HW: Custom Problem, Section 7A #1, 2, 7, 13

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Custom Problems

#1

Let $T \in \mathcal{L}(V)$. Prove T is invertible if and only if T^* is invertible.

"$\Longrightarrow$"
Suppose T is invertible, and that $T^*(w_1) = T^*(w_2)$. Then by the definition of T^*

$$\langle T(v), w_1\rangle = \langle v, T^*(w_1)\rangle \quad \text{and} \quad \langle T(v), w_2\rangle = \langle v, T^*(w_2)\rangle$$

for every $v \in V$. Thus,

$$\langle T(v), w_1\rangle = \langle v, T^*(w_1)\rangle = \langle v, T^*(w_2)\rangle = \langle T(v), w_2\rangle$$

for every $v \in V$. However, since T is invertible, it is surjective, and thus range$(T) = V$, and so

$$\langle z, w_1\rangle = \langle z, w_2\rangle$$

for every $z \in V$. Thus $w_1 = w_2$. Thus T^* is injective. Thus T^* is invertible.

"$\Longleftarrow$"
Suppose T^* is invertible. Then by the above argument, $(T^*)^*$ is invertible. However, $(T^*)^* = T$. Thus T is invertible.

Thus T is invertible if and only if T^* is invertible. $\square$

Section 7A

#1

Suppose n is a positive integer. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by

$$T(z_1, \ldots, z_n) = (0, z_1, \ldots, z_{n-1})$$

Find a formula for $T^*(w_1, \ldots, w_n)$.

Let $w = (w_1, \ldots, w_n) \in \mathcal{L}(\mathbb{F}^n)$. Note the following:

$$\begin{aligned}\langle T(z_1, \ldots, z_n), w\rangle &= \langle (0, z_1, \ldots, z_{n-1}), (w_1, \ldots, w_n)\rangle \\ &= z_1 w_2 + \ldots z_{n-1} w_n \\ &= \langle (z_1, dots, z_n), (w_2, \ldots, w_n, 0)\rangle\end{aligned}$$

for every $v \in V$. Thus, by the definition of T^*,

$$T^*(w_1, \ldots, w_n) = (w_2, \ldots, w_n, 0)$$

□

#2

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Prove that λ is an eigenvalue of T if and only if $\overline{lambda}$ is an eigenvalue of T^*.

"$\Longrightarrow$"
Suppose λ is an eigenvalue of T. Then $T(v) = \lambda v$ for some nonzero $v \in V$. Thus $v \in \text{null}(T - \lambda I)$. In other words, $(T - \lambda I)(v) = 0$. Now note that

$$(T - \lambda I)^* = T^* - (\lambda I)^* = T^* - \overline{\lambda} I^* = T^* - \overline{\lambda} I$$

since I is self-adjoint. Thus,

$$\langle (T - \lambda I)(v), w\rangle = \langle v, (T^* - \overline{\lambda} I)(w)\rangle$$

for all $w \in V$. However, $(T - \lambda I)(v) = 0$. Then

$$\begin{aligned}&\langle (T - \lambda I)(v), w\rangle \langle 0, w\rangle = 0 \\ &\Longrightarrow \ 0 = \langle v, (T^* - \overline{\lambda} I)(w)\rangle\end{aligned}$$

for all $w \in V$. Now suppose $\overline{\lambda}$ is *not* an eigenvalue of T^*. Then $\text{range}(T^* - \overline{\lambda} I) = V$. Then in particular, $\exists z \in V$ such that $(T^* - \overline{\lambda} I)(z) = v$. Thus

$$0 = \langle v, v\rangle$$

Thus $v = 0$, which implies v is not an eigenvalue of T, which is a contradiction. Then $\overline{\lambda}$ is an eigenvalue of T^*.

"$\Longleftarrow$" Suppose $\overline{\lambda}$ is an eigenvalue of T^*. Then since $\overline{\overline{\lambda}} = \lambda$ and $(T^*)^* = T$, then by the above argument, λ is an eigenvalue of T.

Thus λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*. □

#7

Suppose $S, T \in \mathcal{L}(V)$ are self-adjoint. Prove that $ST$is self-adjoint if and only if $ST = TS$.

Since S and T are self-adjoint, $S = S^*$ and $T = T^*$. Note the following:

$$(ST) = (ST)^* \iff ST = (T^*)(S^*) \iff ST = (T)(S) \iff ST = TS$$

Thus, ST is self-adjoint if and only if $ST = TS$. □

#13

Give an example of an operator $T \in \mathbb{C}^{\triangle}$ such that T is normal but not self-adjoint.

In other words, we want to find an operator such that $TT^* = T^*T$, but $T \neq T^*$. Let $e = (e_1, e_2, e_3, e_4)$ be the standard basis in $\mathbb{C}^4$. Then define $T(z_1, z_2, z_3, z_4) = (z_1, z_2, iz_3, -iz_4)$. In other words,

$$\mathcal{M}(T, e) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

Since $\mathcal{M}(T^*, e)$ is the conjugate transpose of $\mathcal{M}(T, e)$,

$$\mathcal{M}(T^*, e) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

Note

$$\begin{aligned} \mathcal{M}(T, e) \cdot \mathcal{M}(T^*, e) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \\ &= \mathcal{M}(T^*, e) \cdot \mathcal{M}(T, e) \\ \implies TT^* &= T^*T \end{aligned}$$

Thus T is normal. However, $\mathcal{M}(T, e) \neq \mathcal{M}(T^*, e)$, and so $T \neq T^*$. Thus T is not self-adjoint.