In[1]:=
```
(*ra is r_min and D is sphere diameter*)
h1[r_, ra_, D_] := 1/(2*(ra + D/2))
h2[r_, ra_, D_] := D/2 - (ra + D/2) - r
h3[r_, ra_, D_] := r - (ra + D/2) - D/2
h4[r_, ra_, D_] := r + D/2 - (ra + D/2)
h5[r_, ra_, D_] := (ra + D/2) + D/2 + r

h[r_, ra_, D_] := h1[r, ra, D] * Sqrt[h2[r, ra, D] * h3[r, ra, D] * h4[r, ra, D] * h5[r, ra, D]]

Theta[r_, ra_, D_] := ArcSin[h[r, ra, D] / r]

S[r_, ra_, D_] := 2 * Pi * (r^2) * (-Cos[Theta[r, ra, D]] + 1)
Clear[ra]
Clear[d]
```

In[11]:=
```
Integrate[S[x, ra, d], {x, ra, ra + d}]
```

Out[11]=

$$\text{ConditionalExpression}\Big[\frac{2}{3}\pi\left(-\frac{3}{4}d\,(d+2\,ra)^2+d\,\left(d^2+3\,d\,ra+3\,ra^2\right)\right),$$
$$\left(\left(\left(\text{Re}\left[\frac{ra}{d}\right]\geq 0\,\&\&\,\frac{ra}{d}\neq 0\right)\,||\,\frac{ra}{d}\notin\mathbb{R}\,||\,\text{Re}\left[\frac{ra}{d}\right]<-1\right)\&\&\right.$$
$$\left(\frac{ra-\sqrt{-ra\,(d+ra)}}{d}\notin\mathbb{R}\,||\,\left(\text{Re}\left[\frac{ra-\sqrt{-ra\,(d+ra)}}{d}\right]<-1\,\&\&\right.\right.$$
$$\left.\left(\text{Re}\left[\frac{-ra+\sqrt{-ra\,(d+ra)}}{d}\right]\geq 1\,||\,\text{Re}\left[\frac{-ra+\sqrt{-ra\,(d+ra)}}{d}\right]==0\right)\right)\,||$$
$$\left.\left(\text{Re}\left[\frac{ra-\sqrt{-ra\,(d+ra)}}{d}\right]\geq 0\,\&\&\,\text{Re}\left[\frac{-ra+\sqrt{-ra\,(d+ra)}}{d}\right]<0\right)\right)\&\&$$
$$\left(\frac{ra+\sqrt{-ra\,(d+ra)}}{d}\notin\mathbb{R}\,||\,\text{Re}\left[\frac{ra+\sqrt{-ra\,(d+ra)}}{d}\right]>0\,||\,\text{Re}\left[\frac{ra+\sqrt{-ra\,(d+ra)}}{d}\right]<-1\right)\&\&$$
$$\left.d+ra\neq 0\right)\Big]$$

In[12]:=

$$\text{Simplify}\left[\frac{2}{3}\pi\left(-\frac{3}{4}d\,(d+2\,ra)^2+d\,\left(d^2+3\,d\,ra+3\,ra^2\right)\right)\right]$$

Out[12]=

$$\frac{d^3\,\pi}{6}$$

In[13]:=
```
Clear[ra]
Clear[d]
```

In[15]:=
```
S1[r_, ra_, D_] := S[r, ra, D] * (1/r)
```

In[16]:= 
```
Integrate[S1[x, ra, d], {x, ra, ra + d}]
```

Out[16]= ConditionalExpression

$$\left[\pi\left(d\,(d+2\,ra)-\frac{2\,(d^3+6\,d^2\,ra+6\,d\,ra^2)}{3\,(d+2\,ra)}\right),\right.$$

$$\left(\left(\mathrm{Re}\left[\frac{ra}{d}\right]\geq 0\,\&\&\,\frac{ra}{d}\neq 0\right)\,||\,\frac{ra}{d}\notin\mathbb{R}\,||\,\mathrm{Re}\left[\frac{ra}{d}\right]<-1\right)\,\&\&$$

$$\left(\frac{ra-\sqrt{-ra\,(d+ra)}}{d}\notin\mathbb{R}\,||\,\left(\mathrm{Re}\left[\frac{ra-\sqrt{-ra\,(d+ra)}}{d}\right]<-1\,\&\&\right.\right.$$

$$\left.\left(\mathrm{Re}\left[\frac{-ra+\sqrt{-ra\,(d+ra)}}{d}\right]\geq 1\,||\,\mathrm{Re}\left[\frac{-ra+\sqrt{-ra\,(d+ra)}}{d}\right]==0\right)\right)\,||$$

$$\left.\left(\mathrm{Re}\left[\frac{ra-\sqrt{-ra\,(d+ra)}}{d}\right]\geq 0\,\&\&\,\mathrm{Re}\left[\frac{-ra+\sqrt{-ra\,(d+ra)}}{d}\right]<0\right)\right)\,\&\&$$

$$\left(\frac{ra+\sqrt{-ra\,(d+ra)}}{d}\notin\mathbb{R}\,||\,\mathrm{Re}\left[\frac{ra+\sqrt{-ra\,(d+ra)}}{d}\right]>0\,||\,\mathrm{Re}\left[\frac{ra+\sqrt{-ra\,(d+ra)}}{d}\right]<-1\right)\,\&\&$$

$$\left.d+ra\neq 0\right]$$

In[17]:= **Simplify**

$$\left[\pi\left(d\,(d+2\,ra)-\frac{2\,(d^3+6\,d^2\,ra+6\,d\,ra^2)}{3\,(d+2\,ra)}\right)\right]$$

