

HW: Custom Problems, Section 8A #7, Section 8B #1, 5, Section 8C #1, 3

Sam Fleischer

April 14, 2015

Custom Problems

#1

Let $U = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 + z_2 + z_3 = 0\}$, and let $f_1 = (1, -1, 0)$, and $f_2 = (1, 0, -1)$. Define $T \in \mathcal{L}(U)$ by $T(f_1) = (1, 1, -2)$ and $T(f_2) = f_2$.

(a) Find the eigenvalues and associated eigenspaces for T .

First note that (f_1, f_2) is a basis of U since they are linearly independent and $\dim(U) = 2$. Thus for any $v \in U$, $v = a_1 f_1 + a_2 f_2$. To find the eigenvalues and eigenvectors of T , set $T(v) = \lambda v$.

$$\begin{aligned} T(v) &= \lambda v \\ \implies T(a_1 f_1 + a_2 f_2) &= \lambda(a_1 f_1 + a_2 f_2) \\ \implies a_1(1, 1, -2) + a_2(1, 0, -1) &= \lambda(a_1(1, -1, 0) + a_2(1, 0, -1)) \\ \implies (a_1 + a_2, a_1, -2a_1 - a_2) &= (\lambda(a_1 + a_2), -\lambda a_1, -\lambda a_2) \\ \implies \begin{cases} a_1 + a_2 &= \lambda(a_1 + a_2) \\ a_1 &= -\lambda a_1 \\ -2a_1 - a_2 &= -\lambda a_2 \end{cases} \end{aligned}$$

$a_1 = -\lambda a_1 \implies a_1 = 0$ or $\lambda = -1$. If $a_1 = 0$, then $a_1 + a_2 = \lambda(a_1 + a_2) \implies a_2 = 0$ or $\lambda = 1$. If $a_2 = 0$, then $v = 0$, and thus λ would not be an eigenvalue. Thus $\lambda = 1$. Thus $-2a_1 - a_2 = \lambda a_2 \implies a_2$ is arbitrary. Thus $\lambda = 1$ is an eigenvalue of T and $E(1) = \{a(1, 0, -1) \mid a \in \mathbb{C}\}$. Now suppose $\lambda = -1$. Thus $a_1 + a_2 = 0$. So -1 is an eigenvalue of T and $E(-1) = \{a_1 f_1 + a_2 f_2 \in U \mid a_1 + a_2 = 0\}$. However, since $a_1 = -a_2$, $E(-1) = \{a(0, -1, 1) \mid a \in \mathbb{C}\}$.

(b) Determine whether there exists a basis for U so that $\mathcal{M}(T)$ is diagonal with respect to that basis. If so, find such a basis, compute $\mathcal{M}(T)$ with respect to that basis, and ignore (c) and (d). If not, move on to parts (c) and (d).

Note $\dim(E(-1)) = 1$ and $\dim(E(1)) = 1$. Let $b_1 = (1, 0, -1) \in E(1)$ and $b_2 = (0, -1, 1) \in E(-1)$. Then $\pi = (b_1, b_2)$ is linearly independent since vectors from different eigenspaces are linearly independent. Since $\dim(U) = 2 = \text{len}(\pi)$, π is a basis for U comprised entirely of eigenvectors. Thus $\mathcal{M}(T, \pi)$ is a diagonal matrix.

$$\begin{aligned} b_1 &= (1, 0, -1) = f_2 \\ \implies T(b_1) &= T(f_2) = f_2 = b_1 \\ b_2 &= (0, -1, 1) = (1, -1, 0) - (1, 0, -1) = f_1 - f_2 \\ \implies T(b_2) &= T(f_1) - T(f_2) = (1, 1, -2) - (1, 0, -1) = (0, 1, -1) = -b_2 \\ \implies \mathcal{M}(T, \pi) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

We can ignore (c) and (d).

