

# FP1: Control of the Variable Length Pendulum

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Control Design and Analysis for Underactuated Robotics:  
Variable Length Pendulum

*Authors:* Xin Xin, Yannian Liu

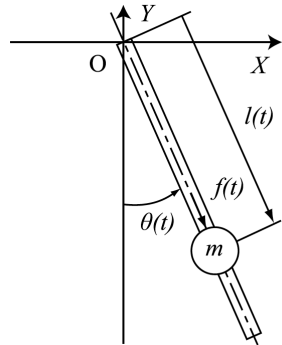
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# Introduction

- the main goal of this work is to study the **Variable Length Pendulum** and present controllers designed to make it achieve a desired swing motion given a desired energy and a desired length of the pendulum
- based on **Control Design and Analysis for Underactuated Robotics: Variable Length Pendulum** by *Xin Xin, Yannian Liu*
- main points in this work
  - motion equation
  - problem formulation
  - controller design
  - motion analysis
  - simulation results

# Motion Equation

- friction at pivot  $O$  and viscous friction of the rod are absent
- the rod is massless
- the angle  $\theta(t)$  to be between the pendulum and the vertical axis
- the length of the pendulum  $l(t)$  starts from the origin  $O$  to the mass of the pendulum  $m$
- $f(t)$  is the force acting on the mass



# Motion Equation

$$x_G = l(t) \sin \theta(t) \quad y_G = -l(t) \cos \theta(t)$$

kinetic energy defined as

$$T = \frac{1}{2}m(\dot{x}_G^2 + \dot{y}_G^2) = \frac{1}{2}m(l(t)\dot{\theta}(t))^2 + \frac{1}{2}m(\dot{l}(t))^2$$

potential energy as

$$P = mgy_G = -mgl(t) \cos \theta(t)$$

the Lagrangian equation of the VLP

$$L = T - P$$

Euler-Lagrangian operator

$$\frac{\partial L}{\partial q} - \frac{dL}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \tau$$

with  $q = [l, \theta]^T$  and  $\tau$  applied generalized forces

# Motion Equation

Euler-Lagrange equation

$$\ddot{\theta}(t) + \frac{2\dot{l}(t)\dot{\theta}(t)}{l(t)} + \frac{g \sin \theta(t)}{l(t)} = 0$$

$$\ddot{l}(t) - ml(t)\dot{\theta}^2(t) - mg \cos \theta(t) = f(t)$$

control input

$$u = \ddot{l}(t)$$

# Problem Formulation

let  $m = 1$ , total mechanical energy

$$E_T = \frac{1}{2}\dot{l}^2(t) + \frac{1}{2}(l(t)\dot{\theta}(t))^2 - gl(t)\cos\theta(t)$$

desired trajectory of swing described by

$$E_r = \frac{1}{2}(l_r\dot{\theta}(t))^2 - gl_r\cos\theta(t)$$

with  $E_r$  and  $l_r$  desired energy and length of the VLP

$$E_r = -gl_r\cos\theta_{max}, \theta_{max} \in (0, \pi]$$

Trajectory tracking control problem

$$\lim_{t \rightarrow \infty} E_T = E_r \quad \lim_{t \rightarrow \infty} \dot{l} = 0 \quad \lim_{t \rightarrow \infty} l = l_r$$

# Controller Design: Total Energy Shaping

Lyapunov candidate with  $k_P > 0$ ,  $k_D > 0$

$$V_c = \frac{1}{2}(E_T - E_r)^2 + \frac{1}{2}k_P(l - l_r)^2 + \frac{1}{2}k_D\dot{l}^2$$

$$\dot{V}_c = \dot{l}((E_T - E_r + k_D)u - (E_T - E_r)(l\ddot{\theta}^2 + g \cos \theta) - k_P(l - l_r))$$

controller

$$u = \frac{(E_T - E_r)(l\ddot{\theta}^2 + g \cos \theta) - k_P(l - l_r) - k_V\dot{l}}{E_T - E_r + k_D}$$

