Homework 3

Mos Kullathon
921425216

1. Find the area of the bounded region enclosed by $y = x^2, x + y = 2$.

$$y = x^2 \qquad \Longrightarrow \qquad f(x) = x^2$$

$$x + y = 2 \qquad \Longrightarrow \qquad g(x) = 2 - x$$

   Finding intersections of f and g.

$$\begin{aligned} x^2 &= 2 - x \\ x^2 + x - 2 &= 0 \\ (x-1)(x+2) &= 0 \\ \therefore x &= -2, 1 \end{aligned}$$

   Since $g(x) \geq f(x)$ for $x \in [-2, 1]$, the area between the curves is:

$$\begin{aligned} \int_1^{-2} g(x) - f(x) \, \mathrm{d}x &= \int_1^{-2} (2 - x - x^2) \, \mathrm{d}x \\ &= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_1^{-2} \\ &= \frac{9}{2}. \end{aligned}$$

2. Find the area of the region enclosed by $y = \tan x$ and the x-axis over the interval $[-\pi/3, \pi/4]$.
   Since $\tan x \leq 0$ for $x = 2\pi n + \pi$ where $n \in \mathbb{Z}$, we find that for values of $x \in [-\frac{\pi}{3}, \frac{\pi}{4}]$:

$$\tan x \leq 0, \qquad x \in \left[-\frac{\pi}{3}, 0\right]$$

$$\tan x \geq 0, \qquad x \in \left[0, \frac{\pi}{4}\right]$$

   As such, the total area of $y = \tan x$ bounded by the x-axis over $[-\frac{\pi}{3}, \frac{\pi}{4}]$ is:

$$
\begin{aligned}
&-\int_{-\pi/3}^{0} \tan x \,\mathrm{d}x + \int_{0}^{\pi/4} \tan x \,\mathrm{d}x \\
&= -\Big[-\ln\cos x\Big]_{-\pi/3}^{0} + \Big[-\ln\cos x\Big]_{0}^{\pi/4} \\
&= \frac{3\ln 2}{2}.
\end{aligned}
$$

3. Find the area of the shaded region.

$$
\begin{aligned}
x = (y-1)^2 \quad &\implies \quad y = \sqrt{x} - 1 \\
x = 3 - y \quad &\implies \quad y = 3 - x \\
x = 2\sqrt{y} \quad &\implies \quad y = \frac{1}{4}x^2
\end{aligned}
$$

$\sqrt{x} - 1 \geq \frac{1}{4}x^2$ for $x \in [0,1]$. As such the area from $x = 0$ to $x = 1$ is:

$$\int_0^1 (\sqrt{x} + 1 - \frac{1}{4}x^2)\,\mathrm{d}x.$$

$3 - x \geq \frac{1}{4}x^2$ for $x \in [1,2]$. As such the area from $x = 1$ to $x = 2$ is:

$$\int_1^2 (3 - x - \frac{1}{4}x^2)\,\mathrm{d}x.$$

Therefore, the area A between the three functions from $x = 0$ to $x = 1$ is the sum of the two integrals.

$$
\begin{aligned}
A &= \int_0^1 (\sqrt{x} + 1 - \frac{1}{4}x^2)\,\mathrm{d}x + \int_1^2 (3 - x - \frac{1}{4}x^2)\,\mathrm{d}x \\
&= \left[\frac{2x^{3/2}}{3} + x - \frac{x^3}{12}\right]_0^1 + \left[3x - \frac{x^2}{2} - \frac{x^2}{12}\right]_1^2 \\
&= \frac{19}{12} + \frac{10}{3} - \frac{29}{12} \\
&= \frac{5}{2}
\end{aligned}
$$

4. Book Section 2.1: 2, 3, 12, 22, 23.

(2) Area between $y = x^2$ and $y = 3x + 4$.

$$x^2 = 3x + 4$$

$$x^2 - 3x - 4 = 0$$
$$(x-4)(x+1) = 0$$
$$\therefore x = -1, 4$$

$y = x^2$ and $y = 3x + 4$ intersects at $x = -1$ and $x = 4$, and $3x + 4 \geq x^2$ for all $x \in [-1, 4]$. Therefore,

$$\begin{aligned} A &= \int_{-1}^{4} 3x + 4 - x^2 \, \mathrm{d}x \\ &= \left[\frac{3x^2}{2} + 4x - \frac{x^3}{3}\right]_{-1}^{4} \\ &= \frac{125}{6}. \end{aligned}$$

(3) Area between $y = x^3$ and $y = x^2 + x$.

$$x^3 = x^2 + x$$
$$x^3 - x^2 - x = 0$$
$$\therefore x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

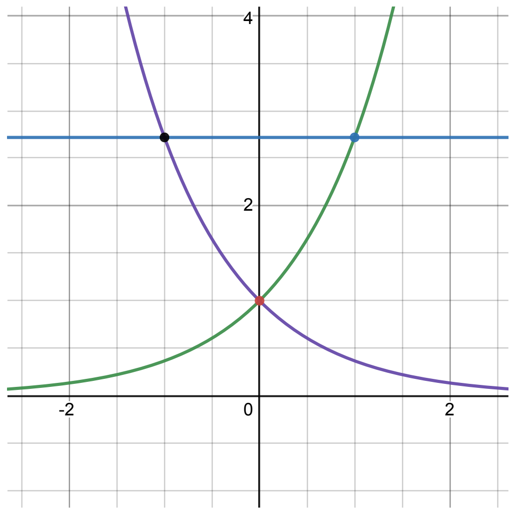
$y = x^3$ and $y = x^2 + x$ intersects at $x = \frac{1-\sqrt{5}}{2}$ and $x = \frac{1+\sqrt{5}}{2}$.

