Homework 10

Mos Kullathon
921425216

1. Determine whether the following series is convergent or divergent. We may use any tests available

(i) $\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}$

As a p-series where $p = 2$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Since $\dfrac{1}{n^2} \geq \dfrac{1}{n^2 2^n}$ for $n \geq 1$, by comparison test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}$ converges.

(ii) $\sum_{n=2}^{\infty} \frac{3n^3+2n^2+2}{n^4-1}$

Let $a_n = \dfrac{3n^3 + 2n^2 + 2}{n^4 - 1}$ and $b_n = \dfrac{1}{n}$.

Then,

$$\frac{a_n}{b_n} = \frac{n(3n^3 + 2n^2 + 2)}{n^4 - 1} = \frac{3n^4 + 2n^3 + 2n}{n^4 - 1}$$
$$\therefore \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{3n^4 + 2n^3 + 2n}{n^4 - 1} = 3.$$

Since $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series such that $p = 1$, $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges. Consequently, by limit comparison test, $\displaystyle\sum_{n=1}^{\infty} a_n$ also diverges.

(iii) $\sum_{n=1}^{\infty} \frac{n^{1/3}}{(n^{3/2}-1)^{1/2}}$

At $n = 1$, the first term is undefined. Therefore, the sum does not exist.

$$\frac{1^{1/3}}{(1^{3/2}-1)^{1/2}} = \frac{1}{0}$$

(iv) $\sum_{n=1}^{\infty} \frac{n^2 \ln n}{n^5+2n-1}$

For sufficiently large n,

$$\frac{n^2 \ln n}{n^5+2n-1} \approx \frac{1}{n^3}.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series such that $p = 3$, the series converges.

(v) $\sum_{n=1}^{\infty} (-1)^{n^2} \frac{n+1}{n!}$

Let $a_n = (-1)^{n^2} \dfrac{n+1}{n!}$.

Then,

$$a_{n+1} = \frac{(-1)^{(n+1)^2}(n+1)+1}{(n+1)!} = \frac{(-1)^{n^2+2n+1} n+2}{(n+1)n!}.$$

and

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{\cancel{n^2}+2n+1} n+2}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{(-1)^{n^2}}(n+1)}\right|$$

$$= \left|\frac{(-1)^{2n+1}(n+2)}{(n+1)^2}\right|$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$$

By ratio test, the series converges absolutely.

(vi) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$

Let $a_n = \dfrac{(n!)^3}{(3n)!}$.

Then,

$$
\begin{aligned}
a_{n+1} &= \frac{((n+1)!)^3}{(3(n+1))!} \\
&= \frac{((n+1)!)^3}{(3n+3)!} \\
&= \frac{((n+1)n!)^3}{(3n+3)(3n+2)(3n+1)(3n)!} \\
&= \frac{(n+1)^3(n!)^3}{(3n+3)(3n+2)(3n+1)(3n)!}
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)^3\cancel{(n!)^3}}{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}} \cdot \frac{\cancel{(3n)!}}{\cancel{(n!)^3}}\right| \\
&= \left|\frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)}\right| \\
\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{1}{3^3} = \frac{1}{27} < 1
\end{aligned}
$$

By ratio test, the series converges absolutely.

(vii) $\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$

Let $a_n = \left(\frac{n}{3n+1}\right)^n$.

Then,

$$
|a_n|^{\frac{1}{n}} = \left|\left(\frac{n}{3n+1}\right)^n\right|^{\frac{1}{n}} = \left|\frac{n}{3n+1}\right| = \frac{n}{3n+1}, \quad n \geq 1
$$

and

$$
\begin{aligned}
\lim_{n\to\infty}|a_n|^{\frac{1}{n}} &= \lim_{n\to\infty}\frac{n}{3n+1} = \frac{1}{3} \\
\therefore \lim_{n\to\infty}|a_n|^{\frac{1}{n}} &< 0.
\end{aligned}
$$

By root test, the series converges absolutely.

(viii) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

Let $a_n = \frac{1}{\ln n}$.

Since $\ln n$ is monotonically increasing, its reciprocal $a_n = \frac{1}{\ln n}$ must be monotonically decreasing.

Then,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln n} = 0.$$

By alternating series test, the series converge.

