Breaking Kuramoto synchronization

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Synchronization is a phenomena found in systems such as flashing fireflies [1], chirping crickets [2], and metabolizing yeast cells [3, 4]. In this paper, I discuss a model for synchronizing oscillators known as the the Kuramoto model [5]. I introduce the Kuramoto model, investigate its dynamics, and show that both decreasing coupling strength between oscillators and adding noise to the system can make synchronization more difficult to achieve.

Synchronization can be found in many natural systems. For example, fireflies can flash in unison [1], crickets can chirp in unison [2], and yeast cells can metabolize synchronously [3, 4]. These systems all feature large populations of individual agents that have different natural frequencies (of flashing in the case of fireflies, or chirping in the case of crickets), yet somehow end up locking on to a common frequency. We can view all these systems as systems of oscillators with unique natural frequencies.

A way this synchronization can occur is if each the frequency of each oscillator is influenced by the frequency of all the other oscillators. This was proposed by Winfree [6], who suggested

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i), i \in [1, N] \tag{1}$$

where N is the number of oscillators, ω_i is the natural frequency of oscillator i, Γ_{ij} is some interaction function between oscillator i and oscillator j, θ_i is the phase angle of oscillator i.

We can visualize the oscillators as points moving around the unit circle. The phase of some oscillator i defines the location of the oscillator on the unit circle as a function of time. In cartesian coordinates, this location is $x_i(t) = \cos(\theta_i(t)), y_i(t) = \sin(\theta_i(t))$. The natural frequency of some oscillator i defines the unique frequency at which oscillator i would move around the unit circle if it was not coupled to the other oscillators.

By inspection of Eq. 1, we can see that the coupling term to the other oscillators causes oscillator i to move faster or slower around the unit circle than its natural frequency. Synchronization occurs when oscillators' phase angles θ_i evolve to the same dynamics (i.e. the frequencies of oscillator orbits evolve such that groups of oscillators orbit around the unit circle together). In order to observe synchronization, we need to choose the form of this coupling term such that at some time after initialization, groups of oscillators are oscillating with the same phase.

The Kuramoto model uses identical sinusoidal coupling functions Γ_{ij} such that Eq. 1 becomes

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \frac{K}{N} \sin(\theta_j - \theta_i), i \in [1, N] \tag{2}$$

where K is a constant determining the strength of the coupling term [5]. The $1/N$ factor ensures that the model is well-behaved as N becomes very large.

In the following, I will 1) discuss the dynamics of the Kuramoto model, and 2) show how adding noise to Eq. 2 affects the those dynamics. Implementation details can be found in the Appendices, and code can be found at `https://github.com/mcmorre/kuramoto`.

Dynamics of the Kuramoto model. — How does the system specified by Eq. 2 evolve in time? If we again visualize the oscillators as points moving around the unit circle, with the phase of some oscillator i defining its location on the unit circle as a function of time, we can imagine the time evolution of a system of oscillators initialized with random phase angles θ_i at $t = 0$. As time progresses, due to the coupling term in Eq. 2, the oscillators begin to synchronize in phase.

We can measure the degree of synchrony by defining the order parameters r and ψ where

$$re^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}. \tag{3}$$

The order parameter $r(t)$ represents the degree of oscillator synchrony at a given time and ranges in value from 0 to 1, with 0 meaning the oscillators are not at all synchronized, and 1 meaning the oscillators are all synchronized. The order parameter ψ represents the average phase of the oscillators, and is not incredibly informative except for its use in the mean field equation for the oscillator system

$$\dot{\theta}_i = \omega_i + Kr \sin(\psi - \theta_i), i \in [1, N]. \tag{4}$$

We can interpret Eq. 4 conceptually as all oscillators being coupled to the mean phase angle of the system. [8]

If $r(t)$ measures the degree of synchrony of the oscillator system, when we initialize our oscillators with randomly distributed phase angles and evolve these phases in time according to Eq. 2, we expect $r(t)$ to start out close to 0 (oscillators are not synchronized) and increase over time to about 1 (groups of oscillators are synchronized). For 100 oscillators, when I draw initial phase angles drawn from $\mathcal{U}[-2\pi, 2\pi]$, natural frequencies drawn from $\mathcal{N}[0, 1]$, make the coupling strength between oscillators strong (set $K = 3.0$), and let the phase angles evolve according to Eq. 2, we indeed observe this increase in r over time (Fig. 1C).