with  $k_V > 0$ , then

$$\dot{V}_c = -k_V\dot{l}^2 \leq 0$$

which holds only if

$$E_T - E_r + k_D \neq 0 \quad \forall t \geq 0$$

$E_T$  satisfies

$$E_T \geq -gl(t)$$

## Total Energy Shaping: Simulation

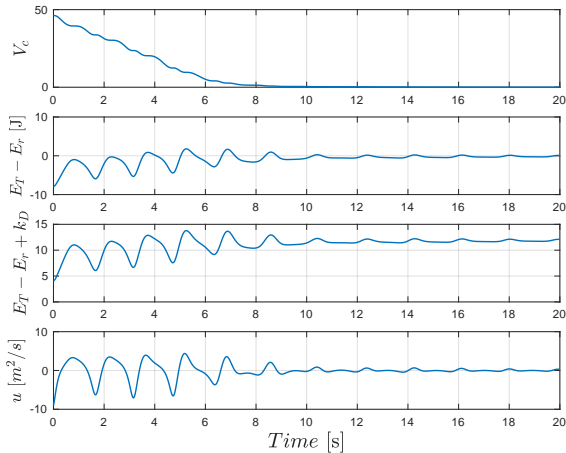
consider the trajectory tracking control problem defined previously  
and  $l_r = 3\text{m}$ ,  $\theta_{\max} = 2\pi/5$ ,  $g = 9.81\text{m/s}^2$

let  $k_D = 12$ ,  $k_P = 30$  and  $k_V = 12$



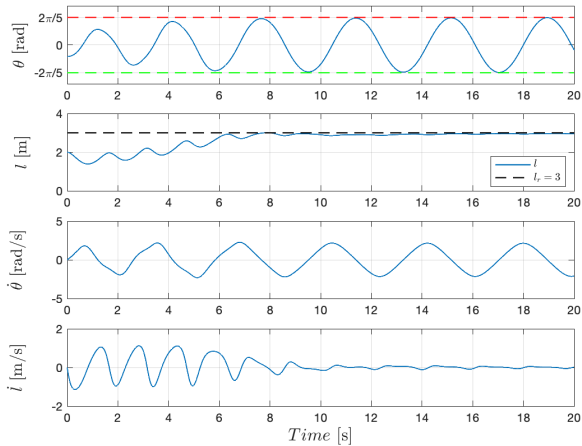
# Total Energy Shaping: Simulation

Time responses of  $V$ ,  $E_T - E_r$ ,  $E_T - E_r + k_D$  and  $u$  with initial state  $(\theta(0), l(0), \dot{\theta}(0), \dot{l}(0)) = (-\pi/6, 2, 0, 0)$



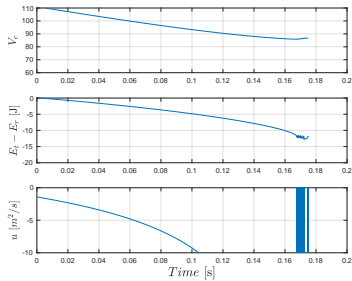
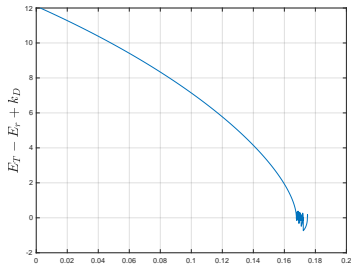
# Total Energy Shaping: Simulation

Time responses of  $(\theta, l, \dot{\theta}, \dot{l})$  with initial state  $(\theta(0), l(0), \dot{\theta}(0), \dot{l}(0)) = (-\pi/6, 2, 0, 0)$



# Total Energy Shaping: Simulation

The controller encountered a singular point for initial state  
 $(\theta(0), l(0), \dot{\theta}(0), \dot{l}(0)) = (-\pi/6, 2, 0, 4)$



