

# FP1: Control of the Variable Length Pendulum

Michele Cipriano, Karim Ghonim, Khaled Wahba

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Control Design and Analysis for Underactuated Robotics:  
Variable Length Pendulum

*Authors:* Xin Xin, Yannian Liu

Elective in Robotics: Underactuated Robotics  
Department of Computer, Control and Management Engineering  
Sapienza University of Rome

# Introduction

- Variable Length Pendulum (VLP)
- Trajectory Tracking Control
- Total Energy Shaping
- Partial Energy Shaping
- Convergence of Energy
- Closed-Loop Equilibrium Points
- Simulink

# Motion Equation

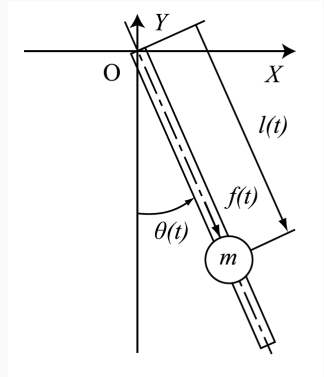
Euler-Lagrange equation:

$$\ddot{\theta}(t) + \frac{2\dot{l}(t)\dot{\theta}(t)}{l(t)} + \frac{g \sin \theta(t)}{l(t)} = 0$$

$$\ddot{l}(t) - ml(t)\dot{\theta}^2(t) - mg \cos \theta(t) = f(t)$$

Control input:

$$u = \ddot{l}(t)$$



# Problem Formulation

Let  $m = 1$ . Total mechanical energy:

$$E_T = \frac{1}{2}\dot{l}^2(t) + \frac{1}{2}(l(t)\dot{\theta}(t))^2 - gl(t)\cos\theta(t)$$

Desired trajectory of swing described by:

$$E_r = \frac{1}{2}(l_r\dot{\theta}(t))^2 - gl_r\cos\theta(t)$$

with  $E_r$  and  $l_r$  desired energy and length of the VLP.

Moreover:  $E_r = -gl_r\cos\theta_{max}$ ,  $\theta_{max} \in (0, \pi]$ .

**Trajectory tracking control problem:**

$$\lim_{t \rightarrow \infty} E_T = E_r \quad \lim_{t \rightarrow \infty} \dot{l} = 0 \quad \lim_{t \rightarrow \infty} l = l_r$$

# Controller Design: Total Energy Shaping

Lyapunov candidate with  $k_P > 0$ ,  $k_D > 0$ :

$$V_c = \frac{1}{2}(E_T - E_r)^2 + \frac{1}{2}k_P(l - l_r)^2 + \frac{1}{2}k_D\dot{l}^2$$
$$\dot{V}_c = \dot{l}((E_T - E_r + k_D)u - (E_T - E_r)(l\dot{\theta}^2 + g \cos \theta) - k_P(l - l_r))$$

Controller:

$$u = \frac{(E_T - E_r)(l\dot{\theta}^2 + g \cos \theta) - k_P(l - l_r) - k_V\dot{l}}{E_T - E_r + k_D}$$

with  $k_V > 0$ , then:

$$\dot{V}_c = -k_V\dot{l}^2 \leq 0$$

which holds only if:

$$E_T - E_r + k_D \neq 0 \quad \forall t \geq 0$$

# Controller Design: Total Energy Shaping

Simulation + Plots.

# Controller Design: Partial Energy Shaping

$E_P$  sum of kinetic energy of rotation and potential energy of VLP:

$$E_P = \frac{1}{2}(l(t)\dot{\theta}(t))^2 - gl(t)\cos\theta(t)$$

Lyapunov candidate:

$$V = \frac{1}{2}(E_P - E_r)^2 + \frac{1}{2}k_P(l - l_r)^2 + \frac{1}{2}k_D\dot{l}^2$$
$$\dot{V} = (-(E_P - E_r)(l\dot{\theta}^2 + g\cos\theta) + k_P(l - l_r) + k_D u)\dot{l}$$

Controller:

$$u = \frac{(E_P - E_r)(l\dot{\theta}^2 + g\cos\theta) - k_P(l - l_r) - k_V\dot{l}}{k_D}$$

which is free of singular points. Moreover:

$$\dot{V} = -k_V\dot{l}^2 \leq 0, \quad k_V > 0$$

# Controller Design: Partial Energy Shaping

Consider:

$$\Gamma_c = \left\{ (\theta, l, \dot{\theta}, \dot{l}) \mid V(\theta, l, \dot{\theta}, \dot{l}) \leq c \right\}, \quad c > 0$$

Since  $\dot{V} \leq 0$ , any closed-loop solution starting in  $\Gamma_c$  remains in  $\Gamma_c$  for all  $t \geq 0$ . Let  $W$  be the largest invariant set in

$$S = \{(\theta, l, \dot{\theta}, \dot{l}) \in \Gamma_c \mid \dot{V} = 0\}$$

Using **LaSalle's invariant principle**, every closed-loop solution starting in  $\Gamma_c$  approaches  $W$  as  $t \rightarrow \infty$ .