- (c) *Compute the generalized eigenspaces for T , and find a basis for U consisting of generalized eigenvectors.*

((IGNORE)))

- (d) *Compute $\mathcal{M}(T)$ with respect to the basis that is your answer in part (c), arranging your answer in block diagonal form.*

((IGNORE)))

#2

Let $T \in \mathcal{L}(\mathbb{C}^3)$ by $T(z_1, z_2, z_3) = (4z_1 + 4z_2 + 4z_3, 5z_2 + 4z_3, 5z_3)$

- (a) *Find the eigenvalues and associated eigenspaces for T .*

Let $e = (e_1, e_2, e_3)$ be the standard basis of \mathbb{C}^3 . Then

$$\begin{aligned} T(e_1) &= (4, 0, 0) \\ T(e_2) &= (4, 5, 0) \\ T(e_3) &= (4, 4, 5) \\ \implies \mathcal{M}(T, e) &= \begin{pmatrix} 4 & 4 & 4 \\ 0 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix} \end{aligned}$$

Since $\mathcal{M}(T, e)$ is an upper-triangular matrix, the eigenvalues of T are the entries on the main diagonal, namely $\lambda = 4$ and $\lambda = 5$. To calculate $E(\lambda)$, find the conditions on v by which $T(z_1, z_2, z_3) = \lambda(z_1, z_2, z_3)$.

$$\begin{aligned} T(z_1, z_2, z_3) &= 4(z_1, z_2, z_3) \\ \implies (4z_1 + 4z_2 + 4z_3, 5z_2 + 4z_3, 5z_3) &= (4z_1, 4z_2, 4z_3) \\ \implies \begin{cases} 4z_1 + 4z_2 + 4z_3 &= 4z_1 \\ 5z_2 + 4z_3 &= 4z_2 \\ 5z_3 &= 4z_3 \end{cases} \end{aligned}$$

$$\begin{aligned}
&\implies z_3 = 0 \\
&\implies z_2 = 0 \\
&\implies z_1 \text{ is arbitrary} \\
&\implies E(4) = \{a(1, 0, 0) \mid a \in \mathbb{C}\}
\end{aligned}$$

$$\begin{aligned}
&T(z_1, z_2, z_3) = 5(z_1, z_2, z_3) \\
&\implies (4z_1 + 4z_2 + 4z_3, 5z_2 + 4z_3, 5z_3) = (5z_1, 5z_2, 5z_3) \\
&\implies \begin{cases} 4z_1 + 4z_2 + 4z_3 = 5z_1 \\ 5z_2 + 4z_3 = 5z_2 \\ 5z_3 = 5z_3 \end{cases} \\
&\implies z_3 = 0 \\
&\implies z_1 = 4z_2, z_2 \text{ arbitrary} \\
&\implies E(5) = \{a(4, 1, 0) \mid a \in \mathbb{C}\}
\end{aligned}$$

- (b) Determine whether there exists a basis for U so that $\mathcal{M}(T)$ is diagonal with respect to that basis. If so, find such a basis, compute $\mathcal{M}(T)$ with respect to that basis, and ignore (c) and (d). If not, move on to parts (c) and (d).

Note $\dim(E(4)) = 1$ and $\dim(E(5)) = 1$. Thus each of these eigenspaces may contribute only one vector to a linearly independent list. Since $\dim(\mathbb{C}^3) = 3$, the by the Pigeonhole Principle, we cannot form basis for \mathbb{C}^3 comprised entirely of vectors from $E(4)$ and $E(5)$. Thus we cannot form a basis comprised entirely of eigenvectors, and thus T is not diagonalizable. We unfortunately cannot ignore (c) and (d).

- (c) Compute the generalized eigenspaces for T , and find a basis for U consisting of generalized eigenvectors.

By definition, $G(4) = \text{null}(T - 4I)^3$

$$\begin{aligned}
&\mathcal{M}((T - 4I), e) = \begin{pmatrix} 0 & 4 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \\
&\implies [\mathcal{M}((T - 4I), e)]^2 = \begin{pmatrix} 0 & 4 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 4 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 20 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{pmatrix} \\
&\implies [\mathcal{M}((T - 4I), e)]^3 = \begin{pmatrix} 0 & 4 & 20 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 4 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 36 \\ 0 & 1 & 12 \\ 0 & 0 & 1 \end{pmatrix} \\
&\implies (T - 4I)^3(e_1) = 0 \\
&\quad (T - 4I)^3(e_2) = 4e_1 + e_2 \\
&\quad (T - 4I)^3(e_3) = 36e_1 + 12e_2 + e_3 \\
&\implies (T - 4I)^3(z_1, z_2, z_3) = z_2(4e_1 + e_2) + z_3(36e_1 + 12e_2 + e_3) \\
&\quad = (4z_2 + 36z_3, z_2 + 12z_3, z_3)
\end{aligned}$$

