Homework 11

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1(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ for the following parametrized curve:

(i) $x = \sec t, y = \tan t$

$$x = \sec t \implies \frac{\mathrm{d}x}{\mathrm{d}t} = \sec t \tan t$$

$$y = \tan t \implies \frac{\mathrm{d}y}{\mathrm{d}t} = \sec^2 t$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t} = \frac{1}{\cos t}\frac{\cos t}{\sin t} = \frac{1}{\sin t} = \csc t$$

$$\implies \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\csc t}{\sec t \tan t} = -\frac{\cos t}{\sin^2 t \sec t \tan t} = -\frac{\cos^2 t}{\sin^2 t \tan t} = -\cot^3 t$$

(ii) $x = 2t^2, y = t^4$

$$x = 2t^2 \implies \frac{\mathrm{d}x}{\mathrm{d}t} = 4t$$

$$y = t^4 \implies \frac{\mathrm{d}y}{\mathrm{d}t} = 4t^3$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4t^3}{4t} = t^2$$

$$\implies \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}t^2}{4t} = \frac{2t}{4t} = \frac{1}{2}$$

(b) For (i) above, find the equation of the tangent line at $\frac{\pi}{4}$

$$
\begin{aligned}
t = \frac{\pi}{4} &\implies \frac{\mathrm{d}y}{\mathrm{d}x} = \csc\frac{\pi}{4} = \sqrt{2} \\
&\implies x = \sec\frac{\pi}{4} = \sqrt{2} \\
&\implies y = \tan\frac{\pi}{4} = 1
\end{aligned}
$$

Then, the equation of the tangent line to the curve at $t = \dfrac{\pi}{4}$ is

$$
\begin{aligned}
y - 1 &= \sqrt{2}(x - \sqrt{2}) \\
y &= \sqrt{2}x - 1.
\end{aligned}
$$

2. Find the arc length of the curve $x = t^3, y = \frac{3t^2}{2}$ when $0 \leq t \leq \sqrt{3}$.

$$
\begin{aligned}
x = t^3 &\implies \frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2 \\
y = \frac{3t^2}{2} &\implies \frac{\mathrm{d}y}{\mathrm{d}t} = 3t
\end{aligned}
$$

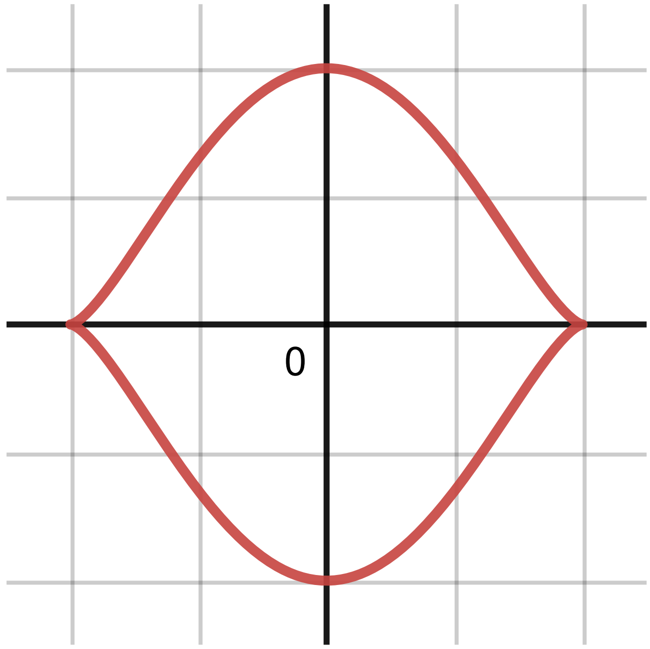
The arc length L of the curve for $0 \leq t \leq \sqrt{3}$ is