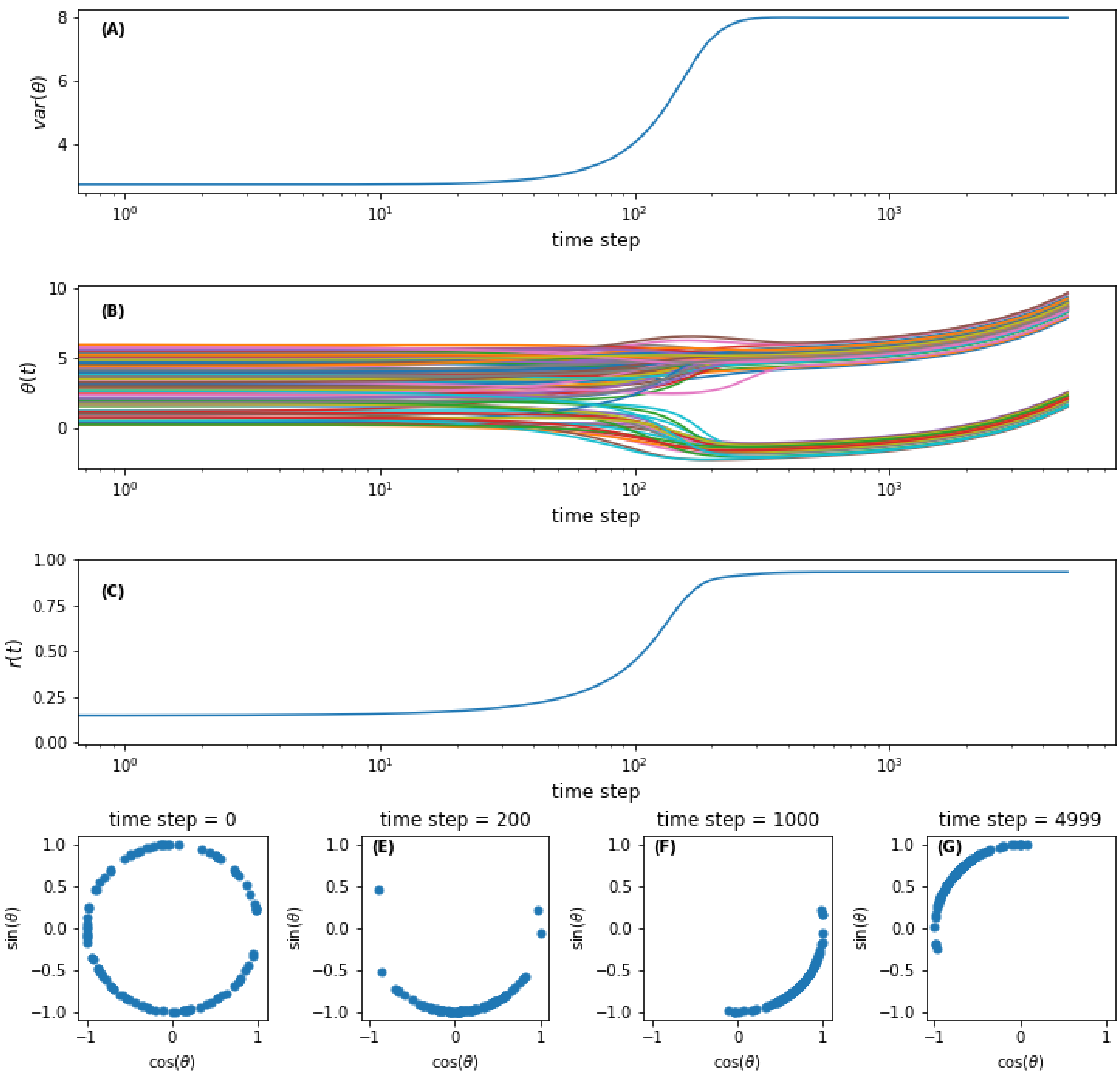

FIG. 1. Results for 100 oscillators with initial phase angles drawn from $\mathcal{U}[-2\pi, 2\pi]$, natural frequencies drawn from $\mathcal{N}[0, 1]$, and strong coupling strength ($K = 3.0$), and phase angle evolution according to Eq. 2. **(A)** Variance of phase angle θ_i over oscillators as a function of time step. **(B)** Phase angle θ_i for each oscillator i as a function of time step. Different oscillators are represented as different colors. **(C)** Order parameter r as a function of time step. **(D)**-**(F)** Position of each oscillator on the unit circle at the following time steps: 0, 200, 1000, and 4999.

We can also see signatures of this increase in synchronization level over time in the stabilization of phase angles $\theta_i(t)$ to clustered trajectories (Fig. 1B) changing in time together, which implies a final state of synchronization. We can see that the variance of the phase angles increases as the phase angles stabilize to these