Homework: Sec. 5B # 3, 9

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Sec. 5B

3

Suppose $T \in \mathcal{L}(V)$ and $T^2 = I$ and -1 is not an eigenvalue ot T. Prove $T = I$.

By the multiplicative properties of polynomials and operators, $T^2 - I = (T - I)(T + I)$. But $T^2 = I \implies T^2 - I = \mathbf{0}$ where $\mathbf{0}$ is the zero operator. Thus $(T - I)(T + I)(v) = \mathbf{0}(v) = 0\ \forall v \in V$. Since -1 is not an eigenvalue of T, $T + I$ is bijective, and thus $\text{range}(T) = V$. Then $\forall w \in V$, $\exists v \in V$ such that $(T + I)(v) = w$. Then $(T - I)(w) = 0$, $\forall w \in V$. Then $T - I = \mathbf{0} \implies T = I$. $\square$

9

Suppose V is finite dimensional, $T \in \mathcal{L}(V)$, and $v \in V$ with $v \neq 0$. Let p be a nonzero polynomial of smallest degree such that $(p(T))(v) = 0$. Prove that every zero of p is an eigenvalue of T.

Let λ be a zero of p. Then $p(x) = (x - \lambda)q(x)$ for some polynomial $q(x)$ with $\deg(q) = \deg(p) - 1$. Then $p(T) = (T - \lambda I)q(T)$.

$$(p(T))(v) = 0 \implies (T - \lambda I)(q(T))(v) = 0$$

Since p is a polynomial of smallest degree such that $(p(T))(v) = 0$ and $\deg(q) < \deg(p)$, then $(q(T))(v) = w$ for some $w \in V$, $w \neq 0$.

$$(T - \lambda I)(q(T))(v) = 0 \implies (T - \lambda I)(w) = 0$$

Thus $T - \lambda I$ is *not* injective. Thus λ is an eigenvalue of T. $\square$