$$
\begin{aligned}
L &= \int_0^{\sqrt{3}} \sqrt{(3t^2)^2 + (3t)^2}\,\mathrm{d}t \\
&= \int_0^{\sqrt{3}} \sqrt{9t^4 + 9t^2}\,\mathrm{d}t \\
&= \int_0^{\sqrt{3}} \sqrt{9t^2(t^2 + 1)}\,\mathrm{d}t \\
&= 3\int_0^{\sqrt{3}} t\sqrt{t^2 + 1}\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
u = t^2 + 1 \quad &\Longrightarrow \quad \mathrm{d}u = 2t\,\mathrm{d}t \\
&\Longleftrightarrow \quad \mathrm{d}t = \frac{\mathrm{d}u}{2t} \\
t = 0 \quad &\Longrightarrow \quad u = 0^2 + 1 = 1 \\
t = \sqrt{3} \quad &\Longrightarrow \quad u = \sqrt{3}^2 + 1 = 4
\end{aligned}
$$

$$
\begin{aligned}
\therefore L = 3\int_0^{\sqrt{3}} t\sqrt{t^2+1}\,\mathrm{d}t &= 3\int_1^4 t\sqrt{u}\frac{\mathrm{d}u}{2t} \\
&= \frac{3}{2}\int_1^4 \sqrt{u}\,\mathrm{d}u \\
&= \frac{3}{2}\left[\frac{2u^{3/2}}{3}\right]_1^4 \\
&= \frac{3}{2}\left(\frac{16}{3} - \frac{2}{3}\right) \\
&= \frac{14}{2} \\
&= 7.
\end{aligned}
$$

3(a) Use Desmos to draw the graph $x(t) = \cos t, y(t) = \sin^3 t$ for $-\pi \leq t \leq \pi$.



(b) Find the area of the bounded region.

$$x(t) = \cos t \quad \Longrightarrow \quad x'(t) = -\sin t$$

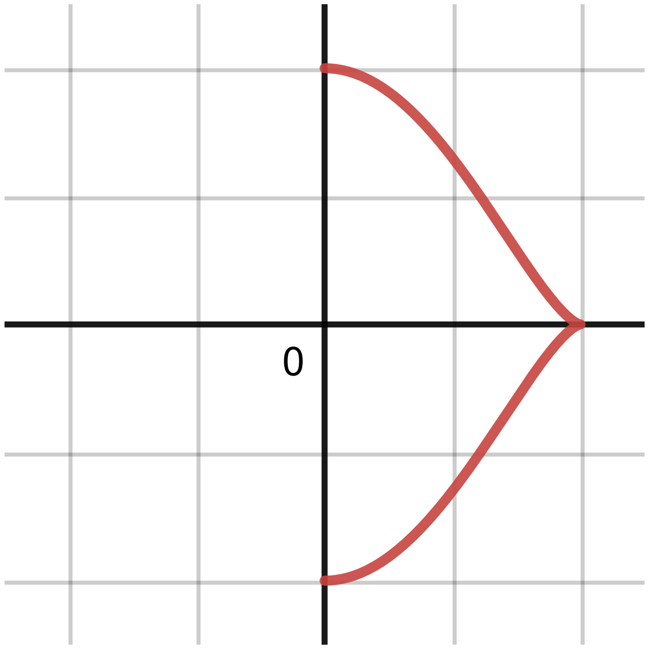
$$y(t) = \sin^3 t \quad \Longrightarrow \quad y'(t) = 3\sin^2 t \cos t$$

For $t \in [-\pi, \pi]$, we have that:

$$y(t) = \sin^3 t = -1 \implies t = -\frac{\pi}{2}$$

$$y(t) = \sin^3 t = 1 \implies t = \frac{\pi}{2}.$$

The graph for the curve $x(t) = \cos t, y(t) = \sin^3 t$ over $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ is as follows:



Integrating with respect to y from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$ yields the area of the right-hand side of the curve.

