The Hanging Picture Problem

Daniel Morrison

March 19, 2018

Problem Description

Suppose we want to hang a picture using 2 nails and some string attached to the corners of the picture. However, we want it hung in a specific way so that

1. When both nails are in the wall the picture is held up
2. When a single nail is removed, the picture falls

Problem Description

Suppose we want to hang a picture using 2 nails and some string attached to the corners of the picture. However, we want it hung in a specific way so that

1. When both nails are in the wall the picture is held up
2. When a single nail is removed, the picture falls

Is this possible? If so, how can this be done?

Problem Description

Suppose we want to hang a picture using 2 nails and some string attached to the corners of the picture. However, we want it hung in a specific way so that

1. When both nails are in the wall the picture is held up
2. When a single nail is removed, the picture falls

Is this possible? If so, how can this be done?

What if there are 3 nails? 4 nails? k nails?

Solving the Problem

Try it Yourself!

Method of Solving

1. Introduction to Algebraic Topology
2. Application of AT to Nail Problem
3. Algebra of the Nail Problem
4. Writing Solutions

Intro to Algebraic Topology

The difficult part of this problem is describing how the string is wrapped around the nails. Algebraic Topology is a tool for precisely this task. It has two uses for us:

Intro to Algebraic Topology

The difficult part of this problem is describing how the string is wrapped around the nails. Algebraic Topology is a tool for precisely this task. It has two uses for us:

- Determining when two string configurations are "the same"

The difficult part of this problem is describing how the string is wrapped around the nails. Algebraic Topology is a tool for precisely this task. It has two uses for us:

- Determining when two string configurations are "the same"
- Describing the distinct configurations

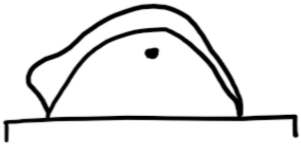
"Sameness" in Algebraic Topology

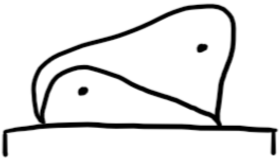
Definition

Two paths in a space are equivalent if one can be continuously deformed into the other.

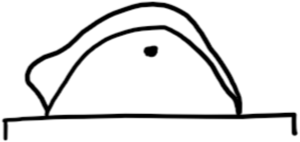
Figure: Equivalent paths & Non-equivalent paths

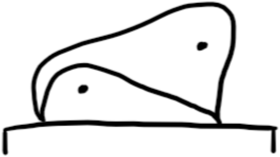
Why are Equivalent Paths Useful?





Why are Equivalent Paths Useful?





If two paths are equivalent, then they have the same result in the Hanging Picture Problem. If not, then they have different results.

Describing Different Paths

- In our problem, paths can wrap around a nail clockwise, counter-clockwise, or not at all.

Describing Different Paths

- In our problem, paths can wrap around a nail clockwise, counter-clockwise, or not at all.
- Say we denote each nail as $n_1, n_2, \ldots, n_k$.

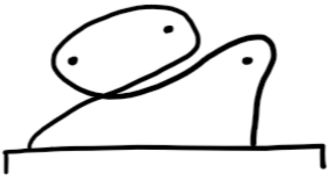
Describing Different Paths

- In our problem, paths can wrap around a nail clockwise, counter-clockwise, or not at all.
- Say we denote each nail as $n_1, n_2, \ldots, n_k$.
- We can describe the first case as n_i and the second as n_i^{-1}.

Figure: n_i and n_i^{-1}

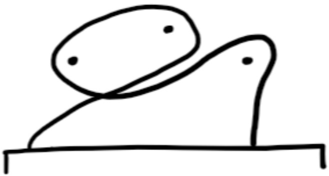
Combining the Notation

Paths in general can wrap around nails any number of times. We write these paths by starting on the left end of the path, and writing down what nails it loops around in order.



Combining the Notation

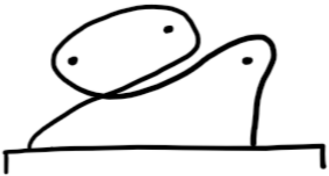
Paths in general can wrap around nails any number of times. We write these paths by starting on the left end of the path, and writing down what nails it loops around in order.



$$n_2^{-1} n_1^{-1} n_3$$

Combining the Notation

Paths in general can wrap around nails any number of times. We write these paths by starting on the left end of the path, and writing down what nails it loops around in order.



$$n_2^{-1} n_1^{-1} n_3$$

Note: If the path wraps around no nails, then we denote it e. Notice $n_i n_i^{-1} = e$.

