Homework 5

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1. Evaluate the following integrals

(a) $\displaystyle\int x\cos(5x)\,\mathrm{d}x$

$$
\begin{aligned}
u &= x & \implies && u' &= 1 \\
v' &= \cos(5x) & \implies && v &= \frac{1}{5}\sin(5x)
\end{aligned}
$$

$$
\begin{aligned}
\int x\cos(5x)\,\mathrm{d}x &= uv - \int vu'\,\mathrm{d}x \\
&= \frac{x}{5}\sin(5x) - \int \frac{1}{5}\sin(5x)\,\mathrm{d}x \\
&= \frac{x}{5}\sin(5x) - \frac{1}{5}\int \sin(5x)\,\mathrm{d}x \\
&= \frac{x}{5}\sin(5x) - \frac{1}{5}\left(-\frac{1}{5}\cos(5x)\right) + c \\
&= \frac{1}{5}\sin(5x) + \frac{1}{25}\cos(5x) + c
\end{aligned}
$$

(b) $\displaystyle\int_0^1 x^2 e^{-x}\,\mathrm{d}x$

$$
\begin{aligned}
u &= x^2 & \implies && u' &= 2x \\
v' &= e^{-x} & \implies && v &= -e^{-x}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 x^2 e^{-x}\,\mathrm{d}x &= \Big[uv\Big]_0^1 - \int_0^1 vu'\,\mathrm{d}x \\
&= \Big[-x^2 e^{-x}\Big]_0^1 - \int_0^1 -2xe^{-x}\,\mathrm{d}x \\
&= \Big[-x^2 e^{-x}\Big]_0^1 + 2\int_0^1 xe^{-x}\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
p &= x &\implies&& p' &= 1 \\
q' &= e^{-x} &\implies&& q &= -e^{-x}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 xe^{-x}\,\mathrm{d}x &= \Big[pq\Big]_0^1 - \int_0^1 qp'\,\mathrm{d}x \\
&= \Big[-xe^{-x}\Big]_0^1 - \int_0^1 -e^{-x}\,\mathrm{d}x \\
&= \Big[-xe^{-x}\Big]_0^1 + \Big[-e^{-x}\Big]_0^1 \\
&= -\frac{1}{e} - \frac{1}{e} + 1 \\
&= \frac{-2+e}{e}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \int_0^1 x^2 e^{-x}\,\mathrm{d}x &= \Big[-x^2 e^{-x}\Big]_0^1 + 2\left(\frac{-2+e}{e}\right) \\
&= -\frac{1}{e} + \frac{-4+2e}{e} \\
&= \frac{-5+2e}{e}
\end{aligned}
$$

(c) $\int x(\ln x)^2\,\mathrm{d}x$

$$
\begin{aligned}
u &= (\ln x)^2 &\implies&& u' &= \frac{2}{x}\ln x \\
v' &= x &\implies&& v &= \frac{1}{2}x^2
\end{aligned}
$$

$$\begin{aligned}\int x(\ln x)^2\,\mathrm{d}x &= uv - \int vu'\,\mathrm{d}x \\ &= \frac{1}{2}x^2\ln^2 x - \int x\ln x\,\mathrm{d}x\end{aligned}$$

$$\begin{aligned} p = \ln x \quad &\implies \quad p' = \frac{1}{x} \\ q' = x \quad &\implies \quad q = \frac{1}{2}x^2 \end{aligned}$$

$$\begin{aligned}\int x\ln x\,\mathrm{d}x &= pq - \int qp'\,\mathrm{d}x \\ &= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,\mathrm{d}x \\ &= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + c\end{aligned}$$

$$\begin{aligned}\therefore \int x(\ln x)^2\,\mathrm{d}x &= \frac{1}{2}x^2\ln^2 x - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + c \\ &= \frac{1}{4}x^2(2\ln^2(x) - 2\ln(x) + 1) + c\end{aligned}$$

(d) $\int x\csc^2 x\,\mathrm{d}x$

$$\begin{aligned} u = x \quad &\implies \quad u' = 1 \\ v' = \csc^2 x \quad &\implies \quad v = -\cot x \end{aligned}$$

$$\begin{aligned}\int x\csc^2 x\,\mathrm{d}x &= uv - \int vu'\,\mathrm{d}x \\ &= -x\cot(x) - \int -\cot x\,\mathrm{d}x \\ &= -x\cot(x) + \ln|\sin x| + c\end{aligned}$$

