Homework 9

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Determine if the following series is convergent or divergent. You can use any tests you have learnt so far.

(a) $\sum_{n=1}^{\infty} \sin(n\frac{\pi}{2})$

$\lim_{n\to\infty} \sin\left(n\frac{\pi}{2}\right)$ does not exist. By divergence test, the series diverges.

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Let $f(x) = \dfrac{1}{x \ln^2 x}$.

Since $f(x) = \dfrac{1}{x \ln^2 x} > 0$ and $x \ln^2 x$ is monotonically increasing for $x \geq 2$, so its reciprocal $f(x) = \dfrac{1}{x \ln^2 x}$ is decreasing for $x \geq 2$.

By integral test:

$$\int_2^\infty \frac{1}{x(\ln x)^2}\,\mathrm{d}x = \lim_{n\to\infty} \int_2^n \frac{1}{x(\ln x)^2}\,\mathrm{d}x$$

$$\begin{aligned}
u = \ln x \quad &\implies \quad \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x \\
&\iff \quad \mathrm{d}x = x\,\mathrm{d}u \\
x = 2 \quad &\implies \quad u = \ln 2 \\
x = n \quad &\implies \quad u = \ln n
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty}\int_{\ln 2}^{\ln n}\frac{1}{x(\ln x)^2}\,\mathrm{d}x &= \lim_{n\to\infty}\int_{\ln 2}^{\ln n}\frac{x\,\mathrm{d}u}{xu^2}\\
&= \lim_{n\to\infty}\int_{\ln 2}^{\ln n} u^{-2}\,\mathrm{d}u\\
&= \lim_{n\to\infty}\left[-\frac{1}{u}\right]_{\ln 2}^{\ln n}\\
&= \lim_{n\to\infty} -\frac{1}{\ln n}+\frac{1}{\ln 2}\\
&= \frac{1}{\ln 2}
\end{aligned}$$

Since $\int_2^\infty \frac{1}{x(\ln x)^2}\,\mathrm{d}x = \frac{1}{\ln 2} < \infty$, the series is convergent.

(c) $\sum_{n=1}^\infty \frac{1}{n^{3/2}+1}$

Let $a_n = \frac{1}{n^{3/2}+1}$ and $b_n = \frac{1}{n^{3/2}}$.

Then,

$$\begin{aligned}
\frac{a_n}{b_n} &= \frac{n^{3/2}}{n^{3/2}+1}.\\
\therefore \lim_{n\to\infty}\frac{a_n}{b_n} &= 1
\end{aligned}$$

By limit comparison test, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ will either converge or diverge together.

Let $f(x) = \frac{1}{x^{3/2}}$.

Since $f(x) > 0$ and $f'(x) = -\frac{3}{2x^{5/2}} < 0$ for all $x \geq 1$, we apply the integral test.

$$\begin{aligned}\int_1^\infty \frac{1}{x^{3/2}}\,\mathrm{d}x &= \lim_{n\to\infty}\int_1^n x^{-3/2}\,\mathrm{d}x \\ &= \lim_{n\to\infty}\left[\frac{x^{-1/2}}{-1/2}\right]_1^n \\ &= \lim_{n\to\infty}\left[-\frac{2}{\sqrt{x}}\right]_1^n \\ &= \lim_{n\to\infty} -\frac{2}{\sqrt{n}}+2 \\ &= 2\end{aligned}$$

By integral test, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is convergent. Subsequently, by the limit comparison test, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}+1}$ must also be convergent.

(d) $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{(n+1)^2}$

Let $a_n=\dfrac{\sqrt{n}}{(n+1)^2}=\dfrac{n^{1/2}}{n^2+2n+1}$ and $b_n=\dfrac{1}{n^{3/2}}$.

Then,

$$\frac{a_n}{b_n}=\frac{n^2}{n^2+2n+1}.$$
$$\therefore \lim_{n\to\infty}\frac{a_n}{b_n}=1$$

Since $b_n=\dfrac{1}{n^{3/2}}$ is nonnegative and decreasing, we apply the integral test.

From (c), we concluded that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is convergent by integral test. Subsequently, by the limit comparison test, the series $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{(n+1)^2}$ must also be convergent.

(e) $\sum_{n=2}^{\infty}\frac{(\ln n)^2}{n}$

Since

$$\ln n = 1 \iff n = e,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}n} \ln n = \frac{1}{n} > 0, \forall n > 3$$

we have that

$$\ln n > 1, \forall n > 3.$$

As such, for all values $n > 3$:

$$\ln n > 1$$
$$\iff \ln^2 n > 1$$
$$\iff \frac{\ln^2 n}{n} > \frac{1}{n}.$$

Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent, by comparison test $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^2}{n}$ must also be divergent.

