Determinants: An Intuitive Approach

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Preface

The Determinant is a concept which is usually taught to undergraduate students in their introductory Linear Algebra classes . As a current university student myself, what has become increasingly clear to me, is the lack of intuitive understanding that most students hold on the concept of the determinant. Most students, including myself, are introduced to the concept of the determinant in a manner that is purely abstract. Students are often told that the determinant can be thought of as a value that is calculated by following a given set of rules, and if they are fortunate they may also learn some applications along the way. In most cases these rules are given to the student with no derivation or intuition presented, which, in my opinion, reduces the students ability to not only retain, but to also appreciate what they have just learnt. Despite the obvious power and utility of abstraction in mathematics, it is my opinion that an initial intuitive approach to a new topic is preferred, for it allows the student to appreciate the mathematical reasoning and experimentation in developing a new concept. Without this, the student may begin to view certain topics as being too esoteric and chose to avoid them entirely. What this paper aims to do is present the reader with a clear and methodical development on the concept of the determinant. I must give credit the excellent book *Higher Algebra* by Aleksandr Kurosh, which has motivated many of the concepts and explanations that I shall present.

1 Systems of Two Linear Equations with Two Unknowns

Let us first consider the general system of equations

$$a_1x + b_1y = c_1, \tag{1}$$
$$a_2x + b_2y = c_2. \tag{2}$$

Suppose we now wish to solve this system of equations. Then, multiplying equation (1) by b_2, equation (2) by b_1, and taking the difference of the resulting equations, we get

$$a_1b_2x - a_2b_1x = b_2c_1 - b_1c_2.$$

Factoring out the x term gives us

$$x(a_1b_2 - a_2b_1) = b_2c_1 - b_1c_2, \tag{3}$$

which yields the solution

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}. \tag{4}$$

Similarly, we are able to obtain the solution

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \tag{5}$$

It is important to note here that we presupposed

$$a_1b_2 - a_2b_1 \neq 0.$$

Clearly, this value must non-zero if we are to require a solution. Hence, we call this value the *determinant*, for it helps us determine our solution as well as the necessary conditions for having a solution. In particular, we say that this value is a *second-order determinant*, since we originally had a system of equations with two unknowns. To quickly compute this value we first construct a coefficient matrix of the original system of equations.

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

It is now obvious that if we find the product of the terms in the principal diagonal and subtract from it the product of the terms in the secondary diagonal* we obtain the value for the determinant. We use vertical bars to denote this value. That is,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1. \tag{6}$$

It is also worth noting that the numerator in (4) can easily be obtained by finding the determinant when replacing the first column in (6) by c_1 and c_2 respectively. Similarly, for the solution to y in (5), the numerator is simply the determinant obtained after replacing the the second column in (6) by c_1 and c_2 respectively. So, in terms of determinants, we can state our solutions as follows:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \tag{7}$$

and

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{8}$$

*The principal diagonal is considered to be all the entries that start at the top left most corner and go down to the bottom right most corner in a diagonal orientation. The secondary diagonal consists of all terms that start at the top right most corner and go down to the bottom left most corner in a diagonal orientation.

2 Systems of Three Linear Equations with Three Unknowns

Suppose now we wish to solve a system of three unknowns with three variables.

$$a_1x + b_1y + c_1z = d_1, \tag{1}$$
$$a_2x + b_2y + c_2z = d_2, \tag{2}$$
$$a_3x + b_3y + c_3z = d_3. \tag{3}$$

Finding the solutions to this system of equations is not as straight forward as the system considered in the previous section. Nevertheless, with some algebraic work, we can find the solutions. Let us first consider (1) and (2) without (3). So, we have

$$a_1x + b_1y + c_1z = d_1, \tag{4}$$
$$a_2x + b_2y + c_2z = d_2. \tag{5}$$

Transposing a_1x in (4) and a_2x in (5) gives us

$$b_1y + c_1z = d_1 - a_1x, \tag{6}$$
$$b_2y + c_2z = d_2 - a_2x. \tag{7}$$

If we now let

$$d_1 - a_1x = m, \tag{8}$$

and

$$d_2 - a_2x = n, \tag{9}$$

we get

$$b_1y + c_1z = m,$$
$$b_2y + c_2z = n.$$

From the previous section we can solve this system of equations which gives the solutions

$$y = \frac{c_2m - c_1n}{b_1c_2 - b_2c_1},$$

and

$$z = \frac{b_1n - b_2m}{b_1c_2 - b_2c_1}.$$

Substituting these values for y and z into (3), we get

$$a_3x + b_3\left(\frac{c_2m - c_1n}{b_1c_2 - b_2c_1}\right) + c_3\frac{b_1n - b_2m}{b_1c_2 - b_2c_1} = d_3$$
$$a_3x + \frac{b_3(c_2m - c_1n)}{b_1c_2 - b_2c_1} + \frac{c_3(b_1n - b_2m)}{b_1c_2 - b_2c_1} = d_3.$$

Multiplying the a_3x term by $\frac{b_1c_2 - b_2c_1}{b_1c_2 - b_2c_1}$, we get

$$\frac{a_3x(b_1c_2 - b_2c_1)}{b_1c_2 - b_2c_1} + \frac{b_3(c_2m - c_1n)}{b_1c_2 - b_2c_1} + \frac{c_3(b_1n - b_2m)}{b_1c_2 - b_2c_1} = d_3$$
$$\frac{a_3x(b_1c_2 - b_2c_1) + b_3(c_2m - c_1n) + c_3(b_1n - b_2m)}{b_1c_2 - b_2c_1} = d_3$$
$$\frac{a_3b_1c_2x - a_3b_2c_1x + b_3c_2m - b_3c_1n + b_1c_3n - b_2c_3m}{b_1c_2 - b_2c_1} = d_3.$$

If we now replace m and n for their respective expressions defined in (8) and (9), and expand then simplify all terms, we obtain

$$x(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2) = \\ d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_3b_2c_1 - d_2b_1c_3 - d_1b_3c_2.$$

Finally, dividing both sides be the coefficient of x, we are able to obtain the solution

$$x = \frac{d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_3b_2c_1 - d_2b_1c_3 - d_1b_3c_2}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2}.$$

By applying the same method above, we can also find that

$$y = \frac{a_1d_2c_3 + a_2d_3c_1 + a_3d_1c_2 - a_3d_2c_1 - a_2d_1c_3 - a_1d_3c_2}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2},$$

and

$$z = \frac{a_1b_2d_3 + a_2b_3d_1 + a_3b_1d_2 - a_3b_2d_1 - a_2b_1d_3 - a_1b_3d_2}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2}.$$

Once again we should notice this solution requires

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2 \neq 0. \tag{10}$$

Hence, we call this expression the *third-order determinant*, for it helps us determine the solutions to the original system of equations as well the conditions for having solutions. Once again we denote this value using vertical bars as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2. \tag{11}$$

This expression seems to be quite involved with no obvious subtleties. However, if we construct a coefficient matrix we can quickly notice some patterns.

