Homework 10

Mos Kullathon
921425216

1. Determine whether the following series is convergent or divergent. We may use any tests available

(i) $\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}$

As a p-series where $p = 2$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Since $\dfrac{1}{n^2} \geq \dfrac{1}{n^2 2^n}$ for $n \geq 1$, by comparison test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}$ converges.

(ii) $\sum_{n=2}^{\infty} \frac{3n^3+2n^2+2}{n^4-1}$

Let $a_n = \dfrac{3n^3 + 2n^2 + 2}{n^4 - 1}$ and $b_n = \dfrac{1}{n}$.

Then,

$$\frac{a_n}{b_n} = \frac{n(3n^3 + 2n^2 + 2)}{n^4 - 1} = \frac{3n^4 + 2n^3 + 2n}{n^4 - 1}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n^4 + 2n^3 + 2n}{n^4 - 1} = 3.$$

Since $\dfrac{a_n}{b_n} \to 3$ and $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series such that $p = 1$ (and as such, diverges), by limit comparison test $\displaystyle\sum_{n=2}^{\infty} \frac{3n^3 + 2n^2 + 2}{n^4 - 1}$ also diverges.

(iii) $\sum_{n=1}^{\infty} \frac{n^{1/3}}{(n^{3/2}-1)^{1/2}}$

The first term is undefined. Therefore, the sum does not exist.

$$n = 1 \implies \frac{1^{1/3}}{(1^{3/2} - 1)^{1/2}} = \frac{1}{0}$$

(iv) $\sum_{n=1}^{\infty} \frac{n^2 \ln n}{n^5+2n-1}$

Let $a_n = \dfrac{n^2 \ln n}{n^5+2n-1}$ and $b_n = \dfrac{n^\epsilon}{n^3}$ for some small values of $\epsilon > 0$.

Then,

$$\frac{a_n}{b_n} = \frac{n^2 \ln n}{n^5+2n-1} \cdot \frac{n^3}{n^\epsilon} = \frac{n^5 \ln n}{n^\epsilon(n^5+2n-1)}$$

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^\epsilon} = 0.$$

Since $\dfrac{a_n}{b_n} \to 0$ and $\displaystyle\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{n^\epsilon}{n^3}$ converges by p-series for $0 < \epsilon < 2$, then by limit comparison test, $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 \ln n}{n^5+2n-1}$ also converges.

(v) $\sum_{n=1}^{\infty} (-1)^{n^2} \frac{n+1}{n!}$

Let $a_n = (-1)^{n^2} \dfrac{n+1}{n!}$.

Then,

$$a_{n+1} = \frac{(-1)^{(n+1)^2}(n+1)+1}{(n+1)!} = \frac{(-1)^{n^2+2n+1}(n+2)}{(n+1)n!}$$

and

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{\cancel{n^2}+2n+1}(n+2)}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{(-1)^{n^2}}(n+1)}\right|$$

$$= \left|\frac{(-1)^{2n+1}(n+2)}{(n+1)^2}\right|$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1.$$

By ratio test, the series converges absolutely.

(vi) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$

Let $a_n = \dfrac{(n!)^3}{(3n)!}$.

Then,

$$\begin{aligned}
a_{n+1} &= \frac{((n+1)!)^3}{(3(n+1))!} \\
&= \frac{((n+1)!)^3}{(3n+3)!} \\
&= \frac{((n+1)n!)^3}{(3n+3)(3n+2)(3n+1)(3n)!} \\
&= \frac{(n+1)^3(n!)^3}{(3n+3)(3n+2)(3n+1)(3n)!}
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)^3\cancel{(n!)^3}}{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}} \cdot \frac{\cancel{(3n)!}}{\cancel{(n!)^3}}\right| \\
&= \left|\frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)}\right| \\
\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{1}{3^3} = \frac{1}{27} < 1.
\end{aligned}$$

By ratio test, the series converges absolutely.

(vii) $\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$

Let $a_n = \left(\dfrac{n}{3n+1}\right)^n$.

Then,

$$|a_n|^{\frac{1}{n}} = \left|\left(\frac{n}{3n+1}\right)^n\right|^{\frac{1}{n}} = \left|\frac{n}{3n+1}\right| = \frac{n}{3n+1}, \quad n \geq 1$$

and

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{3n+1} = \frac{1}{3} < 1.$$

By root test, the series converges absolutely.