clustered trajectories (Fig. 1A), which represents the separation of the initially normally distributed phase angles into groups clustered around distinct phase angle trajectories over time. To visualize the synchronization level over time, I plot the phase angles θ_i on the unit circle in Fig. 1D-G at different times. Again, we can see that

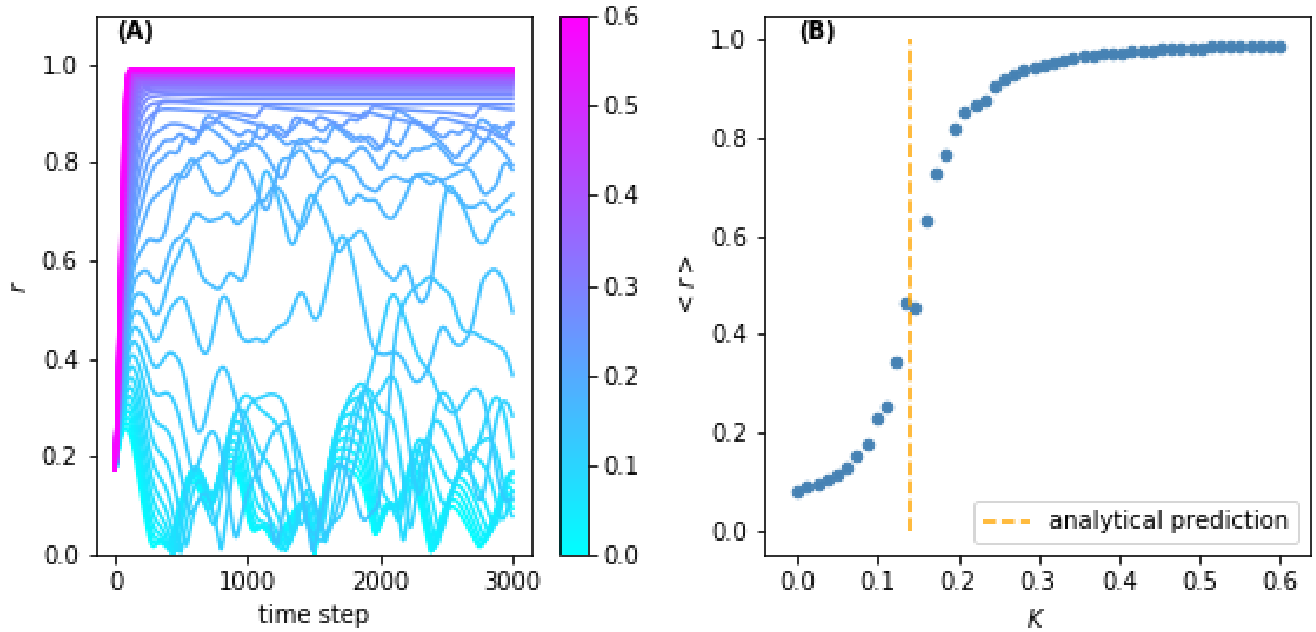

FIG. 2. Order parameter results for 100 oscillators with natural frequencies drawn from $\mathcal{N}(1.0, 0.1)$, initial phase angles drawn from $\mathcal{U}[-2\pi, 2\pi]$, varying coupling strength. **(A)** Order parameter r as a function of time step for coupling constants ranging between 0.0 and 0.6, holding all other parameters constant. **(B)** Time average of order parameter r as a function of coupling constant K, holding all other parameters constant. The theoretical value of the critical coupling separating sychronous and asynchronous states, $K_{crit}^{(theory)} = \frac{8\sigma}{\pi}$ [7], is marked in orange.