(ix) $\sum_{n=2}^{\infty} (-1)^n \ln n$

$$\lim_{n\to\infty} \ln n = \infty$$

By divergence test, the series diverges.

2. Find the radius of convergence and interval of convergence for the following power series

(i) $\sum_{n=0}^{\infty} \frac{1}{n2^n} x^n$

$$a_n = \frac{x^n}{n2^n}, \quad a_{n+1} = \frac{x^{n+1}}{(n+1)2^{n+1}}$$

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{x^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{x^n}\right| \\ &= \left|\frac{xn}{2(n+1)}\right| \\ &= |x|\frac{n}{2n+2}, \quad n \geq 0 \end{aligned}$$

$$\therefore L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{2} \implies L < 1 \iff |x| < 2$$

Since

$$|x| < 2 \iff -2 < x < 2,$$

by ratio test, the series converges absolutely for $-2 < x < 2$ centered at $x = 0$.

$$x = -2 \implies \sum_{n=0}^{\infty} \frac{1}{n2^n}(-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

Since $\frac{1}{n} \to 0$, by alternating series test, the series converges at $x = -2$.

$$x = 2 \implies \sum_{n=0}^{\infty} \frac{1}{n2^n}(2)^n = \sum_{n=0}^{\infty} \frac{1}{n}$$

At $x = 2$, the series is a p-series such that $p = 1$. Therefore, the series diverge at $x = 2$.

As such:

- the radius of convergence is 2
- the interval of convergence is $[-2, 2)$.

(ii) $\sum_{n=0}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$

$$a_n = \frac{(x-1)^n}{\sqrt{n}}, \quad a_{n+1} = \frac{(x-1)^{n+1}}{\sqrt{n+1}}$$

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(x-1)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(x-1)^n}\right| \\ &= \left|\frac{\sqrt{n}(x-1)}{\sqrt{n+1}}\right| \\ &= |x-1|\frac{\sqrt{n}}{\sqrt{n+1}}, \quad n \geq 0 \end{aligned}$$

$$\therefore L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-1| \implies L < 1 \iff |x-1| < 1$$

Since

$$|x-1| < 1 \iff -1 < x-1 < 1 \iff 0 < x < 2,$$

by ratio test, the series converges absolutely for $0 < x < 2$ centered at $x = 1$.

$$x = 0 \implies \sum_{n=0}^{\infty} \frac{(0-1)^n}{\sqrt{n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Since $\frac{1}{\sqrt{n}} \to 0$, by alternating series test, the series converges at $x = 0$.

$$x = 2 \implies \sum_{n=0}^{\infty} \frac{(2-1)^n}{\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1^n}{\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{n^{1/2}}$$

At $x = 2$, the series is a p-series such that $p = \frac{1}{2}$. Therefore, the series diverges at $x = 2$.

As such:

- the radius of convergence is 1
- the interval of convergence is $[0, 2)$.

(iii) $\sum_{n=2}^{\infty} \frac{1}{n \ln n} x^n$

$$a_n = \frac{x^n}{n \ln n}, \quad a_{n+1} = \frac{x^{n+1}}{(n+1)\ln(n+1)}$$

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{x^{n+1}}{(n+1)\ln(n+1)} \cdot \frac{n \ln n}{x^n}\right| \\ &= \left|\frac{nx \ln n}{(n+1)\ln(n+1)}\right| \end{aligned}$$

$$\begin{aligned} \therefore L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{nx \ln n}{(n+1)\ln(n+1)}\right| \\ &= \lim_{n\to\infty} \left|\frac{nx}{n+1}\right| \cdot \lim_{n\to\infty} \left|\frac{\ln n}{\ln(n+1)}\right| \\ &= |x| \cdot \lim_{n\to\infty} \left|\frac{\frac{1}{n}}{\frac{1}{n+1}}\right| \\ &= |x| \end{aligned}$$

$$\implies L < 1 \iff |x| < 1$$

Since

$$|x| < 1 \iff -1 < x < 1,$$

by ratio test, the series converges absolutely for $-1 < x < 1$.

$$x = -1 \implies \sum_{n=2}^{\infty} \frac{1}{n \ln n}(-1)^n$$

Since $\dfrac{1}{n \ln n} \to 0$, by alternating series test, the series converge at $x = -1$.

