HW: Custom Problem, Section 5A #19, 21, Section 5C #14, 15

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Custom Problem

Let $T \in \mathcal{L}(\mathbb{C}^2)$ by $T(z, w) = (2w, -8z)$.

(a) Find a basis for $\mathbb{C}^2$ consisting of eigenvectors of T.

To find the eigenvalues, set $T(z, w) = \lambda(z, w)$.

$$\begin{aligned} T(z, w) &= \lambda(z, w) \\ \implies (2w, -8z) &= (\lambda z, \lambda w) \\ \implies 2w &= \lambda z, \quad \text{and} \quad -8z = \lambda w \\ \implies -8z &= \frac{\lambda^2}{2} z \end{aligned}$$

If $z = 0$, then $w = 0$. Then λ would not be an eigenvalue. So we suppose $z \neq 0$, and thus

$$\begin{aligned} -8 &= \frac{\lambda^2}{2} \\ \implies -16 &= \lambda^2 \\ \implies \lambda &= \pm 4i \end{aligned}$$

Thus the two eigenvalues for T are $4i$ and $-4i$. If $\lambda = 4i$, then $2w = \lambda z \implies w = 2iz \implies E(4i) = \{(z, 2iz) \in \mathbb{C}^2 \mid z \in \mathbb{C}\}$. Specifically, $(1, 2i) \in E(4i)$. If $\lambda = -4i$, then $2w = \lambda z \implies w = -2iz \implies E(-4i) = \{(z, -2iz) \in \mathbb{C}^2 \mid z \in \mathbb{C}\}$. Specifically, $(1, -2i) \in E(-4i)$. Then $\pi = ((1, 2i), (1, -2i))$ is a linearly independent list since it is comprised entirely of eigenvectors corresponding to distinct eigenvalues. Since $\dim(\mathbb{C}^2) = 2 = \text{len}(\pi)$, then π is a basis for $\mathbb{C}^2$ consisting of eigenvectors of T.

(b) Find $\mathcal{M}(T)$ with respect to this basis.

Since

$$\begin{aligned}T(1,2i) &= (2(2i),-8(1))\\ &= (4i,-8)\\ &= 4i(1,2i)+0(1,-2i)\end{aligned}$$

and

$$\begin{aligned}T(1,-2i) &= (2(-2i),-8(1))\\ &= (-4i,-8)\\ &= 0(1,2i)-4i(1,-2i)\end{aligned}$$

then

$$\mathcal{M}(T,\pi)=\begin{pmatrix}4i & 0\\ 0 & -4i\end{pmatrix}$$

5A

#19

Suppose n is a positive integer and $T \in \mathcal{L}(\mathbb{F}^n)$ is defined by

$$T(x_1,\ldots,x_n)=(x_1+\cdots+x_n,\ldots,x_1+\cdots+x_n)$$

in other words, T is the operator whose matrix (with respect to the standard basis) consists of all 1's. Find the eigenvalues and eigenvectors of T.

To find the eigenvalues, set $T(x_1,\ldots,x_n)=\lambda(x_1,\ldots,x_n)$.

$$\begin{aligned}T(x_1,\ldots,x_n) &= \lambda(x_1,\ldots,x_n)\\ \implies (x_1+\cdots+x_n,\ldots,x_1+\cdots+x_n) &= (\lambda x_1,\ldots,\lambda x_n)\\ \implies x_1+\cdots+x_n &= \lambda x_1=\cdots=\lambda x_n\\ \implies \lambda &= \frac{x_1+\cdots+x_n}{x_1}=\cdots=\frac{x_1+\cdots+x_n}{x_n}\end{aligned}$$

If $\lambda \neq 0$, then $x_1=\cdots=x_n$. This implies $\lambda = nx_n$ is an eigenvalue and $((1,\ldots,1))$ is a basis for $E(nx_n)=\{(x,\ldots,x)\in\mathbb{F}^n \mid x\in\mathbb{F}\}$. However, if $\lambda=0$, then the equation is true only if $x_1+\cdots+x_n=0$. Thus 0 is an eigenvalue and $E(0)=\{(x_1,\ldots,x_n)\in\mathbb{F}^n \mid x_1+\cdots+x_n=0\}$. Note $\dim(E(0))=n-1$ and a basis for $E(0)$ is

$$\pi=((1,-1,0,\ldots,0),(1,0,-1,0,\ldots,0),\ldots,(1,0,\ldots,0,-1))$$

Let π' be the concatenation of $(1,\ldots,1)$ and π. In other words,

$$\pi'=((1,\ldots,1),(1,-1,0,\ldots,0),(1,0,-1,0,\ldots,0),\ldots,(1,0,\ldots,0,-1))$$

