FP1: Control of the Variable Length Pendulum

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Control Design and Analysis for Underactuated Robotics:
Variable Length Pendulum

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Introduction

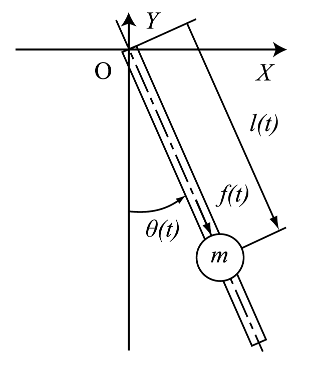
- the main goal of this work is to study the Variable Length Pendulum and present controllers designed to make it achieve a desired swing motion given a desired energy and a desired length of the pendulum

- based on Control Design and Analysis for Underactuated Robotics: Variable Length Pendulum by *Xin Xin*, *Yannian Liu*

- main points in this work
  - motion equation
  - problem formulation
  - controller design
  - motion analysis
  - simulation results

Motion Equation

- friction at pivot O and viscous friction of the rod are absent
- the rod is massless
- the angle $\theta(t)$ to be between the pendulum and the vertical axis
- the length of the pendulum $l(t)$ starts from the origin O to the mass of the pendulum m
- $f(t)$ is the force acting on the mass



Motion Equation

$$x_G = l(t)\sin\theta(t) \quad y_G = -l(t)\cos\theta(t)$$

kinetic energy defined as

$$T = \frac{1}{2}m(\dot{x}_G^2 + \dot{y}_G^2) = \frac{1}{2}m\left(l(t)\dot{\theta}(t)\right)^2 + \frac{1}{2}m\left(\dot{l}(t)\right)^2$$

potential energy as

$$P = mgy_G = -mgl(t)\cos\theta(t)$$

the Lagrangian equation of the VLP

$$L = T - P$$

Euler-Lagrangian operator

$$\frac{\partial L}{\partial q} - \frac{dL}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \tau$$

with $q = [l, \theta]^T$ and τ applied generalized forces

Motion Equation

Euler-Lagrange equation

$$\ddot{\theta}(t) + \frac{2\dot{l}(t)\dot{\theta}(t)}{l(t)} + \frac{g \sin\theta(t)}{l(t)} = 0$$

$$\ddot{l}(t) - ml(t)\dot{\theta}^2(t) - mg\cos\theta(t) = f(t)$$

control input

$$u = \ddot{l}(t)$$

Problem Formulation

let $m = 1$, total mechanical energy

$$E_T = \frac{1}{2}\dot{l}^2(t) + \frac{1}{2}\big(l(t)\dot{\theta}(t)\big)^2 - gl(t)\cos\theta(t)$$

desired trajectory of swing described by

$$E_r = \frac{1}{2}\big(l_r\dot{\theta}(t)\big)^2 - gl_r\cos\theta(t)$$

with E_r and l_r desired energy and length of the VLP
$E_r = -gl_r\cos\theta_{max}, \theta_{max} \in (0, \pi]$

Trajectory tracking control problem

$$\lim_{t\to\infty} E_T = E_r \qquad \lim_{t\to\infty} \dot{l} = 0 \qquad \lim_{t\to\infty} l = l_r$$

Controller Design: Total Energy Shaping

Lyapunov candidate with $k_P > 0$, $k_D > 0$

$$V_c = \frac{1}{2}(E_T - E_r)^2 + \frac{1}{2}k_P(l - l_r)^2 + \frac{1}{2}k_D\dot{l}^2$$
$$\dot{V}_c = \dot{l}\big((E_T - E_r + k_D)u - (E_T - E_r)(l\dot{\theta}^2 + g\cos\theta) - k_P(l - l_r)\big)$$

controller

$$u = \frac{(E_T - E_r)(l\dot{\theta}^2 + g\cos\theta) - k_P(l - l_r) - k_V\dot{l}}{E_T - E_r + k_D}$$