$0 \geq x^3 \geq x^2 + x$ for all $x \in [\frac{1-\sqrt{5}}{2}, 0]$ and $x^2 + x \geq x^3 \geq 0$ for all $x \in [0, \frac{1+\sqrt{5}}{2}]$. Therefore,

$$\begin{aligned} A &= \int_{\frac{1-\sqrt{5}}{2}}^{0} (x^3 - x^2 - x) \, \mathrm{d}x + \int_{0}^{\frac{1+\sqrt{5}}{2}} (x^2 + x - x^3) \, \mathrm{d}x \\ &= \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2}\right]_{\frac{1-\sqrt{5}}{2}}^{0} + \left[\frac{x^3}{3} + \frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{\frac{1+\sqrt{5}}{2}} \\ &= \frac{13}{12}. \end{aligned}$$

(12) Area between $y = e$, $y = e^x$, and $y = e^{-x}$.

$$e = e^x \implies x = 1$$
$$e = e^{-x} \implies x = -1$$
$$\therefore x = \pm 1$$



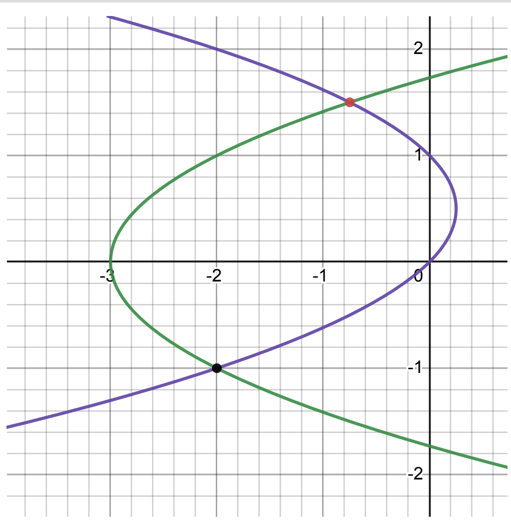
$e = e^x$ at $x = 1$ and

$e = e^{-x}$ at $x = -1$

$$\therefore A = \int_{-1}^{0} e - e^{-x} \, dx + \int_{0}^{1} e - e^{x} \, dx$$
$$= \Big[ex + e^{-x}\Big]_{-1}^{0} + \Big[ex - e^{x}\Big]_{0}^{1}$$
$$= 2$$

(22) Area between $x = -3 + y^2$ and $x = y - y^2$.

$$-3 + y^2 = y - y^2$$
$$2y^2 - y - 3 = 0$$
$$\therefore y = -1, \frac{3}{2}$$



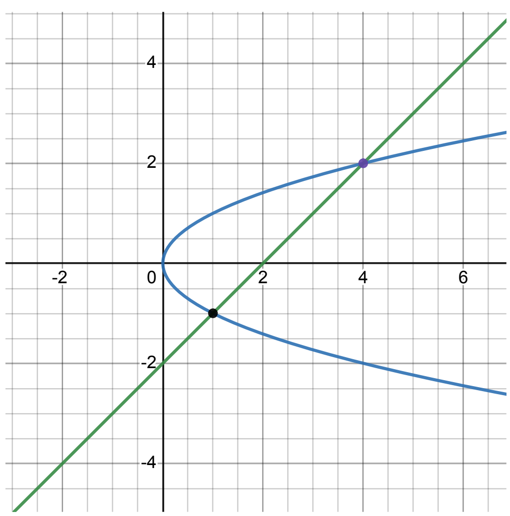
$-3 + y^2 = y - y^2$

at $y = -1$ and $y = \dfrac{3}{2}$

$$\begin{aligned}
\therefore A &= \int_{-1}^{3/2} y - y^2 - (-3 + y^2)\,\mathrm{d}y \\
&= \int_{-1}^{3/2} -2y^2 + y + 3\,\mathrm{d}y \\
&= \left[-\frac{2y^3}{3} + \frac{y^2}{2} + 3y \right]_{-1}^{3/2} \\
&= \frac{125}{24}
\end{aligned}$$

(53) Area between $y^2 = x$ and $x = y + 2$.

$$\begin{aligned}
y^2 &= y + 2 \\
y^2 - y - 2 &= 0 \\
(y+1)(y-2) &= 0 \\
\therefore y &= -1, 2
\end{aligned}$$



$y^2 = y + 2$

at $y = -1$ and $y = 2$

$$\begin{aligned}
\therefore A &= \int_{-1}^{2} y + 2 - y^2\,\mathrm{d}y \\
&= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^{2} \\
&= \frac{9}{2}
\end{aligned}$$