HW: Custom Problems, Section 8A #7, Section 8B #1, 5, Section 8C #1, 3

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Custom Problems

#1

Let $U = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 + z_2 + z_3 = 0\}$*, and let* $f_1 = (1, -1, 0)$*, and* $f_2 = (1, 0, -1)$*. Define* $T \in \mathcal{L}(U)$ *by* $T(f_1) = (1, 1, -2)$ *and* $T(f_2) = f_2$*.*

(a) *Find the eigenvalues and associated eigenspaces for* T*.*

First note that (f_1, f_2) is a basis of U since they are linearly independent and $\dim(U) = 2$. Thus for any $v \in U$, $v = a_1 f_1 + a_2 f_2$. To find the eigenvalues and eigenvectors of T, set $T(v) = \lambda v$.

$$\begin{aligned}
& T(v) = \lambda v \\
\implies & T(a_1 f_1 + a_2 f_2) = \lambda(a_1 f_1 + a_2 f_2) \\
\implies & a_1(1, 1, -2) + a_2(1, 0, -1) = \lambda(a_1(1, -1, 0) + a_2(1, 0, -1)) \\
\implies & (a_1 + a_2, a_1, -2a_1 - a_2) = (\lambda(a_1 + a_2), -\lambda a_1, -\lambda a_2) \\
\implies & \begin{cases} a_1 + a_2 & = \lambda(a_1 + a_2) \\ a_1 & = -\lambda a_1 \\ -2a_1 - a_2 & = -\lambda a_2 \end{cases}
\end{aligned}$$

$a_1 = -\lambda a_1 \implies a_1 = 0$ or $\lambda = -1$. If $a_1 = 0$, then $a_1 + a_2 = \lambda(a_1 + a_2) \implies a_2 = 0$ or $\lambda = 1$. If $a_2 = 0$, then $v = 0$, and thus λ would not be an eigenvalue. Thus $\lambda = 1$. Thus $-2a_1 - a_2 = \lambda a_2 \implies a_2$ is arbitrary. Thus $\lambda = 1$ is an eigenvalue of T and $E(1) = \{a(1, 0, -1) \mid a \in \mathbb{C}\}$. Now suppose $\lambda = -1$. Thus $a_1 + a_2 = 0$. So -1 is an eigenvalue of T and $E(-1) = \{a_1 f_1 + a_2 f_2 \in U \mid a_1 + a_2 = 0\}$. However, since $a_1 = -a_2$, $E(-1) = \{a(0, -1, 1) \mid a \in \mathbb{C}\}$.

(b) *Determine whether there exists a basis for* U *so that* $\mathcal{M}(T)$ *is diagonal with respect to that basis. If so, find suh a basis, compute* $\mathcal{M}(T)$ *with respect to that basis, and ignore (c) and (d). If not, move on to parts (c) and (d).*

Note $\dim(E(-1)) = 1$ and $\dim(E(1)) = 1$. Let $b_1 = (1, 0, -1) \in E(1)$ and $b_2 = (0, -1, 1) \in E(-1)$. Then $\pi = (b_1, b_2)$ is linearly independent since vectors from different eigenspaces are linearly independent. Since $\dim(U) = 2 = \text{len}(\pi)$, π is a basis for U comprised entirely of eigenvectors. Thus $\mathcal{M}(T, \pi)$ is a diagonal matrix.

$$\begin{aligned} b_1 &= (1, 0, -1) = f_2 \\ \implies T(b_1) &= T(f_2) = f_2 = b_1 \\ b_2 &= (0, -1, 1) = (1, -1, 0) - (1, 0, -1) = f_1 - f_2 \\ \implies T(b_2) &= T(f_1) - T(f_2) = (1, 1, -2) - (1, 0, -1) = (0, 1, -1) = -b_2 \\ \implies \mathcal{M}(T, \pi) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

We can ignore *(c)* and *(d)*.

