Breaking Kuramoto synchronization

Mia Morrell

Synchronization

Synchronization can be found in many natural systems.

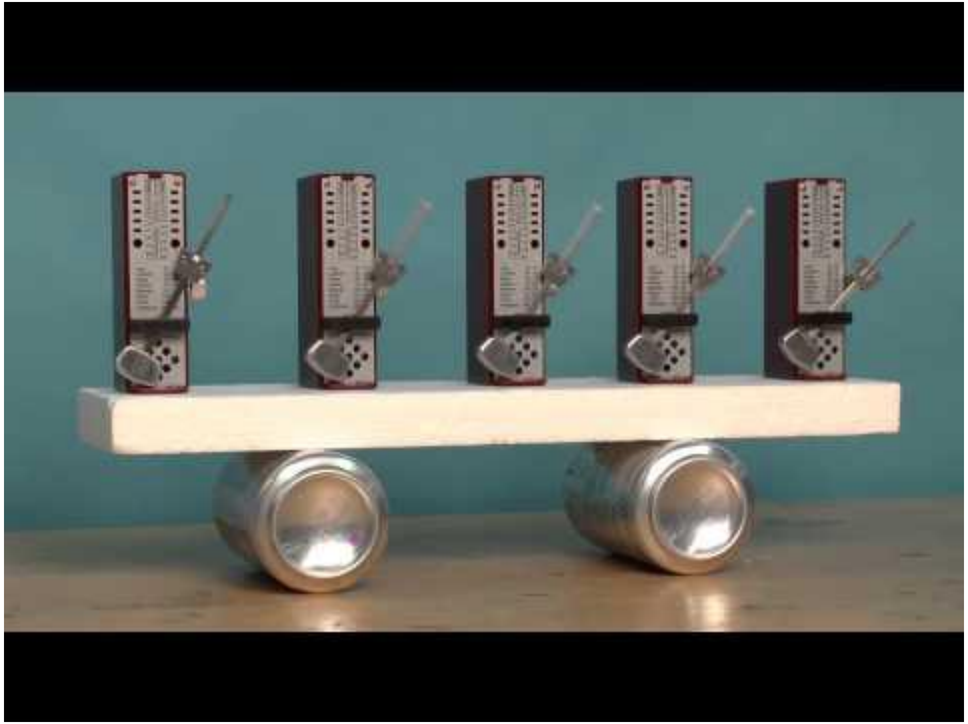
For example:

- fireflies can flash in unison (J. Buck, Q Rev Biol **63**, 265 (1988))
- crickets can chirp in unison (T. J. Walker, Science **166**, 891 (1969))
- yeast cells can metabolize synchronously (J. Aldrige, Nature **259**, 670 (1976))

These systems all feature large populations of individual agents that have different natural frequencies, yet somehow end up locking on to a common frequency.

We can view all these systems as systems of oscillators with unique natural frequencies.

Synchronization



Source: UCLA Physics (https://www.youtube.com/watch?v=T58lGKREubo

Synchronization

A way this synchronization can occur is if each the frequency of each oscillator is influenced by the frequency of all the other oscillators.

This was proposed by Winfree [Journal of Theoretical Biology **16**, 15 (1967)], who suggested

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i), i \in [1, N]$$

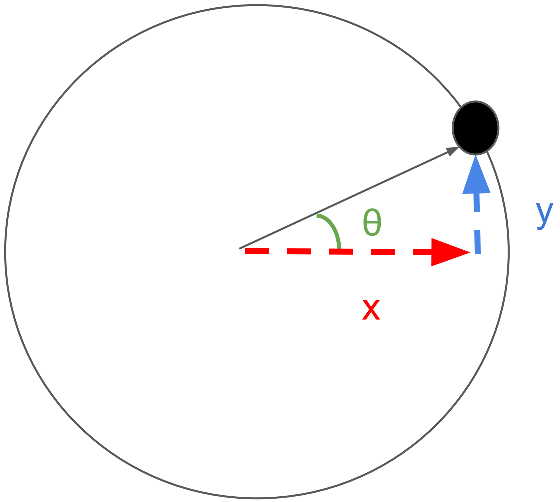
coupling function

natural frequency

phase angle

Synchronization

We can visualize the oscillators as points moving around the unit circle. The phase of a given oscillator defines the location of the oscillator on the unit circle as a function of time.