# Controller Design: Partial Energy Shaping

$E_P$  sum of kinetic energy of rotation and potential energy of VLP

$$E_P = \frac{1}{2}(l(t)\dot{\theta}(t))^2 - gl(t)\cos\theta(t)$$

since  $(E_T, l, \dot{l}) \equiv (E_r, l_r, 0)$  equivalent to  $(E_P, l, \dot{l}) \equiv (E_r, l_r, 0)$

Lyapunov candidate

$$V = \frac{1}{2}(E_P - E_r)^2 + \frac{1}{2}k_P(l - l_r)^2 + \frac{1}{2}k_D\dot{l}^2$$

$$\dot{V} = (- (E_P - E_r)(l\dot{\theta}^2 + g\cos\theta) + k_P(l - l_r)) + k_D u \dot{l}$$

controller

$$u = \frac{(E_P - E_r)(l\dot{\theta}^2 + g\cos\theta) - k_P(l - l_r) - k_V\dot{l}}{k_D}$$

which is free of singular points

$$\dot{V} = -k_V\dot{l}^2 \leq 0, \quad k_V > 0$$

## Controller Design: Partial Energy Shaping

$$\dot{V} \leq 0$$

consider

$$\Gamma_c = \left\{ (\theta, l, \dot{\theta}, \dot{l}) \mid V(\theta, l, \dot{\theta}, \dot{l}) \leq c \right\}, \quad c > 0$$

since  $\dot{V} \leq 0$ , any closed-loop solution starting in  $\Gamma_c$  remains in  $\Gamma_c$  for all  $t \geq 0$

let  $W$  be the largest invariant set in

$$S = \{ (\theta, l, \dot{\theta}, \dot{l}) \in \Gamma_c \mid \dot{V} = 0 \}$$

using **LaSalle's invariant principle**, every closed-loop solution starting in  $\Gamma_c$  approaches  $W$  as  $t \rightarrow \infty$

# Controller Design: Partial Energy Shaping

since  $\dot{V} = 0$  holds for all elements of  $W$ , then

$$V \equiv V^* \quad l \equiv l^*$$

moreover, since  $V$  is a constant in  $W$ ,  $E_p$  is also a constant in  $W$

$$\lim_{t \rightarrow \infty} E_p = E^* \quad \lim_{t \rightarrow \infty} \dot{l} = 0 \quad \lim_{t \rightarrow \infty} l = l^*$$

thus, the largest invariant set  $W$  can be defined as

$$W = \left\{ (\theta, l, \dot{\theta}, \dot{l}) \left| \frac{1}{2} (l^* \dot{\theta})^2 - gl^* \cos \theta \equiv E^*, l \equiv l^* \right. \right\}$$

# Motion Analysis: Convergence of Energy

Trajectory Tracking Control Problem achieved iff  $V^* = 0$ , with

$$V^* = \frac{1}{2}(E^* - E_r)^2 + \frac{1}{2}k_p(l^* - l_r)^2$$

consider  $E_p \equiv E^*$ ,  $l \equiv l^*$ ,  $\dot{l} \equiv 0$  and  $u = \ddot{l} \equiv 0$ , then, from the controller based on partial energy shaping

$$(E^* - E_r)(l^*\dot{\theta}^2 + g \cos \theta) - k_p(l^* - l_r) \equiv 0$$

# Motion Analysis: Convergence of Energy

Case 1:  $E^* = E_r$

$l^* = l_r$ , the largest invariant set  $W$  becomes

$$W_r = \left\{ (\theta, l, \dot{\theta}, \dot{l}) \left| \frac{1}{2}(l_r \dot{\theta})^2 - gl_r \cos \theta \equiv E_r, l \equiv l_r \right. \right\}$$

hence, as  $t \rightarrow \infty$ , the closed-loop solution  $(\theta(t), l(t), \dot{\theta}(t), \dot{l}(t))$  achieves the tracking control objective



