Homework: Custom Problem, Sec. 3C # 4

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Custom Problem

Let $U = \{(x, y, z) \in \mathbb{F}^3 \mid x+y+z = 0\}$, $f_1 = (1, -1, 0)$, *and* $f_2 = (1, 0, -1)$. *Let* $T \in \mathcal{L}(U, U)$ *defined by* $T(f_1) = (1, 1, -2)$, *and* $T(f_2) = (1, 0, -1)$.

(a) *Compute* $\mathcal{M}(T, (f_1, f_2), (f_1, f_2))$, $\mathcal{M}((-4, 2, 2), (f_1, f_2))$, *and* $\mathcal{M}(T(-4, 2, 2), (f_1, f_2))$.

Note $T(f_1) = (1, 1, -2) = -1(1, -1, 0) + 2(1, 0, -1) = (-1)f_1 + (2)f_2$, and $T(f_2) = (1, 0, -1) = 0(1, -1, 0) + 1(1, 0, -1) = (0)f_1 + (1)f_2$. Thus,

$$\mathcal{M}(T, (f_1, f_2), (f_1, f_2)) = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$$

Note $(-4, 2, 2) = -2(1, -1, 0) - 2(1, 0, -1) = (-2)f_1 + (-2)f_2$. Thus,

$$\mathcal{M}((-4, 2, 2), (f_1, f_2)) = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

Note $T(-4, 2, 2) = T((-2)f_1 + (-2)f_2) = (-2)T(f_1) + (-2)T(f_2) = (-2)(1, 1, -2) + (-2)(1, 0, -1) = (-4, -2, 6) = 2(1, -1, 0) + (-6)(1, 0, -1) = (2)f_1 + (-6)f_2$. Thus,

$$\mathcal{M}(T(-4, 2, 2), (f_1, f_2)) = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$$

(b) *Comfirm* $\mathcal{M}(T(-4, 2, 2), (f_1, f_2)) = \mathcal{M}(T, (f_1, f_2), (f_1, f_2)) \cdot \mathcal{M}((-4, 2, 2), (f_1, f_2))$.

$$\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} (-1)(-2) + (0)(-2) \\ (2)(-2) + (1)(-2) \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$$

This helps confirm that if $T \in \mathcal{L}(V, W)$, and if $\mathcal{V}$ and $\mathcal{W}$ are bases for V and W, respectively, and if $v \in V$, that

$$\mathcal{M}(T(v), \mathcal{W}) = \mathcal{M}(T, \mathcal{V}, \mathcal{W}) \cdot \mathcal{M}(v, \mathcal{V})$$

Sec. 3C

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Suppose $\mathcal{V} = (v_1, \ldots, v_m)$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $(w_1, \ldots w_n)$ of W such that all the entries in the first column of $\mathcal{M}(T)$ (with respect to the bases $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_n)$) are 0 except for possibly a 1 in the first row, first column.

Since W is finite dimensional, let $\dim(W) = n$, and let $\mathcal{U} = (u_1, \ldots, u_n)$ be a basis for W. Then since $T \in \mathcal{L}(V, W)$, $T(v_1) = b_1 u_1 + \cdots + b_n u_n$ for some $b_1, \ldots, b_n \in \mathbb{F}$.

Case 1: $b_1 = \cdots = b_n = 0$

Then $T(v_1) = 0u_1 + \cdots + 0u_n$. Thus if $\mathcal{M}(T, \mathcal{V}, \mathcal{U}) = [a_{i,j}]_{n \times m}$, then $a_{i,1} = 0$ for $i = 1, \ldots, n$. In other words, all the entries in the first column of $\mathcal{M}(T, \mathcal{V}, \mathcal{U})$ are 0.

Case 2: $\exists k \in \{1, \ldots, n\}$ such that $b_k \neq 0$

Then construct the list $\mathcal{W} = (w_1, \ldots, w_n)$ where $w_k = T(v_1) = b_1 u_1 + \cdots + b_n u_n$ and $w_i = u_i$ for $i \in \{1, \ldots, k-1, k+1, \ldots, n\}$. Then let

$$a_1 w_1 + \cdots + a_n w_n = 0$$

for some $a_1, \ldots, a_n \in \mathbb{F}$. Thus

$$a_1 u_1 + \ldots a_{k-1} u_{k-1} + a_k(b_1 u_1 + \cdots + b_n u_n) + a_{k+1} u_{k+1} + \cdots + a_n u_n = 0$$

or

$$\begin{aligned}(a_1 + a_k b_1) u_1 + \cdots + (a_{k-1} + a_k b_{k-1}) u_{k-1} + a_k b_k u_k + (a_{k+1} + a_k b_{k+1}) u_{k+1} \\ + \cdots + (a_n + a_k b_n) u_n = 0\end{aligned}$$

But since $\mathcal{U}$ is a basis (and therefore linearly independent), all of the coefficients must be equivalently 0. In particular, $a_k b_k = 0$. By assumption, $b_k \neq 0$. Thus $a_k = 0$ since $\mathbb{F}$ is a field and there are no zero-divisors in fields. $a_k = 0 \implies a_1 = \cdots = a_n = 0$. Thus $\mathcal{W}$ is linearly independent. However, since $\dim(W) = n$, all linearly independent lists in W of length n form a basis of W. Thus $\mathcal{W}$ is a basis of W. Now re-order $\mathcal{W}$ in the following way: $\mathcal{Z} = (w_k, w_1, \ldots, w_{k-1}, w_{k+1}, \ldots, w_n)$. Note $\mathcal{Z}$ is a basis because it is simply a re-ordering of another basis. Then $T(v_1) = w_k = 1w_k + 0w_1 + \cdots + 0w_{k-1} + 0w_{k+1} + \cdots + 0w_n$. Thus if $\mathcal{M}(T, \mathcal{V}, \mathcal{Z}) = [a_{i,j}]_{n \times m}$, then $a_{1,1} = 1$ and $a_{i,1} = 0$ for $i = 2, \ldots, n$. In other words, all the entries in the first column of $\mathcal{M}(T, \mathcal{V}, \mathcal{Z})$ are 0 except for a 1 in the first row, first column.

In either case, there exists a basis $(w_1, \ldots w_n)$ of W such that all the entries in the first column of $\mathcal{M}(T)$ (with respect to the bases $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_n)$) are 0 except for possibly a 1 in the first row, first column. $\square$