$$x = 1 \implies \sum_{n=2}^{\infty} \frac{1}{n \ln n}(1)^n = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Let $g(x) = \dfrac{1}{x \ln x}$. Since $g'(x) < 0$ for $x \geq 2$, we apply the integral test.

$$\begin{aligned}
u = \ln x \quad &\implies \quad \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x \\
&\iff \quad \mathrm{d}x = x\,\mathrm{d}u \\
x = 2 \quad &\implies \quad u = \ln 2 \\
x = n \quad &\implies \quad u = \ln n
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} \int_2^n \frac{1}{x \ln x}\,\mathrm{d}x &= \lim_{n\to\infty} \int_{\ln 2}^{\ln n} \frac{x\,\mathrm{d}u}{xu} \\
&= \lim_{n\to\infty} \Big[\ln u\Big]_{\ln 2}^{\ln n} \\
&= \lim_{n\to\infty} \ln(\ln n) - \ln(\ln 2) \\
&= \infty
\end{aligned}$$

By integral test, the series diverge at $x = 1$.

As such:

- the radius of convergence is 1
- the interval of convergence is $[-1, 1)$.

(iv) $\sum_{n=2}^{\infty}(2+\frac{1}{n})^n x^n$

$$a_n = \left(2+\frac{1}{n}\right)^n x^n$$

$$|a_n|^{\frac{1}{n}} = \left|\left(2+\frac{1}{n}\right)x\right|$$

$$\therefore L = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = |2x| \implies L < 1 \iff |2x| < 1$$

Since

$$|2x| < 1 \iff |x| < \frac{1}{2} \iff -\frac{1}{2} < x < \frac{1}{2},$$

by root test, the series converges absolutely for $-\frac{1}{2} < x < \frac{1}{2}$ centered at $x = 0$.

$$x = -\frac{1}{2} \implies \sum_{n=2}^{\infty}\left(2+\frac{1}{n}\right)^n\left(-\frac{1}{2}\right)^n$$

$$a_n = \left(2+\frac{1}{n}\right)^n\left(-\frac{1}{2}\right)^n$$

$$|a_n|^{\frac{1}{n}} = \left|\left(2+\frac{1}{n}\right)\left(-\frac{1}{2}\right)\right|$$

Since $|a_n|^{\frac{1}{n}} \to -1$, by root test, the series converges at $x = -\frac{1}{2}$.

$$x = \frac{1}{2} \implies \sum_{n=2}^{\infty}\left(2+\frac{1}{n}\right)^n\left(\frac{1}{2}\right)^n$$

$$
\begin{aligned}
\lim_{n\to\infty}\left(2+\frac{1}{n}\right)^n\left(\frac{1}{2}\right)^n &= \lim_{n\to\infty}\left(\left(2+\frac{1}{n}\right)\left(\frac{1}{2}\right)\right)^n \\
&= \lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n \\
&= \lim_{n\to\infty} e^{n\ln\left(1+\frac{1}{2n}\right)} \\
&= \lim_{n\to\infty}\exp\frac{\ln\left(1+\frac{1}{2n}\right)}{1/n} \\
&= \exp\lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{2n}}\cdot-\frac{1}{2n^2}}{-\frac{1}{n^2}} \\
&= \exp\lim_{n\to\infty}\frac{1}{1+\frac{1}{2n}}\cdot\frac{1}{2} \\
&= \exp\frac{1}{2} \\
&= \sqrt{e} \quad \approx 1.64872
\end{aligned}
$$

Since $\left(2+\frac{1}{n}\right)^n\left(\frac{1}{2}\right)^n \to \sqrt{e}$, by divergence test, the series diverge at $x=\frac{1}{2}$.

As such:

- the radius of convergence is $\frac{1}{2}$
- the interval of convergence is $\left[-\frac{1}{2},\frac{1}{2}\right)$.

3. Section 6.1

In the following exercises, given that $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ with convergence in $(-1,1)$, find the power series for each function with the given center *a*, and identify its interval of convergence.

(35) $f(x)=\frac{x}{1-x^2}; a=0$

$$\begin{aligned} f(x) &= \frac{x}{1-x^2} \\ &= \sum_{n=0}^{\infty} x(x^2)^n, \quad |x^2| < 1 \\ &= \sum_{n=0}^{\infty} x^{2n+1}, \quad |x| < 1 \end{aligned}$$

Since

$$|x| < 1 \iff -1 < x < 1,$$

$f(x)$ converges within $-1 < x < 1$.