(c) *Compute the generalized eigenspaces for T, and find a basis for U consisting of generalized eigenvectors.*

(((IGNORE)))

(d) *Compute $\mathcal{M}(T)$ with respect to the basis that is your answer in part (c), arranging your answer in block diagonal form.*

(((IGNORE)))

#2

Let $T \in \mathcal{L}(\mathbb{C}^3$ by $T(z_1, z_2, z_3) = (4z_1 + 4z_2 + 4z_3, 5z_2 + 4z_3, 5z_3)$

(a) *Find the eigenvalues and associated eigenspaces for T.*

Let $e = (e_1, e_2, e_3)$ be the standard basis of $\mathbb{C}^3$. Then

$$\begin{aligned} T(e_1) &= (4, 0, 0) \\ T(e_2) &= (4, 5, 0) \\ T(e_3) &= (4, 4, 5) \\ \implies \mathcal{M}(T, e) &= \begin{pmatrix} 4 & 4 & 4 \\ 0 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix} \end{aligned}$$

Since $\mathcal{M}(T, e)$ is an upper-triangular matrix, the eigenvalues of T are the entries on the main diagonal, namely $\lambda = 4$ and $\lambda = 5$. To calculate $E(\lambda)$, find the conditions on v by which $T(z_1, z_2, z_3) = \lambda(z_1, z_2, z_3)$.

$$\begin{aligned} T(z_1, z_2, z_3) &= 4(z_1, z_2, z_3) \\ \implies (4z_1 + 4z_2 + 4z_3, 5z_2 + 4z_3, 5z_3) &= (4z_1, 4z_2, 4z_3) \\ \implies &\begin{cases} 4z_1 + 4z_2 + 4z_3 &= 4z_1 \\ 5z_2 + 4z_3 &= 4z_2 \\ 5z_3 &= 4z_3 \end{cases} \end{aligned}$$

$$
\begin{aligned}
&\implies z_3 = 0\\
&\implies z_2 = 0\\
&\implies z_1 \text{ is arbitrary}\\
&\implies E(4) = \{a(1,0,0) \mid a \in \mathbb{C}\}
\end{aligned}
$$

$$
\begin{aligned}
T(z_1,z_2,z_3) &= 5(z_1,z_2,z_3)\\
\implies (4z_1+4z_2+4z_3, 5z_2+4z_3, 5z_3) &= (5z_1, 5z_2, 5z_3)\\
&\implies \begin{cases} 4z_1+4z_2+4z_3 &= 5z_1\\ \qquad\quad 5z_2+4z_3 &= 5z_2\\ \qquad\qquad\quad 5z_3 &= 5z_3 \end{cases}\\
&\implies z_3 = 0\\
&\implies z_1 = 4z_2 \text{ , } z_2 \text{ arbitrary}\\
&\implies E(5) = \{a(4,1,0) \mid a \in \mathbb{C}\}
\end{aligned}
$$

(b) *Determine whether there exists a basis for U so that $\mathcal{M}(T)$ is diagonal with respect to that basis. If so, find suh a basis, compute $\mathcal{M}(T)$ with respect to that basis, and ignore (c) and (d). If not, move on to parts (c) and (d).*

Note $\dim(E(4)) = 1$ and $\dim(E(5)) = 1$. Thus each of these eigenspaces may contribute only one vector to a linearly independent list. Since $\dim(\mathbb{C}^3) = 3$, the by the Pigeonhole Principle, we cannot form basis for $\mathbb{C}^3$ comprised entirely of vectors from $E(4)$ and $E(5)$. Thus we cannot form a basis comprised entirely of eigenvectors, and thus T is not diagonalizable. We unfortunately cannot ignore *(c)* and *(d)*.