Since π is a basis for $E(0)$, then π is a linearly independent list. However, $((1, \ldots, 1))$ is a basis for $E(nx_n)$ and any two vectors from different eigenspaces are linearly independent. Thus π' is a linearly independent list. Note

$$\dim(\mathbb{F}^n) = n = 1 + (n-1) = \dim(E(nx_n)) + \dim(E(0)) = \text{len}(\pi')$$

Thus π' is a basis for $\mathbb{F}^n$. Note the following:

$$\begin{aligned} T(1, \ldots, 1) &= (n, \ldots, n) = n(1, \ldots, 1) \\ T(1, -1, 0, \ldots, 0) &= (0, \ldots, 0) \\ T(1, 0, -1, 0, \ldots, 0) &= (0, \ldots, 0) \\ &\vdots \\ (1, 0, \ldots, 0, -1) &= (0, \ldots, 0) \end{aligned}$$

Thus,

$$\mathcal{M}(T, \pi') = \begin{pmatrix} n & 0 & \ldots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \\ 0 & \ldots & & 0 \end{pmatrix}$$

#21

Suppose $T \in \mathcal{L}(V)$ is invertible.

(a) *Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.*

$$\begin{aligned} &\lambda \text{ is an eigenvalue of } T \\ \iff &\exists v \in V \text{such that } T(v) = \lambda v \\ \iff &T^{-1}(T(v)) = T^{-1}(\lambda v) \\ \iff &v = \lambda T^{-1}(v) \\ \iff &T^{-1}(v) = \frac{1}{\lambda} v \text{ for some } v \in V \\ \iff &\frac{1}{\lambda} \text{ is an eigenvalue of } T^{-1} \end{aligned}$$

$\square$

(b) *Prove that T and T^{-1} have the same eigenvectors.*

Let $\hat{v}$ be an eigenvector of T corresponding to an arbitrary eigenvalue of T's, say $\hat{\lambda}$. Then

$$\begin{aligned} T(\hat{v}) &= \hat{\lambda}\hat{v} \\ \implies T^{-1}(T(\hat{v})) &= T^{-1}(\hat{\lambda}\hat{v}) \\ \implies \hat{v} &= \hat{\lambda}T^{-1}(\hat{v}) \\ \implies T^{-1}(\hat{v}) &= \frac{1}{\hat{\lambda}}\hat{v} \end{aligned}$$

Thus $\hat{v}$ is an eigenvector of T^{-1} corresponding to $\frac{1}{\hat{\lambda}}$. Thus any eigenvector of T is an eigenvector of T^{-1}. However, $(T^{-1})^{-1} = T$ implies any eigenvector of T^{-1} is an eigenvector of T. Thus T and T^{-1} have the same eigenvectors. $\square$

5C

#14

Find $T \in \mathcal{L}(\mathbb{C}^3)$ such that 6 and 7 are eigenvalues of T such that T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^3$.

Define $T \in \mathcal{L}(\mathbb{C}^3)$ by

$$T(z_1, z_2, z_3) = (6z_1 + 3z_2 + 4z_3, 6z_2 + z_3, 7z_3)$$

Then let $e = (e_1, e_2, e_3)$ be the standard basis.

$$\begin{aligned} T(e_1) &= (6, 0, 0) = 6e_1 \\ T(e_2) &= (3, 6, 0) = 3e_1 + 6e_2 \\ T(e_3) &= (4, 1, 7) = 4e_1 + 1e_2 + 7e_3 \end{aligned}$$

Thus

$$\mathcal{M}(T, e) = \begin{pmatrix} 6 & 3 & 4 \\ 0 & 6 & 1 \\ 0 & 0 & 7 \end{pmatrix}$$