with $k_V > 0$, then

$$\dot{V}_c = -k_V\dot{l}^2 \leq 0$$

which holds only if

$$E_T - E_r + k_D \neq 0 \quad \forall t \geq 0$$

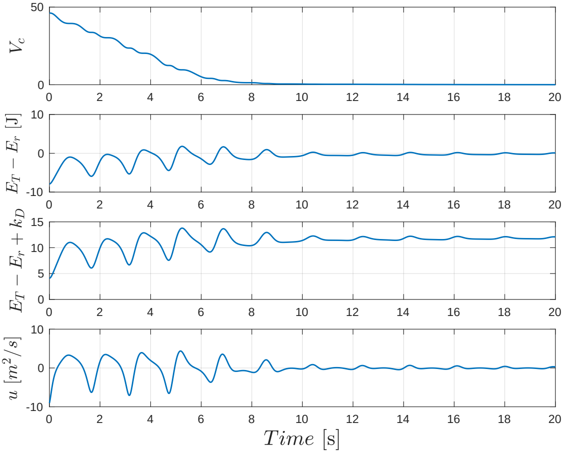
Total Energy Shaping: Simulation

consider the trajectory tracking control problem defined previously and $l_r = 3\text{m}$, $\theta_{\text{max}} = 2\pi/5$, $g = 9.81\text{m/s}^2$

let $k_D = 12$, $k_P = 30$ and $k_V = 12$

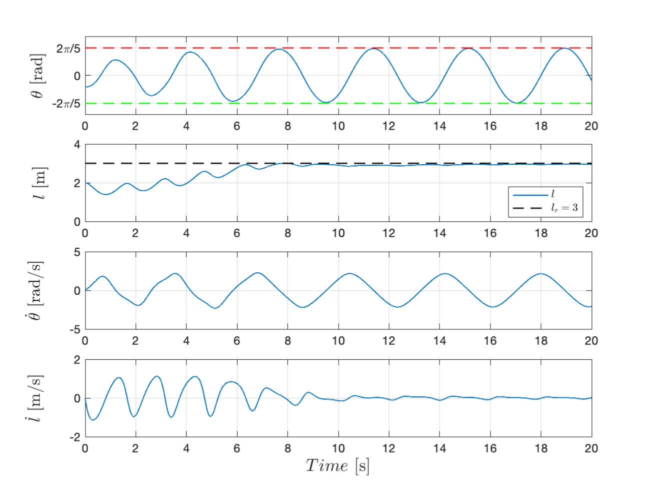
Total Energy Shaping: Simulation

Time responses of V, $E_T - E_r$, $E_T - E_r + k_D$ and u with initial state $(\theta(0), l(0), \dot{\theta}(0), \dot{l}(0)) = (-\pi/6, 2, 0, 0)$



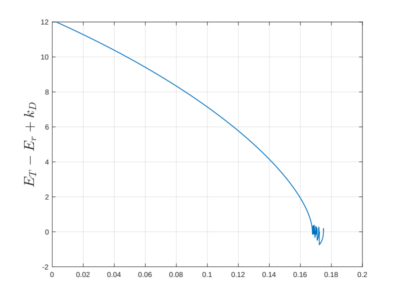
Total Energy Shaping: Simulation

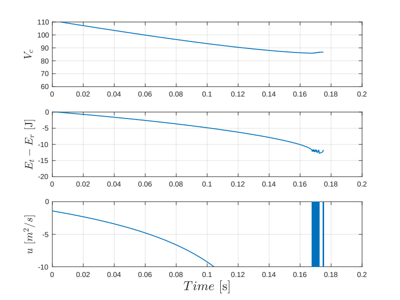
Time reponses of $(\theta, l, \dot{\theta}, \dot{l})$ with initial state $(\theta(0), l(0), \dot{\theta}(0), \dot{l}(0)) = (-\pi/6, 2, 0, 0)$



Total Energy Shaping: Simulation

The controller encountered a singular point for initial state $(\theta(0), l(0), \dot{\theta}(0), \dot{l}(0)) = (-\pi/6, 2, 0, 4)$