(c) *Compute the generalized eigenspaces for T, and find a basis for U consisting of generalized eigenvectors.*

By definition, $G(4) = \text{null}(T-4I)^3$

$$
\begin{aligned}
\mathcal{M}((T-4I),e) &= \begin{pmatrix} 0 & 4 & 4\\ 0 & 1 & 4\\ 0 & 0 & 1 \end{pmatrix}\\
\implies [\mathcal{M}((T-4I),e)]^2 &= \begin{pmatrix} 0 & 4 & 4\\ 0 & 1 & 4\\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 4 & 4\\ 0 & 1 & 4\\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 20\\ 0 & 1 & 8\\ 0 & 0 & 1 \end{pmatrix}\\
\implies [\mathcal{M}((T-4I),e)]^3 &= \begin{pmatrix} 0 & 4 & 20\\ 0 & 1 & 8\\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 4 & 4\\ 0 & 1 & 4\\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 36\\ 0 & 1 & 12\\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\implies (T-4I)^3(e_1) &= 0\\
(T-4I)^3(e_2) &= 4e_1 + e_2\\
(T-4I)^3(e_3) &= 36e_1 + 12e_2 + e_3
\end{aligned}
$$

$$
\begin{aligned}
\implies (T-4I)^3(z_1,z_2,z_3) &= z_2(4e_1+e_2) + z_3(36e_1+12e_2+e_3)\\
&= (4z_2+36z_3, z_2+12z_3, z_3)
\end{aligned}
$$

So $(T-4I)^3(z_1, z_2, z_3) = 0$ only if $z_2 = z_3 = 0$. Thus,

$$G(4) = \{a(1,0,0) \mid a \in \mathbb{C}\} = E(4)$$

By definition, $G(5) = \text{null}(T-5I)^3$

$$
\begin{aligned}
\mathcal{M}((T-5I), e) &= \begin{pmatrix} -1 & 4 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \\
\implies [\mathcal{M}((T-4I), e)]^2 &= \begin{pmatrix} -1 & 4 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 & 4 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -4 & 12 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\implies [\mathcal{M}((T-4I), e)]^3 &= \begin{pmatrix} 1 & -4 & 12 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 & 4 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 4 & -12 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\implies (T-5I)^3(e_1) &= -e_1 \\
(T-4I)^3(e_2) &= 4e_1 \\
(T-4I)^3(e_3) &= -12e_1 \\
\implies (T-4I)^3(z_1, z_2, z_3) &= z_1(-e_1) + z_2(4e_1) + z_3(-12e_1) \\
&= (-z_1 + 4z_2 - 12z_3, 0, 0)
\end{aligned}
$$

So $(T-5I)^3(z_1, z_2, z_3) = 0$ only if $z_1 = 4z_2 - 12z_3$. Thus,

$$G(5) = \{(4a - 12b, a, b) \mid a, b \in \mathbb{C}\} \supset E(5)$$

$\dim(G(4)) = 1 \implies G(4)$ may contribute only one vector to a linearly independent list. $\dim(G(5)) = 2 \implies G(5)$ may contribute up to two vectors to a linearly independent list. Since vectors from different generalized eigenspaces are linearly independent and $\dim(\mathbb{C}^3) = 3$, we can form a basis comprised of one vector from $G(4)$ and two from $G(5)$. Pick $(1,0,0) \in G(4)$, and $(4,1,0), (-12,0,1) \in G(5)$, and let $\pi = \Big((1,0,0), (4,1,0), (-12,0,1)\Big)$. Then π is a basis of $\mathbb{C}^3$ comprised entirely of generalized eigenvectors.

(d) *Compute $\mathcal{M}(T)$ with respect to the basis that is your answer in part (c), arranging your answer in block diagonal form.*

$$
\begin{aligned}
T(1,0,0) &= T(e_1) = 4e_1 = (4,0,0) = 4(1,0,0) \\
T(4,1,0) &= 4T(e_1) + T(e_2) = 4(4e_1) + (4e_1 + 5e_2) \\
&= 20e_1 + 5e_2 = (20, 5, 0) = 5(4,1,0) \\
T(-12,0,1) &= -12T(e_1) + T(e_3) = -12(4e_1) + (4e_1 + 4e_2 + 5e_3) \\
&= -44e_1 + 4e_2 + 5e_3 = (-44, 4, 5) = 4(4,1,0) + 5(-12,0,1) \\
\implies \mathcal{M}(T, \pi) &= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}
\end{aligned}
$$

where

$$D_1 = \begin{pmatrix} 4 \end{pmatrix} \qquad \text{and} \qquad D_2 = \begin{pmatrix} 5 & 4 \\ 0 & 5 \end{pmatrix}$$

8A

#7

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Prove that 0 is the only eigenvalue of N.

