HW: Section 7B #6, 7, 8, 9

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Section 7B

#6

Prove that a normal operator on a complex inner product space is self-adjoint if and only if all of its eigenvalues are real.

Since T is normal, the Complex Spectral Theorem states that T has a basis of orthonormal eigenvectors. Let $e = (e_1, \dots, e_n)$ be one such basis. Then $\mathcal{M}(T, e)$ is a diagonal matrix, and all entries on the diagonal are eigenvalues of T.

Now suppose T is self-adjoint, i.e. $T = T^*$. Then since $\mathcal{M}(T^*, e)$ is the conjugate transpose of $\mathcal{M}(T, e)$, then $\mathcal{M}(T, e)$ is its own conjugate transpose $\iff$ each entry on the diagonal is equal to its conjugate $\iff$ each entry on the diagonal of $\mathcal{M}(T, e)$ is real $\iff$ all of T's eigenvalues are real. $\square$

#7

Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

Since T is normal, the Complex Spectral Theorem states that T has a basis of orthonormal eigenvectors. Let $e = (e_1, \dots, e_n)$ be one such basis. Then $T(e_k) = \lambda_k v$. for some $\lambda_k \in \mathbb{C}$. Then,

$$\begin{aligned}
\lambda_k^9 e_k = T^9(e_k) &= T^8(e_k) = \lambda_k^8 e_k \\
\implies \lambda_k^9 e_k - \lambda_k^8 e_k &= 0 \\
\implies \lambda_k^8(\lambda_k - 1) e_k &= 0 \\
\implies e_k = 0 \quad \text{or} \quad \lambda_k &= 0 \quad \text{or} \quad \lambda_k = 1
\end{aligned}$$

However, since e_k is an eigenvector, $e_k \neq 0$. Then $\lambda_k = 0$ or $\lambda_k = 1$. Since 0 and 1 are real, T may only have real eigenvalues. By the previous problem, any normal operator with real eigenvalues is self-adjoint. Thus T is self-adjoint. Also, since $0^2 = 0$ and $1^2 = 1$, then

$$T^2(e_k) = \lambda_k^2 e_k = \lambda_k e_k = T(e_k)$$

for all $k = 1, \dots, n$. Then let $v = \sum_{k=1}^{n}(a_k e_k)$. Then

$$T^2(v) = T^2\left(\sum_{k=1}^{n}(a_k e_k)\right) = \sum_{k=1}^{n}(a_k T^2(e_k)) = \sum_{k=1}^{n}(a_k T(e_k)) = T\left(\sum_{k=1}^{n}(a_k e_k)\right) = T(v)$$

Thus $T^2 = T$. $\square$

#8

Give an example of an operator T on a complex vector space such that $T^9 = T^8$ but $T^2 \neq T$.

Consider $T \in \mathcal{L}(\mathbb{C}^2)$ defined by $T(z_1, z_2) = (0, z_1)$. Then $T^2(z_1, z_2) = (0, 0) \implies T^2 = \mathbf{0}$, where $\mathbf{0}$ is the zero operator. Thus T is nilpotent. Then $T^k = \mathbf{0}\ \forall k \geq \dim(\mathbb{C}^2) = 2$. Thus $T^9 = T^8 = \mathbf{0}$. $\square$

#9

Suppose V is a complex inner product space. Prove that every normal operator on V has a square root. (An operator $S \in \mathcal{L}(V)$ is called a square root of $T \in \mathcal{L}(V)$ if $S^2 = T$.)

Since T is normal, the Complex Spectral Theorem states that T has a basis of orthonormal eigenvectors. Let $\omega = (\omega_1, \dots, \omega_n)$ be one such basis. Then $T(\omega_k) = \lambda_k \omega_k$ where $\lambda_k = r_k e^{i\theta_k}$, and

$$\mathcal{M}(T, \omega) = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}$$

These λ_k's need not be unique. Since every complex number has a square root (in particular $\sqrt{\lambda_k} = (r_k e^{i\theta_k})^{\frac{1}{2}} = r_k e^{\frac{1}{2}i\theta_k}$), we can define $S \in \mathcal{L}(V)$ by where it maps each element in the basis ω.

$$S(\omega_k) = \sqrt{\lambda_k}\omega_k$$

In other words,

$$\mathcal{M}(S, \omega) = \begin{pmatrix} \sqrt{\lambda_1} & 0 & \dots & 0 \\ 0 & \sqrt{\lambda_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \sqrt{\lambda_n} \end{pmatrix}$$

Since $\mathcal{M}(S, \omega)$ is a diagonal matrix, $[\mathcal{M}(S, \omega)]^2$ is easily computable: $[\mathcal{M}(S, \omega)]^2 = \mathcal{M}(T, \omega)$. Thus $S^2 = T$. Thus every normal operator on a complex inner product space has a square root. $\square$