(e) $\int_0^{\pi/4} e^x\cos(2x)\,\mathrm{d}x$

$$u = \cos(2x) \implies u' = -2\sin(2x)$$

$$v' = e^x \implies v = e^x$$

$$\begin{aligned}\int e^x \cos(2x)\,\mathrm{d}x &= uv - \int vu'\,\mathrm{d}x \\ &= e^x\cos(2x) - \int -2e^x \sin(2x)\,\mathrm{d}x \\ &= e^x\cos(2x) + 2\int e^x \sin(2x)\,\mathrm{d}x\end{aligned}$$

$$p = \sin(2x) \implies p' = 2\cos(2x)$$

$$q' = e^x \implies q = e^x$$

$$\begin{aligned}\int e^x \sin(2x)\,\mathrm{d}x &= pq - \int qp'\,\mathrm{d}x \\ &= e^x\sin(2x) - \int 2e^x \cos(2x) \\ &= e^x\sin(2x) - 2\int e^x \cos(2x)\end{aligned}$$

$$\begin{aligned}\int e^x \cos(2x)\,\mathrm{d}x &= e^x\cos(2x) + 2\left(e^x\sin(2x) - 2\int e^x\cos(2x)\right) \\ &= e^x\cos(2x) + 2e^x\sin(2x) - 4\int e^x\cos(2x)\end{aligned}$$

$$5\int e^x\cos(2x)\,\mathrm{d}x = e^x\cos(2x) + 2e^x\sin(2x)$$

$$\int e^x\cos(2x)\,\mathrm{d}x = \frac{1}{5}(e^x\cos(2x) + 2e^x\sin(2x)) + c$$

$$\begin{aligned}\therefore \int_0^{\pi/4} e^x\cos(2x)\,\mathrm{d}x &= \left[\frac{1}{5}(e^x\cos(2x) + 2e^x\sin(2x))\right]_0^{\pi/4} \\ &= \frac{1}{5}(2e^{\pi/4} - 1)\end{aligned}$$

(f) $\int_0^{\pi} e^{\cos t}\sin(2t)\,\mathrm{d}t$ (Recall $\sin(2t) = 2\sin t\cos t$)

$$
\begin{aligned}
u = \cos t \quad &\implies \quad \mathrm{d}u = -\sin t \, \mathrm{d}t \\
&\iff \quad \mathrm{d}t = \frac{1}{-\sin t} \, \mathrm{d}u \\
t = 0 \quad &\implies \quad u = \cos 0 = 1 \\
t = \pi \quad &\implies \quad u = \cos \pi = -1
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{\pi} e^{\cos t} \sin(2t) \, \mathrm{d}t &= 2 \int_1^{-1} e^{\cos t} \sin t \cos t \, \mathrm{d}t \\
&= 2 \int_1^{-1} u e^u \cancel{\sin t} \left(\frac{1}{\cancel{-\sin t}} \right) \mathrm{d}u \\
&= -2 \int_1^{-1} u e^u \, \mathrm{d}u \\
&= 2 \int_{-1}^{1} u e^u \, \mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
p = u \quad &\implies \quad p' = 1 \\
q' = e^u \quad &\implies \quad q = e^u
\end{aligned}
$$

$$
\begin{aligned}
\int_{-1}^{1} u e^u \, \mathrm{d}u &= \Big[pq \Big]_{-1}^{1} - \int q p' \, \mathrm{d}u \\
&= \Big[u e^u \Big]_{-1}^{1} - \int e^u \, \mathrm{d}u \\
&= \Big[u e^u \Big]_{-1}^{1} - \Big[e^u \Big]_{-1}^{1} \\
&= e + \frac{1}{e} - e + \frac{1}{e} \\
&= \frac{2}{e}
\end{aligned}
$$

$$\therefore \int_0^{\pi} e^{\cos t}\sin(2t)\,\mathrm{d}t = 2\int_1^{-1} e^{\cos t}\sin t\cos t\,\mathrm{d}t$$

$$= 2\cdot\frac{2}{e}$$

$$= \frac{4}{e}$$

2. Show that the reduction formula for $I_n = \displaystyle\int (\ln x)^n\,\mathrm{d}x$ is

$$I_n = x(\ln x)^n - nI_{n-1}$$

$$u = (\ln x)^n \quad\Longrightarrow\quad u' = \frac{n}{x}(\ln x)^{n-1}$$

$$v' = 1 \quad\Longrightarrow\quad v = x$$

$$I_n = \int (\ln x)^n = uv - \int vu'\,\mathrm{d}x$$

$$= x(\ln x)^n - \int \frac{n\cancel{x}}{\cancel{x}}(\ln x)^{n-1}$$

$$= x(\ln x)^n - n\underbrace{\int (\ln x)^{n-1}}_{I_{n-1}}$$

$$= x(\ln x)^n - nI_{n-1}$$

Book Section 3.3:

(135) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}\,\mathrm{d}x$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x$$

$$x = a\sec\theta \quad\Longrightarrow\quad \mathrm{d}x = a\sec\theta\tan\theta\,\mathrm{d}\theta$$

$$\Longleftrightarrow \quad \theta = \sec^{-1}\left(\frac{x}{a}\right)$$

$$
\begin{aligned}
\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{a^2\sec^2\theta-a^2}}(a\sec\theta\tan\theta\,\mathrm{d}\theta)\\
&= \int \frac{1}{\sqrt{a^2(\sec^2\theta-1)}}(a\sec\theta\tan\theta\,\mathrm{d}\theta)\\
&= \int \frac{1}{\sqrt{a^2\tan^2\theta}}(a\sec\theta\tan\theta\,\mathrm{d}\theta)\\
&= \int \frac{1}{\cancel{a\tan\theta}}(\cancel{a}\sec\theta\cancel{\tan\theta}\,\mathrm{d}\theta)\\
&= \int \sec\theta\,\mathrm{d}\theta\\
&= \ln|\tan\theta+\sec\theta|+c\\
&= \ln\left|\tan\left(\sec^{-1}\left(\frac{x}{a}\right)\right)+\sec\left(\sec^{-1}\left(\frac{x}{a}\right)\right)\right|+c\\
&= \ln\left|\tan\left(\sec^{-1}\left(\frac{x}{a}\right)\right)+\frac{x}{a}\right|+c\\
&= \ln|\frac{x}{a}\sqrt{1-\frac{1}{\frac{x^2}{a^2}}}+\frac{x}{a}|+c\\
&= \ln\left|\frac{x}{a}\sqrt{\frac{x^2}{x^2}-\frac{a^2}{x^2}}+\frac{x}{a}\right|+c\\
&= \ln\left|\frac{x}{a}\sqrt{\left(\frac{1}{x^2}\right)x^2-a^2}+\frac{x}{a}\right|+c\\
&= \ln\left|\frac{1}{a}\sqrt{x^2-a^2}+\frac{x}{a}\right|+c\\
&= \ln\left|\frac{1}{a}(\sqrt{x^2-a^2}+x)\right|+c\\
&= \ln\left|\frac{1}{a}\right|+\ln|\sqrt{x^2-a^2}+x|+c\\
&= \ln|\sqrt{x^2-a^2}+x|+c
\end{aligned}
$$

(140) $\int \frac{\mathrm{d}x}{(1+x^2)^2}$

$$
\int \frac{\mathrm{d}x}{(1+x^2)^2} = \int \frac{1}{(1+x^2)^2}\,\mathrm{d}x
$$

$$
\begin{aligned}
x = \tan\theta \quad &\implies \quad \mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta \\
&\iff \quad \theta = \tan^{-1} x \\
&\iff \quad \sin\theta = \frac{x}{\sqrt{1+x^2}} \\
&\iff \quad \cos\theta = \frac{1}{\sqrt{1+x^2}}
\end{aligned}
$$

$$
\begin{aligned}
\int \frac{1}{(1+x^2)^2}\,\mathrm{d}x &= \int \frac{1}{(1+\tan^2\theta)^2}(\sec^2\theta\,\mathrm{d}\theta) \\
&= \int \frac{1}{\sec^4\theta}(\sec^2\,\mathrm{d}\theta) \\
&= \int \frac{1}{\sec^2\theta}\,\mathrm{d}\theta \\
&= \int \cos^2\theta\,\mathrm{d}\theta \\
&= \int \frac{1}{2}(\cos(2\theta)+1)\,\mathrm{d}\theta \\
&= \frac{1}{2}\int \cos(2\theta)+1\,\mathrm{d}\theta \\
&= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + c \\
&= \frac{1}{2}\left(\theta + \frac{1}{2}2\sin\theta\cos\theta\right) + c \\
&= \frac{1}{2}(\theta + \sin\theta\cos\theta) + c \\
&= \frac{1}{2}\left(\tan^{-1}(x) + \frac{x}{1+x^2}\right) + c
\end{aligned}
$$

(142) $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,\mathrm{d}x$

$$
\begin{aligned}
r^2 = 25 \quad &\implies \quad r = 5 \\
x = r\sec\theta \quad &\implies \quad \mathrm{d}x = r\sec\theta\tan\theta\,\mathrm{d}\theta \\
&\iff \quad \theta = \sec^{-1}\left(\frac{x}{r}\right) \\
&\iff \quad \sec\theta = \frac{x}{r} \\
&\iff \quad \tan\theta = \frac{\sqrt{x^2 - r^2}}{r}
\end{aligned}
$$

