Homework 2

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1. Evaluate the following indefinite integrals

   a. $\displaystyle\int \frac{1}{(5-3x)^2}\,\mathrm{d}x$

$$\begin{aligned} u = 5 - 3x \quad &\implies \quad \mathrm{d}u = -3\,\mathrm{d}x \\ &\iff \quad \mathrm{d}x = -\frac{1}{3}\,\mathrm{d}u \end{aligned}$$

$$\begin{aligned} \therefore \int \frac{1}{(5-3x)^2}\,\mathrm{d}x &= \int \frac{1}{u^2}\left(-\frac{1}{3}\,\mathrm{d}u\right) \\ &= -\frac{1}{3}\int u^{-2}\,\mathrm{d}u \\ &= -\frac{1}{3}\cdot\frac{u^{-1}}{-1} + c \\ &= \frac{u^{-1}}{3} + c \\ &= \frac{1}{3(5-3x)} + c \\ &= \frac{1}{15-9x} + c \end{aligned}$$

   b. $\displaystyle\int y(1-y^2)^{2/3}\,\mathrm{d}y$

$$\begin{aligned} u = 1 - y^2 \quad &\implies \quad \mathrm{d}u = -2y\,\mathrm{d}y \\ &\iff \quad \mathrm{d}y = \frac{1}{-2y}\,\mathrm{d}u \end{aligned}$$

$$
\begin{aligned}
\therefore \int y(1-y^2)^{2/3}\,\mathrm{d}y &= \int y u^{2/3}\left(\frac{1}{-2y}\,\mathrm{d}u\right) \\
&= -\frac{1}{2}\int \cancel{y} u^{2/3}\left(\frac{1}{\cancel{y}}\,\mathrm{d}u\right) \\
&= -\frac{1}{2}\int u^{2/3}\,\mathrm{d}u \\
&= -\frac{1}{2}\left(\frac{u^{5/3}}{5/3}\right) + c \\
&= -\frac{1}{2}\left(\frac{3u^{5/3}}{5}\right) + c \\
&= -\frac{3u^{5/3}}{10} + c \\
&= -\frac{3(1-y^2)^{5/3}}{10} + c
\end{aligned}
$$

c. $\int \tan^7 \frac{x}{2} \sec^2 \frac{x}{2}\,\mathrm{d}x$

$$
\begin{aligned}
u = \tan\frac{x}{2} \quad &\implies \quad \mathrm{d}u = \frac{1}{2}\sec^2\frac{x}{2}\,\mathrm{d}x \\
&\iff \quad 2\,\mathrm{d}u = \sec^2\frac{x}{2}\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
\therefore \int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,\mathrm{d}x &= \int 2u^7\,\mathrm{d}u \\
&= 2\int u^7\,\mathrm{d}u \\
&= 2\left(\frac{u^8}{8}\right) + c \\
&= 2\left(\frac{\tan^8\frac{x}{2}}{8}\right) + c \\
&= \frac{2}{8}\tan^8\left(\frac{x}{2}\right) + c \\
&= \frac{1}{4}\tan^8\left(\frac{x}{2}\right) + c
\end{aligned}
$$

d. $\int t\sqrt{4+t}\,\mathrm{d}t$

$$
\begin{aligned}
u = 4 + t \quad &\implies \quad \mathrm{d}u = \mathrm{d}t \\
&\iff \quad t = u - 4
\end{aligned}
$$

$$
\begin{aligned}
\therefore \int t\sqrt{4+t}\,\mathrm{d}t &= \int (u-4)\sqrt{u}\,\mathrm{d}u \\
&= \int u^{1/2}(u-4)\,\mathrm{d}u \\
&= \int (u^{3/2} - 4u^{1/2})\,\mathrm{d}u \\
&= \frac{u^{5/2}}{5/2} - \frac{4u^{3/2}}{3/2} + c \\
&= \frac{2u^{5/2}}{5} - \frac{8u^{3/2}}{3} + c \\
&= \frac{2(4+t)^{5/2}}{5} - \frac{8(4+t)^{3/2}}{3} + c
\end{aligned}
$$