# Controller Design: Partial Energy Shaping

Since  $\dot{V} = 0$  holds for all elements of  $W$ ,  $V$  and  $l$  are constant in  $W$  (let them be  $V^*$  and  $l^*$ ). Moreover, since  $V$  is a constant in  $W$ ,  $E_P$  is also a constant in  $W$ . Consequently:

$$\lim_{t \rightarrow \infty} E_P = E^* \quad \lim_{t \rightarrow \infty} \dot{l} = 0 \quad \lim_{t \rightarrow \infty} l = l^*$$

Thus, the largest invariant set  $W$  can be defined as:

$$W = \left\{ (\theta, l, \dot{\theta}, \dot{l}) \left| \frac{1}{2} (l^* \dot{\theta})^2 - g l^* \cos \theta \equiv E^*, l \equiv l^* \right. \right\}$$

# Motion Analysis: Convergence of Energy

Trajectory Tracking Control Problem achieved iff  $V^* = 0$ , with:

$$V^* = \frac{1}{2}(E^* - E_r)^2 + \frac{1}{2}k_p(l^* - l_r)^2$$

Consider  $E_p \equiv E^*$ ,  $l \equiv l^*$ ,  $\dot{l} \equiv 0$  and  $\ddot{l} \equiv 0$  and  $u = \ddot{l} \equiv 0$ . Then, from the controller based on partial energy shaping:

$$(E^* - E_r)(l^*\dot{\theta}^2 + g \cos \theta) - k_p(l^* - l_r) \equiv 0$$

# Motion Analysis: Convergence of Energy

Case 1:  $E^* = E_r$

In this case  $l^* = l_r$ . The largest invariant set  $W$  becomes:

$$W_r = \left\{ (\theta, l, \dot{\theta}, \dot{l}) \left| \frac{1}{2}(l_r \dot{\theta})^2 - gl_r \cos \theta \equiv E_r, l \equiv l_r \right. \right\}$$

Hence, as  $t \rightarrow \infty$ , the closed-loop solution  $(\theta(t), l(t), \dot{\theta}(t), \dot{l}(t))$  achieves the tracking control objective.

# Motion Analysis: Convergence of Energy

Case 2  $E^* \neq E_r$

In this case:

$$l^* \dot{\theta}^2 + g \cos \theta \equiv \frac{k_p(l^* - l_r)}{E^* - E_r}$$

Considering  $E_p \equiv E^*$  and  $l \equiv l^*$  it is possible to prove that  $\theta$  is a constant. Let  $\theta \equiv \theta^*$ . Moreover, since  $E_r = -gl_r \cos \theta_{\max}$ :

$$l^* = \frac{l_r(k_p + g^2 \cos \theta_{\max} \cos \theta^*)}{k_p + g^2}$$

## Motion Analysis: Convergence of Energy

Since  $\sin \theta^* = 0$  admits a solution only in  $\{0, \pi\}$ , either  $(\theta^*, l^*) = (\pi, l_{ue})$  or  $(\theta^*, l^*) = (0, l_{de})$  where:

$$l_{ue} = l^*|_{\theta^*=\pi} = \frac{l_r(k_p - g^2 \cos \theta_{\max})}{k_p + g^2}$$
$$l_{de} = l^*|_{\theta^*=0} = \frac{l_r(k_p + g^2 \cos \theta_{\max})}{k_p + g^2}$$

To guarantee that the length of the pendulum  $l_{ue}$  and  $l_{de}$  are positive in the two cases respectively assume  $k_p > -g^2 \cos \theta_{\max}$  and  $k_p > g^2 \cos \theta_{\max}$ , which can be rewritten as:

$$k_p > g^2 |\cos \theta_{\max}|$$

# Motion Analysis: Convergence of Energy

Considering also that  $0 < \theta_{\max} \leq \pi$ , it is easy to prove that:

$$0 < l_{ue} \leq l_r$$

$$0 < l_{de} < l_r$$

Let  $\Omega$  be the equilibrium point set (which is an invariant set for the considered system) defined as follows:

$$\Omega = \{(\pi, l_{ue}, 0, 0), (0, l_{de}, 0, 0)\}$$

# Motion Analysis: Closed-Loop Equilibrium Points

Consider the two equilibrium points defined in  $\Omega$ . Let the state variable vector be  $x = (\theta, l, \dot{\theta}, \dot{l})^T$ . The state space representation is  $\dot{x} = f(x)$ , which, considering the dynamics of the VLP and the controller based on partial energy shaping, can be rewritten as:

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = x_4$$

$$\dot{x}_3 = -\frac{2x_3x_4 + g \sin x_1}{x_2}$$

$$\dot{x}_4 = \frac{(E_p - E_r)(x_2x_3^2 + g \cos x_1) - k_p(x_2 - l_r) - k_vx_4}{k_D}$$

where  $E_p = x_2^2x_3^2/2 - gx_2 \cos x_1$ .