Controller Design: Partial Energy Shaping

E_P sum of kinetic energy of rotation and potential energy of VLP

$$E_P = \frac{1}{2}(l(t)\dot{\theta}(t))^2 - gl(t)\cos\theta(t)$$

since $(E_T, l, \dot{l}) \equiv (E_r, l_r, 0)$ equivalent to $(E_T, l, \dot{l}) \equiv (E_r, l_r, 0)$
Lyapunov candidate

$$V = \frac{1}{2}(E_P - E_r)^2 + \frac{1}{2}k_P(l - l_r)^2 + \frac{1}{2}k_D\dot{l}^2$$

$$\dot{V} = \big(-(E_P - E_r)(l\dot{\theta}^2 + g\cos\theta) + k_P(l - l_r)\big) + k_D u)\dot{l}$$

controller

$$u = \frac{(E_P - E_r)(l\dot{\theta}^2 + g\cos\theta) - k_P(l - l_r) - k_V\dot{l}}{k_D}$$

which is free of singular points

$$\dot{V} = -k_V\dot{l}^2 \leq 0, \quad k_V > 0$$

Controller Design: Partial Energy Shaping

consider

$$\Gamma_c = \left\{ (\theta, l, \dot{\theta}, \dot{l}) | V(\theta, l, \dot{\theta}, \dot{l}) \leq c \right\}, \quad c > 0$$

since $\dot{V} \leq 0$, any closed-loop solution starting in Γ_c remains in Γ_c for all $t \geq 0$

let W be the largest invariant set in

$$S = \{(\theta, l, \dot{\theta}, \dot{l})) \in \Gamma_c | \dot{V} = 0\}$$

using **LaSalle's invariant principle**, every closed-loop solution starting in Γ_c approaches W as $t \to \infty$

Controller Design: Partial Energy Shaping

since $\dot{V} = 0$ holds for all elements of W, then

$$V \equiv V^* \quad l \equiv l^*$$

moreover, since V is a constant in W, E_P is also a constant in W

$$\lim_{t\to\infty} E_P = E^* \quad \lim_{t\to\infty} \dot{l} = 0 \quad \lim_{t\to\infty} l = l^*$$

thus, the largest invariant set W can be defined as

$$W = \left\{ (\theta, l, \dot{\theta}, \dot{l}) \middle| \frac{1}{2}(l^*\dot{\theta})^2 - gl^* \cos\theta \equiv E^*, l \equiv l^* \right\}$$

Motion Analysis: Convergence of Energy

Trajectory Tracking Control Problem achieved iff $V^* = 0$, with:

$$V^* = \frac{1}{2}(E^* - E_r)^2 + \frac{1}{2}k_P(l^* - l_r)^2$$

Consider $E_P \equiv E^*$, $l \equiv l^*$, $\dot{l} \equiv 0$ and $u = \ddot{l} \equiv 0$. Then, from the controller based on partial energy shaping:

$$(E^* - E_r)(l^*\dot{\theta}^2 + g\cos\theta) - k_P(l^* - l_r) \equiv 0$$

Motion Analysis: Convergence of Energy

Case 1: $E^* = E_r$

In this case $l^* = l_r$. The largest invariant set W becomes:

$$W_r = \left\{ (\theta, l, \dot{\theta}, \dot{l}) \middle| \frac{1}{2}(l_r\dot{\theta})^2 - gl_r \cos\theta \equiv E_r, l \equiv l_r \right\}$$

Hence, as $t \to \infty$, the closed-loop solution $(\theta(t), l(t), \dot{\theta}(t), \dot{l}(t))$ achieves the tracking control objective.

Motion Analysis: Convergence of Energy

Case 2 $E^* \neq E_r$

In this case:

$$l^*\dot{\theta}^2 + g\cos\theta \equiv \frac{k_P(l^* - l_r)}{E^* - E_r}$$

Considering $E_P \equiv E^*$ and $l \equiv l^*$ it is possible to prove that θ is a constant. Let $\theta \equiv \theta^*$. Moreover, since $E_r = -gl_r\cos\theta_{\max}$:

$$l^* = \frac{l_r(k_P + g^2\cos\theta_{\max}\cos\theta^*)}{k_P + g^2}$$

Motion Analysis: Convergence of Energy

Since $\sin\theta^* = 0$ admits a solution only in $\{0, \pi\}$, either $(\theta^*, l^*) = (\pi, l_{ue})$ or $(\theta^*, l^*) = (0, l_{de})$ where:

$$l_{ue} = l^*|_{\theta^*=\pi} = \frac{l_r(k_P - g^2\cos\theta_{\mathsf{max}})}{k_P + g^2}$$

$$l_{de} = l^*|_{\theta^*=0} = \frac{l_r(k_P + g^2\cos\theta_{\mathsf{max}})}{k_P + g^2}$$