e. $\displaystyle\int \frac{\ln\sqrt{t}}{t}\,\mathrm{d}t$

$$
\begin{aligned}
\int \frac{\ln\sqrt{t}}{t}\,\mathrm{d}t &= \int \frac{\ln t^{1/2}}{t}\,\mathrm{d}t \\
&= \int \frac{\frac{1}{2}\ln t}{t}\,\mathrm{d}t \\
&= \frac{1}{2}\int \frac{\ln t}{t}\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
u = \ln t \quad &\implies \quad \mathrm{d}u = \frac{1}{t}\,\mathrm{d}t \\
&\iff \quad \mathrm{d}t = t\,\mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
\therefore \frac{1}{2}\int \frac{\ln t}{t}\,\mathrm{d}t &= \frac{1}{2}\int \frac{u}{\cancel{t}}(\cancel{t}\,\mathrm{d}u) \\
&= \frac{1}{2}\int u\,\mathrm{d}u \\
&= \frac{1}{2}\left(\frac{u^2}{2}\right) + c \\
&= \frac{1}{2}\left(\frac{\ln^2 t}{2}\right) + c \\
&= \frac{\ln^2 t}{4} + c
\end{aligned}
$$

f. $\int (3x^2 + 2x)e^{x^3+x^2+1}\,\mathrm{d}x$

$$
u = x^3 + x^2 + 1 \quad \Longrightarrow \quad \mathrm{d}u = (3x^2 + 2x)\,\mathrm{d}x
$$

$$
\begin{aligned}
\therefore \int (3x^2 + 2x)e^{x^3+x^2+1}\,\mathrm{d}x &= \int e^u\,\mathrm{d}u \\
&= e^u + c \\
&= e^{x^3+x^2+1} + c
\end{aligned}
$$

2. Evaluate the following definite integrals

a. $\int_0^1 (3t-1)^{50}\,\mathrm{d}t$

$$
\begin{aligned}
u = 3t - 1 \quad &\Longrightarrow \quad \mathrm{d}u = 3\,\mathrm{d}t \\
&\Longleftrightarrow \quad \mathrm{d}t = \frac{1}{3}\,\mathrm{d}u \\
t = 0 \quad &\Longrightarrow \quad u = 3(0) - 1 = -1 \\
t = 1 \quad &\Longrightarrow \quad u = 3(1) - 1 = 2
\end{aligned}
$$

$$\therefore \int_0^1 (3t-1)^{50}\,\mathrm{d}t = \int_{-1}^{2} \frac{1}{3}u^{50}\,\mathrm{d}u$$
$$= \frac{1}{3}\left[\frac{u^{51}}{51}\right]_{-1}^{2}$$
$$= \frac{1}{3}\left(\frac{2^{51}}{51} - \frac{-1^{51}}{51}\right)$$
$$= \frac{1}{3}\left(\frac{2^{51}+1}{51}\right)$$
$$= \frac{250199979298361}{17}$$

b. $\displaystyle\int_0^{\pi/3} (1+e^{\sin x})\cos x\,\mathrm{d}x$

$$u = \sin x \implies \mathrm{d}u = \cos x\,\mathrm{d}x$$
$$x = 0 \implies u = \sin 0 = 0$$
$$x = \frac{\pi}{3} \implies u = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\therefore \int_0^{\pi/3} (1+e^{\sin x})\cos x\,\mathrm{d}x = \int_0^{\frac{\sqrt{3}}{2}} (1+e^u)\,\mathrm{d}u$$
$$= \left[u+e^u\right]_0^{\frac{\sqrt{3}}{2}}$$
$$= \frac{\sqrt{3}}{2} + e^{\frac{\sqrt{3}}{2}} - 0 - e^0$$
$$= \frac{\sqrt{3}}{2} + e^{\frac{\sqrt{3}}{2}} - 1$$

c. $\displaystyle\int_1^e \frac{(\ln x)^2}{x}\,\mathrm{d}x$

$$u = \ln x \implies \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$$
$$x = 1 \implies u = \ln 1 = 0$$
$$x = e \implies u = \ln e = 1$$

$$\begin{aligned} \therefore \int_1^e \frac{(\ln x)^2}{x}\,\mathrm{d}x &= \int_0^1 u^2\,\mathrm{d}u \\ &= \left[\frac{u^3}{3}\right]_0^1 \\ &= \frac{1^3}{3} - \frac{0^3}{3} \\ &= \frac{1}{3} \end{aligned}$$