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \,\mathrm{d}y &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x(t)y'(t)\,\mathrm{d}t \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t(3\sin^2 t\cos t)\,\mathrm{d}t \\ &= \frac{3\pi}{8}\end{aligned}$$

Since the curve is symmetrical about the y-axis, the total area bounded by the curve A is double the resulting area.

$$A = 2 \cdot \frac{3\pi}{8} = \frac{3\pi}{4}$$

(c) Find the volume of solid of revolution by revloving the curve along the y-axis.

$$
\begin{aligned}
V = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \, \mathrm{d}y &= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x(t)^2 y'(t) \, \mathrm{d}t \\
&= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t (3 \sin^2 t \cos t) \, \mathrm{d}t \\
&= \frac{4\pi}{5}
\end{aligned}
$$

(d) Write down the arc length integral of this curve from $t = 0$ to $t = \pi/2$.

$$
\begin{aligned}
L &= \int_0^{\frac{\pi}{2}} \sqrt{x'(t)^2 + y'(t)^2} \, \mathrm{d}t \\
&= \int_0^{\frac{\pi}{2}} \sqrt{\sin^2 t + 9 \sin^4 t \cos^2 t} \, \mathrm{d}t \\
&= \int_0^{\frac{\pi}{2}} \sqrt{\sin^2 t (1 + 9 \sin^2 t \cos^2 t)} \, \mathrm{d}t
\end{aligned}
$$

4. Convert the following points into polar coordinates:

(a) $(-4, 4)$

Point is in second quadrant.

$$
\begin{aligned}
r &= \sqrt{(-4)^2 + 4^2} = 4\sqrt{2} \\
\theta &= \pi + \tan^{-1}(-1) = \frac{3\pi}{4} \\
\therefore (r, \theta) &= \left(4\sqrt{2}, \frac{3\pi}{4}\right)
\end{aligned}
$$

(b) $(3, 3\sqrt{3})$

Point is in first quadrant.

$$
r = \sqrt{3^2 + 3^2 \sqrt{3}^2} = 6
$$

$$
\theta = \tan^{-1}\left(\frac{3\sqrt{3}}{3}\right) = \tan^{-1} \sqrt{3} = \frac{\pi}{3}
$$

$$\therefore (r, \theta) = \left(6, \frac{\pi}{3}\right)$$

(c) $(\sqrt{3}, -1)$

Point is in fourth quadrant.

$$r = \sqrt{\sqrt{3}^2 + (-1)^2} = 2$$
$$\theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$
$$\therefore (r, \theta) = \left(2, -\frac{\pi}{6}\right)$$

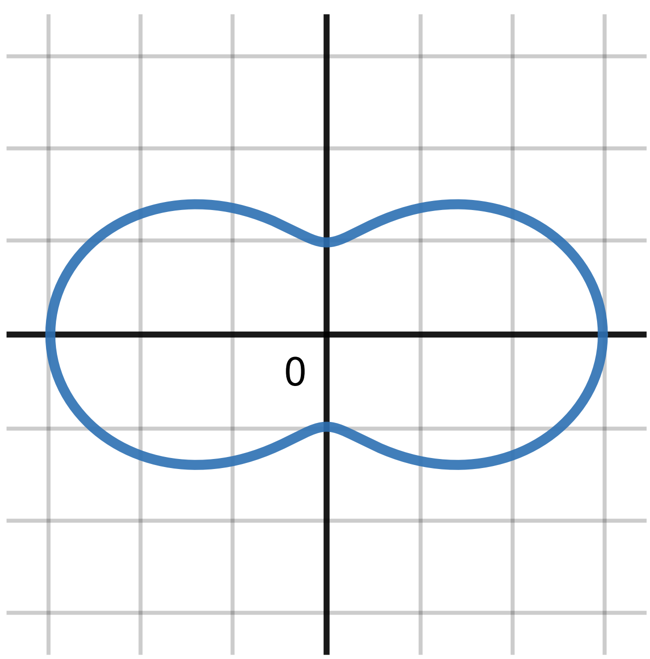
(d) $(-6, 0)$

Point is on the x-axis.

$$r = \sqrt{(-6)^2 + 0^2} = 6$$
$$\theta = \pi$$
$$\therefore (r, \theta) = (6, \pi)$$

5. Consider the polar equation $r = 2 + \cos(2\theta)$.

(a) Use Desmos to sketch the picture.



