GENERIC TEXTBOOK COVER

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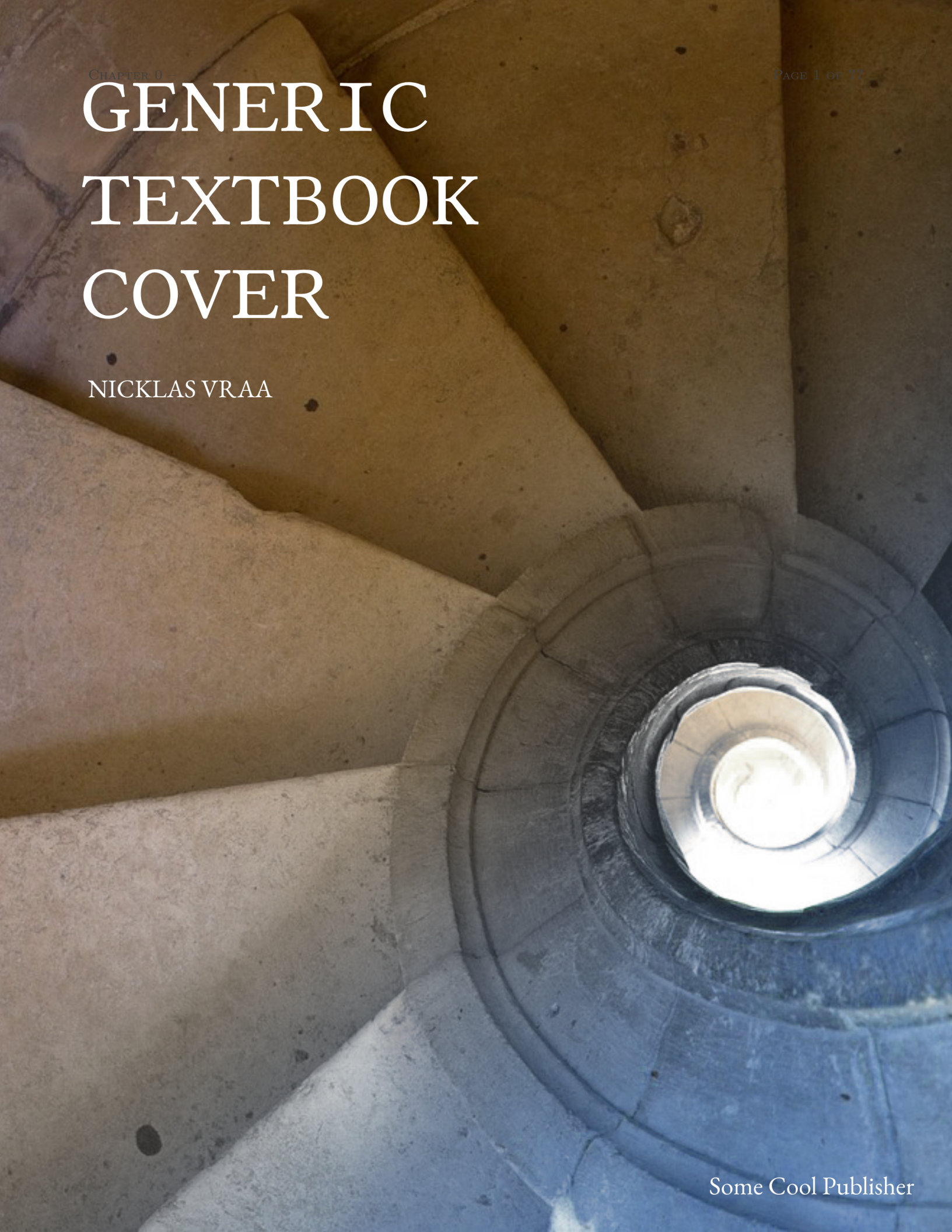

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To Myself

Because I'm cool.

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Contents

Part I

Mathematical Models

Introduction to Mathematical Modeling.

For this chapter, we'll:

- Get our framework from an algorithmic framework of thinking to algebraic framework
- Represent a problem with algebraic expressions.
- Make proper implementations of mathematical models.

1.1. Definitions to mathematical modeling

Before we can begin to think about models, we must understand what they are. And, for the purposes of this book, we can define a mathematical model as a representation of a real life problem with an abstract representation of the phenomenons we intend to study.