Then,

$$f(-1) = \sum_{n=0}^{\infty} (-1)^{2n+1}$$

Since $\lim_{n\to\infty} (-1)^{2n+1}$ does not exist, $f(-1)$ diverges by divergence test.

$$f(1) = \sum_{n=0}^{\infty} 1^{2n+1}$$

Since $\lim_{n\to\infty} 1^{2n+1} = 1$, $f(1)$ diverges by divergence test.

As such,

$$f(x) = \frac{x}{1-x^2} = \sum_{n=0}^{\infty} x^{2n+1}, \quad x \in (-1, 1).$$

(37) $f(x) = \frac{x^2}{1+x^2}; a = 0$

$$\begin{aligned}
f(x) &= \frac{x^2}{1+x^2} \\
&= \frac{x^2}{1-(-x^2)} \\
&= \sum_{n=0}^{\infty} x^2(-x^2)^n, \quad \left|-x^2\right| < 1 \\
&= \sum_{n=0}^{\infty} (-1)^n x^{2n+2}, \quad |x| < 1
\end{aligned}$$

Since

$$|x| < 1 \iff -1 < x < 1,$$

$f(x)$ converges within $-1 < x < 1$.

Then,

$$f(-1) = \sum_{n=0}^{\infty} (-1)^n (-1)^{2n+2} = \sum_{n=0}^{\infty} (-1)^{3n+2}$$

Since $\lim\limits_{n\to\infty} (-1)^{3n+2}$ does not exist, $f(-1)$ diverges by divergence test.

$$f(1) = \sum_{n=0}^{\infty} (-1)^n (1)^{2n+2} = \sum_{n=0}^{\infty} (-1)^n$$

Since $\lim\limits_{n\to\infty} (-1)^n$ does not exist, $f(1)$ diverges by divergence test.

As such,

$$f(x) = \frac{x^2}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n+2}, \quad x \in (-1, 1)$$

(39) $f(x) = \frac{1}{1-2x}; a = 0$

$$
\begin{aligned}
f(x) &= \frac{1}{1-2x} \\
&= \sum_{n=0}^{\infty} (2x)^n, \quad |2x| < 1 \\
&= \sum_{n=0}^{\infty} 2^n x^n, \quad |x| < \frac{1}{2}
\end{aligned}
$$

Since

$$|x| < \frac{1}{2} \iff -\frac{1}{2} < x < \frac{1}{2},$$

$f(x)$ converges within $-\frac{1}{2} < x < \frac{1}{2}$.

Then,

$$f\left(-\frac{1}{2}\right) = \sum_{n=0}^{\infty} 2^n \left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n$$

Since $\lim_{n\to\infty} (-1)^n$ does not exist, $f\left(-\frac{1}{2}\right)$ diverges by divergence test.

$$f\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} 1$$

Since $\lim_{n\to\infty} 1 = 1$, $f(1)$ diverges by divergence test.

As such,

$$f(x) = \frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n, \quad x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

(41) $f(x) = \frac{x^2}{1-4x^2}; a = 0$

$$
\begin{aligned}
f(x) &= \frac{x^2}{1-4x^2} \\
&= \sum_{n=0}^{\infty} x^2 (4x^2)^n, \quad |4x^2| < 1 \\
&= \sum_{n=0}^{\infty} x^2 4^n x^{2n}, \quad |x^2| < \frac{1}{4} \\
&= \sum_{n=0}^{\infty} 4^n x^{2n+2}, \quad |x| < \frac{1}{2}
\end{aligned}
$$

Since

$$|x| < \frac{1}{2} \iff -\frac{1}{2} < x < \frac{1}{2},$$

$f(x)$ converges within $-\frac{1}{2} < x < \frac{1}{2}$.