To guarantee that the length of the pendulum l_{ue} and l_{de} are positive in the two cases respectively assume $k_P > -g^2\cos\theta_{\mathsf{max}}$ and $k_P > g^2\cos\theta_{\mathsf{max}}$, which can be rewritten as:

$$k_P > g^2|\cos\theta_{\mathsf{max}}|$$

Motion Analysis: Convergence of Energy

Considering also that $0 < \theta_{\mathsf{max}} \leq \pi$, it is easy to prove that:

$$0 < l_{ue} \leq l_r$$
$$0 < l_{de} < l_r$$

Let Ω be the equilibrium point set (which is an invariant set for the considered system) defined as follows:

$$\Omega = \{(\pi, l_{ue}, 0, 0), (0, l_{de}, 0, 0)\}$$

Motion Analysis: Closed-Loop Equilibrium Points

Consider the two equilibrium points defined in Ω. Let the state variable vector be $x = (\theta, l, \dot{\theta}, \dot{l})^T$. The state space representation is $\dot{x} = f(x)$, which, considering the dynamics of the VLP and the controller based on partial energy shaping, can be rewritten as:

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -\frac{2x_3x_4 + g\sin x_1}{x_2} \\
\dot{x}_4 &= \frac{(E_P - E_r)(x_2x_3^2 + g\cos x1) - k_P(x_2 - l_r) - k_V x_4}{k_D}
\end{aligned}$$

where $E_P = x_2^2x_3^2/2 - gx_2\cos x_1$.

Motion Analysis: Closed-Loop Equilibrium Points

The characteristic equation of the Jacobian matrix evaluated at the upper equilibrium point $x_{ue} = (\pi, l_{ue}, 0, 0)^T$ yields:

$$det(sI - A_{ue}) = \left(s^2 + \frac{k_V}{k_D}s + \frac{g^2 + k_P}{k_D}\right)\left(s^2 - \frac{g}{l_{ue}}\right)$$

The jacobian A_{ue} has three eigenvalues in the open LHP and one eigenvalue in the open RHP. Thus, the updward equilibrium point is unstable and hyperbolic (since all the eigenvalues of A_{ue} have non-zero real parts).

Motion Analysis: Closed-Loop Equilibrium Points

The characteristic equation of the Jacobian matrix evaluated at the downward equilibrium point $x_{de} = (0, l_{de}, 0, 0)^T$ yields:

$$det(sI - A_{de}) = \left(s^2 + \frac{k_V}{k_D}s + \frac{g^2 + k_P}{k_D}\right)\left(s^2 + \frac{g}{l_{ue}}\right)$$

The Jacobian A_{de} two eigenvalues in the open LHP and two eigenvalues on the imaginary axis.

Thus, the downward equilibrium point is non-hyperbolic (since there is at least one eigenvalue on the imaginary axis).

Motion Analysis: Closed-Loop Equilibrium Points

Consider the Lyapunov function V used for partial energy shaping:

$$V = \frac{1}{2}(E_P - E_r)^2 + \frac{1}{2}k_P(l - l_r)^2 + \frac{1}{2}k_D\dot{l}^2$$

Consider the following set:

$$\Gamma_d = \left\{(\theta, l, \dot{\theta}, \dot{l}) \mid V(\theta, l, \dot{\theta}, \dot{l}) < V(0, l_{de}, 0, 0)\right\}$$

Let the values of V at the downward equilibrium point x_{de} and a state $(\delta, l_{de}, 0, 0)$ be respectively V_{de} and V_δ:

$$V_{de} = V(0, l_{de}, 0, 0) = \frac{1}{2}(gl_{de} + E_r)^2 + \frac{1}{2}k_P(l_{de} - l_r)^2$$
$$V_\delta = V(\delta, l_{de}, 0, 0) = \frac{1}{2}(gl_{de}\cos\delta + E_r)^2 + \frac{1}{2}k_P(l_{de} - l_r)^2$$