# Motion Analysis: Convergence of Energy

Case 2  $E^* \neq E_r$

$$l^* \dot{\theta}^2 + g \cos \theta \equiv \frac{k_p(l^* - l_r)}{E^* - E_r}$$

since  $E_p \equiv E^*$ ,  $l \equiv l^*$ , from the definition of  $E_p$

$$l^* \dot{\theta}^2 - 2g \cos \theta \equiv \frac{2E^*}{l^*}$$

taking the difference between the two shows that  $\theta$  is a constant

$$\theta \equiv \theta^*$$

since  $E_r = -gl_r \cos \theta_{\max}$

$$l^* = \frac{l_r(k_p + g^2 \cos \theta_{\max} \cos \theta^*)}{k_p + g^2}$$

## Motion Analysis: Convergence of Energy

from the dynamics of the VLP  $\sin \theta^* = 0$ , which admits a solution only in  $\{0, \pi\}$ , hence, either  $(\theta^*, l^*) = (\pi, l_{ue})$  or  $(\theta^*, l^*) = (0, l_{de})$  where

$$l_{ue} = l^*|_{\theta^*=\pi} = \frac{l_r(k_p - g^2 \cos \theta_{\max})}{k_p + g^2}$$

$$l_{de} = l^*|_{\theta^*=0} = \frac{l_r(k_p + g^2 \cos \theta_{\max})}{k_p + g^2}$$

to guarantee that  $l_{ue}, l_{de} > 0$  assume

$$k_p > g^2 |\cos \theta_{\max}|$$

considering also that  $0 < \theta_{\max} \leq \pi$ , it is easy to prove that

$$0 < l_{ue} \leq l_r$$

$$0 < l_{de} < l_r$$

let  $\Omega$  be the equilibrium point set (invariant set) defined as

$$\Omega = \{(\pi, l_{ue}, 0, 0), (0, l_{de}, 0, 0)\}$$

# Motion Analysis: Closed-Loop Equilibrium Points

let the state variable vector be  $x = (\theta, l, \dot{\theta}, \dot{l})^T$ , considering the dynamics of the VLP and the controller based on partial energy shaping, the state space representation is

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = x_4$$

$$\dot{x}_3 = -\frac{2x_3x_4 + g \sin x_1}{x_2}$$

$$\dot{x}_4 = \frac{(E_p - E_r)(x_2x_3^2 + g \cos x_1) - k_p(x_2 - l_r) - k_vx_4}{k_D}$$

where  $E_p = x_2^2x_3^2/2 - gx_2 \cos x_1$

# Motion Analysis: Closed-Loop Equilibrium Points

characteristic equation of the Jacobian evaluated at  $x_{ue} = (\pi, l_{ue}, 0, 0)^T$

$$\det(sI - A_{ue}) = \left( s^2 + \frac{k_V}{k_D}s + \frac{g^2 + k_P}{k_D} \right) \left( s^2 - \frac{g}{l_{ue}} \right)$$

$A_{ue}$  has three eigenvalues in open LHP and one in open RHP

$x_{ue}$  is **unstable** and **hyperbolic**

# Motion Analysis: Closed-Loop Equilibrium Points

characteristic equation of the Jacobian evaluated at  $x_{de} = (0, l_{de}, 0, 0)^T$

$$\det(sI - A_{de}) = \left( s^2 + \frac{k_V}{k_D}s + \frac{g^2 + k_P}{k_D} \right) \left( s^2 + \frac{g}{l_{ue}} \right)$$

