Sapienza Università di Roma

Department of Computer, Control and Management Engineering

FP1 Control of the Variable Length Pendulum

Underactuated Robotics

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Contents

1 Introduction

The aim of the project is to study the variable length pendulum (VLP) presented in [1] along with the controllers designed to make it achieve a desired swing motion given a desired energy and a desired length of the pendulum.

The report introduces the theory needed to study the stability of the considered systems in Section 2, describing invariant sets and LaSalle's invariance principle. A detailed description of the variable length pendulum is given in Section 3, where the dynamics equations of the VLP are introduced by using Euler-Lagrange equations. The trajectory control problem is briefly described in Section 4. Section 5 studies two approaches to design the controllers that make the VLP perform a swing trajectory. The first one is based on total energy shaping while the second one is based on partial energy shaping. Section 6 analyzes the behaviour of the VLP under the controller designed with partial energy shaping, discussing the stability of the equilibrium points of the system. Section 7 presents a few simulation experiments which have been performed through a custom Simulink implementation of the controllers studied in this report. Section 8 concludes the report, summarizing the obtained results.

2 Theoretical Background

2.1 LaSalle's Invariance Principle

Let $\dot{x} = f(x)$ be an autonomous system with $f : D \to \mathbb{R}^n$ locally Lipschitz map from a domain $D \subseteq \mathbb{R}^n$ into $\mathbb{R}^n$.

Definition 2.1. (Invariant Set) A set W is called *invariant set* with respect to the above autonomous system $\dot{x} = f(x)$ if

$$x(0) \in W \implies \forall t \in \mathbb{R} : x(t) \in W \tag{1}$$

Intuitively, if a solution belongs to W at a certain instant t, then it belongs to W for all instants. W is called *positively invariant set* if the above holds $\forall t > 0$.

Theorem 1. (LaSalle's Invariance Principle) Let $\Gamma \subset D$ be a closed and bounded set (a compact set) which is invariant with respect to the above autonomous system $\dot{x} = f(x)$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Γ. Let $S = \{x \in \Gamma \mid \dot{V}(x) = 0\}$. Let W be the largest invariant set in S. Then, every solution starting in Γ approaches W as $t \to \infty$.

3 Motion Equation

Consider a variable length pendulum (VLP) as shown in Fig. 1 and assume the following properties in order to define the equations of motion of the system

1. the friction at the pivot O and the viscous friction of the rod are considered absent
2. the angle $\theta(t)$ to be between the pendulum and the vertical axis
3. the length of the pendulum $l(t)$ that starts from the origin O to the mass of the pendulum m
4. the rod is considered massless
5. $f(t)$ is the force acting on the mass

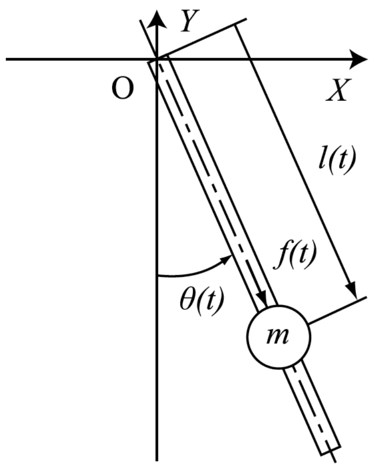

Figure 1: Variable Length Pendulum [1]

Since the rod is massless, the position coordinates (x_G, y_G) of the COM of the VLP is defined such that

$$x_G = l(t)\sin\theta(t) \quad y_G = -l(t)\cos\theta(t) \tag{2}$$

The kinetic energy T of the VLP is defined as

$$T = \frac{1}{2}m(\dot{x}_G^2 + \dot{y}_G^2) = \frac{1}{2}m\big(l(t)\dot{\theta}(t)\big)^2 + \frac{1}{2}m\big(\dot{l}(t)\big)^2 \tag{3}$$

while the potential energy as

$$P = mgy_G = -mgl(t)\cos\theta(t) \tag{4}$$

Using the kinetic and the potential energy, the Lagrangian equation of the VLP can be defined as

$$L = T - P$$

Applying the Euler-Lagrangian operator on the defined Lagrangian equation L, where the Lagrangian operator is given as

$$\frac{\partial L}{\partial q} - \frac{dL}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \tau \tag{5}$$

with $q = [l, \theta]^T$ state vector and τ applied generalized forces on the VLP, it is possible to obtain the equations of motions of the system

$$\ddot{\theta}(t) + \frac{2\dot{l}(t)\dot{\theta}(t)}{l(t)} + \frac{g\sin\theta(t)}{l(t)} = 0 \tag{6}$$

$$\ddot{l}(t) - ml(t)\dot{\theta}^2(t) - mg\cos\theta(t) = f(t) \tag{7}$$

For simplicity, the control input will be chosen to be

$$u = \ddot{l}(t)$$

where $f(t)$ can be computed directly from equation (7).