Synchronization

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i), i \in [1, N]$$

The coupling term to the other oscillators causes a given oscillator to move faster or slower around the unit circle than its natural frequency.

Synchronization occurs when oscillators' phase angles evolve to the same dynamics.

In order to observe synchronization, we need to choose the form of this coupling term such that at some time after initialization, groups of oscillators are oscillating with the same phase.

The Kuramoto model

The Kuramoto model [*Lecture Notes in Physics,* 1975] uses a identical sinusoidal coupling functions such that

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i), i \in [1, N]$$

becomes

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \frac{K}{N} \sin(\theta_j - \theta_i), i \in [1, N]$$

Under what conditions does the Kuramoto model synchronize, and under what conditions do we break the synchronization?

Methods: ODEs

ODE integration: scipy.odeint

General strategy:

1. Define an interaction matrix with elements: $D_{ij} = 1 - \delta_{ij}$
2. Define a matrix holding possible interactions: $M_{ij} = \sin(\theta_j - \theta_i)$
3. The Kuramoto model can then be written as:

$$\dot{\theta}_i = \omega_i + K \frac{\sum_{j=1}^{N} D_{ij} M_{ij}}{\sum_{j=1}^{N} D_{ij}}$$

4. Feed the above into scipy.odeint

Code: https://github.com/mcmorre/kuramoto

Methods: SDEs

SDE integration: sdeint.itoint

General strategy:

1. Define an interaction matrix with elements: $D_{ij} = 1 - \delta_{ij}$
2. Define a matrix holding possible interactions: $M_{ij} = \sin(\theta_j - \theta_i)$
3. The Kuramoto model can then be written as:

$$\dot{\theta}_i = \omega_i + \sigma\xi_i + K\frac{\sum_{j=1}^{N} D_{ij}M_{ij}}{\sum_{j=1}^{N} D_{ij}}$$

4. Recall that white noise is the formal time derivative of a Wiener process:

$$\frac{d\theta_i}{dt} = \omega_i + \sigma\frac{dW_i}{dt} + K\frac{\sum_{j=1}^{N} D_{ij}M_{ij}}{\sum_{j=1}^{N} D_{ij}}$$

5. Feed the above into sdeint.itoint

Code: https://github.com/mcmorre/kuramoto

The mean-field Kuramoto model

We can define the order parameters r and psi:

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

where r(t) represents the degree of oscillator synchrony at a given time and ranges in value from 0 (all unsynchronized) to 1 (all synchronized).

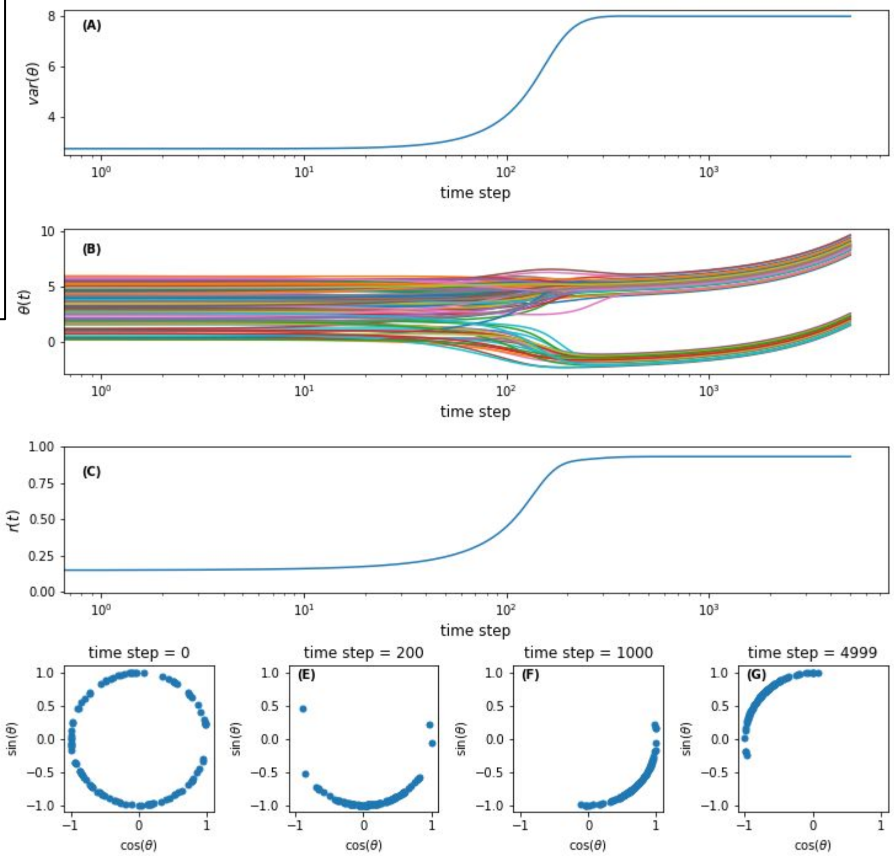
The order parameter psi represents the average phase of the oscillators, and is not incredibly informative except for its use in the mean field equation for the oscillator system