(viii) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

Let $a_n = \dfrac{1}{\ln n}$.

Since $\ln n$ is monotonically increasing, its reciprocal $a_n = \dfrac{1}{\ln n}$ must be monotonically decreasing.

Then,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln n} = 0.$$

By alternating series test, the series converges.

(ix) $\sum_{n=2}^{\infty} (-1)^n \ln n$

$$\lim_{n\to\infty} \ln n = \infty$$

By divergence test, the series diverges.

2. Find the radius of convergence and interval of convergence for the following power series

(i) $\sum_{n=0}^{\infty} \frac{1}{n2^n} x^n$

$$a_n = \frac{x^n}{n2^n}, \quad a_{n+1} = \frac{x^{n+1}}{(n+1)2^{n+1}}$$

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{x^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{x^n}\right| \\ &= \left|\frac{xn}{2(n+1)}\right| \\ &= |x|\frac{n}{2n+2}, \quad n \geq 0\end{aligned}$$

$$\therefore L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{2} \implies L < 1 \iff |x| < 2$$

Since

$$|x| < 2 \iff -2 < x < 2,$$

by ratio test, the series converges absolutely for $-2 < x < 2$ centered at $x = 0$.

$$x = -2 \implies \sum_{n=0}^{\infty} \frac{1}{n2^n}(-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

Since $\dfrac{1}{n} \to 0$, by alternating series test, the series converges at $x = -2$.

$$x = 2 \implies \sum_{n=0}^{\infty} \frac{1}{n2^n}(2)^n = \sum_{n=0}^{\infty} \frac{1}{n}$$

At $x = 2$, the series is a p-series such that $p = 1$. Therefore, the series diverges at $x = 2$.

As such:

- the radius of convergence is 2
- the interval of convergence is $[-2, 2)$.

(ii) $\sum_{n=0}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$

$$a_n = \frac{(x-1)^n}{\sqrt{n}}, \quad a_{n+1} = \frac{(x-1)^{n+1}}{\sqrt{n+1}}$$

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(x-1)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(x-1)^n}\right| \\ &= \left|\frac{\sqrt{n}(x-1)}{\sqrt{n+1}}\right| \\ &= |x-1|\frac{\sqrt{n}}{\sqrt{n+1}}, \quad n \geq 0\end{aligned}$$

$$\therefore L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-1| \implies L < 1 \iff |x-1| < 1$$

Since

$$|x-1| < 1 \iff -1 < x-1 < 1 \iff 0 < x < 2,$$

by ratio test, the series converges absolutely for $0 < x < 2$ centered at $x = 1$.

$$x = 0 \implies \sum_{n=0}^{\infty} \frac{(0-1)^n}{\sqrt{n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Since $\dfrac{1}{\sqrt{n}} \to 0$, by alternating series test, the series converges at $x = 0$.

$$x = 2 \implies \sum_{n=0}^{\infty} \frac{(2-1)^n}{\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1^n}{\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{n^{1/2}}$$

At $x = 2$, the series is a p-series such that $p = \dfrac{1}{2}$. Therefore, the series diverges at $x = 2$.

As such:

- the radius of convergence is 1
- the interval of convergence is $[0, 2)$.

(iii) $\sum_{n=2}^{\infty} \frac{1}{n \ln n} x^n$

$$a_n = \frac{x^n}{n \ln n}, \quad a_{n+1} = \frac{x^{n+1}}{(n+1)\ln(n+1)}$$

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{x^{n+1}}{(n+1)\ln(n+1)} \cdot \frac{n \ln n}{x^n}\right| \\ &= \left|\frac{nx \ln n}{(n+1)\ln(n+1)}\right|\end{aligned}$$

$$\begin{aligned}\therefore L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{nx\ln n}{(n+1)\ln(n+1)}\right| \\ &= \lim_{n\to\infty}\left|\frac{nx}{n+1}\right| \cdot \lim_{n\to\infty}\left|\frac{\ln n}{\ln(n+1)}\right| \\ &= |x| \cdot \lim_{n\to\infty}\left|\frac{\frac{1}{n}}{\frac{1}{n+1}}\right| \\ &= |x|\end{aligned}$$

$$\implies L < 1 \iff |x| < 1$$

Since

$$|x| < 1 \iff -1 < x < 1,$$

by ratio test, the series converges absolutely for $-1 < x < 1$.