Since $\mathcal{M}(T, e)$ is an upper-triangular matrix, the entries on the main diagonal are the eigenvalues of T. Thus 6 and 7 are the eigenvalues of T. To find the eigenvectors of T corresponding to 6, set $T(z_1, z_2, z_3) = 6(z_1, z_2, z_3)$. In other words, $(6z_1 + 3z_2 + 4z_3, 6z_2 + z_3, 7z_3) = (6z_1, 6z_2, 6z_3)$, or

$$\begin{cases} 6z_1 + 3z_2 + 4z_3 & = 6z_1 \\ 6z_2 + z_3 & = 6z_2 \\ 7z_3 & = 6z_3 \end{cases} \implies \begin{cases} z_1 & \text{is arbitrary in } \mathbb{C} \\ z_2 & = 0 \\ z_3 & = 0 \end{cases} \implies E(6) = \{(z, 0, 0) \mid z \in \mathbb{C}\}$$

To find the eigenvectors of T corresponding to 7, set $T(z_1, z_2, z_3) = 7(z_1, z_2, z_3)$. In other words, $(6z_1 + 3z_2 + 4z_3, 6z_2 + z_3, 7z_3) = (7z_1, 7z_2, 7z_3)$, or

$$\begin{cases} 6z_1 + 3z_2 + 4z_3 & = 7z_1 \\ 6z_2 + z_3 & = 7z_2 \\ 7z_3 & = 7z_3 \end{cases} \implies \begin{cases} z_1 & = 7z_3 \\ z_2 & = z_3 \\ z_3 & \text{is arbitrary in } \mathbb{C} \end{cases} \implies E(7) = \{(7z, z, z) \mid z \in \mathbb{C}\}$$

Note $\dim(E(7)) = \dim(E(6)) = 1$, and thus we can never form a linearly independent list comprised entirely of eigenvectors of length more than 2. Since $\dim(\mathbb{C}^3) = 3$, all bases are lists of length 3. Thus there does not exist a basis for $\mathbb{C}^3$ consisting entirely of eigenvectors. This is equivalent to saying there does not exist a basis for $\mathbb{C}^3$ such that $\mathcal{M}(T)$ with respect to that basis is diagonal. $\square$

#15

Suppose $T \in \mathcal{L}(\mathbb{C}^3)$ such that 6 and 7 are eigenvalues of T. Furthermore, suppose T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^3$. Prove that there exists $(x, y, z) \in \mathbb{F}^3$ such that $T(x, y, z) = (17 + 8x, \sqrt{5} + 8y, 2\pi + 8z)$.

Note the following:

$$\begin{aligned} T(x, y, z) &= (17 + 8x, \sqrt{5} + 8y, 2\pi + 8z) \\ \iff T(x, y, z) &= (17, \sqrt{5}, 2\pi) + 8(x, y, z) \\ \iff T(x, y, z) - 8(x, y, z) &= (17, \sqrt{5}, 2\pi) \\ \iff T(x, y, z) - 8I(x, y, z) &= (17, \sqrt{5}, 2\pi) \\ \iff (T - 8I)(x, y, z) &= (17, \sqrt{5}, 2\pi) \end{aligned}$$

It suffices to show $\exists (x, y, z) \in \mathbb{F}^3$ such that $(T - 8I)(x, y, z) = (17, \sqrt{5}, 2\pi)$. To do this, we will show 8 is not an eigenvalue, which will imply $T - 8I$ is surjective, proving the result.

Assume 8 is an eigenvalue. Then $\exists v_1 \neq 0$ such that $v_1 \in E(8)$. Similarly, since 6 and 7 are eigenvalues, there exist non-zero elements v_2 and v_3 such that $v_2 \in E(6)$ and $v_3 \in E(7)$. Let $\tau = (v_1, v_2, v_3)$. Since a list containing elements from distinct eigenspaces is linearly independent, τ is linearly independent. Also, since $\text{len}(\tau) = 3 = \dim(\mathbb{F}^3)$, then τ is a basis for $\mathbb{F}^3$. Since τ consists entirely of eigenvectors, T is diagonalizable $\Longrightarrow\Longleftarrow$. Thus 8 is not an eigenvalue. Thus $(T - 8I)$ is surjective. Thus $\exists (x, y, z) \in \mathbb{F}^3$ such that $(T - 8I)(x, y, z) = (17, \sqrt{5}, 2\pi)$. Thus $\exists (x, y, z) \in \mathbb{F}^3$ such that $T(x, y, z) = (17 + 8x, \sqrt{5} + 8y, 2\pi + 8z)$. $\square$