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i), i \in [1, N]$$

which we can interpret as all oscillators being coupled to the mean phase angle of the system.

Dynamics of the Kuramoto model

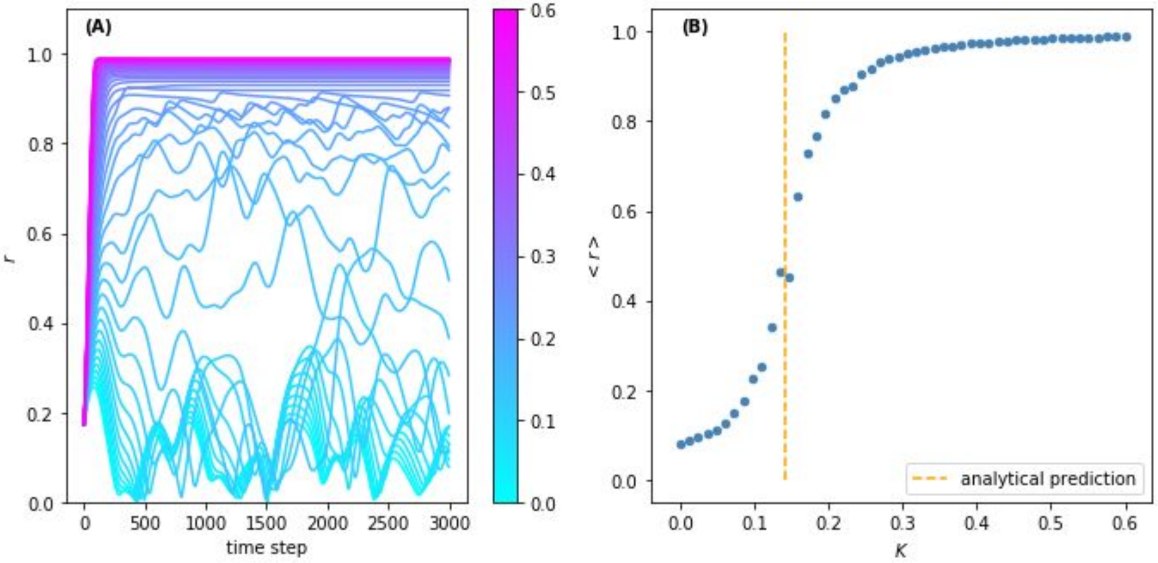
Parameters:
- 100 oscillators
- coupling strength K=3.0
- natural frequencies drawn from standard normal distribution
- initial thetas uniformly drawn from [-2π, 2π]



- r(t) represents the degree of oscillator synchrony at a given time and ranges in value from 0 (not synchronized) to 1 (all synchronized)
- As synchronization increases, r increases, and variance of theta increases.