over time the oscillators synchronize and thus by the end of the simulation, they orbit the unit circle in closely packed groups.

So far, I have shown that as time progresses, strongly coupled oscillators with random phases at $t = 0$ governed by Eq. 2 eventually oscillate in phase with groups of other oscillators. Do we observe this same synchronization behavior if we decrease the coupling strength between oscillators? If we make the coupling term K in Eq. 2 very small, then the oscillators will not feel any pull to synchronize and they will oscillate at their natural frequencies ω_i for all time. Therefore, as we decrease the coupling strength K, we expect that it will become increasingly difficult for the oscillators to synchronize.

When I initialize 100 oscillators with natural frequencies drawn from $\mathcal{N}(1.0, 0.1)$ and initial phase angles drawn from $\mathcal{U}[-2\pi, 2\pi]$, we indeed see that as coupling strength K decreases while holding all other parameters constant, it becomes harder and harder for the oscillators to synchronize (Fig. 2A). Additionally, if we plot the time average of the order parameter r as a function of coupling strength K, holding all other parameters constant, we can see that the mean synchronization increases sigmoidally with K (Fig. 2B). That is, there is a critical coupling value K_{crit} that separates systems with non-synchronizing behavior from systems with synchronizing behavior. This special coupling value $K = K_{crit}$ is the inflection point of the time averaged order parameter vs. coupling strength plot in (Fig. 2B). This inflection point is pretty close to the theoretical value, $K_{crit}^{(theory)} = \frac{8\sigma}{\pi}$ [7], marked in orange on Fig. 2B. The separation of the unsynchronized state and synchronized state by the critical coupling valie K_{crit} is an example of a second order phase transition [7].

Dynamics of the noisy Kuramoto model.— If we add noise to Eq. 2, how does this effect the synchronization described in the previous section? That is, if we change Eq. 2 to

$$\dot{\theta}_i = \omega_i + \sigma\xi_i + \sum_{j=1}^{N} \frac{K}{N} \sin(\theta_j - \theta_i), i \in [1, N] \quad (5)$$

where $\xi_i \in \mathcal{N}(0, 1)$, σ is the standard deviation of the white noise, and ξ_i varies in time, how do the dynamics of phase angles change? Based on intuition, we expect that as we increasingly scramble the signal in Eq. 2, synchronization becomes harder and harder to achieve. In this case, our intuition is correct: in Fig. 3A we observe that while holding all other parameters constant (100 oscillators are simulated with natural frequencies drawn from $\mathcal{N}(0, 1)$, initial phase angles drawn from $\mathcal{U}[-2\pi, 2\pi]$, and coupling $K = 3.0$), injecting white noise according to Eq. 5 results in less synchronization. It is even more apparent that injected noise prevents synchronization when examining Fig. 3B-D: for the "clean" Kuramoto model (Fig. 3B), phase angles separate into clusters around dis-