$A_{de}$  has two eigenvalues in open LHP and two on imaginary axis

$x_{de}$  is non-hyperbolic

# Motion Analysis: Closed-Loop Equilibrium Points

consider the Lyapunov function  $V$  used for partial energy shaping

$$V = \frac{1}{2}(E_P - E_r)^2 + \frac{1}{2}k_P(l - l_r)^2 + \frac{1}{2}k_D\dot{l}^2$$

consider the following set

$$\Gamma_d = \left\{ (\theta, l, \dot{\theta}, \dot{l}) \mid V(\theta, l, \dot{\theta}, \dot{l}) < V(0, l_{de}, 0, 0) \right\}$$

let the values of  $V$  at  $x_{de}$  and at  $(\delta, l_{de}, 0, 0)$  respectively be

$$V_{de} = V(0, l_{de}, 0, 0) = \frac{1}{2}(gl_{de} + E_r)^2 + \frac{1}{2}k_P(l_{de} - l_r)^2$$

$$V_\delta = V(\delta, l_{de}, 0, 0) = \frac{1}{2}(gl_{de} \cos \delta + E_r)^2 + \frac{1}{2}k_P(l_{de} - l_r)^2$$

## Motion Analysis: Closed-Loop Equilibrium Points

$$V_{\delta} - V_{de} = -\frac{g^2 l_{de} l_r (1 - \cos \delta) \Xi}{k_P + g^2}$$

$$\Xi = k_P (1 - \cos \theta_{\max}) - (k_P + g^2 \cos \theta_{\max}) \sin^2 \left( \frac{\delta}{2} \right)$$

using  $k_P > -g^2 \cos \theta_{\max}$  and  $|\sin \delta| < |\delta|$  for  $\delta \neq 0$

$$\Xi > k_P (1 - \cos \theta_{\max}) - \frac{(k_P + g^2 \cos \theta_{\max}) \delta^2}{4}$$

with  $\delta \neq 0$



# Motion Analysis: Closed-Loop Equilibrium Points

if  $\delta$  satisfies

$$0 < |\delta| \leq \delta_m = 2\sqrt{\frac{k_P(1 - \cos \theta_{\max})}{k_P + g^2 \cos \theta_{\max}}}$$

then  $\Xi > 0$ , which, using  $V_\delta - V_{de}$ , implies

$$V_\delta < V_{de}, \quad (\delta, l_{de}, 0, 0) \in \Gamma_d$$

which itself implies that  $\Gamma_d \neq \emptyset$

$x_{de}$  is **unstable** in the Lyapunov sense

# Motion Analysis: Closed-Loop Equilibrium Points

every closed-loop solution under the closed-loop system consisted of the dynamic equation of the VLP and the controller based on partial energy shaping, supposing

$$0 < \theta_{\max} \leq \pi$$
$$k_P > g^2 |\cos \theta_{\max}|, \quad k_D > 0, \quad k_V > 0$$

approaches  $W = W_r \cup \Omega$

## Partial Energy Shaping: Simulation

consider the trajectory tracking control problem defined previously  
and  $l_r = 3\text{m}$ ,  $\theta_{\max} = 2\pi/5$ ,  $g = 9.81\text{m/s}^2$

the condition on  $k_p$  is

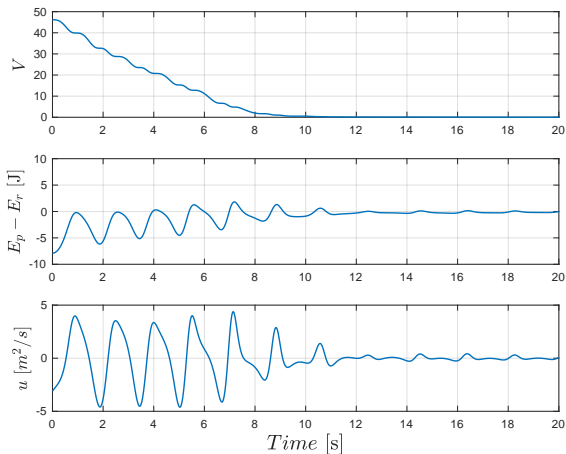
$$k_p > g^2 |\cos \theta_{\max}| = 29.74$$