Effect of coupling strength on synchronization

Parameters:
- 100 oscillators
- natural frequencies drawn from *N*(1, 0.1)
- initial thetas uniformly drawn from [-2π, 2π]

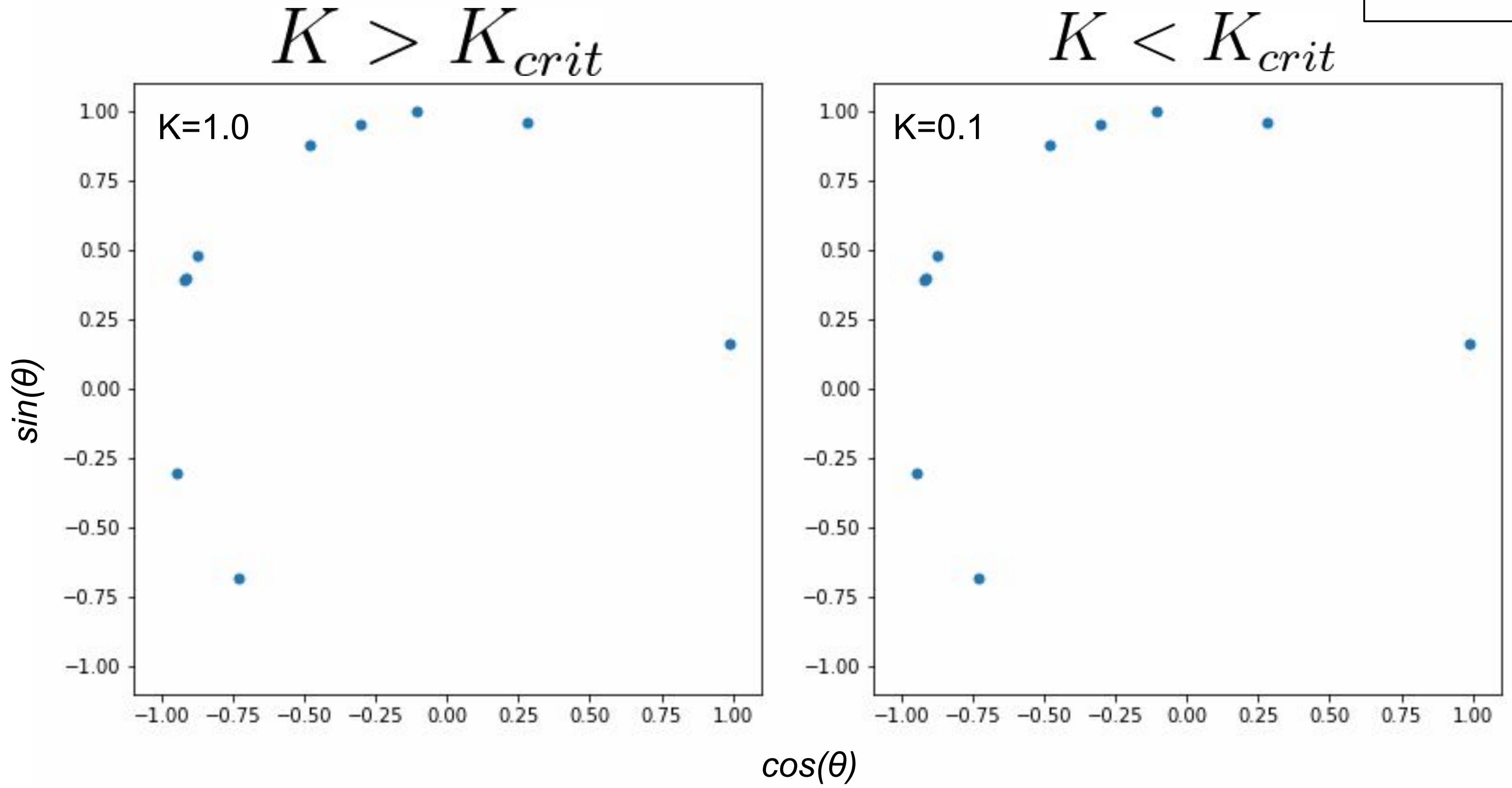


- As coupling strength decreases, synchronization decreases

- There is a second order phase transition between the synchronized and unsynchronized states.