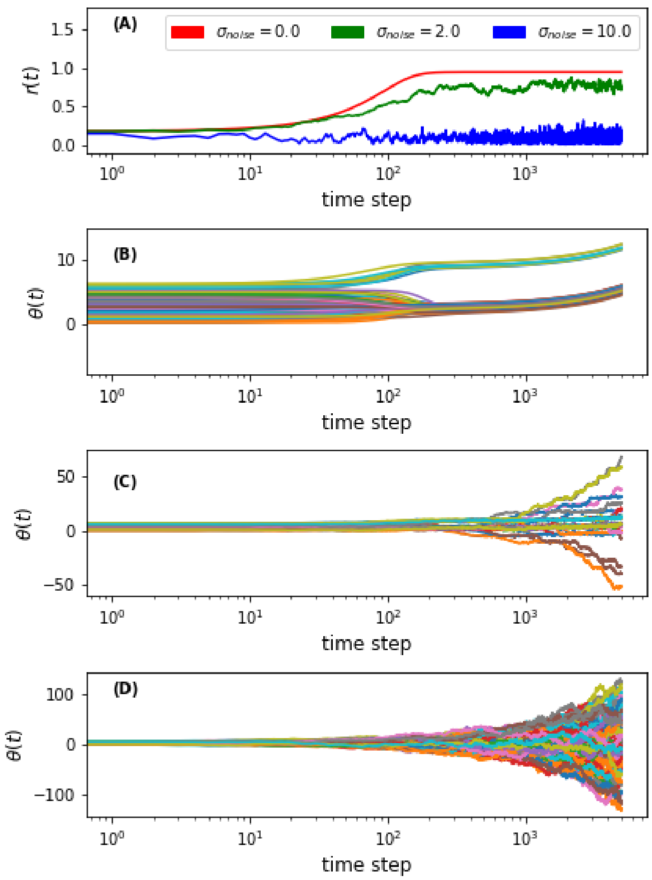

FIG. 3. Result of adding noise to the Kuramoto model. **(A)** Order parameter r as a function of time step for clean Kuramoto model (red), noisy Kuramoto model with white noise standard deviation of 2.0 (green), and noisy Kuramoto model with white noise standard deviation of 10.0 (blue). **(B)**-**(D)** Phase angle θ_i as a function of time step for the clean Kuramoto model **(B)**, noisy Kuramoto model with white noise standard deviation of 2.0 **(C)**, and noisy Kuramoto model with white noise standard deviation of 10.0 **(D)**. For all results in this figure, 100 oscillators are simulated with natural frequencies drawn from $\mathcal{N}(0, 1)$, initial phase angles drawn from $\mathcal{U}[-2\pi, 2\pi]$, and coupling $K = 3.0$.

tinct trajectories over time and synchronization occurs, while the noisy Kuramoto models in Fig. 3C-D do not synchronize. Instead, the noisy Kuramoto models' phase angles disperse in something like a random walk. In Fig. 4 we can also see that injected noise prevents synchronization. Fig. 4A-D show the clean Kuramoto model, which features the oscillators eventually traveling around the unit circle in a cluster, therefore synchronized. However, the noisy Kuramoto models in Fig. 4E-H and Fig. 4I-L do not show this synchronization: the oscillators continue to travel individually around the unit circle. Finally, in Fig. 5 we observe again that injecting noise into the Kuramoto model prevents synchronization, as the second order phase transition behavior observed in Fig. 2 is destroyed as added white noise standard deviation increases. In Fig. 5, 50 oscillators are simulated for each noise level with natural frequencies drawn from $\mathcal{N}(1.0, 0.1)$, initial phase angles drawn from $\mathcal{U}[-2\pi, 2\pi]$, and couplings strengths are varied from 0.0 to 0.6.

Discussion. — In summary, I have shown that oscillators governed by the Kuramoto model synchronize when their coupling K is greater than the critical coupling K_{crit}, and do not synchronize when their coupling K is less than the critical coupling K_{crit}. I have also shown that when white noise is injected into the Kuramoto model, oscillator synchronization becomes more difficult. These results show that 1) the Kuramoto model undergoes a second order phase transition between its unsynchronized and synchronized states, and 2) scrambling the interactions between oscillators prevents synchronization from occurring. Both of these points are critical to understanding how systems of oscillators synchronize.