Let $\dim(V) = n$.

$$\begin{aligned} & N \text{ is nilpotent} \\ \Longrightarrow\ & N^n = \mathbf{0} \\ \Longrightarrow\ & (N - 0I)^n = \mathbf{0} \\ \Longrightarrow\ & \text{null}(N - 0I)^n = V \\ \Longrightarrow\ & G(0) = V. \end{aligned}$$

Now suppose $\lambda \neq 0$ and λ is an eigenvalue of N. Then $G(\lambda)$ is non-trivial, i.e. $G(\lambda) \neq \{0\}$. Suppose $\pi = (v_1, \ldots, v_n)$ is a basis for V. Then π is a basis for $G(0)$. Then pick $w \in G(\lambda)$ such that $w \neq 0$. Since vectors from different generalized eigenspaces are linearly independent, $\phi = (v_1, \ldots, v_n, w)$ is a linearly independent list. But $\text{len}(\phi) = n + 1 > n = \text{len}(\pi)$. This is a contradiction since a linearly independent set cannot have longer length than a basis. Thus there is no eigenvalue of N other than 0. $\square$

8B

#1

Suppose V is a complex vector space, $N \in \mathcal{L}(V)$, and 0 is the only eigenvalue of N. Prove that N is nilpotent.

Since 0 is the only eigenvector of N, $G(0)$ is the only generalized eigenspace of N. Since every complex vector space has a basis consisting entirely of generalized eigenvectors, V must have a basis comprised entirely of vectors from $G(0)$. Thus $\dim(G(0)) \geq \dim(V)$. But $G(0) \subset V$, so $\dim(G(0)) \leq \dim(V)$. Thus $\dim(G(0)) = \dim(V)$, which implies $G(0) = V$.

$$\begin{aligned} & G(0) = V \\ \Longrightarrow\ & \text{null}(N - 0I)^n = V \\ \Longrightarrow\ & (N - 0I)^n = \mathbf{0} \\ \Longrightarrow\ & N^n = \mathbf{0} \\ \Longrightarrow\ & N \text{ is nilpotent} \end{aligned}$$

$\square$

#5

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that V has a basis consisting of eigenvectors of T if and only if every generalized eigenvector of T is an eigenvector of T.

"$\Longrightarrow$"
Suppose V has a basis consisting of eigenvectors of T. Let π be one such basis, listed in such a way so that eigenvectors from the same eigenspace are adjacent. Then T is diagonalizable. In particular, $\mathcal{M}(T, \pi)$ is a diagonal matrix. Supposing T has m eigenvalues,

$$\mathcal{M}(T, \pi) = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix}$$

where A_i is a $d_i \times d_i$ matrix of the form

$$A_i = \lambda_i I_{d_i} = \begin{pmatrix} \lambda_i & 0 & \dots & 0 \\ 0 & \lambda_i & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_i \end{pmatrix}, \qquad i = 1, \dots, m$$

and d_i is the multiplicity of λ_i. Notice $\mathcal{M}(T, \pi)$ is also block diagonal, indicating a decomposition of π into generalized eigenspaces of T. Now let v be a generalized eigenvector corresponding to λ_k, and let $\sigma = (v_1, \dots, v_{d_k})$ be a basis for $E(\lambda_k)$, (i.e. the d_k elements of σ are linearly independent eigenvectors correspoding to λ_k). Then $v = a_1 v_1 + \dots + a_{d_k} v_{d_k}$. Thus,

$$\begin{aligned} T(v) &= a_1 T(v_1) + \dots + a_{d_k} T(v_{d_k}) \\ &= a_1(\lambda_k v_1) + \dots + a_{d_k}(\lambda_k v_{d_k}) \\ &= \lambda_k(a_1 v_1 + \dots + a_{d_k} v_{d_k}) \\ &= \lambda_k v \end{aligned}$$

Thus v is an eigenvector of T corresponding to λ_k. Thus every generalized eigenvector of T is an eigenvector of T.