Effect of coupling strength on synchronization

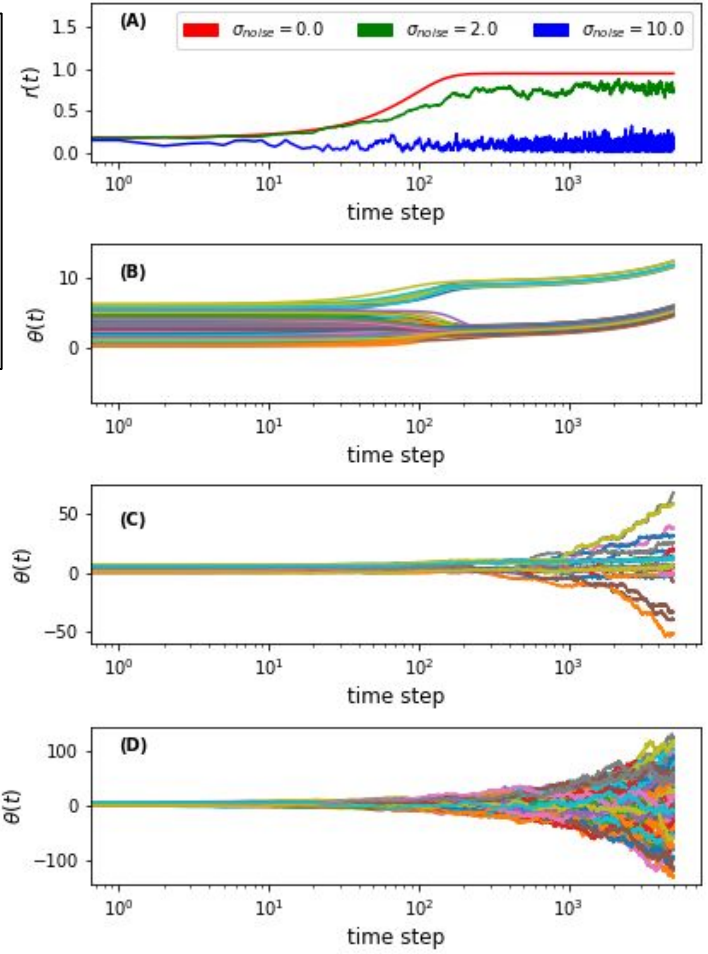
Parameters:
- 50 oscillators
- natural frequencies drawn from *N*(1, 0.1)
- initial thetas uniformly drawn from [-2π, 2π]



Effect of noise on synchronization

Parameters:

- 100 oscillators
- coupling strength K=3.0
- natural frequencies drawn from standard normal distribution
- initial thetas uniformly drawn from [-2π, 2π]