$$x = -1 \implies \sum_{n=2}^{\infty} \frac{1}{n \ln n} (-1)^n$$

Since $\dfrac{1}{n \ln n} \to 0$, by alternating series test, the series converges at $x = -1$.

$$x = 1 \implies \sum_{n=2}^{\infty} \frac{1}{n \ln n} (1)^n = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Let $g(x) = \dfrac{1}{x \ln x}$. Since $g'(x) < 0$ for $x \geq 2$, we apply the integral test.

$$\begin{aligned} u = \ln x \quad &\implies \quad \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x \\ &\iff \quad \mathrm{d}x = x\,\mathrm{d}u \\ x = 2 \quad &\implies \quad u = \ln 2 \\ x = n \quad &\implies \quad u = \ln n \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} \int_2^n \frac{1}{x \ln x}\,\mathrm{d}x &= \lim_{n\to\infty} \int_{\ln 2}^{\ln n} \frac{x\,\mathrm{d}u}{xu} \\ &= \lim_{n\to\infty} \Big[\ln u \Big]_{\ln 2}^{\ln n} \\ &= \lim_{n\to\infty} \ln(\ln n) - \ln(\ln 2) \\ &= \infty \end{aligned}$$

By integral test, the series diverges at $x = 1$.

As such:

- the radius of convergence is 1
- the interval of convergence is $[-1, 1)$.

(iv) $\sum_{n=2}^{\infty}(2+\frac{1}{n})^n x^n$

$$a_n = \left(2+\frac{1}{n}\right)^n x^n$$

$$|a_n|^{\frac{1}{n}} = \left|\left(2+\frac{1}{n}\right)x\right|$$

$$\therefore L = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = |2x| \implies L < 1 \iff |2x| < 1$$

Since

$$|2x| < 1 \iff |x| < \frac{1}{2} \iff -\frac{1}{2} < x < \frac{1}{2},$$

by root test, the series converges absolutely for $-\frac{1}{2} < x < \frac{1}{2}$ centered at $x = 0$.

$$x = -\frac{1}{2} \implies \sum_{n=2}^{\infty}\left(2+\frac{1}{n}\right)^n \left(-\frac{1}{2}\right)^n$$

$$a_n = \left(2+\frac{1}{n}\right)^n \left(-\frac{1}{2}\right)^n$$

$$|a_n|^{\frac{1}{n}} = \left|\left(2+\frac{1}{n}\right)\left(-\frac{1}{2}\right)\right|$$

Since $|a_n|^{\frac{1}{n}} \to -1$, by root test, the series converges at $x = -\frac{1}{2}$.

$$x = \frac{1}{2} \implies \sum_{n=2}^{\infty}\left(2+\frac{1}{n}\right)^n \left(\frac{1}{2}\right)^n$$

$$\begin{aligned}
\lim_{n\to\infty}\left(2+\frac{1}{n}\right)^n\left(\frac{1}{2}\right)^n &= \lim_{n\to\infty}\left(\left(2+\frac{1}{n}\right)\left(\frac{1}{2}\right)\right)^n \\
&= \lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n \\
&= \lim_{n\to\infty} e^{n\ln\left(1+\frac{1}{2n}\right)} \\
&= \lim_{n\to\infty}\exp\frac{\ln\left(1+\frac{1}{2n}\right)}{1/n} \\
&= \exp\lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{2n}}\cdot-\frac{1}{2n^2}}{-\frac{1}{n^2}} \\
&= \exp\lim_{n\to\infty}\frac{1}{1+\frac{1}{2n}}\cdot\frac{1}{2} \\
&= \exp\frac{1}{2} \\
&= \sqrt{e} \quad \approx 1.64872
\end{aligned}$$

Since $\left(2+\frac{1}{n}\right)^n\left(\frac{1}{2}\right)^n \to \sqrt{e}$, by divergence test, the series diverges at $x=\frac{1}{2}$.