Out[17]=

$$\frac{d^3\,\pi}{3\,d+6\,ra}$$

In[18]:= 
```
Clear[ra]
Clear[d]
```

In[20]:= 
```
S2[r_, ra_, D_, α_, λ_] := S[r, ra, D] * (α * Exp[-r / λ]) / r
```

In[21]:= `Integrate[S2[x, ra, d, α, λ], {x, ra, ra + d}]`

Out[21]= $\text{ConditionalExpression}\Big[-\frac{2\,e^{-\frac{d+ra}{\lambda}}\,\pi\,\alpha\,\left(-d-e^{d/\lambda}\,(d-2\,\lambda)-2\,\lambda\right)\,\lambda^2}{d+2\,ra},\ d+ra\neq 0\ \&\&\ \left(\text{Re}\Big[\frac{ra+\sqrt{-ra\,(d+ra)}}{d}\Big]>0\ ||\ 1+\text{Re}\Big[\frac{ra+\sqrt{-ra\,(d+ra)}}{d}\Big]<0\ ||\ \frac{ra+\sqrt{-ra\,(d+ra)}}{d}\notin\mathbb{R}\right)\ \&\&\ \left(\left(1+\text{Re}\Big[\frac{ra-\sqrt{-ra\,(d+ra)}}{d}\Big]<0\ \&\&\ \left(\text{Re}\Big[\frac{-ra+\sqrt{-ra\,(d+ra)}}{d}\Big]\geq 1\ ||\ \text{Re}\Big[\frac{-ra+\sqrt{-ra\,(d+ra)}}{d}\Big]==0\right)\right)\ ||\ \left(\text{Re}\Big[\frac{ra-\sqrt{-ra\,(d+ra)}}{d}\Big]\geq 0\ \&\&\ \text{Re}\Big[\frac{-ra+\sqrt{-ra\,(d+ra)}}{d}\Big]<0\right)\ ||\ \frac{ra-\sqrt{-ra\,(d+ra)}}{d}\notin\mathbb{R}\right)\ \&\&\ \left(\left(\text{Re}\Big[\frac{ra}{d}\Big]\geq 0\ \&\&\ \frac{ra}{d}\neq 0\right)\ ||\ 1+\text{Re}\Big[\frac{ra}{d}\Big]<0\ ||\ \frac{ra}{d}\notin\mathbb{R}\right)\Big]$

In[22]:= `Clear[α, λ]`

In[23]:= $\textbf{Simplify}\Big[-\frac{2\,e^{-\frac{d+ra}{\lambda}}\,\pi\,\alpha\,\left(-d-e^{d/\lambda}\,(d-2\,\lambda)-2\,\lambda\right)\,\lambda^2}{d+2\,ra}\Big]$

Out[23]= $-\frac{2\,e^{-\frac{d+ra}{\lambda}}\,\pi\,\alpha\,\left(-d-e^{d/\lambda}\,(d-2\,\lambda)-2\,\lambda\right)\,\lambda^2}{d+2\,ra}$

In[24]:= `Clear[α, λ, ra]`

In[25]:= $\textbf{D}\Big[-\frac{2\,e^{-\frac{d+ra}{\lambda}}\,\pi\,\alpha\,\left(-d-e^{d/\lambda}\,(d-2\,\lambda)-2\,\lambda\right)\,\lambda^2}{d+2\,ra},\ ra\Big]$

Out[25]= $\frac{2\,e^{-\frac{d+ra}{\lambda}}\,\pi\,\alpha\,\left(-d-e^{d/\lambda}\,(d-2\,\lambda)-2\,\lambda\right)\,\lambda}{d+2\,ra}+\frac{4\,e^{-\frac{d+ra}{\lambda}}\,\pi\,\alpha\,\left(-d-e^{d/\lambda}\,(d-2\,\lambda)-2\,\lambda\right)\,\lambda^2}{(d+2\,ra)^2}$

In[26]:= `Clear[α, λ, ra]`

In[27]:= $\textbf{Simplify}\Big[\frac{2\,e^{-\frac{d+ra}{\lambda}}\,\pi\,\alpha\,\left(-d-e^{d/\lambda}\,(d-2\,\lambda)-2\,\lambda\right)\,\lambda}{d+2\,ra}+\frac{4\,e^{-\frac{d+ra}{\lambda}}\,\pi\,\alpha\,\left(-d-e^{d/\lambda}\,(d-2\,\lambda)-2\,\lambda\right)\,\lambda^2}{(d+2\,ra)^2}\Big]$

Out[27]= $-\frac{2\,e^{-\frac{d+ra}{\lambda}}\,\pi\,\alpha\,\lambda\,\left(d+d\,e^{d/\lambda}+2\,\lambda-2\,e^{d/\lambda}\,\lambda\right)\,\left(d+2\,(ra+\lambda)\right)}{(d+2\,ra)^2}$

In[28]:= `Clear[α, λ, ra]`

In[29]:= $\textbf{Limit}\Big[-\frac{2\,e^{-\frac{d+ra}{\lambda}}\,\pi\,\alpha\,\lambda\,\left(d+d\,e^{d/\lambda}+2\,\lambda-2\,e^{d/\lambda}\,\lambda\right)\,\left(d+2\,(ra+\lambda)\right)}{(d+2\,ra)^2},\ \lambda\to\textbf{Infinity}\Big]$

In[30]:= $-\frac{2 d^3 \pi \alpha}{3 \left(d + 2 ra\right)^2}$

Out[30]= $-\frac{2 d^3 \pi \alpha}{3 \left(d + 2 ra\right)^2}$