So $(T - 4I)^3(z_1, z_2, z_3) = 0$ only if $z_2 = z_3 = 0$. Thus,

$$G(4) = \{a(1, 0, 0) \mid a \in \mathbb{C}\} = E(4)$$

By definition, $G(5) = \text{null}(T - 5I)^3$

$$\begin{aligned} \mathcal{M}((T - 5I), e) &= \begin{pmatrix} -1 & 4 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \\ \implies [\mathcal{M}((T - 4I), e)]^2 &= \begin{pmatrix} -1 & 4 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 & 4 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -4 & 12 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \implies [\mathcal{M}((T - 4I), e)]^3 &= \begin{pmatrix} 1 & -4 & 12 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 & 4 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 4 & -12 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \implies (T - 5I)^3(e_1) &= -e_1 \\ (T - 4I)^3(e_2) &= 4e_1 \\ (T - 4I)^3(e_3) &= -12e_1 \\ \implies (T - 4I)^3(z_1, z_2, z_3) &= z_1(-e_1) + z_2(4e_1) + z_3(-12e_1) \\ &= (-z_1 + 4z_2 - 12z_3, 0, 0) \end{aligned}$$

So $(T - 5I)^3(z_1, z_2, z_3) = 0$ only if $z_1 = 4z_2 - 12z_3$. Thus,

$$G(5) = \{(4a - 12b, a, b) \mid a, b \in \mathbb{C}\} \supset E(5)$$

$\dim(G(4)) = 1 \implies G(4)$ may contribute only one vector to a linearly independent list. $\dim(G(5)) = 2 \implies G(5)$ may contribute up to two vectors to a linearly independent list. Since vectors from different generalized eigenspaces are linearly independent and $\dim(\mathbb{C}^3) = 3$, we can form a basis comprised of one vector from $G(4)$ and two from $G(5)$. Pick $(1, 0, 0) \in G(4)$, and $(4, 1, 0), (-12, 0, 1) \in G(5)$, and let $\pi = ((1, 0, 0), (4, 1, 0), (-12, 0, 1))$. Then π is a basis of \mathbb{C}^3 comprised entirely of generalized eigenvectors.

(d) Compute $\mathcal{M}(T)$ with respect to the basis that is your answer in part (c), arranging your answer in block diagonal form.

$$\begin{aligned} T(1, 0, 0) &= T(e_1) = 4e_1 = (4, 0, 0) = 4(1, 0, 0) \\ T(4, 1, 0) &= 4T(e_1) + T(e_2) = 4(4e_1) + (4e_1 + 5e_2) \\ &= 20e_1 + 5e_2 = (20, 5, 0) = 5(4, 1, 0) \\ T(-12, 0, 1) &= -12T(e_1) + T(e_3) = -12(4e_1) + (4e_1 + 4e_2 + 5e_3) \\ &= -44e_1 + 4e_2 + 5e_3 = (-44, 4, 5) = 4(4, 1, 0) + 5(-12, 0, 1) \\ \implies \mathcal{M}(T, \pi) &= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \end{aligned}$$

where

$$D_1 = \begin{pmatrix} 4 \end{pmatrix} \quad \text{and} \quad D_2 = \begin{pmatrix} 5 & 4 \\ 0 & 5 \end{pmatrix}$$

8A

#7

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Prove that 0 is the only eigenvalue of N .