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}. \tag{12}$$

We immediately notice that the first term in (10) is obtained by taking the product of all the terms in the principal diagonal. The second and third terms are obtained by first taking the product of the terms that lie parallel to the main diagonal with a third factor being included from the opposite corner*. Similarly, the negative terms in (10) are obtained in a near identical manner but relative to the secondary diagonal.

Once again we notice that the numerator in the solution for x is the determinant obtained after replacing the first column of (11) by d_1, d_2, and d_3 respectively. A similar result holds for the solutions to y and z. So, our solutions in terms of determinants are:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

At this point we should notice the similarities in obtaining solutions for linear systems with two equations and two unknowns, and three equations and three unknowns. In both cases we encountered expressions that were

*One of the parallels to the main diagonal will include the terms a_2 and b_3. The opposite corner of this parallel will be the term c_1. Similarly, another parallel to the main diagonal will have the terms b_1 and c_2 with a_3 in the opposite corner of great importance in finding solutions. We called these expressions *determinants*. Furthermore, in both cases, solutions could easily be found by replacing the appropriate columns in the determinant. This does seem to imply that similar results could be found for systems with four equations and four unknowns, and even to higher order systems. However, one thing that became apparently clear is that in going from a system with two equations and two unknowns to a system with three equations and three unknowns, the amount of work that went into finding a solution increased dramatically. Therefore, if we wish to generalise these results to linear systems of a higher order we should seek to develop alternative means to finding determinants.

3 n-th Order Determinants

As we have mentioned, the previous method employed for finding determinants of order higher than three would be incredibly tedious. Further, this method does not encompass one of the great powers of mathematics, that is the power of generalisation. An appropriate response to this is to re-define the process of finding a determinant and then seek to prove that such a definition still shares all the properties of our previous definition.

We begin by attempting to notice any common features in computing determinants of order two and three. Let us remind ourselves what these values are.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1,$$

and

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

Due to the involved nature of calculating these determinants, the first thing that becomes apparently obvious is that none of the terms in either determinant contain factors that are in the same row, or the same column. This seems to be a good starting point in establishing a new method in finding determinants. Further, we notice that any possible combination of terms in different rows and columns appears in the determinant. For example in the third order determinant we can pick the factor b_1 in the first row and second column, c_2 in the second row and third column, and a_3 from the third row and first column. Clearly, all these factors lie in different rows and columns from each other, and the product of these factors appears in the determinant, that is $a_3b_1c_2$. From this, let us propose the following as a new method finding determinants of order n.

We first pick a term in the first row. Let us suppose it is in the j-th column. Next, we pick another term in the second row that is not in the j-th column, say it is in the k-th column. Once more, we pick a term in the third row that is not in the j-th or k-th column. We keep doing this until we have reached the final row of the determinant. In the first row, we had n possible terms to choose. Then, for the second row, there is one less option since we *eliminate* a column. Therefore, in the second row there are $n-1$ options. So for now, we have $n \times (n-1)$ possible terms. Similarly, for the third row there are $n-2$ options. Therefore we now have $n \times (n-1) \times (n-2)$ possible terms. Doing this for all rows, we find that we have $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ possible terms. This simply evaluates to the factorial of n. So, the determinant will have $n!$ terms.

With this new definition we are easily able to determine the terms in a determinant of any order with no reliance on tedious algebra. However, what we still have not found is a method of determining the signs of the terms in the determinant, that is, whether they are positive or negative.

Let us make a table to map the rows and columns of each factor in a term of the determinant. We shall structure the table such that the rows are in ascending order. Let us first consider the first term in a third-order determinant, $a_1b_2c_3$. The first factor, a_1 lies in the first row and the first column. So, we write down the row as 1, and the column as 1 in the below table. Similarly, b_2 lies in the second row and the second column, and we likewise note it down in the below table. We do this again with the final term, c_3, to complete the table.

Row	1	2	3
Column	1	2	3

Table 1: Row and Columns of factors in $a_1b_2c_3$

Something we should notice is that the bottom row of the table has its numbers expressed in order. We should keep note of this.

Let us now proceed to the next term in a third-order determinant, that is, the term $a_2b_3c_1$. The first factor, a_2 lies in the second row and the first column. So, we write down the row as 2, and the column as 1 in the below table. The second factor, b_3 lies in the third row and the second column. So, we write a 3 for the row, and a 2 for the column. Doing this for the final term, c_1, we write a 1 for the row and a 3 for the column.

Row	1	2	3
Column	3	1	2

Table 2: Row and Columns of factors in $a_2b_3c_1$

We should now notice that the bottom row is unordered. To re-order this row in the table let us repeatedly perform swaps between the terms. To make things clearer, let us isolate the bottom row.

$$\begin{pmatrix} 3 & 1 & 2 \end{pmatrix}.$$

Swapping the term '1' with the term '3', we get

$$\begin{pmatrix} 1 & 3 & 2 \end{pmatrix}.$$

Performing one final swap between the term '2' and the term '3', we get

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}.$$

Here, we should stop and make important note of the fact that we performed a total of two swaps to reorder the column numbers in ascending order.

Let us now look at the final term in the third-order determinant, that is, $a_1b_3c_2$. Constructing the a table for the rows and columns of its factors, we get

Row	1	2	3
Column	1	3	2

Table 3: Row and Columns of factors in $a_2b_3c_1$

To reorder the row that represents the columns, we only need to perform a single swap, that is the '2' term and the '3' term.

If we repeat this process of constructing tables and keeping count of how many swaps must be made to attain order order in the bottom row of each table, then we notice the following. All positive terms require an even number of swaps, while all negative terms require an odd number of swaps. With this, we can now complete our new process for finding a determinant.

Determinant: Each term in an n-th order determinant is obtained by taking the product of entries into a matrix such that no two factors share the same row or column. Further, each term must have a factor from each row and each column. The determinant contains *all* possible such terms. There are $n!$ such terms. To find the sign of each terms we construct a table with row numbers in ascending order and their corresponding column numbers for each factor in each term. The number of swaps needed to order the column numbers in ascending order is noted. If the number of swaps is even, then the term is positive, otherwise, if it is odd, the term is negative. After determining all the terms and their signs, the determinant is obtained by summing all the terms with their respective signs*.

This is a good time to stop and consider what we have done. We first considered linear systems of equations and found an important value that revealed solutions to the system as well as some conditions for solutions. We called this term the *determinant*. It then became clearly obvious that calculating this value for higher order linear systems would become increasingly tedious and algebraically involved. Hence we sought to redefine the process in finding determinants such that its value for second and third order linear systems remained the same yet was much less involved. What we must now seek to show is that such a definition still gives us solutions and conditions for linear systems of higher order. However, before we can do this, we must first establish some important properties with our new definition of the determinant.