4 Problem Formulation

The total mechanical energy of the system can be expressed as the sum of the kinetic and the potential energy

$$E_T = T + P = \frac{1}{2}\dot{l}^2(t) + \frac{1}{2}\big(l(t)\dot{\theta}(t)\big)^2 - gl(t)\cos\theta(t) \tag{8}$$

where, without loss of generality, it is assumed that $m = 1$. Consider the desired trajectory of swing to be described by

$$E_r = \frac{1}{2}\big(l_r\dot{\theta}(t)\big)^2 - gl_r\cos\theta(t) \tag{9}$$

where, for a given desired length of the pendulum l_r and a maximal angle θ_{max} of desired swing. E_r is the desired energy, which satisfies the following

$$E_r = -gl_r\cos\theta_{max}, \qquad \theta_{max} \in (0, \pi] \tag{10}$$

The trajectory tracking problem proposed in [1] is to design a control law u in order to achieve the following

$$\lim_{t\to\infty} E_T = E_r \qquad \lim_{t\to\infty} \dot{l} = 0 \qquad \lim_{t\to\infty} l = l_r \tag{11}$$

5 Controller Design

To propose a design for control input u for the given control tracking problem, in order to achieve the required control objective in equation (11), two approaches for the controller design have been considered in [1]. The first one applies the conventional energy-based control approach, while the second one was proposes modifications for the aforementioned method.

5.1 Total Energy Shaping

Consider the conventional Lyapunov candidate

$$V_c = \frac{1}{2}(E_T - E_r)^2 + \frac{1}{2}k_P(l - l_r)^2 + \frac{1}{2}k_D\dot{l}^2 \tag{12}$$

where $k_D, k_P > 0$ are scalar control parameters. Computing the time derivative of E_T defined in equation (8)

$$\dot{E}_T = (u - l\dot{\theta}^2 - g l\cos\theta)\dot{l} \tag{13}$$

then the time derivative of the Lyapunov candidate V_c along the trajectories of (6) is computed as

$$\dot{V}_c = \dot{l}\big((E_T - E_r + k_D)u - (E_T - E_r)(l\dot{\theta}^2 + g\cos\theta) - k_P(l - l_r)\big) \tag{14}$$

Consider the following controller

$$u = \frac{(E_T - E_r)(l\dot{\theta}^2 + g\cos\theta) - k_P(l - l_r) - k_V\dot{l}}{E_T - E_r + k_D} \tag{15}$$

for some positive k_V, then

$$\dot{V}_c = -k_V\dot{l}^2 \leq 0 \tag{16}$$

which holds only if

$$E_T - E_r + k_D \neq 0 \qquad \forall t \geq 0 \tag{17}$$

Given Eq. (8), since $E_T \geq -gl(t)$, the total mechanical energy is not bounded from below. [THIS BOUNDED FROM BELOW PART MUST BE EXPLAINED BETTER]. Hence, it is not possible to choose a k_D which would ensure the proposed controller to be free of singular points.

5.2 Partial Energy Shaping

Let E_P be the sum of kinetic energy of rotation and potential energy of the VLP

$$E_P = \frac{1}{2}(l(t)\dot{\theta}(t))^2 - gl(t)\cos\theta(t) \tag{18}$$

and consider the following Lyapunov candidate

$$V = \frac{1}{2}(E_P - E_r)^2 + \frac{1}{2}k_P(l - l_r)^2 + \frac{1}{2}k_D\dot{l}^2 \tag{19}$$