Let $\dim(V) = n$.

$$\begin{aligned} & N \text{ is nilpotent} \\ \implies & N^n = \mathbf{0} \\ \implies & (N - 0I)^n = \mathbf{0} \\ \implies & \text{null}(N - 0I)^n = V \\ \implies & G(0) = V. \end{aligned}$$

Now suppose $\lambda \neq 0$ and λ is an eigenvalue of N . Then $G(\lambda)$ is non-trivial, i.e. $G(\lambda) \neq \{0\}$. Suppose $\pi = (v_1, \dots, v_n)$ is a basis for V . Then π is a basis for $G(0)$. Then pick $w \in G(\lambda)$ such that $w \neq 0$. Since vectors from different generalized eigenspaces are linearly independent, $\phi = (v_1, \dots, v_n, w)$ is a linearly independent list. But $\text{len}(\phi) = n + 1 > n = \text{len}(\pi)$. This is a contradiction since a linearly independent set cannot have longer length than a basis. Thus there is no eigenvalue of N other than 0. \square

8B

#1

Suppose V is a complex vector space, $N \in \mathcal{L}(V)$, and 0 is the only eigenvalue of N . Prove that N is nilpotent.

Since 0 is the only eigenvalue of N , $G(0)$ is the only generalized eigenspace of N . Since every complex vector space has a basis consisting entirely of generalized eigenvectors, V must have a basis comprised entirely of vectors from $G(0)$. Thus $\dim(G(0)) \geq \dim(V)$. But $G(0) \subset V$, so $\dim(G(0)) \leq \dim(V)$. Thus $\dim(G(0)) = \dim(V)$, which implies $G(0) = V$.

$$\begin{aligned} & G(0) = V \\ \implies & \text{null}(N - 0I)^n = V \\ \implies & (N - 0I)^n = \mathbf{0} \\ \implies & N^n = \mathbf{0} \\ \implies & N \text{ is nilpotent} \end{aligned}$$

\square

#5

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that V has a basis consisting of eigenvectors of T if and only if every generalized eigenvector of T is an eigenvector of T .

“ \implies ”

Suppose V has a basis consisting of eigenvectors of T . Let π be one such basis, listed in such a way so that eigenvectors from the same eigenspace are adjacent. Then T is diagonalizable. In particular, $\mathcal{M}(T, \pi)$ is a diagonal matrix.

$$\mathcal{M}(T, \pi) = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix}$$

where $A_i = \lambda_i I_{d_i}$ and $d_i = \dim(E(\lambda_i))$, $i = 1, \dots, m$. However, this is also a block diagonal matrix. **something something proof.**

“ \impliedby ”

Suppose every generalized eigenvector of T is an eigenvector of T . Since V is a complex vector space, $\exists \pi$ such that π is a basis of V consisting of generalized eigenvectors of T . Since every generalized eigenvector is an eigenvector, π is consisting of eigenvectors of T . Thus V has a basis of consisting of eigenvectors of T .

Thus, V has a basis consisting of eigenvectors of T if and only if every generalized eigenvector of T is an eigenvector of T . \square

8C

#1

Suppose that $T \in \mathcal{L}(\mathbb{C}^4)$ is such that the eigenvalues of T are 3, 5, and 8. Prove that $(T - 3I)^2(T - 5I)^2(T - 8I)^2 = \mathbf{0}$.

Since the eigenvalues of T are 3, 5, and 8, the characteristic polynomial of T is $(T - 3I)^{d_3}(T - 5I)^{d_5}(T - 8I)^{d_8}$ where d_λ is the multiplicity of eigenvalue λ . Since $\dim(\mathbb{C}^4) = 4$, $d_3 + d_5 + d_8 = 4$. But $d_\lambda \geq 1$ for each eigenvalue. Thus $d_\lambda = 1$ for two of the three eigenvalues, and $d_\lambda = 2$ for exactly one of the three eigenvalues. Without loss of generality, suppose $d_3 = 2$, and $d_5 = d_8 = 1$. Then the characteristic polynomial is $p = (z - 3)^2(z - 5)(z - 8)$. Then $q = (z - 3)^2(z - 5)^2(z - 8)^2$ is a polynomial multiple of the p . However, p is a polynomial multiple of the minimal polynomial m , and thus q is a polynomial multiple of m . Thus $q(T) = (T - 3I)^2(T - 5I)^2(T - 8I)^2 = \mathbf{0}$. \square

#3

Give an example of an operator of \mathbb{C}^4 whose characteristic polynomial equals $(z - 7)^2(z - 8)^2$.

SOMETHING