*This method of finding the determinant is technically known as the *Leibniz Formula*. We have simplified much of the technical language in our definition.

4 Properties of the Determinant.

With our newly established method of finding the determinant, we can seek to discover some properties associated with this value, in the hope of allowing us to demonstrate its connection with the solutions and conditions of a linear system. We should note that it will be useful if we change our current method of denoting the elements of a matrix by a letter and a single subscript. If we instead denote all the elements of a matrix with the same letter, but use a two digit subscript to differentiate them, we will find this of great benefit in later work. It is common to write the first digit of the subscript as the row the element is in, and the second digit of the subscript as the column the element is in.

Let us now introduce the concept of taking the *transpose* of a matrix. This is a matrix that is obtained by interchanging the rows for the columns (or vice-versa) of a given matrix. For example, if we consider a general matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

Then the transpose of A is

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}.$$

We should note that in taking the transpose, the digits in the subscript of each element is swapped. For example, a_{12} became a_{21}. So, in the transpose the first digit of the subscript represents the column, while the second represents the row. Let us now prove the following properties of the determinant.

Property 1: The determinant of a matrix is equal to the determinant of its transpose matrix.

Proof:
Consider some term in the determinant of A. Let this term have the general form

$$a_{1i}a_{2j}\cdots a_{nk} \tag{1}$$

where $i \neq j \neq \cdots \neq n$ and $1 \leq i, j, \cdots, k \leq n$. If we take each of the factors in (1) and map it to its corresponding position in A^T, then since $i \neq j \neq \cdots \neq n$, each factor lies in a unique row in A^T. Also, since $1 \neq 2 \neq \cdots \neq n$, each factor also lies in a unique column in A^T. Therefore, each term in the determinant of A is also a term in the determinant A^T. Now we must prove that the sign of each term in is the same in A as it is in A^T.

Let us construct a table so that we can obtain the sign of the term (1).

Row	1	2	.	.	.	n
Column	i	j	.	.	.	k

Table 1: Row and Columns of some term in A

Suppose that i, j, $\cdots$,k are not in ascending order. Now, let us make some swaps such that i, j, and k are in ascending order. This time however, whenever we swap any term representing a column, we shall also swap the corresponding term representing the row value to keep track of the changes being made. For example, if we must swap i and k, we shall also swap the values 1 and n. Without loss of generality, suppose that after making m swaps to achieve order in the bottom row (the row in the table that represents the columns), we get the following table.

Row	2	n	.	.	.	1
Column	j	k	.	.	.	i

Table 2: Row and Columns of some term in A after making m swaps

If we again consider the same term, however this time in A^T, we will get the following table.

However, there is a slight problem with our current construction. The issue is, we do not know if it is necessarily true that $i = 1$, $j = 2$, $\cdots$, $k = n$. Therefore, the row values may not actually be in ascending order

Row	i	j	.	.	.	k
Column	1	2	.	.	.	n

Table 3: Incorrect table for row and columns of some term in A^T

as we require. So, we must first make some swaps with the row values to get them in ascending order before we can proceed. In table 1 we made m swaps to achieve order with the values $i, j, \cdots, k$, and at the same time we also kept track of the corresponding row values. Therefore, if we make the exact same swaps, also keeping track of the corresponding values in the bottom row, we can easily reconstruct our table for the term in A^T. Doing so, we get the following table.

Row	j	k	.	.	.	i
Column	2	n	.	.	.	1

Table 4: Row and Columns of some term in A^T

Now that the top row is in ascending order, we can go about ordering the bottom row to determine the sign of the term. We saw in table 3 that we did actually have ascending order in the bottom row. However, we had to make m swaps to achieve ascending order in the top row, and in doing so we destroyed the order in the bottom row. So, all we must do to achieve order in the bottom row is reverse the number of swaps initially made. So, to achieve order in the bottom row, we must make a total of m swaps.

Therefore, for any term, in both tables, for A and A^T, we must make m swaps. So the sign of any term in A is the same in A^T. Therefore,

$$|A| = |A^T|.$$

From this, it is perfectly clear that any future property we establish in terms of the rows of a determinant, will also hold true for the columns of the determinant. Therefore, for brevity, we shall only prove certain properties in terms of the rows of a determinant, keeping in mind that the same must obviously hold true in terms of the columns.

Property 2: If one of the rows or columns of a matrix consists entirely of zeros, then the determinant is zero.

Proof:

Suppose that there exists a matrix with a row consisting entirely of zeros. Then, since each term in the determinant must contain a factor from each row, every term will have a factor of zero. Therefore, each term in the determinant will evaluate to zero meaning that the determinant ultimately evaluates to zero.

Property 3: Interchanging the rows or columns of a matrix will reverse the sign of the determinant.

Proof:

Consider some general matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

After swapping the i-th and j-th row, we obtain the following matrix.

$$A' = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Let us consider some term in the determinant of A,

$$a_{1\alpha}a_{2\beta}\cdots a_{ik}\cdots a_{jl}\cdots a_{n\gamma} \tag{2}$$

where $\alpha \neq \beta \neq \cdots \neq k \neq \cdots \neq l \neq \cdots\gamma$ and $1 \leq \alpha, \beta, \cdots, k, \cdots, l, \cdots, \gamma \leq n$. Then, every factor in (2) obviously exists in A'. Further, every term in (2) also lies in a different row and different column in A'. Therefore, every term in A is also in A'. Now let us find the sign of (2) in A and A'.

Constructing a table for the rows and columns of each factor in (2) we get the following.

Row	1	2	.	.	.	i	.	.	.	j	.	.	.	n
Column	α	β	.	.	.	k	.	.	.	l	.	.	.	γ

Table 1: Table for row and columns for some term in A

Let us now suppose that to achieve ascending order for the column numbers, we must make m swaps. Now, let us construct a similar table for A'. We get,

Row	1	2	.	.	.	j	.	.	.	i	.	.	.	n
Column	α	β	.	.	.	l	.	.	.	k	.	.	.	γ

Table 2: Table for row and columns for some term in A'

If we swap the terms j and i and their corresponding column numbers, we will get an identical result as table 1. Then to achieve order we must likewise make m swaps. So, in total, we must make $m + 1$ swaps. Therefore, if m is even $m + 1$ is odd. Conversely, if m is odd, then $m + 1$ is even. So, every term in the determinant of A is present in A' but with opposite sign. So, in general,

$$|A| = -|A'|.$$

Property 4: If a matrix contains two identical rows or columns, the determinant is zero.

