1}\mathcal{J} \cdot \mathcal{O}_{\widetilde{X}}$.
- We define a positive current D which is the integration current along the divisor $\{s = 0\}$. By the Poincaré-Lelong formula, D can be locally written as $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|s|^2$.
- Let E be the reduced (full) exceptional divisor (locus) of σ.
- Fix a (positive definite) Hermitian form ω on X such that $g_i^*T \geq \omega$ for each i.
- Let u_E be some closed smooth $(1,1)$-form in the class $[E]$ such that $\sigma^*\omega - \epsilon u_E$ is a positive definite Hermitian form when $\epsilon > 0$ is sufficiently small (cf. [DP04, Proof of Lemma 3.5]).

We show that $\widetilde{X}$ is Kähler by the following claim.

Claim 3.3. *The class $[\widetilde{\alpha}] := [\sigma^*\alpha - cD - \epsilon u_E]$ contains a Kähler form for sufficiently small $\epsilon > 0$.*

Let $T_i' := \sigma^* g_i^* T$ and $f_{j,i}' := f_j \circ g_i \circ \sigma$. Then the potential function of T_i' is locally of the form

$$\varphi_i' = c\log\Big(\sum_{j=1}^{n}|f_{j,i}'|^2\Big) + h \circ g_i \circ \sigma$$

and $(f_{j,i}')_j$ are local generators of $\mathcal{J}_i'$. Let s_i be the g.c.d. of the $(f_{j,i}')$'s. Then we can write down the Siu's decomposition

$$T_i' = R_i + cD_i$$

where D_i is the integration current along the divisor $\{s_i = 0\}$ which can be locally written as $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|s_i|^2$ and $R_i \geq \sigma^*\omega$ such that the Lelong super-level (analytic) set $E_c(R_i)$ has codimension at least 2 (cf. [Bou04, Section 2.2.1–2.2.2] and [Bou02, Section 2.2]). Note that s is a factor of s_i since $\mathcal{J}_i' \subseteq \sigma^{-1}\mathcal{J} \cdot \mathcal{O}_{\widetilde{X}}$. Then we have $D_i \geq D$ and hence

$$T_i' - cD \geq R_i \geq \sigma^*\omega.$$

We now construct

$$\widetilde{T}_i := T_i' - cD - \epsilon u_E = (\sigma^* g_i^*\alpha - \epsilon u_E) + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_i' - cD \in [\widetilde{\alpha}]$$

with $\epsilon > 0$ sufficiently small. By the construction, the potential function of $\widetilde{T}_i$ can be locally written as

$$\begin{aligned}\widetilde{\varphi}_i &= c\log\Big(\sum_{j=1}^{n}|f'_{j,i}|^2\Big) + h\circ g_i\circ\sigma - c\log|s|^2\\ &= c\log\Big(\sum_{j=1}^{n}|\widetilde{f}_{j,i}|^2\Big) + h\circ g_i\circ\sigma\end{aligned}$$

where $f'_{j,i} = \widetilde{f}_{j,i}\cdot s$. Note that

$$\widetilde{T}_i \geq \sigma^*\omega - \epsilon u_E$$

while the right-hand side is positive definite by the construction before Claim 3.3. Therefore, $\widetilde{T}_i$ is also a Kähler current.

Note that $\widetilde{\mathcal{J}}_i$ is locally generated by $(\widetilde{f}_{j,i})_j$. Then

$$\bigcap_{i=1}^{m}\mathrm{Sing}(\widetilde{T}_i) = \bigcap_{i=1}^{m}V(\widetilde{\mathcal{J}}_i) = V\Big(\sum_{i=1}^{m}\widetilde{\mathcal{J}}_i\Big) = \emptyset$$

where $V(-)$ is the zeros of the ideal sheaf. In particular, $E_{nK}(\widetilde{\alpha}) = \emptyset$ and the claim is proved by Theorem 2.4.

Finally, note that the induced action of $\mathrm{Aut}_{[\alpha]}(X)$ on $\widetilde{X}$ is holomorphic by Lemma 3.2. □

4. Boundedness on group components

In this section, we are going to prove Corollaries 1.3 and 1.5. Our Corollary 1.3 plays a key role in the reduction to the Kähler case.

