Homework: Sec. 3A # 10, Sec. 3B # 3, 5, 6

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Sec. 3A

#10

Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$. Define $T : V \to W$ by

$$T(v) = \begin{cases} S(v) & \text{if } v \in U \\ 0 & \text{if } v \in V \setminus U \end{cases}$$

Prove that T is not a linear map on V.

(The notation $\oplus_V$ and $\oplus_W$ represent the addition operations in V and W, respectively. Since U is a subspace of V, the addition operation on U is inherited from V.) Let $u \in U$ such that $S(u) = w \neq 0$, and let $v \in V \setminus U$. Then since $U \subset V$, $u \in V$. Then $u \oplus_V v \in V$ since V is closed under $\oplus_V$. Note $u \oplus_V v \notin U$.

> *Suppose $u \oplus_V v \in U$. Then $u \oplus_V v = \overline{u}$ for some $\overline{u} \in U$. Then $v = (\text{-}u) \oplus_V \overline{u} \implies v \in U \Longrightarrow\Longleftarrow$ since $v \in V \setminus U$.*

Thus $u \oplus_V v \in V \setminus U$. Then $T(u \oplus_V v) = 0$. But $T(u) \oplus_W T(v) = S(u) \oplus_W 0 = w + 0 = w \neq 0 = T(x \oplus_V v)$. Thus T does not preserve addition, and hence is not a linear map. $\square$

Sec. 3B

#3

Suppose $(v_1, \ldots v_m)$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by

$$T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m$$

(a) *What property of T corresponds to $(v_1, \ldots, v_m)$ spanning V?* $\boxed{\text{Surjectivity}}$

Suppose $(v_1, \ldots, v_m)$ spans V. Then $\forall v \in V$, $\exists (z_1, \ldots, z_m) \in \mathbb{F}^m$ such that $z_1 v_1 + \cdots + z_m v_m = v$. Then $\exists z \in \mathbb{F}^m$ such that $T(z) = v$. Thus T is surjective. Now suppose T is surjective. Then $\forall v \in V$, $\exists z \in \mathbb{F}^m$ such that $T(z) = v$. Since $z \in \mathbb{F}^m$, $z = (z_1, \ldots, z_m)$ for some $z_1, \ldots, z_m \in \mathbb{F}$. Then $v = z_1 v_1 + \ldots z_m v_m$. Since v was arbitrary in V, $(v_1, \ldots, v_m)$ spans V.

(b) *What property of T corresponds to $(v_1, \ldots, v_m)$ being linearly independent?* $\boxed{\text{Injectivity}}$

Suppose $(v_1, \ldots, v_m)$ is linearly independent. Then $\forall v \in \text{span}(v_1, \ldots, v_m)$, $\exists! (z_1, \ldots, z_m) \in \mathbb{F}^m$ such that $v = z_1 v_1 + \cdots + z_m v_m$. Then suppose $z, y \in \mathbb{F}^m$ (so $z = (z_1, \ldots, z_m)$ and $y = (y_1, \ldots, y_m)$ for some $z_1, \ldots, z_m, y_1, \ldots, y_m \in \mathbb{F}$), and suppose $T(z) = T(y)$. Then $z_1 v_1 + \cdots + z_m v_m = y_1 v_1 + \cdots + y_m v_m \implies (z_1 - y_1) v_1 + \cdots + (z_m - y_m) v_m = 0$. Thus $z_1 = y_1, \ldots, z_m = y_m \implies z = y$. Thus T is injective. Now suppose T is injective. Then $T(z) = T(y) \implies z = y$. Then let $z = (z_1, \ldots, z_m)$ and $z_1 v_1 + \cdots + z_m v_m = 0$. But $0 = 0 v_1 + \cdots + 0 v_m$. Then since T is injective, $z_1 = \cdots = z_m = 0$. Thus $(v_1, \ldots, v_m)$ is linearly independent.

#5

Give an example of a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$\text{range } T = \text{null } T$$

Define $T : \mathbb{R}^4 \to \mathbb{R}^4$ by $T(w, x, y, z) = (y, z, 0, 0)$. Since y and z are arbitrary in $\mathbb{R}$,

$$\text{range } T = \{(a, b, 0, 0) \in \mathbb{R}^4 \mid a, b, \in \mathbb{R}\}$$

Now suppose $T(w, x, y, z) = 0_{\mathbb{R}^4} = (0, 0, 0, 0)$. Then $(y, z, 0, 0) = (0, 0, 0, 0) \implies y = z = 0$. Since w and x are arbitrary in $\mathbb{R}^4$,

$$\text{null } T = \{(a, b, 0, 0) \in \mathbb{R}^4 \mid a, b, \in \mathbb{R}\} = \text{range } T$$

#6

Prove that there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that

$$\text{range } T = \text{null } T$$

Let $T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^5)$. Then $\dim \mathbb{R}^5 = \dim \text{range } T + \dim \text{null } T$. Suppose range T = null T. Then dim range T = dim null T. Thus

$$2(\dim \text{range } T) = \dim \mathbb{R}^5 = 5$$
$$\implies \dim \text{range } T = \frac{5}{2}$$

This is a contradiction since the dimension of a vector space must be a natural number. Thus there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that range T = null T. $\square$