For an immediately relevant example, a lot of problems in graph theory can be explained in algebraic terms. Although these problems can be a bit more mathematical in nature, then we can use them to explain real-life problems. Such as:

- Shortest path in a graph, for computer networking and pathfinding.
- The traveling salesman problem, for pathfinding, DNA sequencing and microchip manufacturing
- The vehicular routing problem, for finding multiple instances of a TSP-esque solution on a same graph

As a practical example of such a graph-based problem, let's analyze the following:

1.1.1. Example 1: The transport problem

Let a finite number of sources and a finite number of destinations, we can determine the number of elements we can route to every destination with a minimal cost for every source. For example, if we said such sources are factories and the destinations are warehouses, we could see a 3x3 graph as it follows:

1.1.2. Example 2: Cover problems

More than a specific problem, this is a subdivisio of problems meant to cover a certain group of demands or coverage criteria according to a specific defined problem.

1.1.2.1. The hospital problem

For example, say we got a group of neighborhoods that are of euclidian behavior, and we intend to put the minimum amount of hospitals that can cover the entire region we're considering. A hospital covers the neighborhood is located on and those who share a boundary with it. A boundary is a non-zero

For an example, imagine the following instance of this problem:

1.1.2.2. The four color theorem

The four color theorem is a very famous application of a cover problem.

1.1.3. Example 3: Minimum spanning tree

If we imagine a tree data structure such as a graph with no cycles, and a graph as an interlocked set of nodes, a minimum spanning tree would be the minimum cost path that can connect all nodes as a tree.

This problem has a set of requirements for it to be considered correct:

- There must be no cycles or subcycles in the solution
- the number of connections must be the number of nodes minus one

1.2. Classification of mathematical models

These models can be, as most things in mathematics, categorized and calificated in

1.2.1. Deterministic mathematical models

These are models that can be predicted in a way that is certain.

1.2.2. Stochastic mathematicam models

A stochastic model is one that involves a certain amount of randomness. For example,

1.2.3. Static mathematical models

A static model is one that does not depend on time, implying that whatever time passes, the result of such a system should stay constant. Generally this will happen for models where we don't even consider time as a relevant variable.

1.2.4. Dynamic mathematical models

1.2.4.1. Continuous time models

A continuous time model is one where we assume time as a continuous function, that meaning, it is flowing in somewhat real time. Although the way computation has it should bring it to be a finite level of statuses, this should be a big enough sample such as we can assume the variable can be statistically generalized into a continuous system

1.2.4.2. Discrete time models

A fairly interesting application of discrete time models comes from finance, where there exists capitalization in a set time interval. For example, we could imagine the free cash flow of an economic project as a mathematical problem to be optimized, such as this:

1.2.4.3. Continuous status models

In a continuous status model, the dependant variable can assume any value in the range of a specific interval.

1.2.4.4. Discrete status models

In these models, the difference is analogous to the way a discrete time model varies from a continuous time model. Values for the dependant variable shouls only assume specific values in an interval. For example, a model where every value the dependant variable can take is in the range of natural numbers ($\mathbb{N}$) would be a discrete status.

1.2.4.5. Examples of model classification

1.2.4.5.1. Number of attendants in a bank every hour This example would be a dynamic mathematical model where every

1.2.4.5.2. Temperature of a classroom every hour For this,

1.2.5. Open mathematical models

An open mathematical model is one where the input that is recieved will be external to this model and independent from itself. For example a translator, such as Google Translate or DeepL will model natural language such as the input recieved is not eventually retroactively giving feedback to the model. Although it is possible that in ML models there are eventually expected use cases introduced into the model, for out purposes, they will be considered open as long as at runtime the previous example does not feedback into further answers.

1.2.6. Linear mathematical models

Assume the following definition of linearity

A function 'f(x)' is linear if:

- Ex. 1: $f(u+v) = f(u) + f(v)$
- Ex. 2: $f(k \cdot u) = k \cdot f(u); k \textit{ is constant}$
- $\nexists\ x \in \mathbb{R} \implies (Ex.1\ f(x+y); x, y \in \mathbb{R} \neq f(x) + f(y)) \vee (Ex.2\ f(k \cdot x); k, x \in \mathbb{R} \neq k \cdot f(x))$

a linear mathematical model should have every single function following these constraints.

Optimization Solutions in Mathematical Models

Now, given that we know what a problem in modeling entails, how can we begin to solve it? As it might be evident by now, this question will be the crux of most our problems, and yet there is not much to fear. As it comes, we really have tought about quite an important amount of problems already, and they provide a baseline for our analysis. As with many problems in computer science, we stand in the shoulders of giants, really.

