Homework: Sec. 3B # 17, Sec. 3C # 2

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Sec. 3B

#17

Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if dim$(V) \leq$ dim(W).

Suppose $T \in \mathcal{L}(V, W)$ such that T is injective. Since $T(0) = 0$ and T is injective, $\text{null}(T) = \{0\}$. Thus $\dim(\text{null}(T)) = 0$. Thus,

$$\begin{aligned} & \dim(\text{range}(V)) = \dim(V) - \dim(\text{null}(T)) \\ \implies & \dim(\text{range}(V)) = \dim(V) - 0 \\ \implies & \dim(\text{range}(V)) = \dim(V) \end{aligned}$$

Since $\text{range}(V) \subset W$, $\dim(\text{range}(V)) \leq \dim(W)$. Thus $\dim(V) \leq \dim(W)$.

Now suppose $m = \dim(V) \leq \dim(W) = n$. Then let $(v_1, \dots, v_m)$ be a basis for V and let $(w_1, \dots, w_n)$ be a basis for W. Define $T : V \to W$ by $T(v_i) = w_i$ for $i \in \{1, \dots, m\}$. Note this is a valid map since $m \leq n$ (so $w_i \in W$ for $i \in \{1, \dots, m\}$). Also, this is a linear map because every linear map is uniquely defined by where it sends basis vectors. Thus $T \in \mathcal{L}(V, W)$. Now suppose $a, b \in V$ and $T(a) = T(b)$. Then $\exists! a_1, \dots, a_m, b_1, \dots, b_m \in \mathbb{F}$ such that $a = a_1 v_1 + \dots + a_m v_m$ and $b = b_1 v_1 + \dots + b_m v_m$. Then by the additivity and homogeneity of T, $T(a) = T(a_1 v_1 + \dots + a_m v_m) = a_1 w_1 + \dots + a_m w_m$ and $T(b) = T(b_1 v_1 + \dots + b_m v_m) = b_1 w_1 + \dots + b_m w_m$. However, since $T(a) = T(b)$, $a_1 w_1 + \dots + a_m w_m = b_1 w_1 + \dots + b_m w_m \implies (a_1 - b_1) w_1 + \dots (a_m - b_m) w_m = 0$. Since $(w_1, \dots, w_n)$ is a basis, $(w_1, \dots, w_m)$ is linearly independent. And thus $(a_i - b_i) = 0$ and $a_i = b_i$ for $i \in \{1, \dots, m\}$. Thus $a = b$. Thus T is injective.

Thus for finite dimensional vector spaces V and W, $\exists T \in \mathcal{L}(V, W)$ such that T is injective $\iff \dim(V) \leq \dim(W)$. $\square$

Sec. 3C

#2

Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbb{R})$ and a basis of $\mathcal{P}_2(\mathbb{R})$ such that the matrix of D with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Consider the list of vectors in $\mathcal{P}_3(\mathbb{R})$, $\mathcal{X} = (x^3, x^2, x, 1)$, and let $p = p_3x^3 + p_2x^2 + p_1x + p_0 \in \mathcal{P}_3(\mathbb{R})$. Then let $a_i = p_i$ for $i \in \{0, 1, 2, 3\}$. Then $a_3(x^3) + a_2(x^2) + a_1(x) + a_0(1) = p$, and thus $\mathcal{X}$ spans $\mathcal{P}_3(\mathbb{R})$. Then suppose $a_3x^3 + a_2x^2 + a_1x + a_0 = 0 = 0x^3 + 0x^2 + 0x + 0$. This implies $a_3 = a_2 = a_1 = a_0 = 0$, and thus $\mathcal{X}$ is linearly independent and a basis of $\mathcal{P}_3(\mathbb{R})$. Consider the list of vectors in $\mathcal{P}_2(\mathbb{R})$, $\mathcal{Y} = (3x^2, 2x, 1)$, and let $q = q_2x^2 + q_1x + q_0 \in \mathcal{P}_2(\mathbb{R})$. Then let $b_2 = \frac{1}{3}q_2$, $b_1 = \frac{1}{2}q_1$, and $b_0 = q_0$. Then $b_2(3x^2) + b_1(2x) + b_0(1) = q$, and thus $\mathcal{Y}$ spans $\mathcal{P}_2(\mathbb{R})$. Then suppose $b_2(3x^2) + b_1(2x) + b_0(1) = 0 = 0x^2 + 0x + 0$. This implies $3b_2 = 2b_1 = b_0 = 0$, which implies $b_2 = b_1 = b_0 = 0$. Thus $\mathcal{Y}$ is linearly independent and a basis of $\mathcal{P}_2(\mathbb{R})$. Then,

$$\begin{aligned} D(x_3) &= 3x^2 = 1(3x^2) + 0(2x) + 0(1) \\ D(x_2) &= 2x = 0(3x^2) + 1(2x) + 0(1) \\ D(x) &= 1 = 0(3x^2) + 0(2x) + 1(1) \\ D(1) &= 0 = 0(3x^2) + 0(2x) + 0(1) \end{aligned}$$

Thus,

$$\mathcal{M}(D, \mathcal{X}, \mathcal{Y}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$