Recall that Fujiki [Fuj78, Theorem 4.8] and Lieberman [Lie78, Proposition 2.2] showed separately that

$$[\mathrm{Aut}_{[\alpha]}(X) : \mathrm{Aut}_0(X)] < \infty$$

for a compact Kähler manifold X and a Kähler form α. For the convenience of our proof for Corollary 1.3, we give a version on a big class generalized by Dinh, Hu and Zhang; see [DHZ15, Theorem 2.1] for a more generalized setting. Note however their proof still requires the existence of a Kähler form. We refer to [MZ18, Propositions 2.9 and 3.6] for a generalized explanation by cone analysis and linear algebra.

Theorem 4.1. *Let X be a compact Kähler manifold. Then*

$$[\mathrm{Aut}_{[\alpha]}(X) : \mathrm{Aut}_0(X)] < \infty$$

for any big $(1,1)$-class $[\alpha]$.

We need the following lemma concerning the descending of connected complex Lie group action via a bimeromorphic holomorphic map.

Lemma 4.2. *Let $\sigma\colon X' \to X$ be a bimeromorphic holomorphic map of normal compact complex spaces. Then the map*

$$\tau\colon \operatorname{Aut}_0(X') \to \operatorname{Aut}_0(X), \quad g \mapsto \sigma \circ g \circ \sigma^{-1}$$

is an injective complex Lie group homomorphism.

Proof. Note that $\sigma_* \mathcal{O}_{X'} = \mathcal{O}_X$ and $\operatorname{Aut}_0(X')$ is connected. Then the lemma is essentially a corollary of the rigidity lemma (cf. [Akh95, § 2.4, Lemmas 1 and 2]). Note that τ is injective because σ is bimeromorphic. □

Theorem 4.3. *Let X be a normal compact complex space in Fujiki's class $\mathcal{C}$. Then there is a Kähler model $\sigma\colon \widetilde{X} \to X$ such that the map*

$$\tau\colon \operatorname{Aut}_0(\widetilde{X}) \to \operatorname{Aut}_0(X)$$

via $\tau(g) \coloneqq \sigma \circ g \circ \sigma^{-1}$ is a complex Lie group isomorphism.

Proof. Let $\pi\colon X' \to X$ be an $\operatorname{Aut}(X)$-equivariant resolution of singularities (cf. [BM97, Theorem 13.2]). It follows from Lemmas 3.2 and 4.2 that the map

$$\operatorname{Aut}_0(X') \to \operatorname{Aut}_0(X), \quad g \mapsto \pi \circ g \circ \pi^{-1}$$

is a complex Lie group isomorphism since X is normal (see also [Fuj78, Lemma 2.5]). Note that X' is also in Fujiki's class $\mathcal{C}$. So we may replace X with X' and assume that X is smooth.

Since $\operatorname{Aut}_0(X)$ is connected, its pullback action on $H^2(X, \mathbb{Z})$ is trivial. Note that the $\partial\bar{\partial}$-lemma holds for compact complex manifolds in Fujiki's class $\mathcal{C}$. So $H^{1,1}(X, \mathbb{R})$ is a subspace of $H^2(X, \mathbb{R})$ and $\operatorname{Aut}_0(X)$ acts trivially on $H^{1,1}(X, \mathbb{R})$. Let $[\alpha] \in H^{1,1}(X, \mathbb{R})$ be a big $(1, 1)$-class. Note that $\operatorname{Aut}_0(X) \le \operatorname{Aut}_{[\alpha]}(X)$.

We take σ as in Theorem 1.1. Then the map

$$\phi\colon \operatorname{Aut}_0(X) \times \widetilde{X} \to \widetilde{X}$$

via $(g, \widetilde{x}) \mapsto (\sigma^{-1} \circ g \circ \sigma)(\widetilde{x})$ is well-defined and holomorphic. In particular, $\operatorname{Aut}_0(X)$ lifts to a (unique) subgroup of $\operatorname{Aut}_0(\widetilde{X})$. By Lemma 4.2, the map

$$\tau\colon \operatorname{Aut}_0(\widetilde{X}) \to \operatorname{Aut}_0(X)$$

via $\tau(g) \coloneqq \sigma \circ g \circ \sigma^{-1}$ is an injective complex Lie group homomorphism. We just see the surjectivity of τ by the lifting property. So τ is isomorphic. □

