FP1: Control of the Variable Length Pendulum

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Control Design and Analysis for Underactuated Robotics:
Variable Length Pendulum
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Introduction

- Variable Length Pendulum (VLP)
- Trajectory Tracking Control
- Total Energy Shaping
- Partial Energy Shaping
- Convergence of Energy
- Closed-Loop Equilibrium Points
- Simulink

Motion Equation

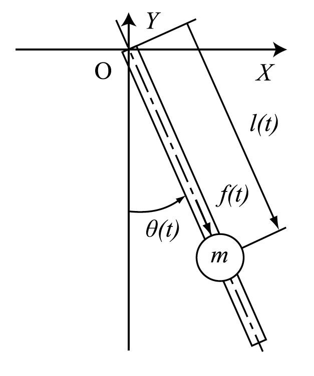
Euler-Lagrange equation:

$$\ddot{\theta}(t) + \frac{2\dot{l}(t)\dot{\theta}(t)}{l(t)} + \frac{g \sin\theta(t)}{l(t)} = 0$$

$$\ddot{l}(t) - ml(t)\dot{\theta}^2(t) - mg\cos\theta(t) = f(t)$$

Control input:

$$u = \ddot{l}(t)$$



Problem Formulation

Let $m = 1$. Total mechanical energy:

$$E_T = \frac{1}{2}\dot{l}^2(t) + \frac{1}{2}\big(l(t)\dot{\theta}(t)\big)^2 - gl(t)\cos\theta(t)$$

Desired trajectory of swing described by:

$$E_r = \frac{1}{2}\big(l_r\dot{\theta}(t)\big)^2 - gl_r\cos\theta(t)$$

with E_r and l_r desired energy and length of the VLP.
Moreover: $E_r = -gl_r\cos\theta_{max}, \theta_{max} \in (0, \pi]$.

Trajectory tracking control problem:

$$\lim_{t\to\infty} E_T = E_r \qquad \lim_{t\to\infty} \dot{l} = 0 \qquad \lim_{t\to\infty} l = l_r$$

Controller Design: Total Energy Shaping

Lyapunov candidate with $k_P > 0$, $k_D > 0$:

$$V_c = \frac{1}{2}(E_T - E_r)^2 + \frac{1}{2}k_P(l - l_r)^2 + \frac{1}{2}k_D\dot{l}^2$$

$$\dot{V}_c = \dot{l}\big((E_T - E_r + k_D)u - (E_T - E_r)(l\dot{\theta}^2 + g\cos\theta) - k_P(l - l_r)\big)$$

Controller:

$$u = \frac{(E_T - E_r)(l\dot{\theta}^2 + g\cos\theta) - k_P(l - l_r) - k_V\dot{l}}{E_T - E_r + k_D}$$

with $k_V > 0$, then:

$$\dot{V}_c = -k_V\dot{l}^2 \leq 0$$

which holds only if:

$$E_T - E_r + k_D \neq 0 \quad \forall t \geq 0$$

Controller Design: Total Energy Shaping

Simulation + Plots.

Controller Design: Partial Energy Shaping

E_P sum of kinetic energy of rotation and potential energy of VLP:

$$E_P = \frac{1}{2}(l(t)\dot{\theta}(t))^2 - gl(t)\cos\theta(t)$$

Lyapunov candidate:

$$V = \frac{1}{2}(E_P - E_r)^2 + \frac{1}{2}k_P(l - l_r)^2 + \frac{1}{2}k_D\dot{l}^2$$

$$\dot{V} = \big(-(E_P - E_r)(l\dot{\theta}^2 + g\cos\theta) + k_P(l - l_r)\big) + k_D u)\dot{l}$$

Controller:

$$u = \frac{(E_P - E_r)(l\dot{\theta}^2 + g\cos\theta) - k_P(l - l_r) - k_V\dot{l}}{k_D}$$

which is free of singular points. Moreover:

$$\dot{V} = -k_V\dot{l}^2 \leq 0, \quad k_V > 0$$

Consider:

$$\Gamma_c = \left\{(\theta, l, \dot{\theta}, \dot{l}) | V(\theta, l, \dot{\theta}, \dot{l}) \leq c\right\}, \quad c > 0$$

Since $\dot{V} \leq 0$, any closed-loop solution starting in Γ_c remains in Γ_c for all $t \geq 0$. Let W be the largest invariant set in

$$S = \{(\theta, l, \dot{\theta}, \dot{l})) \in \Gamma_c | \dot{V} = 0\}$$

Using **LaSalle's invariant principle**, every closed-loop solution starting in Γ_c approaches W as $t \to \infty$.

Controller Design: Partial Energy Shaping

Since $\dot{V} = 0$ holds for all elements of W, V and l are constant in W (let them be V^* and l^*). Moreover, since V is a constant in W, E_P is also a constant in W. Consequently:

$$\lim_{t\to\infty} E_P = E^* \qquad \lim_{t\to\infty} \dot{l} = 0 \qquad \lim_{t\to\infty} l = l^*$$

Thus, the largest invariant set W can be defined as:

$$W = \left\{ (\theta, l, \dot{\theta}, \dot{l}) \middle| \frac{1}{2}(l^*\dot{\theta})^2 - gl^* \cos\theta \equiv E^*, l \equiv l^* \right\}$$

Motion Analysis: Convergence of Energy

Convergence of Energy.

Closed-Loop Equilibrium Points.

Conclusion

Conclusion.

Q&A

References

X. Xin and Y. Liu, *Control Design and Analysis for Underactuated Robotic Systems.* Springer Science & Business Media, 2014.