d. $\displaystyle\int_\pi^{\pi/2} \frac{\cos x}{\sin^4 x}\,\mathrm{d}x$

$$\begin{aligned} u = \sin x \quad &\Longrightarrow \quad \mathrm{d}u = \cos x\,\mathrm{d}x \\ x = \pi \quad &\Longrightarrow \quad u = \sin \pi = 0 \\ x = \frac{\pi}{2} \quad &\Longrightarrow \quad u = \sin \frac{\pi}{2} = 1 \end{aligned}$$

$$\begin{aligned} \therefore \int_\pi^{\pi/2} \frac{\cos x}{\sin^4 x}\,\mathrm{d}x &= \int_0^1 \frac{1}{u^4}\,\mathrm{d}u \\ &= \int_0^1 u^{-4}\,\mathrm{d}u \\ &= \left[\frac{u^{-3}}{-3}\right]_0^1 \\ &= \frac{1}{-3(1^3)} - \frac{1}{-3(0^3)} \\ &= -\frac{1}{3} + \frac{1}{0} \\ &= \text{DNE} \end{aligned}$$

3. Suppose that $\displaystyle\int_0^a xe^{-x^2}\,\mathrm{d}x = \frac{1}{3}$. Find a.

$$
\begin{aligned}
u = -x^2 \quad &\implies \quad \mathrm{d}u = -2x\,\mathrm{d}x \\
&\iff \quad \mathrm{d}x = \frac{1}{-2x}\,\mathrm{d}u = -\frac{1}{2}\cdot\frac{1}{x}\,\mathrm{d}u \\
x = 0 \quad &\implies \quad u = -0^2 = 0 \\
x = a \quad &\implies \quad u = -a^2
\end{aligned}
$$

$$
\begin{aligned}
\therefore \int_0^a xe^{-x^2}\,\mathrm{d}x &= \int_0^{-a^2} xe^u\,\mathrm{d}x\left(-\frac{1}{2}\cdot\frac{1}{x}\,\mathrm{d}u\right) \\
&= -\frac{1}{2}\int_0^{-a^2} e^u\,\mathrm{d}u \\
&= -\frac{1}{2}\Big[e^u\Big]_0^{-a^2} \\
&= -\frac{1}{2}\left(e^{-a^2} - e^0\right) \\
&= -\frac{1}{2}e^{-a^2} + \frac{1}{2} \\
&= \frac{1}{3} \\
-\frac{1}{2}e^{-a^2} + \frac{1}{2} &= \frac{1}{3} \\
-\frac{1}{2e^{a^2}} &= \frac{1}{3} - \frac{1}{2} \\
-\frac{1}{2e^{a^2}} &= -\frac{1}{6} \\
2e^{a^2} &= 6 \\
e^{a^2} &= 3 \\
a^2 &= \ln 3 \\
a &= \pm\sqrt{\ln 3}
\end{aligned}
$$

4. Find $\int \cot x\,\mathrm{d}x$ and $\int \csc x\,\mathrm{d}x$.

$$\begin{aligned}\int \cot x \,\mathrm{d}x &= \int \frac{1}{\tan x}\,\mathrm{d}x \\ &= \int \frac{\cos x}{\sin x}\,\mathrm{d}x\end{aligned}$$

$$u = \sin x \quad \Longrightarrow \quad \mathrm{d}u = \cos x \,\mathrm{d}x$$

$$\begin{aligned}\therefore \int \frac{\cos x}{\sin x}\,\mathrm{d}x &= \int \frac{1}{u}\,\mathrm{d}u \\ &= \int u^{-1}\,\mathrm{d}u \\ &= \ln(u) + c \\ &= \ln(\sin x) + c\end{aligned}$$

$$\begin{aligned}\int \csc x \,\mathrm{d}x &= \int \frac{\csc x(\cot x + \csc x)}{\cot x + \csc x}\,\mathrm{d}x \\ &= \int \frac{\csc(x)\cot(x) + \csc^2 x}{\cot x + \csc x}\,\mathrm{d}x\end{aligned}$$