Proof of Corollary 1.3. Let $[\alpha]$ be a big $(1,1)$-class. By Theorem 1.1, there is a Kähler model $\sigma\colon \widetilde{X} \to X$ such that $\mathrm{Aut}_{[\alpha]}(X)$ lifts to a group $G \le \mathrm{Aut}(\widetilde{X})$ via σ. By Theorem 4.3, $\mathrm{Aut}_0(\widetilde{X}) \le G$. Note that $G \le \mathrm{Aut}_{\sigma^*[\alpha]}(\widetilde{X})$ and $\sigma^*[\alpha]$ is still big. Since $\widetilde{X}$ is Kähler, we have that

$$[\mathrm{Aut}_{\sigma^*[\alpha]}(\widetilde{X}) : \mathrm{Aut}_0(\widetilde{X})] < \infty$$

by Theorem 4.1. Finally, note that

$$\mathrm{Aut}_{[\alpha]}(X)/\,\mathrm{Aut}_0(X) \cong G/\,\mathrm{Aut}_0(\widetilde{X}) \le \mathrm{Aut}_{\sigma^*[\alpha]}(\widetilde{X})/\,\mathrm{Aut}_0(\widetilde{X}).$$

So the corollary is proved. $\square$

Remark 4.4. Lemma 4.2 and also Theorem 4.3 fail in general if X has non-normal singularities such that $\sigma_*\mathcal{O}_{X'} = \mathcal{O}_X$ does not hold. A simple example is by taking $X' = \mathbb{P}^1$, $X = \{y^2z - x^3 = 0\}$ the cuspidal curve, and σ just the normalization of X. Note that $\mathrm{Aut}(X') = \mathrm{Aut}_0(X') = \mathrm{PGL}(2)$ while $\mathrm{Aut}_0(X)$ is (conjugate to) the subgroup of upper triangular matrices in $\mathrm{PGL}(2)$.

5. Torsion group actions

In this section, we will prove Corollaries 1.4 and 1.5. Our Theorem 1.1 plays a key role in the reduction to the connected Lie group action.

The following result holds true for algebraic groups defined over an algebraically closed field of characteristic 0; see [Jav21, Lemma 5.4]. The proof there cannot be applied well to the case of connected real Lie groups, e.g., we do not have the Chevalley decomposition. So we give a fundamental Lie-group theoretical proof for the convenience of the readers.

Lemma 5.1. *A torsion connected real Lie group G is trivial.*

Proof. Consider the adjoint representation

$$\rho\colon G \to \mathrm{GL}(T_{G,e})$$

where e the identity element of G and $T_{G,e}$ is the tangent space of G at e. Note that $\operatorname{Ker}\rho$ is just the centre of G since G is connected.

We claim that $\rho(G)$ is trivial. Suppose the contrary. Then $\rho(G)$ is infinite since it is connected. Note that $\rho(G)$ is a torsion connected Lie subgroup of $\mathrm{GL}(T_{G,e})$. So any finitely generated subgroup of $\rho(G)$ is finite by Schur's theorem (cf. [Lam01, Chapter 3, § 9, Theorem (9.9)]). Then we may choose a sequence of $g_i \in \rho(G)$ such that $g_{n+1} \notin G_n$ where G_n is the (finite) group generated by $(g_1, \ldots, g_n)$. Note that the order $|G_n|$ increases strictly. Let M be a maximal compact subgroup of $\rho(G)$ which is unique up to conjugation. Then G_n can be viewed as a subgroup of M via a conjugation. In particular, M is an infinite group. Let M_0 be the neutral component of M which is also infinite since M is compact. Let T be a maximal torus contained in M_0. Since T is torsion, T is trivial. However, every element of M_0 is conjugate to an element of T by the torus theorem. Then M_0 is trivial, a contradiction. So the claim is proved.

Now $\rho(G)$ is trivial and hence $G = \operatorname{Ker}\rho$ is abelian. Note that an abelian connected real Lie group G is isomorphic to $\mathbb{R}^m \times (S^1)^n$. The latter is easily seen to be non-torsion unless $m = n = 0$. Therefore, G is trivial. □

For general linear groups over number fields, we may even have boundedness on their torsion subgroups.