As such:

- the radius of convergence is $\frac{1}{2}$
- the interval of convergence is $\left[-\frac{1}{2},\frac{1}{2}\right)$.

3. Section 6.1

In the following exercises, given that $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ with convergence in $(-1,1)$, find the power series for each function with the given center *a*, and identify its interval of convergence.

(35) $f(x)=\frac{x}{1-x^2}; a=0$

$$
\begin{aligned}
f(x) &= \frac{x}{1-x^2} \\
&= \sum_{n=0}^{\infty} x(x^2)^n, \quad |x^2| < 1 \\
&= \sum_{n=0}^{\infty} x^{2n+1}, \quad |x| < 1
\end{aligned}
$$

Since

$$|x| < 1 \iff -1 < x < 1,$$

$f(x)$ converges within $-1 < x < 1$.

Then,

$$f(-1) = \sum_{n=0}^{\infty} (-1)^{2n+1}$$

Since $\lim\limits_{n \to \infty} (-1)^{2n+1}$ does not exist, $f(-1)$ diverges by divergence test.

$$f(1) = \sum_{n=0}^{\infty} 1^{2n+1}$$

Since $\lim\limits_{n \to \infty} 1^{2n+1} = 1$, $f(1)$ diverges by divergence test.

As such,

$$f(x) = \frac{x}{1-x^2} = \sum_{n=0}^{\infty} x^{2n+1}, \quad x \in (-1, 1).$$

(37) $f(x) = \frac{x^2}{1+x^2}; a = 0$

$$
\begin{aligned}
f(x) &= \frac{x^2}{1+x^2} \\
&= \frac{x^2}{1-(-x^2)} \\
&= \sum_{n=0}^{\infty} x^2(-x^2)^n, \quad \left|-x^2\right| < 1 \\
&= \sum_{n=0}^{\infty} (-1)^n x^{2n+2}, \quad |x| < 1
\end{aligned}
$$

Since

$$|x| < 1 \iff -1 < x < 1,$$

$f(x)$ converges within $-1 < x < 1$.

Then,

$$f(-1) = \sum_{n=0}^{\infty} (-1)^n (-1)^{2n+2} = \sum_{n=0}^{\infty} (-1)^{3n+2}$$

Since $\lim\limits_{n\to\infty} (-1)^{3n+2}$ does not exist, $f(-1)$ diverges by divergence test.

$$f(1) = \sum_{n=0}^{\infty} (-1)^n (1)^{2n+2} = \sum_{n=0}^{\infty} (-1)^n$$

Since $\lim\limits_{n\to\infty} (-1)^n$ does not exist, $f(1)$ diverges by divergence test.

As such,

$$f(x) = \frac{x^2}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n+2}, \quad x \in (-1, 1)$$

(39) $f(x) = \frac{1}{1-2x}; a = 0$

$$
\begin{aligned}
f(x) &= \frac{1}{1-2x} \\
&= \sum_{n=0}^{\infty} (2x)^n, \quad |2x| < 1 \\
&= \sum_{n=0}^{\infty} 2^n x^n, \quad |x| < \frac{1}{2}
\end{aligned}
$$

Since

$$
|x| < \frac{1}{2} \iff -\frac{1}{2} < x < \frac{1}{2},
$$

$f(x)$ converges within $-\dfrac{1}{2} < x < \dfrac{1}{2}$.

Then,

$$
f\left(-\frac{1}{2}\right) = \sum_{n=0}^{\infty} 2^n \left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n
$$

Since $\lim\limits_{n\to\infty} (-1)^n$ does not exist, $f\left(-\frac{1}{2}\right)$ diverges by divergence test.

$$
f\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} 1
$$

Since $\lim\limits_{n\to\infty} 1 = 1$, $f(1)$ diverges by divergence test.

As such,

$$
f(x) = \frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n, \quad x \in \left(-\frac{1}{2}, \frac{1}{2}\right)
$$

(41) $f(x) = \frac{x^2}{1-4x^2}; a = 0$

$$
\begin{aligned}
f(x) &= \frac{x^2}{1-4x^2} \\
&= \sum_{n=0}^{\infty} x^2 (4x^2)^n, \quad |4x^2| < 1 \\
&= \sum_{n=0}^{\infty} x^2 4^n x^{2n}, \quad |x^2| < \frac{1}{4} \\
&= \sum_{n=0}^{\infty} 4^n x^{2n+2}, \quad |x| < \frac{1}{2}
\end{aligned}
$$

Since

$$|x| < \frac{1}{2} \iff -\frac{1}{2} < x < \frac{1}{2},$$

$f(x)$ converges within $-\frac{1}{2} < x < \frac{1}{2}$.

