

# HW: Section 7B #6, 7, 8, 9

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## Section 7B

### #6

*Prove that a normal operator on a complex inner product space is self-adjoint if and only if all of its eigenvalues are real.*

Since  $T$  is normal, the Complex Spectral Theorem states that  $T$  has a basis of orthonormal eigenvectors. Let  $e = (e_1, \dots, e_n)$  be one such basis. Then  $\mathcal{M}(T, e)$  is a diagonal matrix, and all entries on the diagonal are eigenvalues of  $T$ .

Now suppose  $T$  is self-adjoint, i.e.  $T = T^*$ . Then since  $\mathcal{M}(T^*, e)$  is the conjugate transpose of  $\mathcal{M}(T, e)$ , then  $\mathcal{M}(T, e)$  is its own conjugate transpose  $\iff$  each entry on the diagonal is equal to its conjugate  $\iff$  each entry on the diagonal of  $\mathcal{M}(T, e)$  is real  $\iff$  all of  $T$ 's eigenvalues are real.  $\square$

### #7

*Suppose  $V$  is a complex inner product space and  $T \in \mathcal{L}(V)$  is a normal operator such that  $T^9 = T^8$ . Prove that  $T$  is self-adjoint and  $T^2 = T$ .*

Since  $T$  is normal, the Complex Spectral Theorem states that  $T$  has a basis of orthonormal eigenvectors. Let  $e = (e_1, \dots, e_n)$  be one such basis. Then  $T(e_k) = \lambda_k e_k$  for some  $\lambda_k \in \mathbb{C}$ . Then,

$$\begin{aligned} \lambda_k^9 e_k &= T^9(e_k) = T^8(e_k) = \lambda_k^8 e_k \\ \implies \lambda_k^9 e_k - \lambda_k^8 e_k &= 0 \\ \implies \lambda_k^8 (\lambda_k - 1) e_k &= 0 \\ \implies e_k = 0 \text{ or } \lambda_k = 0 \text{ or } \lambda_k = 1 \end{aligned}$$

However, since  $e_k$  is an eigenvector,  $e_k \neq 0$ . Then  $\lambda_k = 0$  or  $\lambda_k = 1$ . Since 0 and 1 are real,  $T$  may only have real eigenvalues. By the previous problem, any normal operator with real eigenvalues is self-adjoint. Thus  $T$  is self-adjoint. Also, since  $0^2 = 0$  and  $1^2 = 1$ , then

$$T^2(e_k) = \lambda_k^2 e_k = \lambda_k e_k = T(e_k)$$

for all  $k = 1, \dots, n$ . Then let  $v = \sum_{k=1}^n (a_k e_k)$ . Then

$$T^2(v) = T^2\left(\sum_{k=1}^n (a_k e_k)\right) = \sum_{k=1}^n (a_k T^2(e_k)) = \sum_{k=1}^n (a_k T(e_k)) = T\left(\sum_{k=1}^n (a_k e_k)\right) = T(v)$$

Thus  $T^2 = T$ . □

## #8

Give an example of an operator  $T$  on a complex vector space such that  $T^9 = T^8$  but  $T^2 \neq T$ .

Consider  $T \in \mathcal{L}(\mathbb{C}^2)$  defined by  $T(z_1, z_2) = (0, z_1)$ . Then  $T^2(z_1, z_2) = (0, 0) \implies T^2 = \mathbf{0}$ , where  $\mathbf{0}$  is the zero operator. Thus  $T$  is nilpotent. Then  $T^k = \mathbf{0} \ \forall k \geq \dim(\mathbb{C}^2) = 2$ . Thus  $T^9 = T^8 = \mathbf{0}$ . □

## #9

Suppose  $V$  is a complex inner product space. Prove that every normal operator on  $V$  has a square root. (An operator  $S \in \mathcal{L}(V)$  is called a square root of  $T \in \mathcal{L}(V)$  if  $S^2 = T$ .)

Since  $T$  is normal, the Complex Spectral Theorem states that  $T$  has a basis of orthonormal eigenvectors. Let  $\omega = (\omega_1, \dots, \omega_n)$  be one such basis. Then  $T(\omega_k) = \lambda_k \omega_k$  where  $\lambda_k = r_k e^{i\theta_k}$ , and

$$\mathcal{M}(T, \omega) = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}$$

These  $\lambda_k$ 's need not be unique. Since every complex number has a square root (in particular  $\sqrt{\lambda_k} = (r_k e^{i\theta_k})^{\frac{1}{2}} = r_k e^{\frac{1}{2}i\theta_k}$ ), we can define  $S \in \mathcal{L}(V)$  by where it maps each element in the basis  $\omega$ .

$$S(\omega_k) = \sqrt{\lambda_k} \omega_k$$

In other words,

$$\mathcal{M}(S, \omega) = \begin{pmatrix} \sqrt{\lambda_1} & 0 & \dots & 0 \\ 0 & \sqrt{\lambda_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \sqrt{\lambda_n} \end{pmatrix}$$

Since  $\mathcal{M}(S, \omega)$  is a diagonal matrix,  $[\mathcal{M}(S, \omega)]^2$  is easily computable:  $[\mathcal{M}(S, \omega)]^2 = \mathcal{M}(T, \omega)$ . Thus  $S^2 = T$ . Thus every normal operator on a complex inner product space has a square root. □