Taking the time derivative of E_P and using the Euler-Lagrange equation (6) yields

$$\dot{E}_P = -(l\dot{\theta}^2 + g\cos\theta)\dot{l} \tag{20}$$

which, by taking the time derivative of V, shows that

$$\dot{V} = \big(- (E_P - E_r)(l\dot{\theta}^2 + g\cos\theta) + k_P(l - l_r)\big) + k_D u)\dot{l} \tag{21}$$

Consider the following controller

$$u = \frac{(E_P - E_r)(l\dot{\theta}^2 + g\cos\theta) - k_P(l - l_r) - k_V\dot{l}}{k_D} \tag{22}$$

which is free of singular points. Substituting (22) back into (21) shows that

$$\dot{V} = -k_V\dot{l}^2 \leq 0, \quad k_V > 0 \tag{23}$$

Consider the following set

$$\Gamma_c = \left\{(\theta, l, \dot{\theta}, \dot{l}) | V(\theta, l, \dot{\theta}, \dot{l}) \leq c\right\}, \quad c > 0 \tag{24}$$

Since $\dot{V} \leq 0$ from (23), any closed-loop solution starting in Γ_c remains in Γ_c for all $t \geq 0$. Let W be the largest invariant set in

$$S = \{(\theta, l, \dot{\theta}, \dot{l})) \in \Gamma_c | \dot{V} = 0\} \tag{25}$$

Using LaSalle's invariant principle (Theorem 1), every closed-loop solution starting in Γ_c approaches W as $t \to \infty$.

Since $\dot{V} = 0$ holds for all elements of W, V and l are constant in W (let them be V^* and l^*). Moreover, since V is a constant in W, E_P is also a constant in W (from (19)). Consequently

$$\lim_{t\to\infty} E_P = E^* \qquad \lim_{t\to\infty} \dot{l} = 0 \qquad \lim_{t\to\infty} l = l^* \tag{26}$$

Thus, the largest invariant set W can be defined as

$$W = \left\{ (\theta, l, \dot{\theta}, \dot{l}) \middle| \frac{1}{2}(l^*\dot{\theta})^2 - gl^* \cos\theta \equiv E^*, l \equiv l^* \right\} \tag{27}$$

6 Motion Analysis

The tracking control objective in (11) is achieved if and only if $V^* = 0$, where V^* is defined as

$$V^* = \frac{1}{2}(E^* - E_r)^2 + \frac{1}{2}k_P(l^* - l_r)^2 \tag{28}$$

6.1 Convergence of Energy

Consider $E_P \equiv E^*$, $l \equiv l^*$, $\dot{l} \equiv 0$ and $\dot{l} \equiv 0$ and $u = \ddot{l} \equiv 0$, then, from (22)

$$(E^* - E_r)(l^*\dot{\theta}^2 + g\cos\theta) - k_P(l^* - l_r) \equiv 0 \tag{29}$$

Case 1: $E^* = E_r$

In this case, $l^* = l_r$ directly follows from (29). The largest invariant set W defined in (27) becomes

$$W_r = \left\{ (\theta, l, \dot{\theta}, \dot{l}) \middle| \frac{1}{2}(l_r\dot{\theta})^2 - gl_r \cos\theta \equiv E_r, l \equiv l_r \right\} \tag{30}$$

hence, as $t \to \infty$, the closed-loop solution $(\theta(t), l(t), \dot{\theta}(t), \dot{l}(t))$ achieves the tracking control objective in (11).

Case 2 $E^* \neq E_r$

In this case, from (29)

$$l^*\dot{\theta}^2 + g\cos\theta \equiv \frac{k_P(l^* - l_r)}{E^* - E_r} \tag{31}$$

Moreover, since by hypothesis $E_P \equiv E^*$ and $l \equiv l^*$

$$l^*\dot{\theta}^2 - 2g\cos\theta \equiv \frac{2E^*}{l^*} \tag{32}$$

Taking difference between the above equations shows that $\cos\theta$ is a constant, hence θ is a constant. Let's denote it as $\theta \equiv \theta^*$, which yields $\sin\theta^* = 0$.