Then,

$$f\left(-\frac{1}{2}\right) = \sum_{n=0}^{\infty} 4^n \left(-\frac{1}{2}\right)^{2n+2}.$$

Since $\lim_{n\to\infty} 4^n \left(-\frac{1}{2}\right)^{2n+2}$ does not exist, by divergence test $f\left(-\frac{1}{2}\right)$ diverges.

$$f\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} 4^n \left(\frac{1}{2}\right)^{2n+2}$$

Since

$$
\begin{aligned}
\lim_{n\to\infty} 4^n \left(\frac{1}{2}\right)^{2n+2} &= \lim_{n\to\infty} \frac{4^n}{2^{2n+2}} \\
&= \lim_{n\to\infty} \frac{4^n}{2^{2n}\cdot 2^2} \\
&= \lim_{n\to\infty} \frac{4^n}{(2^2)^n\cdot 2^2} \\
&= \lim_{n\to\infty} \frac{1}{4} \\
&= \frac{1}{4}
\end{aligned}
$$

by divergence test $f\left(\frac{1}{2}\right)$ diverges.

As such,

$$
f(x) = \frac{x^2}{1-4x^2} = \sum_{n=0}^{\infty} 4^n x^{2n+2}, \quad x \in \left(-\frac{1}{2}, \frac{1}{2}\right)
$$

4. Find

(i) The Taylor polynomial of degree 3 of $f(x) = x^{2/3}$ at $x = 8$

$$
\begin{aligned}
f(x) = x^{2/3} &\implies f'(x) = \frac{2}{3}x^{-1/3} \\
&\implies f''(x) = -\frac{2}{9}x^{-4/3} \\
&\implies f'''(x) = \frac{8}{27}x^{-7/3}
\end{aligned}
$$

$$
\begin{aligned}
f(8) &= 8^{2/3} &&= 4 \\
f'(8) &= \frac{2}{3}\cdot 8^{-1/3} &&= \frac{1}{3} \\
f''(8) &= -\frac{2}{9}\cdot 8^{-4/3} &&= -\frac{1}{72} \\
f'''(8) &= \frac{8}{27}\cdot 8^{-7/3} &&= \frac{1}{432}
\end{aligned}
$$

$$\therefore \sum_{n=0}^{3} \frac{f^{(n)}(x)}{n!}(x-8)^n = f(8) + f'(8)(x-8) + \frac{f''(8)(x-8)^2}{2!} + \frac{f''(8)(x-8)^3}{3!}$$

$$= 4 + \frac{x-8}{3} - \frac{(x-8)^2}{72 \cdot 2!} + \frac{(x-8)^3}{432 \cdot 3!}$$

$$= 4 + \frac{x-8}{3} - \frac{(x-8)^2}{144} + \frac{(x-8)^3}{2592}$$

(ii) The Taylor polynomial of degree 2 of $f(x) = \sec x$ at $x = 0$

$$f(x) = \sec x \implies f'(x) = \sec x \tan x$$

$$\implies f''(x) = (\sec x \tan^2 x) + \sec^3 x$$

$$\begin{aligned} f(0) &= \sec 0 && = 1 \\ f'(0) &= \sec 0 \tan 0 && = 0 \\ f''(0) &= (\sec 0 \tan^2 0) + \sec^3 0 && = 1 \end{aligned}$$

$$\therefore \sum_{n=0}^{2} \frac{f^{(n)}(x)}{n!}(x-0)^n = f(0) + f'(0)x + \frac{f''(0)x^2}{2!}$$

$$= 1 + 0x + \frac{x^2}{2!}$$

$$= 1 + \frac{x^2}{2}$$

5. Find the Taylor series expansion of the following functions using the formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Identify their interval of convergence. You may need to use differentiation and integration of the Taylor series.