Motion Analysis: Closed-Loop Equilibrium Points

Compute the difference between V_δ and V_{de} yields:

$$V_\delta - V_{de} = -\frac{g^2 l_{de} l_r (1 - \cos\delta)\Xi}{k_P + g^2}$$

where:

$$\Xi = k_P(1 - \cos\theta_{\max}) - (k_P + g^2\cos\theta_{\max})\sin^2\left(\frac{\delta}{2}\right)$$

Using $k_P > -g^2\cos\theta_{\max}$ and $|\sin\delta| < |\delta|$ for $\delta \neq 0$:

$$\Xi > k_P(1 - \cos\theta_{\max}) - \frac{(k_P + g^2\cos\theta_{\max})\delta^2}{4}$$

with $\delta \neq 0$.

Motion Analysis: Closed-Loop Equilibrium Points

If δ satisfies:

$$0 < |\delta| \leq \delta_m = 2\sqrt{\frac{k_P(1 - \cos\theta_{\text{max}})}{k_P + g^2\cos\theta_{\text{max}}}}$$

then $\Xi > 0$ (proof by simply noticing that $-\delta^2 \geq -\delta_m^2$), which, using $V_\delta - V_{de}$, implies:

$$V_\delta < V_{de}, \quad (\delta, l_{de}, 0, 0) \in \Gamma_d$$

which itself impliesthat $\Gamma_d \neq \varnothing$.

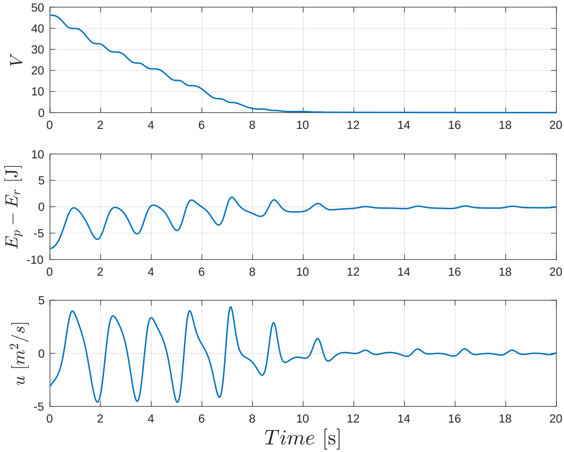
Motion Analysis: Closed-Loop Equilibrium Points

Since V is monotonically decreasing under the controller based on partial energy shaping, for any δ satisfying $0 < |\delta| \leq \delta_m$, closed-loop solution starting from Γ_d approaches W_r due to the results obtained in Convergence of Energy (Case 1). Thus, the downward equilibrium point $(0, l_{de}, 0, 0)$ is unstable in the Lyapunov sense.

Every closed-loop solution under the closed-loop system consisted of the dynamic equation of the VLP and the controller based on partial energy shaping, supposing $k_P > g^2|\cos\theta_{\max}|$, $k_D > 0$, $k_V > 0$, $0 < \theta_{\max} \leq \pi$, approaches $W = W_r \cup \Omega$.

Partial Energy Shaping: Simulation

Time responses of V, $E_P - E_r$ and u with initial state $(\theta(0), l(0), \dot{\theta}(0), \dot{l}(0)) = (-\pi/6, 2, 0, 0)$