(f) $\sum_{n=1}^{\infty} \frac{n^2}{(n^2+10)^2}$

Let $a_n = \dfrac{n^2}{(n^2+10)^2} = \dfrac{n^2}{n^4 + 20n^2 + 100}$ and $b_n = \dfrac{1}{n^2}$.

Then,

$$\frac{a_n}{b_n} = \frac{n^4}{n^4 + 20n^2 + 100}$$
$$\therefore \lim_{n \to \infty} \frac{a_n}{b_n} = 1$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then by limit comparison $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(n^2+10)^2}$ also converges.

(g) $\sum_{n=1}^{\infty} \frac{n \ln n}{n^3+2}$

For sufficiently large n,

$$\frac{n \ln n}{n^3 + 2} \approx \frac{n}{n^3 + 2}$$

Let $a_n = \dfrac{n}{n^3+2}$ and $b_n = \dfrac{n^\epsilon}{n^2} = \dfrac{1}{n^{2-\epsilon}}$ for some small $\epsilon > 0$.

Then,

$$\frac{a_n}{b_n} = \frac{n^{3-\epsilon}}{n^3 + 2}$$
$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

For small values of $\epsilon > 0$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2-\epsilon}}$ converges. As such, by limit comparison test $\displaystyle\sum_{n=1}^{\infty} \frac{n \ln n}{n^3 + 2}$ converges.

(h) $\sum_{n=2}^{\infty} \frac{1}{n^2 (\ln n)^2}$

$$\ln e = 1 \implies \ln n > 1, \forall n > 3$$

So for $n > 3$:

$$\ln n > 1$$
$$\iff \ln^2 n > 1$$
$$\iff n^2 \ln^2 n > n^2$$
$$\iff \frac{1}{n^2 \ln^2 n} < \frac{1}{n^2}$$

Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, by comparison test $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2 (\ln n)^2}$ also converges.

Section 5.5

State whether each of the following series converges absolutely, conditionally, or not at all.

(251) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n} + 1}{\sqrt{n} + 3}$

Let $a_n = \dfrac{\sqrt{n} + 1}{\sqrt{n} + 3}$.

$$\lim_{n\to\infty} a_n = \frac{\sqrt{n} + 1}{\sqrt{n} + 3} = 1 \neq 0$$

By divergence test, the series diverges.

(252) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n+3}}$

Let $a_n = \frac{1}{\sqrt{n+3}}$.

Since $\sqrt{n+3}$ is monotonically increasing, its reciprocal $a_n = \frac{1}{\sqrt{n+3}}$ must subsequently be decreasing.

And

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+3}} = 0.$$

By alternating series test, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n+3}}$ is convergent.

Testing for absolute convergence

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{\sqrt{n+3}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+3}}$$

Let $f(x) = \frac{1}{\sqrt{x+3}}$.

We note that $f(x) > 0$ and $f'(x) < 0$ for $x \in [1, \infty)$.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{\sqrt{x+3}}\,\mathrm{d}x &= \lim_{n\to\infty}\int_1^n (x+3)^{-1/2}\,\mathrm{d}x \\
&= \lim_{n\to\infty}\left[2\sqrt{x+3}\right]_1^n \\
&= \lim_{n\to\infty} 2\sqrt{n+3} - 4 \\
&= \infty
\end{aligned}$$

By integral test, we find that the sum of the absolute values of the terms do not converge.

As such, we conclude that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n+3}}$ exhibits conditional convergence.

(260) $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2(1/n)$

Since $\sin x \leq x$ for $x > 0$. Then, where $a_n = \sin^2(1/n)$, we have that $0 \leq a_{n+1} \leq a_n$ for $n \geq 1$.

$$\lim_{n\to\infty}\sin^2(1/n) = \sin^2\left(\lim_{n\to\infty}\frac{1}{n}\right) = \sin^2 0 = 0$$

By alternating series test, the series is convergent.

Testing for absolute convergence

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\sin^2(1/n)\right| = \sum_{n=1}^{\infty}\sin^2(1/n)$$

For $n > 0$:

$$\begin{aligned} \sin n &\leq n \\ \iff \sin\frac{1}{n} &\leq \frac{1}{n} \\ \iff \sin^2\frac{1}{n} &\leq \frac{1}{n^2} \end{aligned}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, by comparison test $\sum_{n=1}^{\infty}\sin^2\frac{1}{n}$ also converges. As such, we conclude that $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2(1/n)$ exhibits absolute convergence.

(261) $\sum_{n=1}^{\infty}(-1)^{n+1}\cos^2(1/n)$

$$\lim_{n\to\infty}\cos^2(1/n) = \cos^2\left(\lim_{n\to\infty}\frac{1}{n}\right) = \cos^2 0 = 1 \neq 0$$

By divergence test, the series is divergent.