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Appendix A: Implementing the Kuramoto model

My strategy of implementing the Kuramoto model is to define a function that calculates Eq. 2 and to feed that equation into the scipy ordinary differential equation `scipy.integrate.odeint` to obtain the phase angle θ_i for each oscillator i as a function of time. I implement

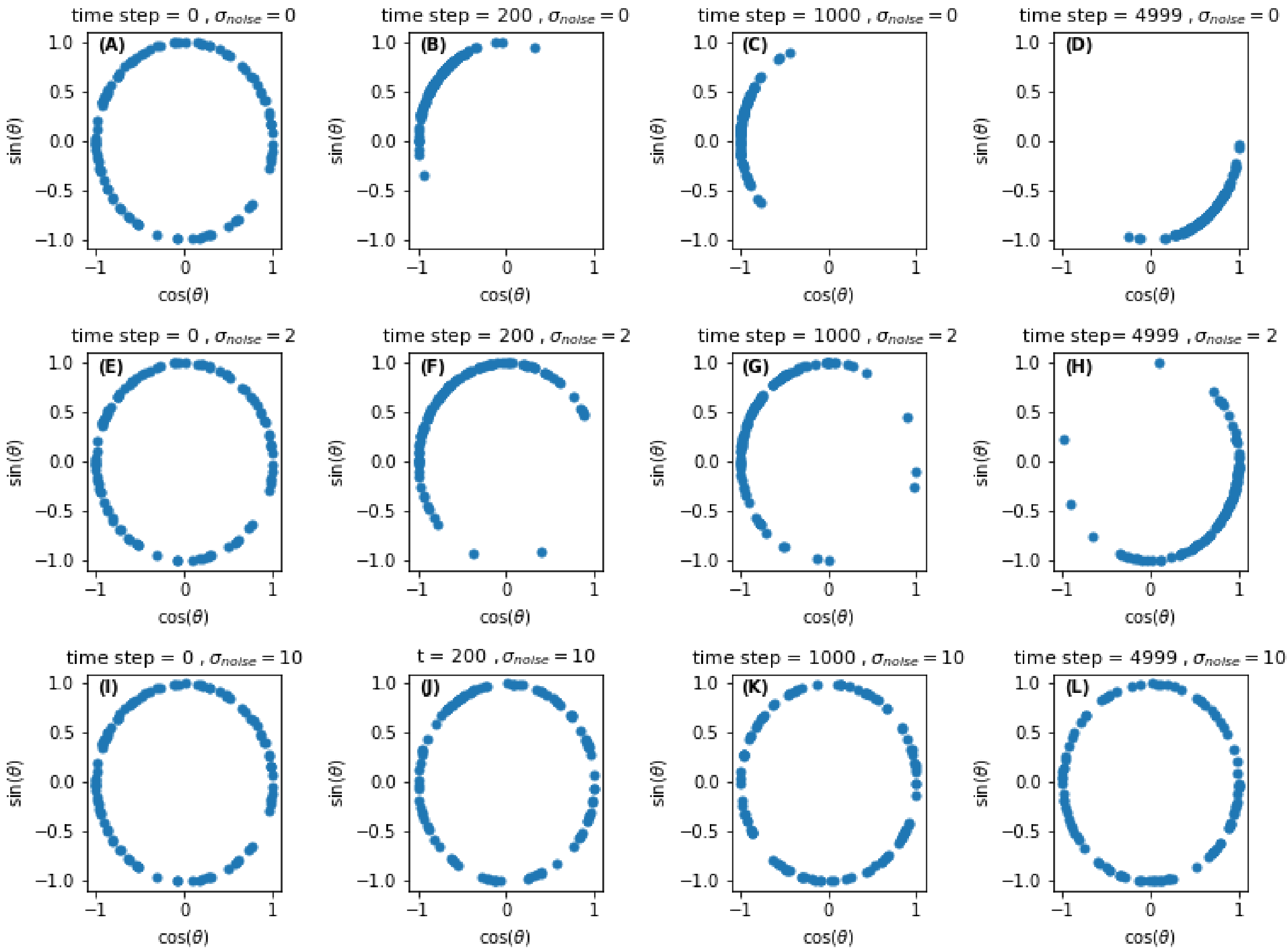

FIG. 4. Position of each oscillator on the unit circle at the following time steps: 0, 200, 1000, and 4999 for clean Kuramoto model **(A)**-**(D)**, noisy Kuramoto model with white noise standard deviation of 2.0 **(E)**-**(H)**, and noisy Kuramoto model with white noise standard deviation of 10.0 **(I)**-**(L)**. For all results in this figure, 100 oscillators are simulated with natural frequencies drawn from $\mathcal{N}(0, 1)$, initial phase angles drawn from $\mathcal{U}[-2\pi, 2\pi]$, and coupling $K = 3.0$.