(b) Find the slope of the tangent line at $\theta = \pi/4$.

For $r = f(\theta)$ where $f(\theta) = 2 + \cos(2\theta)$.

$$\begin{aligned}
x(\theta) &= f(\theta)\cos\theta \\
&= (2+\cos(2\theta))\cos\theta \\
&= 2\cos\theta + \cos(2\theta)\cos\theta \\
y(\theta) &= f(\theta)\sin\theta \\
&= (2+\cos(2\theta))\sin\theta \\
&= 2\sin\theta + \sin\theta\cos(2\theta).
\end{aligned}$$

Then,

$$\begin{aligned}
x'(\theta) &= -2\sin\theta - 2\sin(2\theta)\cos\theta - \cos(2\theta)\sin\theta \\
y'(\theta) &= 2\cos\theta + \cos\theta\cos(2\theta) - 2\sin\theta\sin(2\theta).
\end{aligned}$$

As such, the slope of the tangent line at $\theta = \dfrac{\pi}{4}$ is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y'(\frac{\pi}{4})}{x'(\frac{\pi}{4})} = \frac{2\cos\frac{\pi}{4} + \cos\frac{\pi}{4}\cos(2\frac{\pi}{4}) - 2\sin\frac{\pi}{4}\sin(2\frac{\pi}{4})}{-2\sin\frac{\pi}{4} - 2\sin(2\frac{\pi}{4})\cos\frac{\pi}{4} - \cos(2\frac{\pi}{4})\sin\frac{\pi}{4}} = \frac{0}{-2\sqrt{2}} = 0.$$

(c) Find the area of the bounded region of the graph.

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi} (2+\cos(2\theta))^2 \,\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2\pi} 4 + \cos^2(2\theta) + 4\cos(2\theta)\,\mathrm{d}\theta
\end{aligned}$$

