HW: Custom Problem, Section 5A #19, 21

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March 17, 2015

Custom Problem

Let $T \in \mathcal{L}(\mathbb{C}^2)$ by $T(z,w) = (2w, -8z)$.

(a) *Find a basis for $\mathbb{C}^2$ consisting of eigenvectors of T.*

To find the eigenvalues, set $T(z,w) = \lambda(z,w)$.

$$\begin{aligned} & T(z,w) = \lambda(z,w) \\ \implies & (2w,-8z) = (\lambda z, \lambda w) \\ \implies & 2w = \lambda z, \quad \text{and} \quad -8z = \lambda w \\ \implies & -8z = \frac{\lambda^2}{2} z \end{aligned}$$

If $z = 0$, then $w = 0$. Then λ would not be an eigenvalue. So we suppose $z \neq 0$, and thus

$$\begin{aligned} & -8 = \frac{\lambda^2}{2} \\ \implies & -16 = \lambda^2 \\ \implies & \lambda = \pm 4i \end{aligned}$$

Thus the two eigenvalues for T are $4i$ and $-4i$. To find their eigenspaces, set $T(z,w) = 4i(z,w)$ and $T(z,w) = -4i(z,w)$, respectively. So,

$$\begin{aligned} & T(z,w) = 4i(z,w) \\ \implies & (2w,-8z) = (4iz, 4iw) \\ \implies & w = 2iz \end{aligned}$$

Thus $(1, 2i)$ is an eigenvector corresponding to $4i$. Furthermore, $(1,2i)$ is a basis for $E(T, 4i)$. Also,

$$\begin{aligned} & T(z,w) = -4i(z,w) \\ \implies & (2w,-8z) = (-4iz, -4iw) \\ \implies & w = -2iz \end{aligned}$$

Thus $(1, -2i)$ is an eigenvector corresponding to $-4i$. Furthermore, $(1, -2i)$ is a basis for $E(T, -4i)$. Then $\pi = ((1, 2i), (1, -2i))$ is a linearly independent set since eigenvectors corresponding to different eigenvalues are linearly independent. Since $\dim(\mathbb{C}^2) = 2$ and $\operatorname{len}(\pi) = 2$, π is a basis for $\mathbb{C}^2$ consisting of eigenvectors of T.

(b) *Find $\mathcal{M}(T)$ with respect to this basis.*

Since

$$\begin{aligned} T(1, 2i) &= (2(2i), -8(1)) \\ &= (4i, -8) \\ &= 4i(1, 2i) + 0(1, -2i) \end{aligned}$$

and

$$\begin{aligned} T(1, -2i) &= (2(-2i), -8(1)) \\ &= (-4i, -8) \\ &= 0(1, 2i) - 4i(1, -2i) \end{aligned}$$

then

$$\mathcal{M}(T, \pi) = \begin{pmatrix} 4i & 0 \\ 0 & -4i \end{pmatrix}$$

5A

#19

Suppose n is a positive integer and $T \in \mathcal{L}(\mathbb{F}^n)$ is defined by

$$T(x_1, \ldots, x_n) = (x_1 + \ldots x_n, \ldots, x_1 + \ldots, x_n)$$

in other words, T is the operator whose matrix (with respect to the standard basis) consists of all 1's. Find the eigenvalues and eigenvectors of T.

To find the eigenvalues, set $T(x_1, \ldots, x_n) = \lambda(x_1, \ldots, x_n)$.

$$\begin{aligned} T(x_1, \ldots, x_n) &= \lambda(x_1, \ldots, x_n) \\ \implies (x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n) &= (\lambda x_1, \ldots, \lambda x_n) \\ \implies x_1 + \cdots + x_n &= \lambda x_1 = \cdots = \lambda x_n \\ \implies \lambda &= \frac{x_1 + \cdots + x_n}{x_1} = \cdots = \frac{x_1 + \cdots + x_n}{x_n} \end{aligned}$$

If $\lambda \neq 0$, then $x_1 = \cdots = x_n$, and thus $(1, \ldots, 1)$ is a basis for $E(T, \lambda)$. However, if $\lambda = 0$, then $x_1 + \cdots + x_n = 0$. Then $E(T, 0) = \{(x_1, \ldots, x_n) \in \mathbb{F}^n \mid x_1 + \cdots + x_n = 0\}$. Note $\dim(E(T, 0)) = n - 1$ and a basis for $E(T, 0)$ is

$$\pi = ((1, -1, 0, \ldots, 0), (1, 0, -1, 0, \ldots, 0), \ldots, (1, 0, \ldots, 0, -1))$$

#21

Suppose $T \in \mathcal{L}(V)$ is invertible.

(a) *Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\dfrac{1}{\lambda}$ is an eigenvalue of T^{-1}.*

$$\begin{aligned}
&\lambda \text{ is an eigenvalue of } T \\
\iff &\exists v \in V \text{such that } T(v) = \lambda v \\
\iff &T^{-1}(T(v)) = T^{-1}(\lambda v) \\
\iff &v = \lambda T^{-1}(v) \\
\iff &T^{-1}(v) = \frac{1}{\lambda} v \text{ for some } v \in V \\
\iff &\frac{1}{\lambda} \text{ is an eigenvalue of } T^{-1}
\end{aligned}$$

$\square$

(b) *Prove that T and T^{-1} have the same eigenvectors.*

Let $\hat{v}$ be an eigenvector of T corresponding to an arbitrary eigenvalue of T's, say $\hat{\lambda}$. Then

$$\begin{aligned}
T(\hat{v}) &= \hat{\lambda}\hat{v} \\
\implies T^{-1}(T(\hat{v})) &= T^{-1}(\hat{\lambda}\hat{v}) \\
\implies \hat{v} &= \hat{\lambda} T^{-1}(\hat{v}) \\
\implies T^{-1}(\hat{v}) &= \frac{1}{\hat{\lambda}} \hat{v}
\end{aligned}$$

Thus $\hat{v}$ is an eigenvector of T^{-1} corresponding to $\dfrac{1}{\hat{\lambda}}$. Thus any eigenvector of T is an eigenvector of T^{-1}. However, the same argument and $(T^{-1})^{-1} = T$ implies any eigenvector of T^{-1} is an eigenvector of T. Thus T and T^{-1} have the same eigenvectors. $\square$