Eq. 2 by defining an interaction matrix with elements

$$D_{ij} = 1 - \delta_{ij} \tag{A1}$$

and defining a matrix holding possible interactions

$$M_{ij} = \sin(\theta_j - \theta_i) \tag{A2}$$

so that Eq. 2 can be written as

$$\dot{\theta}_i = \omega_i + K \frac{\sum_{j=1}^{N} D_{ij} M_{ij}}{\sum_{j=1}^{N} D_{ij}}. \tag{A3}$$

Eq. A3 can be easily fed into `scipy.integrate.odeint` to obtain the phase angle θ_i for each oscillator i as a function of time. Note that when Eq. 2 is written as Eq. A3, we can change the interaction matrix elements D_{ij} freely. However, in this paper we will keep D_{ij} as defined in Eq. A1: a given oscillator i couples to all other oscillators except itself. This implementation of the Kuramoto model was inspired by an implementation of the Kuramoto model on a graph [9].

Appendix B: Implementing the noisy Kuramoto model

How do we implement adding white noise to the Kuramoto model? That is, how do we integrate the Eq. 5? Eq. 5 is a stochastic differential equation (SDE), so we cannot use the usual tool `scipy.integrate.odeint` to integrate it. Instead, we must use `sdeint.itoint`, an integrator for SDEs. The same procedure is followed for the noisy Kuramoto model as for the "clean" Kuramoto model: an interaction matrix with elements given by Eq. A1 is defined and a matrix holding possible interactions with elements given by Eq. A2 is defined. Rewrit-

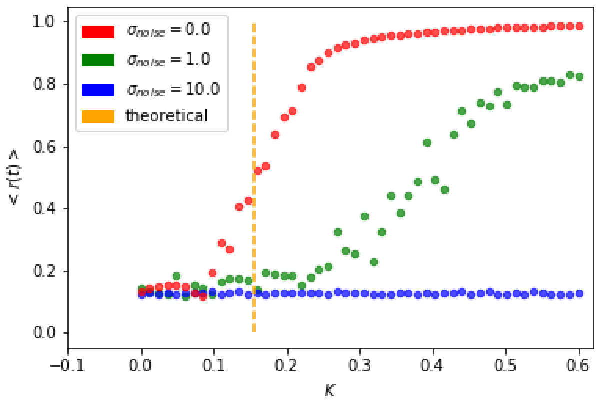

FIG. 5. Time average of order parameter r as a function of coupling constant K, holding all other parameters constant, for clean Kuramoto model (red), and Kuramoto models with added noise (green, blue). The theoretical value of the critical coupling separating sychronous and asynchronous states, $K_{crit}^{(theory)} = \frac{8\sigma}{\pi}$ [7], is marked in orange. For all results on this figure, 50 oscillators are simulated with natural frequencies drawn from $\mathcal{N}(1.0, 0.1)$, initial phase angles drawn from $\mathcal{U}[-2\pi, 2\pi]$, varying coupling strength from 0.0 to 0.6.

ing Eq. 5 in terms of Eq. A1 and Eq. A2, the resulting equation is

$$\dot{\theta}_i = \omega_i + \sigma\xi_i + K\frac{\sum_{j=1}^{N} D_{ij}M_{ij}}{\sum_{j=1}^{N} D_{ij}}. \tag{B1}$$

If we recall that white noise is the formal time derivative of a Wiener process, we can rewrite Eq. B1 as follows:

$$\frac{d\theta_i}{dt} = \omega_i + \sigma\frac{dW_i}{dt} + K\frac{\sum_{j=1}^{N} D_{ij}M_{ij}}{\sum_{j=1}^{N} D_{ij}}. \tag{B2}$$

where $\frac{dW_i}{dt}$ is the time derivative of the Wiener process W_i. Eq. B2 is ready to be plugged into the SDE integrator `sdeint.itoint`, from which the phase angle θ_i for a given oscillator i as a function of time is obtained.