which yields  $l_{de} = 1.42\text{m}$

let  $k_D = 12$ ,  $k_p = 30$  and  $k_V = 12$

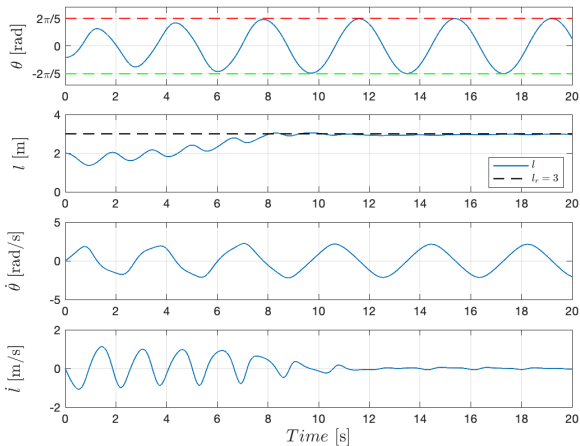
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Time responses of  $V$ ,  $E_p - E_r$  and  $u$  with initial state  $(\theta(0), l(0), \dot{\theta}(0), \dot{l}(0)) = (-\pi/6, 2, 0, 0)$



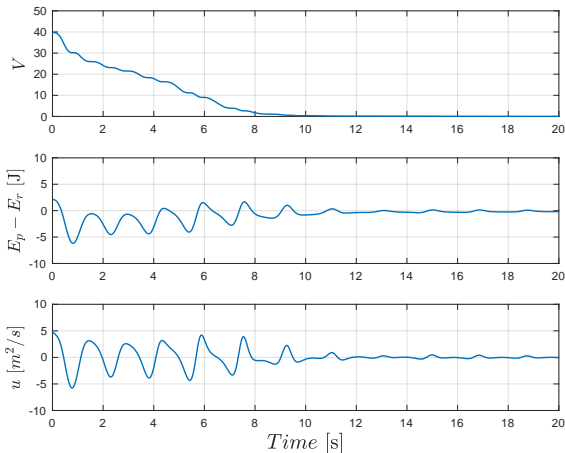
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Time responses of  $(\theta, l, \dot{\theta}, \dot{l})$  with initial state  $(\theta(0), l(0), \dot{\theta}(0), \dot{l}(0)) = (-\pi/6, 2, 0, 0)$



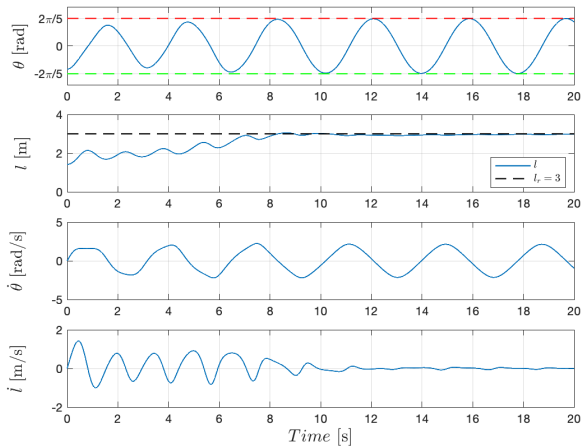
# Partial Energy Shaping: Simulation

Time responses of  $V$ ,  $E_p - E_r$  and  $u$  with initial state  $(\theta(0), l(0), \dot{\theta}(0), \dot{l}(0)) = (-\pi/3, l_{de}, 0, 0)$ , which satisfied  $V_\delta < V_{de}$



# Partial Energy Shaping: Simulation

Time responses of  $(\theta, l, \dot{\theta}, \dot{l})$  with initial state  $(-\pi/3, l_{de}, 0, 0)$



Q&A



# References



X. Xin and Y. Liu, *Control Design and Analysis for Underactuated Robotic Systems*.  
Springer Science & Business Media, 2014.