Exercise 1.13. Prove that $Fib(n)$ is the closest integer to $\emptyset^n/\sqrt{5}$, where $\emptyset = (1+\sqrt{5})/2$. Hint: Let $\psi = (1-\sqrt{5})/2$. Use induction and the definition of the Fibonacci numbers (see section 1.2.2) to prove that $Fib(n) = (\phi^n - \psi^n)/\sqrt{5}$.

To prove $Fib(n)$ is the closest integer to $\emptyset^n/\sqrt{5}$, let's prove $Fib(n) = (\phi^n - \psi^n)/\sqrt{5}$ where $\psi = (1-\sqrt{5})/2$ by induction first.

Basis:

$Fib(0) = 0$, and

$(\phi^0 - \psi^0)/\sqrt{5} = 0$. So $Fib(n) = (\phi^n - \psi^n)/\sqrt{5}$ is true for $n = 1$.

Induction:

Assume $Fib(n) = (\phi^n - \psi^n)/\sqrt{5}$ holds (for some unspecified value of n). It must then be shown that $Fib(n+1) = (\phi^{n+1} - \psi^{n+1})/\sqrt{5}$ holds.

By definition:

$$Fib(n+1) = Fib(n) + Fib(n-1)$$

Using the induction hypothesis that $Fib(n) = (\phi^n - \psi^n)/\sqrt{5}$ holds, the right-hand side can be rewritten to:

$$\frac{\phi^n - \psi^n}{\sqrt{5}} + \frac{\phi^{n-1} - \psi^{n-1}}{\sqrt{5}} = \frac{\phi^n + \phi^{n-1} - \psi^n - \psi^{n-1}}{\sqrt{5}} = \frac{\phi^{n-1}(\phi+1) - \psi^{n-1}(\psi+1)}{\sqrt{5}}$$

Since $\emptyset = (1+\sqrt{5})/2$, so

$$\phi^2 = (\frac{1+\sqrt{5}}{2})^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2} = 1+\phi$$

The same for ψ,

$$\psi^2 = (\frac{1-\sqrt{5}}{2})^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2} = 1 + \frac{1-\sqrt{5}}{2} 1+\psi$$

So

$$Fib(n+1) = \frac{\phi^{n-1}(\phi+1) - \psi^{n-1}(\psi+1)}{\sqrt{5}}$$

$$= \frac{\phi^{n-1}(\phi^2) - \psi^{n-1}(\psi^2)}{\sqrt{5}} = \frac{\phi^{n+1} - \psi^{n+1}}{\sqrt{5}}$$

Thereby $Fib(n+1) = (\phi^{n+1} - \psi^{n+1})/\sqrt{5}$ holds.

Since both the basis and the inductive step have been proved, it has now been proved by mathematical induction that $Fib(n) = (\phi^n - \psi^n)/\sqrt{5}$ holds for all natural n.

Since $Fib(n) = (\phi^n - \psi^n)/\sqrt{5}$,

$$Fib(n) - \frac{\phi^n}{\sqrt{5}} = \frac{-\psi^n}{\sqrt{5}},$$

And

$$\left|\frac{-\psi^n}{\sqrt{5}}\right| < \frac{1}{2}\ for\ n \geq 0$$

So $Fib(n)$ is the closest integer to $\emptyset^n/\sqrt{5}$.

Q.E.D