Proof:

Let there be some matrix A with two identical rows. After swapping the two identical rows, we get another matrix, say A'. But obviously, both matrices are the same since we swapped two identical rows. So,

$$|A| = |A'|. \tag{3}$$

However, from property 3, since we made a swap between two rows in a matrix, we must also have

$$|A| = -|A'|. \tag{4}$$

Adding (3) and (4), we get

$$2|A| = 0.$$

From this, we can easily deduces that $|A|$ must be zero.

Property 5: If all the elements in some row or column of a matrix are multiples of some number k, then then the determinant of the matrix is itself a multiple of k

Proof:

Consider some matrix A where all the terms of the i-th row are a multiple of some number k.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Then, since each term in the determinant will contain a factor from each row, each term will have a factor k. So, in general, the determinant will be something like this.

$$|A| = \underbrace{\pm(a_{1j}a_{2k}\cdots ka_{il}\cdots ka_{nm}) \pm \cdots \pm (a_{1\alpha}a_{2\beta}\cdots ka_{i\gamma}\cdots ka_{n\delta})}_{n \text{ terms}} \tag{5}$$

Factoring out the k in each term in the determinant, we get

$$|A| = k(\pm a_{1j}a_{2k}\cdots a_{il}\cdots a_{nm} \pm \cdots \pm a_{1\alpha}a_{2\beta}\cdots a_{i\gamma}\cdots a_{n\delta}) \tag{6}$$

If it was not already clear, we can now quite obviously see that the determinant is a multiple of k. While the proof is technically complete at this point, we can continue to find some further subtleties with our current construction.

Every term inside the parentheses in (6) is identical to its corresponding term in (5), expect there is a missing factor of k. To be more exact, every element in the i-th row loses its factor k. For instance, in going from (5) to (6), ka_{il} turns into a_{il}, and $ka_{i\gamma}$ turns into $a_{i\gamma}$. From this we can easily deduce that the expression in parentheses in (6) is obtained by taking the determinant of $|A|$ with the i-th row having no terms with a factor k. So, in general

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \pm a_{1j}a_{2k}\cdots a_{il}\cdots a_{nm} \pm \cdots \pm a_{1\alpha}a_{2\beta}\cdots a_{i\gamma}\cdots a_{n\delta}$$

With this, we can express our determinant as follows:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

The value of this corollary cannot be understated for in the computation of determinants we will undoubtedly reduce the amount of arithmetic done with large numbers.

Property 6: If a matrix contains two proportional rows or columns, the determinant is zero.

Proof:

Consider a matrix A with two proportional rows. Suppose that every term in i-th row is identical to the corresponding term in the j-th except it contains a single factor k. Then, from the previous property, we can *factor* out the k term from the i-th row when calculating the determinant. However, in doing so, the i-th and j-th become identical, and so from property 4, the determinant must be zero.

Property 7: If a row or column in a matrix has all its elements expressed as a sum of of two terms, then the determinant of the matrix can be expressed as the sum of two determinants. The first determinant is obtained from an identical matrix to the original, expect its i-th row or column is replaced with the first summand of each respective element in the first matrix. Likewise, second determinant is obtained from an identical matrix to the original, expect its i-th row or column is replaced with the second summand of each respective element in the first matrix.

Proof:
Consider a matrix A where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{i1}+c_{i1} & b_{i2}+c_{i2} & \cdots & b_{in}+c_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Let us take some general term in the determinant of A.

$$a_{1j}a_{2k}\cdots(b_{il}+c_{il})\cdots a_{nm}. \tag{7}$$

Expanding, we can re-write this term as

$$a_{1j}a_{2k}\cdots b_{il}\cdots a_nm + a_{1j}a_{2k}\cdots c_{il}\cdots a_{nm}. \tag{8}$$

We should notice that the first summand in (8) is identical to (7) except that $b_{il}+c_{il}$ has now become b_{il}. Similarly, the second summand in (8) is identical to (7) except that $b_{il}+c_{il}$ has now become c_{il}. From this, we can deduce that the first summand in (8) is a term in the determinant of a matrix identical to A, expect the i-th row is replaced with the first summand of each respective element. A similar result follows for the second summand in (8). Since we considered a general term of the determinant of A, then this must be true for *all* the terms in the determinant. We therefore arrive at the following property:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{i1}+c_{i1} & b_{i2}+c_{i2} & \cdots & b_{in}+c_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

This property is also extended to the case that each element in the i-th row is the sum of more than two terms. A similar proof as the one given above can be used to verify this.

Property 8: If one of the rows in a matrix is a linear-combination of other rows or columns, then the determinant of the matrix is zero.

Proof:
We should note here that what we mean by a *linear-combination* of a row is that each element in the given row can be expressed as a scaled sum of corresponding terms from other rows. For instance, if we have an $n \times n$ matrix where every term in the i-th row can be expressed as

$$a_{ij} = k_1 a_{\alpha j} + k_2 a_{\beta j} + \cdots + k_n a_{\gamma l},$$

then the i-th row is said to be a *linear-combination* of the other rows. We should note that $i \neq \alpha, \beta, \cdots, \gamma$.

Let us now suppose that the i-th row of an $n \times n$ matrix A is a linear-combination of say, m other rows, where $1 \leq m \leq n-1$. Then each element in the i-th row will be a sum with m summands. So, from the previous property, the determinant of A can be expressed as the sum of m determinants. However, each determinant will clearly have one of its rows proportional to another. From property 6, each of these determinants will evaluate to zero. Therefore, the determinant of A will be zero.

Property 9: Adding the elements of one row or column of a matrix to another multiplied by some scalar k will not change the value of the determinant.

Proof:

Consider some matrix A where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Now suppose that the i-th row has the j-th row added to it multiplied by some number k. So, we have

$$A' = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1}+ka_{j1} & a_{i2}+ka_{j2} & \cdots & a_{in}+ka_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Then, from property 7, we have

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1}+ka_{j1} & a_{i2}+ka_{j2} & \cdots & a_{in}+ka_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ ka_{j1} & ka_{j2} & \cdots & ka_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

However, the second summand has one row proportional to another, namely, the i-th row is proportional to the j-th. So, from property 6, the second summand will evaluate to zero. Therefore,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1}+ka_{j1} & a_{i2}+ka_{j2} & \cdots & a_{in}+ka_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Therefore, in adding to one row of a matrix another row multiplied by some number, the determinant remains unchanged.

At this point, we have obtained a number of useful properties of the determinant. With these properties we are able to simplify the process of computing some determinants. We also hope that such properties may aid us in showing that our new definition of the determinant is still linked to systems of linear equations.

5 Sub-matrices, Minors, and Co-factors

With our new definition of the determinant, we were able to simplify the computation that we were initially required to do. Further, we were able to derive some seemingly useful properties that may be of use for us later on. However, this new method is still quite cumbersome and has not yet revealed itself to be linked to systems of linear equations in any way. We should remind ourselves that this goal is what we ultimately seek to accomplish.

