Exercise 1.14. Draw the tree illustrating the process generated by the count-change procedure of section 1.2.2 in making change for 11 cents. What are the orders of growth of the space and number of steps used by this process as the amount to be changed increases?

Part 2

Take n as the amount of money, k as the number of the kinds of coins, and d_k as the first denomination of the k kinds of coins. $T(n,k)$is the mathematical function of the procedure `(cc n k)`.

The order of growth of the space is $\Phi(n)$.
Because the longest path in the tree is $k + n/d_{min}$ where d_{min} is the smallest denomination of all the kinds of coins.

The order of growth of the number of steps is $\Phi(n^5)$.
When $k = 1$

$T(n, 1) = 3n/d_1$ where $d_1$is the denomination of the coin.
For each iteration, three steps are calculated:

`(cc a 1)`
`(cc a 0)`
`(cc a 0)+(cc a-d1 1)` where a is the amount of money left.

So $T(n, 1) = 3n/d_1$, and the order of growth is $\Phi(n)$

When $k = 2$

$$T(n,2) = T(n - d_2, 2) + T(n, 1)$$
$$= T(n - 2d_2, 2) + T(n - d_2, 1) + T(n, 1)$$
$$= T\left(n - \left\lfloor\frac{n+1}{d_2}\right\rfloor d_2, 2\right) + T\left(n - \left\lfloor\frac{n}{d_2}\right\rfloor d_2, 1\right) + \cdots + T(n - d_2, 1) + T(n, 1)$$

Since $n - \left\lfloor\frac{n+1}{d_2}\right\rfloor d_2 < 0$, so $T\left(n - \left\lfloor\frac{n+1}{d_2}\right\rfloor d_2, 2\right) = 0$

$$T(n,2) = 0 + \sum_{i=0}^{\left\lfloor\frac{n}{d_2}\right\rfloor} T\left(\frac{n - id_2}{d_1}, 1\right)$$
$$= \sum_{i=0}^{\left\lfloor\frac{n}{d_2}\right\rfloor} \frac{3(n - id_2)}{d_1} = \frac{3\left(\left\lfloor\frac{n}{d_2}\right\rfloor + 1\right)(n + (n - \left\lfloor\frac{n}{d_2}\right\rfloor d_2))}{2}$$
$$= \frac{3\left(\left\lfloor\frac{n}{d_2}\right\rfloor + 1\right)(n + c)}{2} \quad where\ c \ge 0$$
$$= \frac{3\left(\left\lfloor\frac{n^2}{d_2}\right\rfloor + n + \left\lfloor\frac{cn}{d_2}\right\rfloor + c\right)}{2}$$

So the order of growth of $T(n,2)$ is $\Phi(n^2)$.

Similarly, when $k = 3$

$$T(n,3) = T(n - d_3, 3) + T(n, 2)$$

$$= T\left(n - \left\lfloor\frac{n+1}{d_3}\right\rfloor d_3, 3\right) + T\left(n - \left\lfloor\frac{n}{d_3}\right\rfloor d_3, 2\right) + \cdots + T(n - d_3, 2) + T(n, 2)$$

$$= 0 + \sum_{i=0}^{\left\lfloor\frac{n}{d_3}\right\rfloor} T(\frac{n - id_3}{d_3}, 2) = \frac{\left(\left\lfloor\frac{n}{d_3}\right\rfloor + 1\right)(T\left(\frac{n}{d_3}, 2\right) + T\left(\frac{n - \left\lfloor\frac{n}{d_3}\right\rfloor d_3}{d_3}, 2\right))}{2}$$

So the order of growth of $T(n, 3)$ is $\left(\left\lfloor\frac{n}{d_3}\right\rfloor + 1\right)\Phi(n^2) = \Phi(n^3)$.

When $k = 5$, the order of growth is $\Phi(n^5)$.

More generally, the order of growth of the number of steps is $\Phi(n^k)$, where k is number of kinds of coins.