$$
\begin{aligned}
\int \frac{\sqrt{x^2 - 25}}{x}\,\mathrm{d}x &= \int \frac{\sqrt{r^2\sec^2\theta - r^2}}{r\sec\theta}(r\sec\theta\tan\theta\,\mathrm{d}\theta) \\
&= \int \frac{\sqrt{r^2(\sec^2\theta - 1)}}{r\sec\theta}(r\sec\theta\tan\theta\,\mathrm{d}\theta) \\
&= \int \frac{\sqrt{r^2\tan^2\theta}}{r\sec\theta}(r\sec\theta\tan\theta\,\mathrm{d}\theta) \\
&= \int \frac{r\tan\theta}{\cancel{r\sec\theta}}(\cancel{r\sec\theta}\tan\theta\,\mathrm{d}\theta) \\
&= r\int \tan^2\theta\,\mathrm{d}\theta \\
&= r\int \frac{\sin^2\theta}{\cos^2\theta}\,\mathrm{d}\theta \\
&= r\int \frac{1 - \cos^2\theta}{\cos^2\theta}\,\mathrm{d}\theta \\
&= r\int \frac{1}{\cos^2\theta} - 1\,\mathrm{d}\theta \\
&= r\int \sec^2(\theta) - 1\,\mathrm{d}\theta \\
&= r(\tan(\theta) - \theta) + c \\
&= r\left(\frac{\sqrt{x^2 - r^2}}{r} - \sec^{-1}\left(\frac{x}{r}\right)\right) + c \\
&= \sqrt{x^2 - r^2} - r\sec^{-1}\left(\frac{x}{r}\right) + c \\
&= \sqrt{x^2 - 25} - 5\sec^{-1}\left(\frac{x}{5}\right) + c
\end{aligned}
$$

(153) $\int_{-1}^{1} (1 - x^2)^{3/2}\,\mathrm{d}x$

$$\int_{-1}^{1} (1 - x^2)^{3/2}\,\mathrm{d}x = \int_{-1}^{1} \sqrt{(1 - x^2)^3}\,\mathrm{d}x$$

$$\begin{aligned}
x = \sin\theta \quad &\implies \quad \mathrm{d}x = \cos\theta\,\mathrm{d}\theta \\
&\iff \quad \theta = \sin^{-1}(x) \\
x = \sin\theta = -1 \quad &\implies \quad \theta = -\frac{\pi}{2} \\
x = \sin\theta = 1 \quad &\implies \quad \theta = \frac{\pi}{2}
\end{aligned}$$

$$
\begin{aligned}
\int_{-1}^{1} \sqrt{(1-x^2)^3}\,\mathrm{d}x &= \int_{-\pi/2}^{\pi/2} \sqrt{(1-\sin^2\theta)^3}\cdot(\cos\theta\,\mathrm{d}\theta) \\
&= \int_{-\pi/2}^{\pi/2} \sqrt{(\cos^2\theta)^3}\cdot(\cos\theta\,\mathrm{d}\theta) \\
&= \int_{-\pi/2}^{\pi/2} \sqrt{(\cos^2\theta)^2\cos^2\theta}\cdot(\cos\theta\,\mathrm{d}\theta) \\
&= \int_{-\pi/2}^{\pi/2} \cos^2\theta\cos\theta\cdot(\cos\theta\,\mathrm{d}\theta) \\
&= \int_{-\pi/2}^{\pi/2} \cos^2\theta\cos^2\theta\,\mathrm{d}\theta \\
&= \int_{-\pi/2}^{\pi/2} \frac{1}{2}(\cos(2\theta)+1)\cdot\frac{1}{2}(\cos(2\theta)+1)\,\mathrm{d}\theta \\
&= \frac{1}{4}\int_{-\pi/2}^{\pi/2} (\cos(2\theta)+1)^2\,\mathrm{d}\theta \\
&= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \cos^2(2\theta)+1+2\cos(2\theta)\,\mathrm{d}\theta \\
&= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \frac{1}{2}(\cos(4\theta)+1)+1+2\cos(2\theta)\,\mathrm{d}\theta \\
&= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \frac{1}{2}\cos(4\theta)+\frac{1}{2}+1+2\cos(2\theta)\,\mathrm{d}\theta \\
&= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \frac{1}{2}(\cos(4\theta)+1+2+4\cos(2\theta))\,\mathrm{d}\theta \\
&= \frac{1}{8}\int_{-\pi/2}^{\pi/2} (\cos(4\theta)+4\cos(2\theta)+3)\,\mathrm{d}\theta \\
&= \frac{1}{8}\left[\frac{\sin(4\theta)}{4}+4\sin(2\theta)+3\theta\right]_{-\pi/2}^{\pi/2} \\
&= \frac{1}{8}\left(\frac{3}{2}\pi-\left(-\frac{3}{2}\pi\right)\right) \\
&= \frac{3}{8}\pi
\end{aligned}
$$