The Shannon Switching and n-Cop Games

Based on the work of Kimberly Wood (2012)

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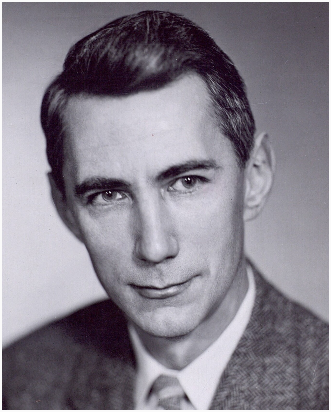
Introduction

- **Network Resilience:** How many backup routes does a power grid or communication network need to survive a coordinated attack?
- **Historical Context:**
  - Created by Claude Shannon (1951) to model electrical circuits.
  - Generalized to matroids and solved by Alfred Lehman (1964).



The Playing Field

Imagine a map of cities (points) and roads (lines).

- **Start and End:** We pick two special cities, u and v.
- **The Connection:** The whole game is about whether a traveler can get from u to v.
- **Complete Networks (K_m):** A map where *every* city has a direct road to every other city.

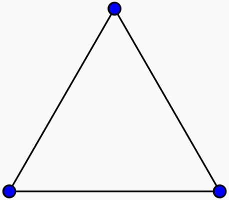


Figure 2: K_3

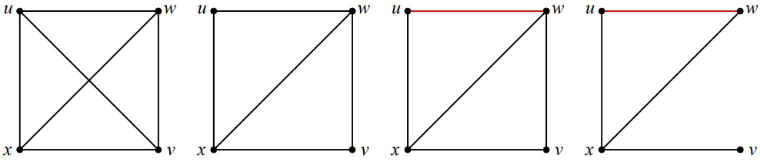
Rules of the Game

The Shannon Switching Game is a 1-on-1 strategy game:

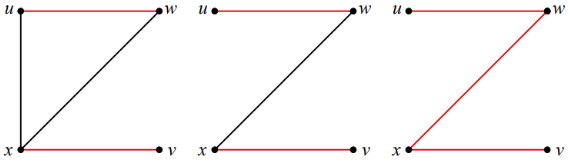
- **The Robber:** Wants to build a path between u and v. They "claim" one road per turn.
- **The Cop:** Wants to block the robber. They "delete" one road per turn from the map forever.

Outcome: The Robber wins if they complete a path. The Cop wins if u and v are permanently disconnected.

Example on K_4



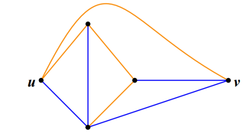
(a) Graph (b) Cop's first turn (c) Robber's first turn (d) Cop's second turn



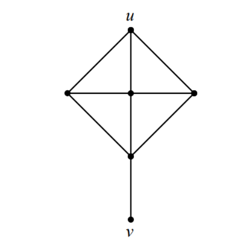
(e) Robber's second turn (f) Cop's third turn (g) Robber's third turn

Who Wins?

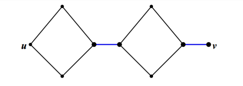
Look at these three different graph structures. Can you identify which one favors the Cop and which favors the Robber?



Graph A



Graph B

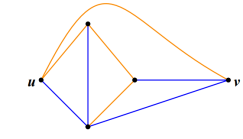


Graph C

Categorizing Game States

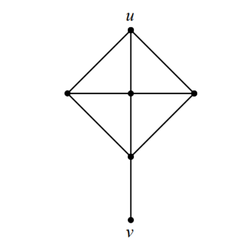
Before calculating results, we classify how 'safe' a graph is:

Positive



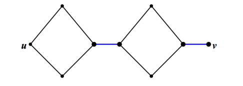
Robber wins regardless of who starts.
Property: High redundancy (two disjoint spanning trees).

Neutral



Winner is determined by who moves first.

Negative



Cop wins regardless of who starts.
Property: Contains "bottlenecks."

Formalizing the Game

We define the game as (G, u, v), where $G = (V, E)$.

Definition 1 (Recursive Win Conditions)

- **Base Case:** *If $u = v$, the game is* **positive***.*
- **Non-Negative:** *If there exists an edge e such that contracting it (merging its two endpoints) creates a positive game.*
- **Positive:** *If for* **any** *n edges the cop deletes, the remaining graph is still a non-negative game.*

If the Cop gets n moves, the graph must be dense enough to survive any n deletions.

The n-Cop Game

The Question: In a complete network K_m, how many cities (m) are required to guarantee a Robber win against a Cop who deletes n roads per turn?

This "threshold" size is our function:

$$\phi(n) = \min\{m \mid K_m \text{ is a positive game against } n \text{ cops}\}$$

Main Result: Bounds on $\phi(n)$

Kimberly Wood proved that the "tipping point" size falls in this range:

Theorem 1 (Wood, 2012)

For $n \geq 1$ cops:

$$n + 4 \leq \phi(n) \leq 2n^2 + n + 1$$

- **Implication:** We now have a guaranteed "safety limit" for complete networks.
- **The Gap:** As n grows, the upper bound grows quadratically, while the lower bound is linear.

Lower Bound: Why K_{n+3} is Not Enough

Result: $\phi(n) > n + 3$

The Cop's Isolation Strategy:

- **Turn 1:** Cop deletes the direct road (u, v).
- **Starvation:** With n deletions per turn, the Cop can systematically delete roads connected to whichever city the Robber moves to.
- **Outcome:** In a graph this small, the Robber runs out of "neighboring cities" before they can reach the target v.

Upper Bound: Why K_{2n^2+n+1} is Guaranteed

Result: $\phi(n) \leq 2n^2 + n + 1$

The Vertex Counting Argument:

- **Robber's Greedy Strategy:** Always move to a new, "unspoiled" city.
- **The Math of Plenty:** Each turn, the Cop only "spoils" n cities. Even after n turns, the sheer number of vertices $(2n^2 + n + 1)$ ensures the Robber still has safe paths.
- **Turn $n + 2$:** By this turn, the Robber is mathematically guaranteed to find a path through the remaining unblocked cities.

Refining the Result

n (Cops)	Lower Bound ($n+4$)	Upper Bound ($2n^2+n+1$)
1	5*	4
2	6	11
3	7	22

*Note: For $n = 1$, the actual value is 4.

Conjecture 1

Wood conjectures $\phi(2) \leq 9$ and $\phi(3) \leq 14$.
My observation: For $n = 2$, the threshold appears to be exactly $\phi(2) = 7$.

- This suggests $\phi(n)$ might be linear, meaning networks are even more resilient than the upper bound suggests.
- Can the properties of the Shannon Switching Game be generalized to the n-Cop game?
- Further research involves studying the game on complete bipartite graphs $K_{m,m}$.