"$\Longleftarrow$"
Suppose every generalized eigenvector of T is an eigenvector of T. Since V is a complex vector space, $\exists \pi$ such that π is a basis of V consisting of generalized eigenvectors of T. Since every generalized eigenvector is an eigenvector, π is a basis of V consisting of eigenvectors of T. Thus V has a basis of consisting of eigenvectors of T.

Thus, V has a basis consisting of eigenvectors of T if and only if every generalized eigenvector of T is an eigenvector of T. $\square$

8C

#1

Suppose that $T \in \mathcal{L}(\mathbb{C}^4)$ is such that the eigenvalues of T are 3, 5, and 8. Prove that

$(T-3I)^2(T-5I)^2(T-8I)^2 = \mathbf{0}$.

Since the eigenvalues of T are 3, 5, and 8, the characteristic polynomial of T is $(T-3I)^{d_3}(T-5I)^{d_5}(T-8I)^{d_8}$ where d_λ is the multiplicity of eigenvalue λ. Since $\dim(\mathbb{C}^4) = 4$, $d_3 + d_5 + d_8 = 4$. But $d_\lambda \geq 1$ for each eigenvalue. Thus $d_\lambda = 1$ for two of the three eigenvalues, and $d_\lambda = 2$ for exactly one of the three eigenvalues. Without loss of generality, suppose $d_3 = 2$, and $d_5 = d_8 = 1$. Then the characteristic polynomial is $p = (z-3)^2(z-5)(z-8)$. Then $q = (z-3)^2(z-5)^2(z-8)^2$ is a polynomial multiple of the p. However, p is a polynomial multiple of the minimal polynomial m, and thus q is a polynomial multiple of m. Thus $q(T) = (T-3I)^2(T-5I)^2(T-8I)^2 = \mathbf{0}$. □

#3

Give an example of an operator of $\mathbb{C}^4$ whose characteristic polynomial equals $(z-7)^2(z-8)^2$.

Any upper-triangular 4×4 matrix with two 7s and two 8s on the main diagonal will define an operator with that characteristic polynomial. Let $e = (e_1, \ldots, e_4)$ be the standard basis of $\mathbb{C}^4$ and let $T \in \mathcal{L}(\mathbb{C}^4)$ such that

$$\mathcal{M}(T, e) = \begin{pmatrix} 7 & a_{12} & a_{13} & a_{14} \\ 0 & 7 & a_{23} & a_{24} \\ 0 & 0 & 8 & a_{34} \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

In other words, define $T \in \mathcal{L}(\mathbb{C}^4)$ by its action on the standard basis:

$$\begin{aligned} T(e_1) &= 7e_1 \\ T(e_2) &= a_{12}e_1 + 7e_2 \\ T(e_3) &= a_{13}e_1 + a_{23}e_2 + 8e_3 \\ T(e_4) &= a_{14}e_1 + a_{24}7e_2 + a_{34}e_3 + 8e_4 \end{aligned}$$

Finally, we can provide an explicit formula for T through its defintion above:

$$T(z_1, z_2, z_3, z_4) = (7z_1 + a_{12}z_2 + z_{13}z_3 + a_{14}z_4, 7z_2 + z_{23}z_3 + a_{24}z_4, 8z_3 + a_{34}z_4, 8z_4)$$

The simplest case is when $a_{ij} = 0$ when $i \neq j$, giving

$$T(z_1, z_2, z_3, z_4) = (7z_1, 7z_2, 8z_3, 8z_4)$$