Lemma 5.2. *Let G be a torsion subgroup of $\mathrm{GL}_n(K)$ where K is a number field. Then $|G| \leq N$ for some constant N depending only on n and K.*

Proof. By the Minkowski's theorem (cf. [Ser06, Theorem 5, and § 4.3]), there is a constant M depending only on n and K such that the order $o(g) \leq M$ for any $g \in \mathrm{GL}_n(K)$ with finite order. By the Burnside's first theorem (cf. [Lam01, Chapter 3, § 9, Theorem (9.4)]), $|G| \leq N := M^{n^3}$ is finite. □

Proof of Corollary 1.4. Let G be a torsion subgroup of $\operatorname{Aut}(X)/\operatorname{Aut}_0(X)$. Let $\pi\colon X' \to X$ be an $\operatorname{Aut}(X)$-equivariant resolution of singularities (cf. [BM97, Theorem 13.2]), with $\operatorname{Aut}(X)$ lifts to a (unique) subgroup of $\operatorname{Aut}(X')$ via π. Note that $\operatorname{Aut}_0(X)$ lifts (isomorphically) to $\operatorname{Aut}_0(X')$ (cf. [Fuj78, Lemma 2.5]). Then G also lifts to a (unique) torsion subgroup of $\operatorname{Aut}(X')/\operatorname{Aut}_0(X')$. Note that X' is also in Fujiki's class $\mathcal{C}$. Therefore, we may replace X with X' and assume that X is smooth.

Since the pullback action $\operatorname{Aut}_0(X)|_{H^2(X,\mathbb{Q})}$ is trivial, we have the following exact sequence

$$1 \longrightarrow G_\tau \longrightarrow G \longrightarrow G|_{H^2(X,\mathbb{Q})} \longrightarrow 1$$

where G_τ is the kernel. Note that $G|_{H^2(X,\mathbb{Q})}$ is torsion and hence finite with order bounded by some N depending only on $H^2(X,\mathbb{Q})$ (and hence only on X) by Lemma 5.2. Denote by

$$\operatorname{Aut}_\tau(X) := \{g \in \operatorname{Aut}(X) \mid g^*|_{H^2(X,\mathbb{Q})} = \mathrm{id}\}.$$

Note that

$$G_\tau \leq \operatorname{Aut}_\tau(X)/\operatorname{Aut}_0(X) \leq \operatorname{Aut}_{[\alpha]}(X)/\operatorname{Aut}_0(X)$$

for any big $(1,1)$-class $[\alpha]$. By Corollary 1.3, we have

$$|G_\tau| \leq C := [\operatorname{Aut}_\tau(X) : \operatorname{Aut}_0(X)],$$

where C depends only on X. Then $|G| \leq N \cdot C$ and we get an upper bound.

Finally, if $\operatorname{Aut}(X)$ is torsion, then $\operatorname{Aut}_0(X)$ is trivial by Lemma 5.1 and hence $\operatorname{Aut}(X)$ is finite. □

Remark 5.3. Currently, the normality assumption on X is required in the proof of Corollary 1.4. The reason is that the equivariant resolution of singularities may enlarge $\mathrm{Aut}_0(X)$; see Remark 4.4.

It is well known that any torsion subgroup of the general linear group is almost abelian by the Jordan–Schur lemma. Lee generalized it to the case of connected real Lie groups; see [Lee76].

Theorem 5.4. *Let G be a connected real Lie group. Then any torsion subgroup $H \le G$ has an abelian subgroup $H' \le H$ with $[H : H'] \le J$ where J is a constant depending only on G.*

Proof of Corollary 1.5. First, by taking an $\mathrm{Aut}(X)$-equivariant resolution of singularities (cf. [BM97, Theorem 13.2]) which is still in Fujiki's class $\mathcal{C}$, we may assume X is smooth.

Let $G \le \mathrm{Aut}(X)$ be a torsion subgroup. Denote by

$$\mathrm{Aut}_\tau(X) := \{g \in \mathrm{Aut}(X) \mid g^*|_{H^2(X,\mathbb{Q})} = \mathrm{id}\}.$$

Note that $G/G \cap \mathrm{Aut}_\tau(X)$ can be viewed as a torsion subgroup of $\mathrm{GL}(H^2(X,\mathbb{Q}))$. By Lemma 5.2, we have

$$[G : G \cap \mathrm{Aut}_\tau(X)] \le N$$

for some constant N depending only on $H^2(X,\mathbb{Q})$ (and hence only on X).