# Motion Analysis: Closed-Loop Equilibrium Points

The characteristic equation of the Jacobian matrix evaluated at the upper equilibrium point  $x_{ue} = (\pi, l_{ue}, 0, 0)^T$  yields:

$$\det(sI - A_{ue}) = \left( s^2 + \frac{k_V}{k_D}s + \frac{g^2 + k_P}{k_D} \right) \left( s^2 - \frac{g}{l_{ue}} \right)$$

The jacobian  $A_{ue}$  has three eigenvalues in the open LHP and one eigenvalue in the open RHP. Thus, the upward equilibrium point is unstable and **hyperbolic** (since all the eigenvalues of  $A_{ue}$  have non-zero real parts).



# Motion Analysis: Closed-Loop Equilibrium Points

The characteristic equation of the Jacobian matrix evaluated at the downward equilibrium point  $x_{de} = (0, l_{de}, 0, 0)^T$  yields:

$$\det(sI - A_{de}) = \left( s^2 + \frac{k_V}{k_D}s + \frac{g^2 + k_P}{k_D} \right) \left( s^2 + \frac{g}{l_{ue}} \right)$$

The Jacobian  $A_{de}$  has two eigenvalues in the open LHP and two eigenvalues on the imaginary axis. Thus, the downward equilibrium point is **non-hyperbolic** (since there is at least one eigenvalue on the imaginary axis).

# Motion Analysis: Closed-Loop Equilibrium Points

Consider the Lyapunov function  $V$  used for partial energy shaping:

$$V = \frac{1}{2}(E_P - E_r)^2 + \frac{1}{2}k_P(l - l_r)^2 + \frac{1}{2}k_D\dot{l}^2$$

Consider the following set:

$$\Gamma_d = \left\{ (\theta, l, \dot{\theta}, \dot{l}) \mid V(\theta, l, \dot{\theta}, \dot{l}) < V(0, l_{de}, 0, 0) \right\}$$

Let the values of  $V$  at the downward equilibrium point  $x_{de}$  and a state  $(\delta, l_{de}, 0, 0)$  be respectively  $V_{de}$  and  $V_\delta$ :

$$V_{de} = V(0, l_{de}, 0, 0) = \frac{1}{2}(gl_{de} + E_r)^2 + \frac{1}{2}k_P(l_{de} - l_r)^2$$

$$V_\delta = V(\delta, l_{de}, 0, 0) = \frac{1}{2}(gl_{de} \cos \delta + E_r)^2 + \frac{1}{2}k_P(l_{de} - l_r)^2$$

# Motion Analysis: Closed-Loop Equilibrium Points

Compute the difference between  $V_\delta$  and  $V_{de}$  yields:

$$V_\delta - V_{de} = -\frac{g^2 l_{de} l_r (1 - \cos \delta) \Xi}{k_P + g^2}$$

where:

$$\Xi = k_P (1 - \cos \theta_{\max}) - (k_P + g^2 \cos \theta_{\max}) \sin^2 \left( \frac{\delta}{2} \right)$$

Using  $k_P > -g^2 \cos \theta_{\max}$  and  $|\sin \delta| < |\delta|$  for  $\delta \neq 0$ :

$$\Xi > k_P (1 - \cos \theta_{\max}) - \frac{(k_P + g^2 \cos \theta_{\max}) \delta^2}{4}$$

with  $\delta \neq 0$ .

# Motion Analysis: Closed-Loop Equilibrium Points

If  $\delta$  satisfies:

$$0 < |\delta| \leq \delta_m = 2\sqrt{\frac{k_P(1 - \cos \theta_{\max})}{k_P + g^2 \cos \theta_{\max}}}$$

then  $\Xi > 0$  (proof by simply noticing that  $-\delta^2 \geq -\delta_m^2$ ), which, using  $V_\delta - V_{de}$ , implies:

$$V_\delta < V_{de}, \quad (\delta, l_{de}, 0, 0) \in \Gamma_d$$

which itself implies that  $\Gamma_d \neq \emptyset$ .

# Motion Analysis: Closed-Loop Equilibrium Points

Since  $V$  is monotonically decreasing under the controller based on partial energy shaping, for any  $\delta$  satisfying  $0 < |\delta| \leq \delta_m$ , closed-loop solution starting from  $\Gamma_d$  approaches  $W_r$  due to the results obtained in Convergence of Energy (Case 1). Thus, the downward equilibrium point  $(0, l_{de}, 0, 0)$  is unstable in the Lyapunov sense.

Every closed-loop solution under the closed-loop system consisted of the dynamic equation of the VLP and the controller based on partial energy shaping, supposing  $k_P > g^2 |\cos \theta_{\max}|$ ,  $k_D > 0$ ,  $k_V > 0$ ,  $0 < \theta_{\max} \leq \pi$ , approaches  $W = W_r \cup \Omega$ .

# Conclusion

- Total Energy Shaping
- Partial Energy Shaping
- Motion Analysis
- Unstable Equilibrium Points

Q&A

# References



X. Xin and Y. Liu, *Control Design and Analysis for Underactuated Robotic Systems*.

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