Since $\theta \equiv \theta^*$ constant (hence $\dot{\theta} = 0$) the two above equations respectively become

$$g\cos\theta^* = \frac{k_P(l^* - l_r)}{E^* - E_r} \tag{33}$$

$$E^* = -l^* g\cos\theta^* \tag{34}$$

Considering $E_r = -gl_r\cos\theta_{\max}$ from (10), by substituting E^* and E_r into (33) and considering $\cos^2\theta^* = 1 - \sin^2\theta^* = 1$ it is possible to obtain the following

$$l^* = \frac{l_r(k_P + g^2\cos\theta_{\max}\cos\theta^*)}{k_P + g^2} \tag{35}$$

Therefore, since $\sin\theta^* = 0$ admits a solution only in $\{0, \pi\}$, either $(\theta^*, l^*) = (\pi, l_{ue})$ or $(\theta^*, l^*) = (0, l_{de})$ where

$$l_{ue} = l^*|_{\theta^*=\pi} = \frac{l_r(k_P - g^2\cos\theta_{\max})}{k_P + g^2} \tag{36}$$

$$l_{de} = l^*|_{\theta^*=0} = \frac{l_r(k_P + g^2\cos\theta_{\max})}{k_P + g^2} \tag{37}$$

where "ue" and "de" respectively denote upright equilibrium point and downward equilibrium point. To guarantee that the length of the pendulum l_{ue} and l_{de} are positive in the two cases respectively assume $k_P > -g^2\cos\theta_{\max}$ and $k_P > g^2\cos\theta_{\max}$, which can be rewritten as

$$k_P > g^2|\cos\theta_{\max}| \tag{38}$$

By considering the above with $0 < \theta_{\max} \leq \pi$ in (10), it is easy to prove (e.g. by contradiction) that

$$0 < l_{ue} \leq l_r \tag{39}$$

$$0 < l_{de} < l_r \tag{40}$$

Let Ω be the equilibrium point set (which is an invariant set for the considered system) defined as follows

$$\Omega = \{(\pi, l_{ue}, 0, 0), (0, l_{de}, 0, 0)\} \tag{41}$$

6.2 Closed-Loop Equilibrium Points

Consider the two equilibrium points defined in Ω. Let the state variable vector be $x = (\theta, l, \dot{\theta}, \dot{l})^T$. The state space representation is

$$\dot{x} = f(x) \tag{42}$$

which, considering the dynamics described by (6) and the controller described by (22), can be rewritten as

$$\dot{x}_1 = x_3 \tag{43}$$

$$\dot{x}_2 = x_4 \tag{44}$$

$$\dot{x}_3 = -\frac{2x_3x_4 + g\sin x_1}{x_2} \tag{45}$$

$$\dot{x}_4 = \frac{(E_P - E_r)(x_2x_3^2 + g\cos x1) - k_P(x_2 - l_r) - k_V x_4}{k_D} \tag{46}$$

where $E_P = x_2^2 x_3^2/2 - gx_2\cos x_1$ from (18).

By using the indirect method of Lyapunov (also known as first method of Lyapunov) it is possible to study the stability of the equilibrium points of the system.

The characteristic equation of the Jacobian matrix evaluated at the upper equilibrium point $x_{ue} = (\pi, l_{ue}, 0, 0)^T$ yields

$$det(sI - A_{ue}) = \left(s^2 + \frac{k_V}{k_D}s + \frac{g^2 + k_P}{k_D}\right)\left(s^2 - \frac{g}{l_{ue}}\right) \tag{47}$$

where $I \in \mathbb{R}^{4\times4}$ identity matrix and

$$A_{ue} = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_{ue}} \tag{48}$$

hence, the jacobian A_{ue} has three eigenvalues in the open LHP and one eigenvalue in the open RHP. Thus, the updward equilibrium point is unstable and hyperbolic (since all the eigenvalues of A_{ue} have non-zero real parts). SOMETHING ABOUT TH. 2.10 AND LEBESGUE MEASURE ZERO.