So for example, we can imagine the following case:

2.1. Parts of an Optimization problem

Example 1: Let the following problem:

"We have 8 projects that return a certain profit, so we want to select as many projects as possible that will return a certain profit. we want to select as many projects as possible that will generate the highest profit. generate the highest profit. Suppose that the profit of each project is and we have a constraint for acquiring the projects, which is that we can only choose 2 projects. that we can only choose 2 projects. Propose a mathematical optimization model to solve the problem."

Note: We assume equal spending for them.

$$F_j(x); \forall j \in \mathbb{N} \tag{2.1}$$

The first thing to do is to determine the different parts of this program, for those purposes, we can determine them in the following categories:

- Sets: the group of sets that can be applied to this problem, assuming sets as a group of mathematical variables.
- Indexes: the identifier for which we call every item inside of a set
- Parameters: the numerical and mathematical significance of every index inside of a set.
- Decision Variable: the variable we mean to analyze in our problem.
- Objective function: The expected behavior of a decision variable
- Restrictions: The rules under which we analyze the previous ideas.

as such, for this problem we can imagine the categories proposed previously as such:

- Sets: P is a set of projects, that can be represented as

  $$P = 1, 2 \dots 8$$

- Indexes: i is a number from 1 to 8 that represents the order of the project to be selected
- Parameters: g_i will be our representation of profit under this system.
- Decision Variable: x_i will be a binary (either 1 or 0) constant that will represent the selection of a project.
- Objective function:

  $$max(\sum_{i \in P} g_i \cdot x_i)$$

  This means we intend to optimize 'g', which represents profit, and is bound to an 'x' binary constant.

- Restrictions:

  $$\sum_{i \in P} x_i = 2$$

  This means that we can only choose two projects, it makes sense because those values will multiply g_i in our objective function

Now, say we want to represent this model in Pyomo, a python library for this sort of optimization. Say

```
1
2  from __future__ import division
3  from pyomo.environ import *
4
5  from pyomo.opt import SolverFactory
6
7  Model = ConcreteModel()
8
9  # Data de entrada
10 numProjects=8
11
12 p=RangeSet(1, numProjects)
13
14 value={1:2, 2:5, 3:4, 4:2, 5:6, 6:3, 7:1, 8:4}
15
16 # Decision Variable
17 Model.x = Var(p, domain=Binary)
18
19 # Objective Functions
20 Model.obj = Objective(expr = sum(Model.x[i]*value[i] for i in p), sense=maximize)
21
22 # Restrictions
23 Model.res1 = Constraint(expr = sum(Model.x[i] for i in p) == 2)
24
25 # Solver specifications
26 SolverFactory('glpk').solve(Model)
27
28 Model.display()
```

Snippet 2.1: Representation of the proposed solution in code.

a script for this program can be found in the repository for this book. The resulting output will be as follows:

```
1  Model unknown
2  Variables:
3      x : Size=8, Index=[1:8]
4          Key : Lower : Value : Upper : Fixed : Stale : Domain
5          1   :     0 :   0.0 :     1 : False : False : Binary
6          2   :     0 :   1.0 :     1 : False : False : Binary
7          3   :     0 :   0.0 :     1 : False : False : Binary
8          4   :     0 :   0.0 :     1 : False : False : Binary
9          5   :     0 :   1.0 :     1 : False : False : Binary
10         6   :     0 :   0.0 :     1 : False : False : Binary
11         7   :     0 :   0.0 :     1 : False : False : Binary
12         8   :     0 :   0.0 :     1 : False : False : Binary
13
14 Objectives:
15         obj  : Size=1, Index=None, Active=True
16         Key  : Active : Value
17         None :   True :  11.0
18
19 Constraints:
20         res1 : Size=1
21         Key  : Lower : Body : Upper
22         None :   2.0 :  2.0 :   2.0
```

Snippet 2.2: Output of the provided problem

this implies that for the example program, we will choose projects 2 and 5.

2.2. Mathematical management of expressions.

Now as it might be evident, the fact is we'll be using mathematical expressions in order to make sense of our problems. This means we need to understand how

Generally,

- Sumatory: $\sum_{i \in \mathbb{N}} x_i = j; j \in \mathbb{R}$
- For every item: $x_i = j; \forall_{i \in \mathbb{N}}$

2.2.1. Conditional Operations

The before noted expressions are general, that means, when ran over a subset of data, they will take all items and apply an operation over them. however, we won't always want this sort of behaviour in our systems.