$$\begin{aligned}u = \cot x + \csc x \quad \Longrightarrow \quad \mathrm{d}u &= (-\csc^2(x) - \csc(x)\cot(x))\,\mathrm{d}x \\ \Longleftrightarrow \quad -\mathrm{d}u &= (\csc^2(x) + \csc(x)\cot(x))\,\mathrm{d}x\end{aligned}$$

$$\begin{aligned}\therefore \int \csc x \,\mathrm{d}x &= \int \frac{-\mathrm{d}u}{u} \\ &= \int -\frac{1}{u}\,\mathrm{d}u \\ &= -\int u^{-1}\,\mathrm{d}u \\ &= -\ln(u) + c \\ &= -\ln(\cot x + \csc x) + c\end{aligned}$$

5. Section 1.6: 351, 353, 357.

351. $\int x \csc(x^2)\,\mathrm{d}x$

$$\begin{aligned} u = x^2 \quad &\implies \quad \mathrm{d}u = 2x\,\mathrm{d}x \\ &\iff \quad \mathrm{d}x = \frac{1}{2x}\,\mathrm{d}u \end{aligned}$$

$$\begin{aligned} \therefore \int x \csc(x^2)\,\mathrm{d}x &= \int \cancel{x} \csc(u) \left(\frac{1}{2\cancel{x}}\,\mathrm{d}u\right) \\ &= \frac{1}{2}\int \csc u\,\mathrm{d}u \end{aligned}$$

> From (4):
>
> $$\int \csc x\,\mathrm{d}x = -\ln(\cot x + \csc x) + c$$

$$\begin{aligned} \iff \frac{1}{2}\int \csc u\,\mathrm{d}u &= \frac{1}{2}(-\ln(\cot u + \csc u)) + c \\ &= -\frac{1}{2}\ln|\cot x^2 + \csc x^2| + c \end{aligned}$$

353. $\int \ln(\csc x)\cot x\,\mathrm{d}x$

$$\begin{aligned} u = \ln(\csc x) \quad &\implies \quad \mathrm{d}u = \frac{-\cancel{\csc x}\cot x}{\cancel{\csc x}}\,\mathrm{d}x \\ &\iff \quad \mathrm{d}x = -\frac{1}{\cot x}\,\mathrm{d}u \end{aligned}$$

$$\begin{aligned} \therefore \int \ln(\csc x)\cot x\,\mathrm{d}x &= \int -u\,\mathrm{d}u \\ &= -\int u\,\mathrm{d}u \\ &= -\frac{u^2}{2} + c \\ &= -\frac{\ln^2(\csc x)}{2} + c \end{aligned}$$

357. $\int_0^{\pi/3} \frac{\sin x - \cos x}{\sin x + \cos x}\,\mathrm{d}x$

$$
\begin{aligned}
u = \sin x + \cos x \quad &\implies \quad & \mathrm{d}u &= (\cos x - \sin x)\,\mathrm{d}x \\
&\iff & -\mathrm{d}u &= (-\cos x + \sin x)\,\mathrm{d}x \\
&& &= (\sin x - \cos x)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
x = 0 \quad &\implies \quad u = \sin 0 + \cos 0 &&= 1 \\
x = \frac{\pi}{3} \quad &\implies \quad u = \sin\frac{\pi}{3} + \cos\frac{\pi}{3} &&= \frac{1+\sqrt{3}}{2}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \int_1^{\pi/3} \frac{\sin x - \cos x}{\sin x + \cos x}\,\mathrm{d}x &= \int_1^{\frac{1+\sqrt{3}}{2}} \frac{-\mathrm{d}u}{u} \\
&= \int_1^{\frac{1+\sqrt{3}}{2}} -\frac{1}{u}\,\mathrm{d}u \\
&= -\int_1^{\frac{1+\sqrt{3}}{2}} u^{-1}\,\mathrm{d}u \\
&= -\Big[\ln u\Big]_1^{\frac{1+\sqrt{3}}{2}} \\
&= -\left(\ln\frac{1+\sqrt{3}}{2} - \ln 1\right) \\
&= -\ln\frac{1+\sqrt{3}}{2} \\
&= \ln(\sqrt{3}-1)
\end{aligned}
$$