With our new definition of the determinant, in going from finding the determinant of an $n \times n$ to an $(n+1) \times (n+1)$ matrix, we get $(n+1)$ times more terms. So, for a 3×3 determinant there are 6 total terms, whereas for a 4×4 determinant there are 24 total terms. The point here is that in considering a slightly more complex determinant, the amount of work that needs to be done increases dramatically. It is for this reason that we seek to express the determinant of a matrix in terms of the determinants of its *sub-matrices*. For example, if we have a 4×4 matrix, then perhaps we should try and find its determinant in terms of its 3×3 sub-matrices, which we could then seek to express in terms of its 2×2 sub-matrices. With this thought, we can proceed in our investigation.

Let us consider a 5×5 matrix A, and let us suppose that we are required to find its determinant.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix}.$$

To consider the sub-matrices of A, we will *choose* an equal number of rows and columns and consider the elements that intersect them. For example, choosing the first, third, and fifth rows as well as the second, third, and fourth columns, we get the sub-matrix

$$\begin{pmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{52} & a_{53} & a_{54} \end{pmatrix}.$$

We shall call the determinant of this sub-matrix a *minor* of order 3. Alternatively, if we *strike out* the elements in the chosen rows and columns, then what remains is also a square matrix. Taking these elements we get a second sub-matrix, namely,

$$\begin{pmatrix} a_{21} & a_{25} \\ a_{45} & a_{45} \end{pmatrix}.$$

We call the determinant of this sub-matrix a *complementary-minor* of order 2.

Now it would be a beautiful result if the determinant of the minor multiplied by the determinant of the complementary-minor were equal to the determinant of the initial matrix*. Unfortunately, this is not the case. We should however still investigate this thought and see if we learn something of any value. Let us consider a general 3×3 matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

We have already seen that the determinant of this matrix is

$$a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}. \tag{1}$$

Let us take the elements that intersect the first and second rows and columns to get the sub-matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The determinant of this matrix, that is the minor, is

$$a_{11}a_{22} - a_{12}a_{21}.$$

If we now strike out the first and second rows and columns, we get the sub-matrix

$$\begin{pmatrix} a_{33} \end{pmatrix}$$

The determinant of this matrix is quite obviously just

$$a_{33}.$$

*We are ironically reminded of the disappointment in discovering that $\cos(x+y)$ does not equal $\cos(x) + \cos(y)$, or that $(a+b)^2$ does not equal $a^2 + b^2$.

Multiplying these two determinants we get

$$a_{33}(a_{11}a_{22} - a_{12}a_{21}) = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33}. \tag{2}$$

We should immediately notice that the result of multiplying the minor and complementary-minor gives terms that are actually present in the determinant (1). That is, the first term in (2) is the first term in (1), while the second term in (2) is the fifth term in (1). So we could speculate that multiplying a minor with its respective complementary-minor of a matrix will give terms that are in the determinant of the initial matrix. However, as is the case with scientific inquiries, we should repeat our investigations to reveal any possible misconceptions.

Let us this time choose the first and third rows, and the second and third columns. We get the sub-matrix

$$\begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix}.$$

Therefore the minor is, as we have already seen,

$$a_{12}a_{33} - a_{13}a_{32}.$$

If we now strike out the first and third rows, and the second and third columns, we get the sub-matrix

$$\begin{pmatrix} a_{21} \end{pmatrix}.$$

With this, the complementary-minor is

$$a_{21}.$$

Multiplying the minor and complementary-minor we get

$$a_{21}(a_{12}a_{33} - a_{13}a_{32}) = a_{12}a_{21}a_{33} - a_{13}a_{21}a_{32}. \tag{3}$$

This result gives terms that are *almost* like the terms in (1). The difference is our terms have opposite sign. For instance, the first term in (3) has opposite sign to the fifth term in (1) while the second term in (3) has opposite sign to the second term in (1). We could get (3) to contain the exact terms in (1) with correct sign if our complementary-minor were $-a_{21}$ and not a_{21}. However, there currently seems to be no intuitive reason for taking the complementary-minor with a positive or negative sign. That is, we do not know in advance if we should invert the sign of the complementary-minor or not.

When we chose the first and second rows and columns for the minor we did not need to invert the sign of the complementary-minor to get terms in the actual determinant. However, when choosing the first and third rows, and the second and third columns, we did need to invert the sign of the complementary-minor. We may be lead to believe if we choose the same number rows and columns then we do not need to invert the sign of the complementary-minor, otherwise we do. However, this is not necessarily the case. Let us take one more example to illustrate this, this time, choosing only one row and one column to get the minor.

If we choose the first row and third column, we get the sub-matrix

$$\begin{pmatrix} a_{13}. \end{pmatrix}$$

With this, the minor is

$$a_{13}.$$

Striking out the first row and third column, we get the sub-matrix

$$\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32}. \end{pmatrix}$$

From this, we find the complementary-minor to be

$$a_{21}a_{32} - a_{22}a_{31}.$$

Multiplying the minor with its respective complementary-minor, we get

$$a_{13}(a_{21}a_{32} - a_{22}a_{31}) = a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}. \tag{4}$$

Clearly, the terms in (4) are included in the determinant in (1), with no need to invert the sign of the complementary-minor. However, we chose a row number that was different the column number, namely, 1 and 3 respectively. In hope of revealing some clues, let us write down the chosen rows and columns, and whether we needed to invert the sign of the complementary-minor or not to get terms present in the determinant.

Chosen rows	Chosen columns	Invert sign of complementary-minor
1, 2	1, 2	NO
1, 3	2, 3	YES
1	3	NO

Let us take the sum of the chosen rows and columns and see if we notice anything. In the first case we chose the first and second rows and columns. So,

$$1 + 2 + 1 + 2 = 6.$$

In the second case we chose the first and third rows, and the second and third columns. So,

$$1 + 3 + 2 + 3 = 9.$$

In the third case we chose the first and third rows and columns respectively. So,

$$1 + 3 = 4.$$

It seems as though when the sum of the rows and columns are even, we do not need to invert the sign of the complementary-minor. Conversely, when the sum of the rows and columns are odd, we do need to invert the sign of the complementary-minor. However, this fact has great numeric value. We know that negative one raised to any even power will become positive one, while negative one raised to any odd power remains as negative one. So, if we denote the minor by M, the complementary-minor by M', and s as the sum of the chosen rows and columns to get the minor, we can speculate that

$$(-1)^s MM'$$

will contain terms that are present in the determinant of the initial matrix. We call the value $(-1^s)M'$ the *co-factor* of M.

Despite the experimental nature in which we proposed this result, a formal proof is still required.