No noise

White noise, standard deviation = 2.0

White noise, standard deviation = 10.0

Effect of noise on synchronization

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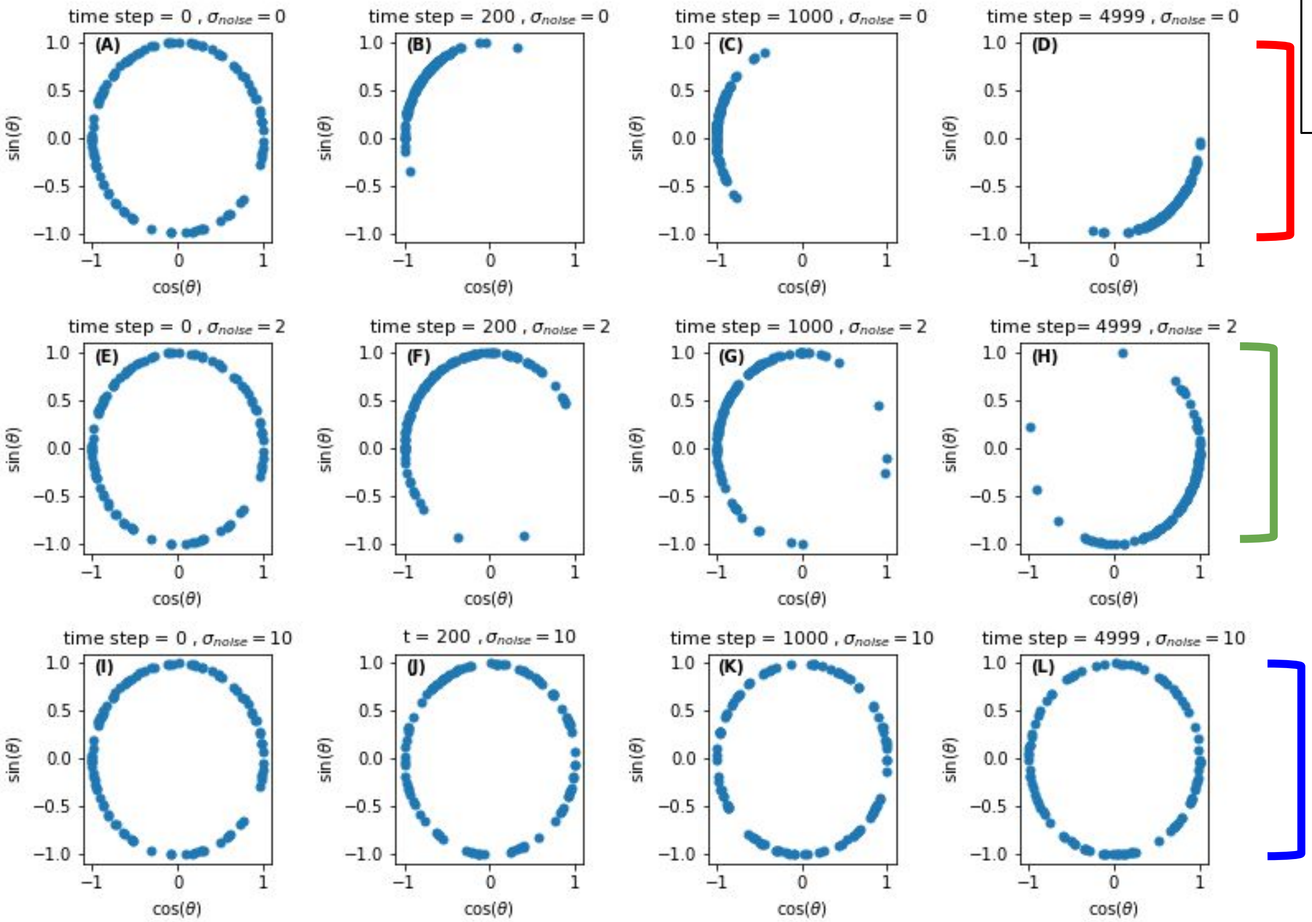

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Effect of noise on synchronization

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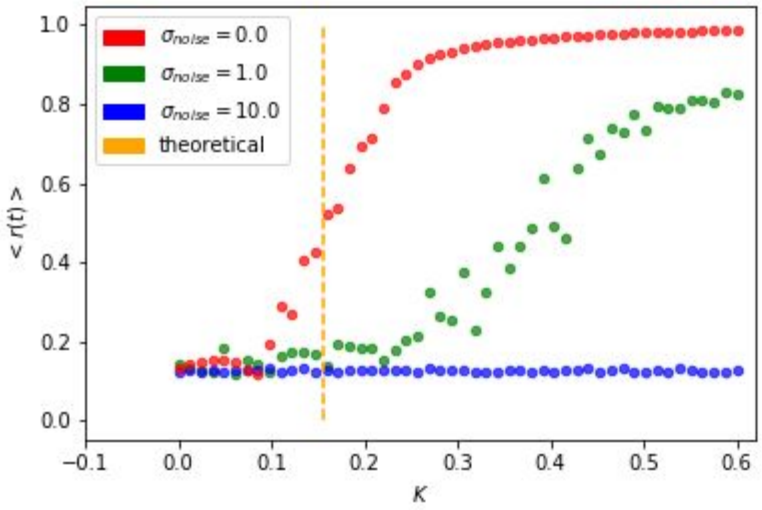

- As coupling strength decreases, synchronization decreases
- The second order phase transition between the synchronized and unsynchronized states is destroyed by sufficient added noise strength.

Summary

- Oscillators governed by the Kuramoto model synchronize when their coupling constant is greater than the critical coupling constant, and do not synchronize when their coupling constant is less than the critical coupling constant.
    - This is an example of a second order phase transition.
- When white noise is injected into the Kuramoto model, oscillator synchronization becomes more difficult.
    - Scrambling the interactions between oscillators prevents synchronization from occurring.