Partial Energy Shaping: Simulation

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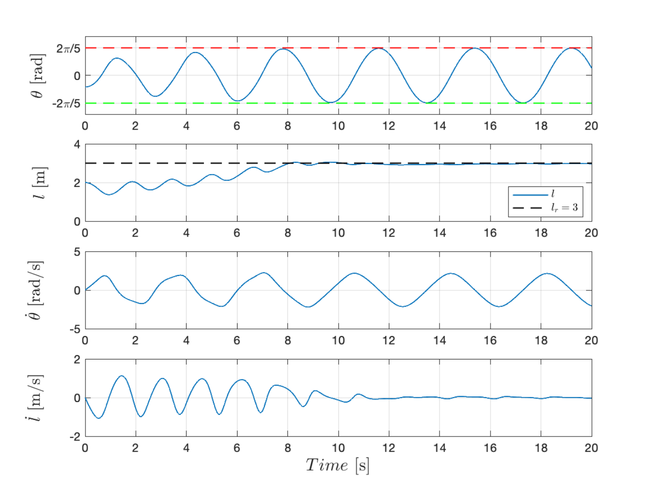
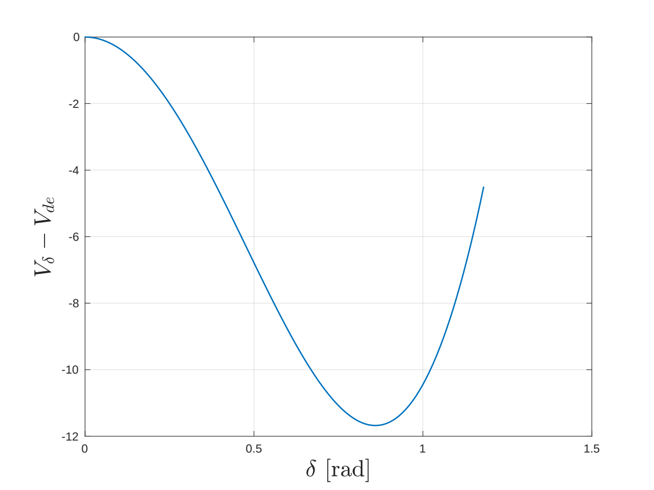


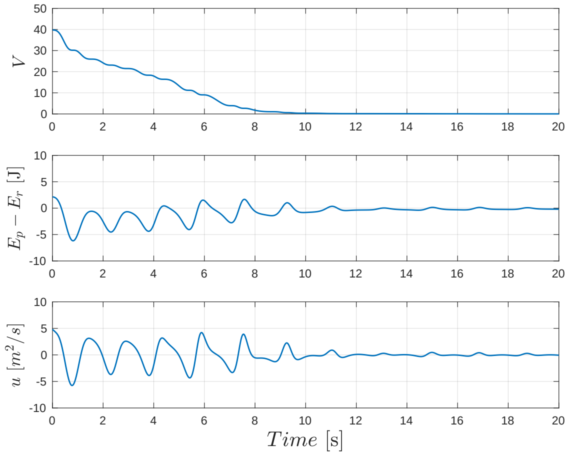
Partial Energy Shaping: Simulation

$V_\delta - V_{de}$



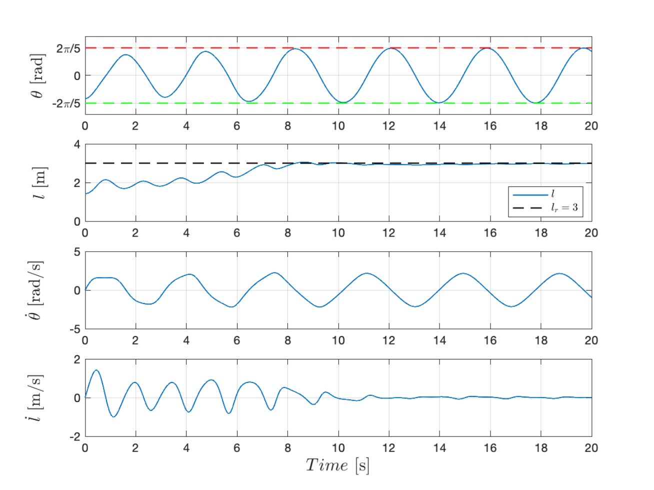
Partial Energy Shaping: Simulation

Time responses of V, $E_P - E_r$ and u with initial state $(\theta(0), (0), \dot{\theta}(0), \dot{l}(0)) = (-\pi/3, l_{de}, 0, 0)$, which satisfied $V_\delta < V_{de}$



Partial Energy Shaping: Simulation

Time responses of $(\theta, l, \dot{\theta}, \dot{l})$ with initial state $(-\pi/3, l_{de}, 0, 0)$



Conclusion

- Total Energy Shaping
- Partial Energy Shaping
- Motion Analysis
- Unstable Equilibrium Points

Q&A

References

X. Xin and Y. Liu, *Control Design and Analysis for Underactuated Robotic Systems.*
Springer Science & Business Media, 2014.