$$u = 2\theta \implies \mathrm{d}u = 2\,\mathrm{d}\theta$$

$$\iff \mathrm{d}\theta = \frac{1}{2}\,\mathrm{d}u$$

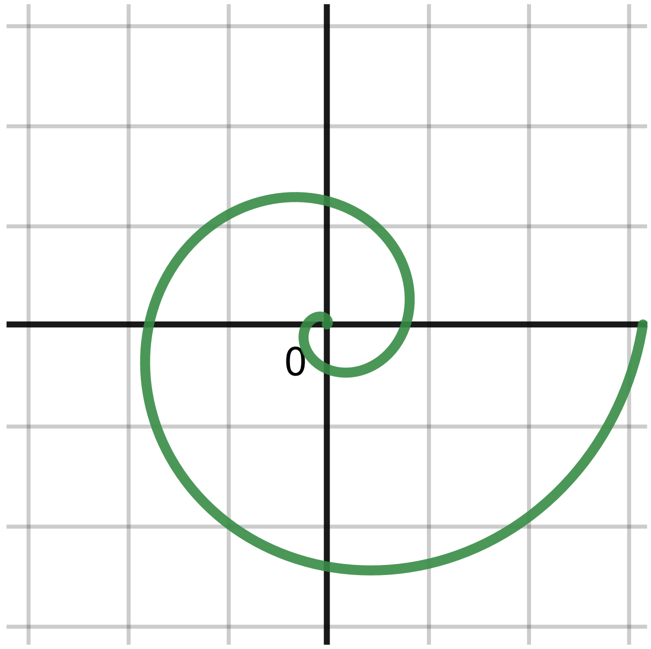
$$\theta = 0 \implies u = 2(0) = 0$$

$$\theta = 2\pi \implies u = 2(2\pi) = 4\pi$$

$$\begin{aligned}
\frac{1}{2}\int_0^{2\pi} 4 + \cos^2(2\theta) + 4\cos(2\theta)\,\mathrm{d}\theta &= \frac{1}{4}\int_0^{4\pi} 4 + \cos^2(u) + 4\cos u\,\mathrm{d}u \\
&= \frac{1}{4}\int_0^{4\pi} 4 + \frac{1}{2}\cos(2u) + \frac{1}{2} + 4\cos u\,\mathrm{d}u \\
&= \frac{1}{8}\int_0^{4\pi} 8 + \cos(2u) + 1 + 8\cos u\,\mathrm{d}u \\
&= \frac{1}{8}\left[9u + \frac{\sin(2u)}{2} + 8\sin u\right]_0^{4\pi} \\
&= \frac{1}{8}(36\pi) \\
&= \frac{9\pi}{2}
\end{aligned}$$

6. Consider the polar equation $r = \theta^2$.

(a) Use Desmos to sketch the picture for $0 \le \theta \le 4\pi$.



(b) Find the slope of the tangent line at $\theta = 3\pi/4$.

For $r = f(\theta)$ where $f(\theta) = \theta^2$,

$$x(\theta) = f(\theta)\cos\theta = \theta^2\cos\theta$$
$$y(\theta) = f(\theta)\sin\theta = \theta^2\sin\theta.$$

Then,

$$x'(\theta) = 2\theta\cos\theta - \theta^2\sin\theta$$
$$y'(\theta) = 2\theta\sin\theta + \theta^2\cos\theta.$$

As such, the slope of the tangent line at $\theta = \dfrac{3\pi}{4}$ is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y'(\frac{3\pi}{4})}{x'(\frac{3\pi}{4})} = \frac{2(\frac{3\pi}{4})\sin(\frac{3\pi}{4}) + (\frac{3\pi}{4})^2\cos(\frac{3\pi}{4})}{2(\frac{3\pi}{4})\cos(\frac{3\pi}{4}) - (\frac{3\pi}{4})^2\sin(\frac{3\pi}{4})} = 1 - \frac{16}{8+3\pi}.$$

(c) Find the arc length of the curve for $0 \le \theta \le 4\pi$.

$$
\begin{aligned}
L &= \int_0^{4\pi} \sqrt{x'(\theta)^2 + y'(\theta)^2}\,\mathrm{d}\theta \\
&= \int_0^{4\pi} \sqrt{(2\theta\cos\theta - \theta^2\sin\theta)^2 + (2\theta\sin\theta + \theta^2\cos\theta)^2}\,\mathrm{d}\theta \\
&= \int_0^{4\pi} \sqrt{\theta^2(\theta^2+4)}\,\mathrm{d}\theta \\
&= \int_0^{4\pi} \theta\sqrt{\theta^2+4}\,\mathrm{d}\theta
\end{aligned}
$$

$$
\begin{aligned}
u = \theta^2 + 4 \quad &\implies \quad \mathrm{d}u = 2\theta\,\mathrm{d}\theta \\
&\iff \quad \mathrm{d}\theta = \frac{\mathrm{d}u}{2\theta} \\
\theta = 0 \quad &\implies \quad u = 0^2 + 4 = 4 \\
\theta = 4\pi \quad &\implies \quad u = (4\pi)^2 + 4 = 4 + 16\pi^2
\end{aligned}
$$

$$
\begin{aligned}
\therefore L = \int_0^{4\pi} \theta\sqrt{\theta^2+4}\,\mathrm{d}\theta &= \int_4^{4+16\pi^2} \theta\sqrt{u}\frac{\mathrm{d}u}{2\theta} \\
&= \frac{1}{2}\int_4^{4+16\pi^2} \sqrt{u}\,\mathrm{d}u \\
&= \frac{1}{2}\left[\frac{2u^{3/2}}{3}\right]_4^{4+16\pi^2} \\
&= \frac{8}{3}((1-4\pi^2)^{3/2} - 1)
\end{aligned}
$$