- Conditional Sumatory: $\sum i \in \mathbb{N} x_i; \forall i \in \mathbb{N} | i = x$
- Conditional for every item: $\forall x_i = j; i \in \mathbb{N} | i = y$

Example:

Given the following cases:

given the set: $N = \{1..10\}$ *and the specific restriction* $x_2 + x_3 + x_6 + x_8 + x_1 0 = 1$*, how would we write the generic restriction?*

We can use a conditional sumatory to represent this restriction, such as it follows:

$$\sum_{i \in N} x_i = 1; \forall i \in N | i \in i \mod 2 = 0 | N = \{1..10\} \tag{2.2}$$

Given the set $N = \{1, 2, 3\}$*, How would we expand the generic expression:*

$\sum_{i \in N} \sum_{j \in N} \sum_{k \in N} = x_{ijk} = 1$

This would be basically a matrix sum, written as such:

$$x_{1,1,1} + x_{1,1,2} + x_{1,1,3} \tag{2.3}$$
$$x_{1,2,1} + x_{1,2,2} + x_{1,2,3} \tag{2.4}$$
$$\dots \tag{2.5}$$
$$x_{3,3,1} + x_{3,3,2} + x_{3,3,3} = 1 \tag{2.6}$$
$$\tag{2.7}$$

The Amount of expressions is a combination from the different elements in N. In this case, it is a sum of 27 elements (3x3x3)

Given the set $N = \{1, 2, 3\}$ *and the specific restrictions:*

$$x_{12} + x_{13} = 1$$

This is a 'for all' expression, such as:

$$\sum_{i,j \in N} x_{i,j} = 1 | \forall_{i,j} \in N | j \neq 1 \tag{2.8}$$

2.2.2. Usual errors when managing mathematical expressions.

We can sometimes screw up, and that's okay when we're practicing. However, it is also important to know why we could do so.

2.2.2.1. Not controlling an index

On the examples here shown, we have implicitly expected an 'index' to be found somewhere. that is where 'i' and 'j' come from.

2.3. graph problems

We will be working extensibly on graph problems on the duration of this

Examples

3.1. Homework 1: The diet problem

3.1.1. Solution to the problem

```
from __future__ import division
from pyomo.environ import *

from pyomo.opt import SolverFactory

'''
# Definicion variables alimento

c = carne = 1
a = arroz = 2
l = leche = 3
p = pan = 4

Todos los valores numericos vienen del enunciado.
'''
Model = ConcreteModel(name='dieta')

'''
N = Productos.

1 = Carne
2 = Arroz
3 = Leche
4 = Pan
'''
N = RangeSet(1,4)

# Variable de decisin, ligada a N
↪  ********************************************************************************************************************************
↪

Model.c = Var(N, domain=NonNegativeReals)

'''
M = Caracteristicas de los productos.

1 = Calorias
2 = Proteinas
3 = Azucar
4 = Grasas
5 = Carbohidratos

'''
M = RangeSet(1,5)

'''
Precios, los valores de las llaves corresponden a aquellos en N
'''
p =  {1:3000,2:1000,3:600,4:700}

'''
Variable objetivo,
'''
Model.obj = Objective(expr = sum(Model.c[i]*p[i] for i in N), sense=minimize)


'''
Datos de las comidas, cada dato esta organizado de forma que pueda leerse en el orden
v[i][j]. donde 'i' es un valor de N (es decir, un alimento) y donde 'j' es un valor de M
(es decir, una caracteristica)
```

```
'''
v = {
    1:{
        1:287,
        2:26,
        3:0,
        4:19.3,
        5:0
    },
    2:{
        1:204,
        2:4.2,
        3:0.01,
        4:0.5,
        5:44.1
    },
    3:{
        1:146,
        2:8,
        3:13,
        4:8,
        5:11
    },
    4:{
        1:245,
        2:6,
        3:25,
        4:0.8,
        5:55
    }
}

'''
Limites.

Ligados a M
'''
L = {
    1: 1500,
    2: 63,
    3: 25,
    4: 50,
    5: 200
}

def inferior(Model, j):
    if j != 1 and j != 2:
        return (sum(Model.c[i]*v[i][j] for i in N ) <= L[j])
    else:
        return Constraint.Skip
Model.inferior = Constraint(M, rule=inferior)

def superior(Model, j):
    if j == 1 or j == 2:
        return (sum(Model.c[i]*v[i][j] for i in N) >= L[j])
    else:
        return Constraint.Skip
Model.superior = Constraint(M, rule=superior)

# Especificacin  del solver
SolverFactory('glpk').solve(Model)

Model.display()
```

Snippet 3.1: Solution to the problem