Proof:

We shall first consider a particular case and then prove the general case. Let us consider an $n \times n$ matrix and let us choose the first k rows and the first k columns to create our minor.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} & a_{1(k+1)} & a_{1(k+2)} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2k} & a_{2(k+1)} & a_{2(k+2)} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} & a_{k(k+1)} & a_{k(k+2)} & \cdots & a_{kn} \\ a_{(k+1)1} & a_{(k+1)2} & \cdots & a_{(k+1)k} & a_{(k+1)(k+1)} & a_{(k+1)(k+2)} & \cdots & a_{(k+1)n} \\ a_{(k+2)1} & a_{(k+2)2} & \cdots & a_{(k+2)k} & a_{(k+2)(k+1)} & a_{(k+2)(k+2)} & \cdots & a_{(k+2)n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} & a_{n(k+1)} & a_{n(k+2)} & \cdots & a_{nn} \end{pmatrix}$$

The sub-matrix enclosed in the top left corner will be used for a minor M, while the sub-matrix in the bottom right corner will be used for the corresponding complementary-minor M'. Let us consider some general term in the minor M.

$$a_{1i}a_{2j} \cdots a_{kl}. \tag{5}$$

Clearly, $1 \leq i, j, \cdots l \leq k$ and $i \neq j \neq \cdots \neq l$. Further, the sign of this term in the minor depends on the number of swaps that must be made to achieve ascending order for the column numbers when constructing our table. Let the number of swaps be m. If the number of swaps are even, the sign is positive, otherwise, the sign is negative. So, the sign of (5) is simply $(-1)^m$.

Let us now consider some general term in the complementary-minor M'.

$$a_{(k+1)\alpha}a_{(k+2)\beta} \cdots a_{(n\gamma)}. \tag{6}$$

We will have $(k+1) \leq \alpha, \beta, \gamma \leq n$ and $\alpha \neq \beta \neq \cdots \neq \gamma$. Additionally, the sign of this term in the complementary-minor depends on the number of swaps that need to be made for the column numbers to be in ascending order when constructing our table. Again, if the number of swaps are even, the sign is positive, otherwise, the sign is negative. Let the number of swaps be m'. So, the sign of (6) is simply $(-1)^{m'}$.

Let us not forget that we intend to take the product of a minor with its *co-factor* and not its complementary-minor. So, our co-factor is

$$(-1)^s M'.$$

The value of s is simply the sum of the chosen rows and columns. So,

$$s = 1 + 2 + \cdots + k + 1 + 2 + \cdots + k = 2(1 + 2 + \cdots + k).$$

Since s is a factor of two s itself must be even. So, negative one raised to the power of s evaluates to one, which means our co-factor is simply M'. Therefore, the product of our minor M with its corresponding co-factor evaluates to

$$MM'.$$

Now, if we multiply the minor M by its corresponding co-factor M', the result will obviously include a term that is simply the product of (5) and (6). Let us consider this product below.

$$a_{1i}a_{2j}\cdots a_{kl}a_{(k+1)\alpha}a_{(k+2)\beta}\cdots a_{(n\gamma)}. \tag{7}$$

Every factor in this term lies in a different row and a different column. If this is not obvious, we see that the row numbers are

$$1, 2, \cdots, k, (k+1), (k+2), \cdots, n.$$

Clearly, all this values are unique. Additionally, the column numbers are

$$i, j, \cdots, l, \alpha, \beta, \cdots, \gamma.$$

Now we know that $i \neq k \neq \cdots \neq l$ and that $\alpha \neq \beta \neq \cdots \neq \gamma$. But what if some column in the first group is equal to the second. That is, what if for instance $j = \gamma$. With little contemplation, we can reject this thought for we saw that $1 \leq i, j, \cdots l \leq k$ and $(k+1) \leq \alpha, \beta, \gamma \leq n$. If we had $j = \gamma$ then we would have $j \leq k$ while $j \geq (k+1)$, which is an obvious contradiction. So, every factor in (7) lies in a different row and column. Further, (7) contains a factor in *each* row and column of A. Therefore, (7) is a term in the determinant of A.

We must now show the sign of (7) obtained by taking the product of (5) and (6) is the same as the sign of (7) in the determinant of A. The sign of (7) after taking the product of (5) and (6) is

$$(-1)^m \times (-1)^{m'} = (-1)^{m+m'}.$$

If we now consider the sign of (7) in the determinant of A, we must first construct the following table.

Row	1	2	.	.	.	k	$k+1$	$k+2$	.	.	.	n
Column	i	j	.	.	.	l	α	β	.	.	.	γ

Now, if we must make, say, p swaps to achieve ascending order for the column values, then the sign of (7) in the determinant of A is simply $(-1)^p$. We do however already know that

$$i, j, \cdots, l < \alpha, \beta, \cdots, \gamma.$$

If we consider the first half of the column values to be $i, j, \cdots, l$, and the second half of the column values to be $\alpha, \beta, \cdots, \gamma$, then it is clear that we will never have to swap any value in the first half with some other value in the second half. So, we can first make swaps solely in the first half to get it into ascending order, and then make swaps solely in the second half to get it into ascending order. Making our swaps in these two distinct stages will achieve overall ascending order. However, we already know to achieve ascending order in the first half, we must make m swaps. Further, to achieve ascending order in the second half, we must make m' swaps. So, the total number of swaps that must be made is $m + m'$. This means that $p = m + m'$ which means that the sign of (7) in the determinant of A is

$$(-1)^{m+m'}.$$

Therefore, the sign of (7) in the product of (5) and (6) is the same when taken in the determinant of A. So, for this particular case, we can say that taking the product of a minor M with its corresponding co-factor M', produces terms that are present in the determinant of the initial matrix.

With this, we can now proceed to the general case. Suppose that instead of picking the first k rows and columns to get a sub-matrix for a minor in A, we choose the rows $i_1, i_2, i_3, \cdots, i_k$, and the columns $j_1, j_2, \cdots, j_k$, where

$$i_1 < i_2 < \cdots < i_k,$$

and

$$j_1 < j_2 < \cdots < j_k.$$

In hope of making use of the result obtained for the particular case, let us make swaps in the matrix A such that i_1-th row becomes the first row, i_2-th becomes the second row, $\cdots$, and i_k-th becomes the k-th row. Similarly, let us make swaps such that that j_1-th becomes the first column, j_2-th becomes the second column, $\cdots$, and j_k-th becomes the k-th column. Now, the i_1-th row is $i_1 - 1$ rows beneath the first row. So, to get the i_1-th row to become the first row, we must make $i_1 - 1$ consecutive swaps. Similarly, to get the i_2-th row to be the second row, we must make $i_2 - 2$ consecutive swaps. In general, to get the i_n-th row to become the n-th row, we must make $i_n - n$ consecutive swaps, and to get the j_n-th column to become the n-th column, we must make $j_n - n$ consecutive swaps. So, in total we make

$$(i_1 - 1) + (i_2 - 2) + \cdots + (i_k - k) + (j_1 - 1) + (j_2 - 2) + \cdots + (j_k - k) = \\ (i_1 + i_2 + \cdots + i_k + j_1 + j_2 + \cdots + j_k) - 2(1 + 2 + \cdots + k)$$

swaps in our matrix A to get the chosen rows and columns to be in the upper left hand corner. Let us call this newly obtained matrix A'.