The characteristic equation of the Jacobian matrix evaluated at the downward equilibrium point $x_{de} = (0, l_{de}, 0, 0)^T$ yields

$$det(sI - A_{de}) = \left(s^2 + \frac{k_V}{k_D}s + \frac{g^2 + k_P}{k_D}\right)\left(s^2 + \frac{g}{l_{ue}}\right) \tag{49}$$

where $I \in \mathbb{R}^{4\times4}$ identity matrix and

$$A_{de} = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_{de}} \tag{50}$$

hence, the Jacobian A_{de} two eigenvalues in the open LHP and two eigenvalues on the imaginary axis (hence, non-hyperbolic since there is at least one eigenvalue on the imaginary axis). In case of non-hyperbolic equilibrium points, linearization fails to determine its stability from the Jacobian matrix. A possible solution is to use CENTER MANIFOLD THEOREM 2.9 (2.1.4), which is however too hard for this system. Consider the Lyapunov function V defined in (19). Consider the following set

$$\Gamma_d = \left\{(\theta, l, \dot{\theta}, \dot{l}) \mid V(\theta, l, \dot{\theta}, \dot{l}) < V(0, l_{de}, 0, 0)\right\} \tag{51}$$

Let the values of V at the downward equilibrium point x_{de} and a state $(\delta, l_{de}, 0, 0)$ be respectively V_{de} and V_δ

$$V_{de} = V(0, l_{de}, 0, 0) = \frac{1}{2}(gl_{de} + E_r)^2 + \frac{1}{2}k_P(l_{de} - l_r)^2 \tag{52}$$

$$V_\delta = V(\delta, l_{de}, 0, 0) = \frac{1}{2}(gl_{de}\cos\delta + E_r)^2 + \frac{1}{2}k_P(l_{de} - l_r)^2 \tag{53}$$

Compute the difference between the two above values yields

$$V_\delta - V_{de} = -\frac{g^2 l_{de} l_r (1 - \cos\delta)\Xi}{k_P + g^2} \tag{54}$$

where

$$\Xi = k_P(1 - \cos\theta_{\max}) - (k_P + g^2\cos\theta_{\max})\sin^2\left(\frac{\delta}{2}\right) \tag{55}$$

Using $k_P > -g^2\cos\theta_{\max}$ from (38) and $|\sin\delta| < |\delta|$ for $\delta \neq 0$ (hence $k_P + g^2\cos\theta_{\max} > 0$ and $-\sin^2(\delta/2) > -\delta^2/4$)

$$\Xi > k_P(1 - \cos\theta_{\max}) - \frac{(k_P + g^2\cos\theta_{\max})\delta^2}{4} \tag{56}$$

with $\delta \neq 0$. Moreover, if δ satisfies

$$0 < |\delta| \leq \delta_m = 2\sqrt{\frac{k_P(1 - \cos\theta_{\max})}{k_P + g^2\cos\theta_{\max}}} \tag{57}$$

then $\Xi > 0$ (proof by simply noticing that $-\delta^2 \geq -\delta_m^2$), which, using (54), implies

$$V_\delta < V_{de}, \quad (\delta, l_{de}, 0, 0) \in \Gamma_d \tag{58}$$

which implies, because of the definition of Γ_d in (51), that $\Gamma_d \neq \varnothing$.

Since the considered Lyapunov function (19) is monotonically decreasing under the controller (22), for any δ satisfying (57), closed-loop solution starting from an initial state belonging to Γ_d approaches W_r due to the results obtained in subsection 6.1 (Case 1), hence, the downward equilibrium point $(0, l_{de}, 0, 0)$ is unstable in the Lyapunov sense.

It is, hence, possible to conclude by saying that every closed-loop solution under the closed-loop system consisted of the dynamic equation (6) and the controller (22), supposing $k_P > g^2|\cos\theta_{\max}|$, $k_D > 0$, $k_V > 0$, $0 < \theta_{\max} \leq \pi$, approaches $W = W_r \cup \Omega$, where W_r is defined in (30) and Ω is defined in (41).

7 Simulation Results

7.1 Total Energy Shaping

Description with plots.

7.2 Partial Energy Shaping

Description with plots.

8 Conclusion

The project presented a detailed analysis of the variable length pendulum, studying two controllers based on total and partial energy shaping. After introducing LaSalle's Invariance Principle and the motion equation of the VLP with the formulation of the trajectory tracking problem, the two controllers are presented. It is immediate to notice how partial energy shaping eliminates the presence of singular points, making it possible to correctly perform a swing trajectory. The considered system presents two equilibrium points, both unstable. Theoretical results are validated with a Simulink implementation of the controllers, reflecting the results obtained in [1].

References

[1] Xin Xin and Yannian Liu. *Control Design and Analysis for Underactuated Robotic Systems*. Springer Science & Business Media, 2014.