Then,

$$f\left(-\frac{1}{2}\right) = \sum_{n=0}^{\infty} 4^n \left(-\frac{1}{2}\right)^{2n+2}.$$

Since $\lim_{n\to\infty} 4^n \left(-\frac{1}{2}\right)^{2n+2}$ does not exist, by divergence test $f\left(-\frac{1}{2}\right)$ diverges.

$$f\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} 4^n \left(\frac{1}{2}\right)^{2n+2}$$

Since

$$
\begin{aligned}
\lim_{n\to\infty} 4^n \left(\frac{1}{2}\right)^{2n+2} &= \lim_{n\to\infty} \frac{4^n}{2^{2n+2}} \\
&= \lim_{n\to\infty} \frac{4^n}{2^{2n}\cdot 2^2} \\
&= \lim_{n\to\infty} \frac{4^n}{(2^2)^n\cdot 2^2} \\
&= \lim_{n\to\infty} \frac{1}{4} \\
&= \frac{1}{4}
\end{aligned}
$$

by divergence test $f\left(\frac{1}{2}\right)$ diverges.

As such,

$$
f(x) = \frac{x^2}{1-4x^2} = \sum_{n=0}^{\infty} 4^n x^{2n+2}, \quad x \in \left(-\frac{1}{2}, \frac{1}{2}\right)
$$

4. Find

(i) The Taylor polynomial of degree 3 of $f(x) = x^{2/3}$ at $x = 8$

$$
\begin{aligned}
f(x) = x^{2/3} &\implies f'(x) = \frac{2}{3}x^{-1/3} \\
&\implies f''(x) = -\frac{2}{9}x^{-4/3} \\
&\implies f'''(x) = \frac{8}{27}x^{-7/3}
\end{aligned}
$$

$$\begin{aligned} f(8) &= 8^{2/3} &&= 4 \\ f'(8) &= \frac{2}{3} \cdot 8^{-1/3} &&= \frac{1}{3} \\ f''(8) &= -\frac{2}{9} \cdot 8^{-4/3} &&= -\frac{1}{72} \\ f'''(8) &= \frac{8}{27} \cdot 8^{-7/3} &&= \frac{1}{432} \end{aligned}$$

$$\begin{aligned} \therefore \sum_{n=0}^{3} \frac{f^{(n)}(x)}{n!}(x-8)^n &= f(8) + f'(8)(x-8) + \frac{f''(8)(x-8)^2}{2!} + \frac{f''(8)(x-8)^3}{3!} \\ &= 4 + \frac{x-8}{3} - \frac{(x-8)^2}{72 \cdot 2!} + \frac{(x-8)^3}{432 \cdot 3!} \\ &= 4 + \frac{x-8}{3} - \frac{(x-8)^2}{144} + \frac{(x-8)^3}{2592} \end{aligned}$$

(ii) The Taylor polynomial of degree 2 of $f(x) = \sec x$ at $x = 0$

$$\begin{aligned} f(x) = \sec x &\implies f'(x) = \sec x \tan x \\ &\implies f''(x) = (\sec x \tan^2 x) + \sec^3 x \end{aligned}$$

$$\begin{aligned} f(0) &= \sec 0 &&= 1 \\ f'(0) &= \sec 0 \tan 0 &&= 0 \\ f''(0) &= (\sec 0 \tan^2 0) + \sec^3 0 &&= 1 \end{aligned}$$

$$\begin{aligned} \therefore \sum_{n=0}^{2} \frac{f^{(n)}(x)}{n!}(x-0)^n &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} \\ &= 1 + 0x + \frac{x^2}{2!} \\ &= 1 + \frac{x^2}{2} \end{aligned}$$

5. Find the Taylor series expansion of the following functions using the formula

$$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$$

Identify their interval of convergence. You may need to use differentiation and integration of the Taylor series.

(i) $\frac{1}{2+3x}$

$$\begin{aligned}
\frac{1}{2+3x} &= \frac{1}{2+3(x-1)+3} \\
&= \frac{1}{5+3(x-1)} \\
&= \frac{1}{5}\cdot\frac{1}{1+3(x-1)} \\
&= \frac{1}{5}\cdot\frac{1}{1-(-3(x-1))} \\
&= \sum_{n=0}^{\infty}\frac{1}{5}(-3(x-1))^n, \quad |-3(x-1)|<1 \\
&= \sum_{n=0}^{\infty}\frac{1}{5}(-3)^n(x-1)^n, \quad |x-1|<\frac{1}{3}
\end{aligned}$$

> ### (ii) $\frac{1}{(1-2x)^2}$

> ### (iii) $\ln(1+x)$