Now we have our chosen sub-matrix for the minor in the upper left hand corner of A', just like we did for the particular case. We should note, that the both the matrices chosen for the minor and the complementary-minor in A' are the same in the original matrix A, for the elements in the chosen rows and columns did not change. Let us call the minor M, and the complementary-minor M'. From the result obtained in our particular case, we can conclude that the product MM' gives terms in the determinant of A'. However, we do not yet know if the product of the minor with its co-factor in A produces terms present in the determinant of A.

If we let

$$s = (i_1 + i_2 + \cdots + i_k + j_1 + j_2 + \cdots + j_k),$$

then the total number of swaps made is

$$s - 2(1 + 2 + \cdots + k).$$

Since s represents the chosen rows and columns, then the product of the minor M with its corresponding co-factor M' in A evaluates to

$$(-1)^s MM'.$$

Let us consider two distinct cases, the first when s is even, and the second when s is odd.

Case 1:

Suppose that s is even. Then, we can let $s = 2l$, where l is some positive integer. So, the total number of swaps made is

$$2l - 2(1 + 2 + \cdots + k) = 2(l - 1 + 2 + \cdots + k).$$

Now l is either equal to, or greater than $1 + 2 + \cdots + k$. So, either zero swaps have been made, or some even positive number of swaps have been made. From property 3 of the preceding section, we saw that swapping a row or column of a matrix inverts the sign of its determinant. However, because we have made either zero, or an even number of swaps, the overall sign is not inverted. So,

$$|A| = |A'|.$$

Now, the product of the minor M with its co-factor $(-1)^s M'$ in A evaluates to

$$MM'$$

since s is even. However, this is identical to the product of the minor M with its corresponding co-factor M' in A'. So, the product of M with its co-factor M' in A gives terms present in the determinant of A'. But we just proved that the determinant of A is equal to the determinant of A'. Therefore, we can conclude that the product of a minor in A multiplied by its corresponding co-factor gives terms present in the determinant of A.

Case 2:

Suppose that s is odd. Then, we can let $s = 2l + 1$, where l is some positive integer. So, the total number of swaps made is

$$2l + 1 - 2(1 + 2 + \cdots + k) = 2(l - 1 + 2 + \cdots + k) + 1.$$

The value in parentheses is either zero, or some positive number, meaning that the total number of swaps made is some positive odd number. Once again, from property 3 of the preceding section, swapping any rows or columns of a matrix inverts the sign of its determinant. Since we make an odd number of swaps, the sign of the determinant is ultimately inverted. So,

$$|A| = -|A'|.$$

So, any term in the determinant of A has its sign inverted in the determinant of A' and vice-versa. Therefore, since the product MM' in A' gives terms in the determinant of A', $-MM'$ gives a term in the determinant of A. Now, the product of the minor M with its co-factor $(-1)^s M'$ in A evaluates to

$$-MM'$$

since s is odd. But we have already deduced that this product gives terms present in the determinant of A. Therefore, the product of a minor in A with its corresponding co-factor gives terms present in A.

So in any case, the product of a minor M with its corresponding co-factor $(-1)^s M'$ in A gives terms present in the determinant of A, regardless of whether s is odd or even. Thus, we have proved our hypotheses true for the general case and can conclude the following:

The product of any chosen minor in a matrix A with its corresponding co-factor, gives terms that are present in the determinant of A.

Thus far, we have made some considerable progress. We have not only proved some important properties of the determinant, but also discovered that terms in the determinant can be obtained by breaking up a matrix into smaller sub-matrices. If we can further refine this result, then it will be of undoubted benefit to us. However, we are yet to show the link that our new definition of determinants have with linear systems of equations. While this may seem discouraging, we are getting very close to establishing this link.

6 Evaluating Determinants in Terms of Their Minors and Co-Factors

With our recent discoveries regarding the minor and its co-factor, we can make use of the following simple, yet effective trick.

Suppose we have an $n \times n$ matrix A, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Now suppose that we choose some row or column in A. Let us say we choose the i-th row. Then, we can choose any single column in A to get a 1×1 sub-matrix that can used to find a minor of A. If we choose all the columns in the matrix respectively, we will get n minors, all along the i-th row. Let us note the minors along the i-th row.

$$a_{i1}, a_{i2}, \cdots, a_{in}.$$

Now, let us consider the co-factors corresponding to each minor along the i-th row. We shall denote the co-factor of a_{ij} by A_{ij}. We know that the product of any minor with its corresponding co-factor gives terms present in the determinant of the matrix. So, $a_{i1}A_{i1}$ gives terms in the determinant of A. Likewise, so does $a_{i2}A_{i2}$, and so do all the remaining products of minors with their co-factors. Let us now take the sum of the products of each minor with its respective co-factor along the i-th row. So, we have

$$a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}.$$

Now, $a_{i1}A_{i1}$ will have the factor a_{i1} in all of its terms. However, no other term in the above sum will contain this factor, since neither their minors nor their co-factors contain the element a_{i1}. Likewise, $a_{i2}A_{i2}$ will have the factor a_{i2} in all of its terms, however no other term in the above sum will contain this factor. This fact will extend to all other terms in the above sum. Therefore, the above sum will not contain any duplicate terms.

Let us now consider how many total terms are in the above sum when expanded. Each co-factor is obtained from an $(n-1) \times (n-1)$ matrix. Therefore, each co-factor will have $(n-1)!$ terms, meaning the product of each minor with its co-factor will have $(n-1)!$ terms. The sum contains n total minor co-factor products. Therefore, there are a total of $n(n-1)!$ terms, or by simplifying, $n!$ total terms. However, the determinant of A also contains $n!$ total terms, and the above sum gives terms in the determinant with no duplicates. Thus the above sum simply evaluates to the determinant of A. By a similar procedure, we can show that doing the same for the i-th *column* of A will produce the same result.

What we have just discovered is called a *co-factor* expansion of a matrix. With this, we can find an $n \times n$ determinant in terms of $(n-1) \times (n-1)$ determinants. Further, if we notice that a row or column in a matrix consists of simple numbers, say zeros and ones, then we can simplify calculations by using a co-factor expansion about the given row or column. This result also provides us with an alternative method of proving property 2 of determinants.

7 Determinants and Systems of Linear Equations

At this point, we have built enough theory to try and link the determinant with systems of linear equations. Without further ado, let us proceed with this final section.

Let us make note of one important fact about co-factor expansion that will be of great value to us in our attempt to reveal the link determinants have with systems of linear equations.