Since $\mathrm{Aut}_\tau(X) \subseteq \mathrm{Aut}_{[\alpha]}(X)$ and by Corollary 1.3, we see that

$$C := [\mathrm{Aut}_\tau(X) : \mathrm{Aut}_0(X)] < \infty$$

and hence

$$[G : G \cap \mathrm{Aut}_0(X)] \le N \cdot C.$$

By Theorem 5.4, there is an abelian subgroup $H \le G \cap \mathrm{Aut}_0(X)$ such that

$$[G \cap \mathrm{Aut}_0(X) : H] \le J_0$$

where J_0 is a constant depending only on $\mathrm{Aut}_0(X)$ (and hence only on X).

Together, we have

$$[G : H] \le J := N \cdot C \cdot J_0$$

as desired. $\square$

References

[Akh95] Dmitri N. Akhiezer, *Lie group actions in complex analysis* (Klas Diederich, ed.), Aspects of Mathematics, vol. 27, Friedr. Vieweg & Sohn, Braunschweig, Wiesbaden, 1995. ↑9

[BM97] Edward Bierstone and Pierre D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Inventiones Mathematicae **128** (1997), no. 2, 207–302. ↑5, 9, 11, 12

[Bou02] Sébastien Boucksom, *On the volume of a line bundle*, Int. J. Math. **13** (2002), no. 10, 1043–1063. ↑4, 7

[Bou04] ______, *Divisorial Zariski decompositions on compact complex manifolds*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 1, 45–76. ↑5, 7

[Dem92] Jean-Pierre Demailly, *Regularization of closed positive currents and intersection theory*, J. Algebraic Geom. **1** (1992), 361–409. ↑4

[Dem97] ______, *Complex analytic and differential geometry*, Université de Grenoble I, 1997. ↑6

[DGMS75] Pierre Deligne, Phillip Griffiths, John Morgan, and Dennis Sullivan, *Real homotopy theory of kähler manifolds*, Invent. Math. **29** (1975), 245–274. ↑2

[DHZ15] Tien-Cuong Dinh, Fei Hu, and De-Qi Zhang, *Compact Kähler manifolds admitting large solvable groups of automorphisms*, Advances in Mathematics **281** (2015), 333–352. ↑8

[Dou66] Adrien Douady, *Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné*, Ann. Inst. Fourier (Grenoble) **16** (1966), no. 1, 1–95. ↑2

[DP04] Jean-Pierre Demailly and Mihai Paun, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. Math. **159** (2004), no. 3, 1247–1274. ↑1, 2, 4, 7

[Fuj78] Akira Fujiki, *On automorphism groups of compact Kähler manifolds*, Invent. Math. **44** (1978), 225–258. ↑1, 2, 8, 9, 11

[Jav21] Ariyan Javanpeykar, *Arithmetic hyperbolicity: automorphisms and persistence*, Math. Ann. **381** (2021), no. 1-2, 439–457. ↑3, 10

[Lam01] Tsit-Yuen Lam, *A first course in noncommutative rings*, Graduate Texts in Mathematics, vol. 131, Springer, New York, 2001. ↑10, 11

[Lee76] Dong Hoon Lee, *On torsion subgroups of Lie groups*, Proc. Amer. Math. Soc. **55** (1976), 3. ↑12

[Lie78] David I. Lieberman, *Compactness of the Chow scheme: applications to automorphisms and deformations of Kahler manifolds*, Fonctions de Plusieurs Variables Complexes III, 1978, pp. 140–186. ↑2, 8

[MPZ20] Sheng Meng, Fabio Perroni, and De-Qi Zhang, *Jordan property for automorphism groups of compact spaces in Fujiki's class C*, arXiv (2020), available at 2011.09381. ↑3

[MZ18] Sheng Meng and De-Qi Zhang, *Building blocks of polarized endomorphisms of normal projective varieties*, Adv. Math. **325** (2018), 243–273. ↑8

[Ser06] Jean-Pierre Serre, *Bounds for the orders of the finite subgroups of G(k)*, Group representation theory, 2006, pp. 405–450. ↑11

[Var89] Jean Varouchas, *Kähler spaces and proper open morphisms*, Math. Ann. **283** (1989), no. 1, 13–52. ↑1

[Wlo09] Jaroslaw Wlodarczyk, *Resolution of singularities of analytic spaces*, Proceedings of Gökova Geometry-Topology Conference, 2009, pp. 31–63. ↑5

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