Applying the Notation to the Hanging Picture Problem

When will the picture fall?

What happens when we remove the nail n_i?

Applying the Notation to the Hanging Picture Problem

When will the picture fall?

- When the path is e.

What happens when we remove the nail n_i?

Applying the Notation to the Hanging Picture Problem

When will the picture fall?

- When the path is e.

What happens when we remove the nail n_i?

- n_i is deleted from the path.

Applying the Notation to the Hanging Picture Problem

When will the picture fall?

- When the path is e.

What happens when we remove the nail n_i?

- n_i is deleted from the path.

So the solution to the problem is a path that is not equal to e, but when we delete <u>any</u> nail the path becomes e.

Writing Solutions: $k = 1, 2$

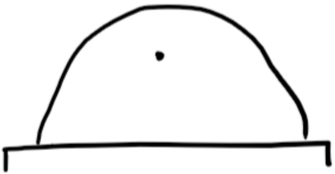
We start with the smallest cases and build up:

Writing Solutions: $k = 1, 2$

We start with the smallest cases and build up:

$k = 1$:

$S_1 = n_1$

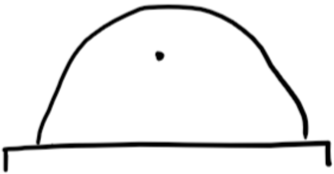


Writing Solutions: $k = 1, 2$

We start with the smallest cases and build up:

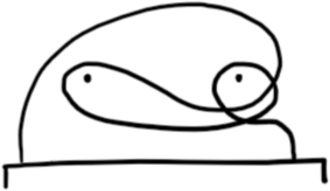
$k = 1$:

$S_1 = n_1$



$k = 2$:

$S_2 = n_1 n_2 n_1^{-1} n_2^{-1}$



Writing Solutions: $k = 3, \ldots$

$k = 3$:
$S_3 = n_3 S_2 n_3^{-1} S_2^{-1}$

Writing Solutions: $k = 3, \ldots$

$k = 3$:
$S_3 = n_3 S_2 n_3^{-1} S_2^{-1}$

$S_3 = n_3(n_1 n_2 n_1^{-1} n_2^{-1}) n_3^{-1} (n_2 n_1 n_2^{-1} n_1^{-1})$

Writing Solutions: $k = 3, \ldots$

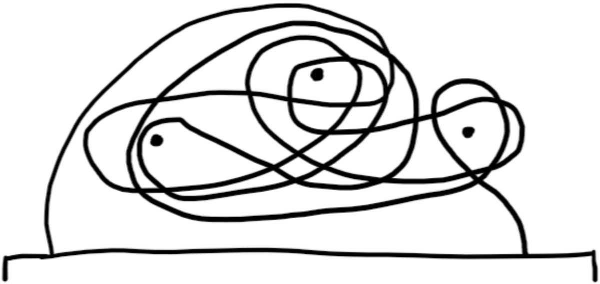
$k = 3$:
$S_3 = n_3 S_2 n_3^{-1} S_2^{-1}$

$S_3 = n_3 (n_1 n_2 n_1^{-1} n_2^{-1}) n_3^{-1} (n_2 n_1 n_2^{-1} n_1^{-1})$

Generalizing this pattern,

$$S_1 = n_1$$
$$S_k = n_k S_{k-1} n_k^{-1} S_{k-1}^{-1} \qquad k > 1$$

Illustrating $k = 3$



Further Work

- Understand the math underlying Algebraic Topology

Further Work

- Understand the math underlying Algebraic Topology
- Improve images of solutions

Further Work

- Understand the math underlying Algebraic Topology
- Improve images of solutions
- Are there shorter solutions?

Further Work

- Understand the math underlying Algebraic Topology
- Improve images of solutions
- Are there shorter solutions?
- Suppose we want the picture to fall after removing m of the k nails with $1 \leq m \leq k$. What do solutions look like now?