Let us consider the expression

$$a_{1k}A_{1j} + a_{2k}A_{2j} + \cdots + a_{nk}A_{nj} \tag{1}$$

where $j \neq k$. This looks similar to the co-factor expansion of some matrix, however it seems as the though the minors are taken from a different column than the co-factors. That is, the minors are taken from the k-th column, while the co-factors are taken from the j-th column. So, perhaps our expression is obtained from the following matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1k} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2k} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nk} & \cdots & a_{nn} \end{pmatrix}.$$

If we replace all the terms in j-th column with the corresponding terms in the k-th column, then we will get some other matrix A', where

$$A' = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} & \cdots & a_{1k} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2k} & \cdots & a_{2k} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} & \cdots & a_{nk} & \cdots & a_{nn} \end{pmatrix}.$$

Now, the value of (1) in both A and A' is the same, for in replacing the j-th column with the k-th column, none of the co-factors actually change. That is, the co-factors along j-th column in both A and A' are the same. Since A' contains two identical rows, its determinant must be zero. If we now perform a co-factor expansion along the j-th column of A', we will get

$$a_{1k}A_{1j} + a_{2k}A_{2j} + \cdots + a_{nk}A_{nj} = |A'|.$$

But we just deduced that the determinant of A' must be zero. So,

$$a_{1k}A_{1j} + a_{2k}A_{2j} + \cdots + a_{nk}A_{nj} = 0.$$

Therefore, in general, if $j \neq k$, then

$$a_{1k}A_{1j} + a_{2k}A_{2j} + \cdots + a_{nk}A_{nj} = 0. \tag{2}$$

Let us now consider some system of linear equations with n unknowns and n equations. Let us denote the unknowns with $x_1, x_2, \cdots, x_n$. So,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1i}x_i + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2i}x_i + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{ni}x_i + \cdots + a_{nn}x_n &= b_n. \end{aligned}$$

Let us also construct a coefficient matrix for this system of linear equations.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} & \cdots & a_{nn} \end{pmatrix}.$$

Let us multiply the first equation by the co-factor A_{1i}, the second equation by the co-factor A_{2i}, $\cdots$, and the n-th equation by the co-factor A_{ni}. So, we get the following equivalent linear system:

$$\begin{aligned} A_{1j}(a_{11}x_1 + a_{12}x_2 + \cdots a_{1i}x_i + \cdots + a_{1n}x_n) &= b_1 A_{1i}, \\ A_{2i}(a_{21}x_1 + a_{22}x_2 + \cdots a_{2i}x_i + \cdots + a_{2n}x_n) &= b_2 A_{2i}, \\ &\vdots \\ A_{ni}(a_{n1}x_1 + a_{n2}x_2 + \cdots a_{ni}x_i + \cdots + a_{nn}x_n) &= b_n A_{ni}. \end{aligned}$$

Expanding, our linear system becomes

$$\begin{aligned} a_{11}A_{1i}x_1 + a_{12}A_{1i}x_2 + \cdots a_{1i}A_{1i}x_i + \cdots + a_{1n}A_{1i}x_n &= b_1 A_{1i}, \\ a_{21}A_{2i}x_1 + a_{22}A_{2i}x_2 + \cdots a_{2i}A_{2i}x_i + \cdots + a_{2n}A_{2i}x_n &= b_1 A_{1i}, \\ &\vdots \\ a_{n1}A_{ni}x_1 + a_{n2}A_{ni}x_2 + \cdots a_{ni}A_{ni}x_i + \cdots + a_{nn}A_{ni}x_n &= b_n A_{ni}. \end{aligned}$$

Let us now add all the equations in our system together to get the equation

$$\begin{aligned} (a_{11}A_{1i} + a_{21}A_{2i} + \cdots + a_{n1}A_{ni})x_1 + (a_{12}A_{1i} + a_{22}A_{2i}x_2 + \cdots + a_{n2}A_{ni})x_2 + \cdots + (a_{1i}A_{1i} + a_{2i}A_{2i} + \cdots + a_{ni}A_{ni})x_i \\ + \cdots + (a_{1n}A_{1i} + a_{2n}A_{2i} + \cdots + a_{n2}A_{ni})x_n = b_1A_{1i} + b_2A_{2i} + \cdots + b_nA_{ni}. \end{aligned}$$

Now, the coefficient of x_i is the coefficient expansion of the matrix A along the i-th row. Thus, the coefficient of x_i is the determinant of A. However, from (3), all the coefficients of the remaining variables are zero. Now, the right hand side of the equation is simply the coefficient expansion about the i-th column after replacing $a_{1i}, a_{2i}, \cdots, a_{ni}$ with $b_1, b_2, \cdots, b_n$ respectively. So, if we let

$$A_i = \begin{pmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & b_2 & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & \cdots & a_{nn} \end{pmatrix},$$

then we will have

$$|A|x_i = |A_i|.$$

Now, suppose that we do not have all the b values as zero. When this is the case, our system of linear equations is said to be *non-homogeneous.* By solving for x_i we get

$$x_i = \frac{|A_i|}{|A|}.$$

Now, clearly $|A|$ cannot be zero if we require solutions. So, if we have a non-homogeneous system of linear equations, then if the determinant of our co-efficient matrix is zero, we have no solutions. In other words, our system is inconsistent.

What if our system of linear equations are *homogeneous*, that is all the b values are zero. Then, the value of $|A_i|$ will be zero for there will be a column consisting entirely of zeros. So, we will have

$$|A|x_i = 0.$$

Now, if $|A| \neq 0$, then x_i will be the trivial solution, that is, $x_i = 0$. However, what if $|A| = 0$? Then, x_i can take on *any* value. That is, there will be an infinite number of solutions. We can summarise our findings in the following statements.

Suppose we have a system of n linear equations with n unknowns of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n. \end{aligned}$$

If the system is *non-homogeneous* then the system will have solutions if, and only if the determinant of the coefficient matrix is non-zero. Further, the solutions can be found for x_i by we take the quotient of the determinant of the matrix with the determinant of a new matrix obtained by replacing the i-th column with $b_1, b_2, \cdots, b_n$ respectively.

If the system is *homogeneous* then the system will have trivial solutions if, and only if the determinant of the coefficient matrix is zero. Otherwise, there will be an *infinite* number of solutions.

8 Conclusion

I hope that this paper has given you a better and more fundamental understanding of the determinant. I have, where possible, tried to present the determinant in the least esoteric way possible. I have tried to show that the determinant is not, and should not be treated as some numeric value that has no apparent insight or motivation. I hope that this will paper will motivate the reader to increase the depth of their mathematical knowledge and change the way they think about mathematics.

To whoever had the patience and fortitude to have made it to this point of the paper, I would like to say thank you. I eagerly await any criticisms and possible improvements that can be made.