▪

> **Center of convergence not specified**
> The following answers for questions 5(i) through 5(iii) assumes that the center of convergence is at $x = 0$.

(i) $\frac{1}{2+3x}$

Let the center of convergence be at $x = 0$.

$$
\begin{aligned}
\frac{1}{2+3x} &= \frac{1}{2(1+\frac{3}{2}x)} \\
&= \frac{1}{2} \cdot \frac{1}{1-(-\frac{3}{2}x)} \\
&= \sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{3}{2}x\right)^n, \quad \left|-\frac{3}{2}x\right| < 1 \\
&= \sum_{n=0}^{\infty} \frac{(-3)^n}{2(2)^n} x^n, \quad |x| < \frac{2}{3} \\
&= \sum_{n=0}^{\infty} \frac{(-3)^n}{2^{n+1}} x^n, \quad |x| < \frac{2}{3}
\end{aligned}
$$

Since

$$
|x| < \frac{2}{3} \iff -\frac{2}{3} < x < \frac{2}{3},
$$

the series converges within $-\frac{2}{3} < x < \frac{2}{3}$.

Then,

$$
x = -\frac{2}{3} \implies \sum_{n=0}^{\infty} \frac{(-3)^n}{2^{n+1}} \left(-\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} \frac{2^n}{2^{n+1}}
$$

Since $\frac{2^n}{2^{n+1}} \to \frac{1}{2}$, the series diverges by divergence test at $x = -\frac{2}{3}$.

$$
x = \frac{2}{3} \implies \sum_{n=0}^{\infty} \frac{(-3)^n}{2^{n+1}} \left(\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{2^{n+1}}
$$

Since $\frac{2^n}{2^{n+1}} \to \frac{1}{2}$, the series diverges by divergence (and therefore the limit of the summand does not exist) test at $x = \frac{2}{3}$.

As such, the interval of convergence for a series centered at $x = 0$ is $\left(-\frac{2}{3}, \frac{2}{3}\right)$.

(ii) $\frac{1}{(1-2x)^2}$

Let the center of convergence be at $x = 0$.

$$\begin{aligned}\frac{1}{1-2x} &= \sum_{n=0}^{\infty}(2x)^n, \quad |2x| < 1 \\ &= \sum_{n=0}^{\infty} 2^n x^n, \quad |x| < \frac{1}{2}\end{aligned}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-2x}\right) = \frac{1}{(1-2x)^2},$$

then

$$\begin{aligned}\frac{1}{(1-2x)^2} &= \sum_{n=1}^{\infty} 2n(2x)^{n-1}, \quad |x| < \frac{1}{2} \\ &= \sum_{n=1}^{\infty} 2^n n x^{n-1}, \quad |x| < \frac{1}{2}.\end{aligned}$$

And since

$$|x| < -\frac{1}{2} \iff -\frac{1}{2} < x < \frac{1}{2},$$

the series converges within $-\frac{1}{2} < x < \frac{1}{2}$.

Then,

$$x = -\frac{1}{2} \implies \sum_{n=1}^{\infty}(-1)^{n-1} 2n$$

Since $2n \to \infty$, the series diverges by divergence test at $x = -\frac{1}{2}$.

$$x = \frac{1}{2} \implies \sum_{n=1}^{\infty} 2n$$

Since $2n \to \infty$, the series diverges by divergence test at $x = \frac{1}{2}$.

As such, the interval of convergence for a series centered at $x = 0$ is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

(iii) $\ln(1 + x)$

Let the center of convergence be at $x = 0$.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$$

Since

$$\int \frac{1}{1+x} \, \mathrm{d}x = \ln(1+x),$$

then

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}, \quad |x| < 1.$$

And since

$$|x| < 1 \iff -1 < x < 1,$$

the series converges within $-1 < x < 1$.

Then,

$$x = -1 \implies \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{1}{n+1} = \sum_{n=0}^{\infty} -\frac{1}{n+1}$$

Since $-\frac{1}{n+1} \approx -\frac{1}{n}$ for large values of n, the series diverges at $x = -1$ by p-series test.

$$x = 1 \implies \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$$

Since $\frac{1}{n+1} \to 0$, the series converge at $x = 1$ by alternating series test.

As such, the interval of convergence for a series centered at $x = 0$ is $(-1, 1]$.