Homework 4

Mos Kullathon
921425216

1. In lecture, we found the volume of a donut by revolving the cricles

$$(x-(R-r))^2+y^2=r^2$$

along the y-axis where $R>r$.

(a) Derive the integral formula for the volume of the donut. (It was covered in lass, but please rewrite it on your own)

First, we solve for x to get an expression in terms of y.

$$\begin{aligned}(x-(R-r))^2+y^2&=r^2\\(x-(R-r))^2&=r^2-y^2\\x-(R-r)&=\pm\sqrt{r^2-y^2}\\x&=(R-r)\pm\sqrt{r^2-y^2}\end{aligned}$$

We can then split the expression for x to produce two semicicles, where the positive square root component will produce the right side of the circle, and the negative component for the left side.

$$\begin{aligned}f_L(y)&=(R-r)-\sqrt{r^2-y^2}\\f_R(y)&=(R-r)+\sqrt{r^2-y^2}\end{aligned}$$

Revolving the right-hand component ($f_R(y)$) around the x-axis would give us the outside shape of the donut. Similarly, revolving the left-hand component ($f_L(y)$) around the x-axis would produce the inside portion (the hole) of the donut.

As such, the volume of the donut V would be the volume produced by revolving the outside minus the inside.

$$\begin{aligned}V&=\pi\int_{-r}^{r}f_R(y)^2-f_L(y)^2\,\mathrm{d}y\\&=\pi\int_{-r}^{r}\left((R-r)+\sqrt{r^2-y^2}\right)^2-\left((R-r)-\sqrt{r^2-y^2}\right)^2\,\mathrm{d}y\end{aligned}$$

Let $A = R - r$ and $B = \sqrt{r^2 - y^2}$. Then,

$$\begin{aligned}
&\left((R-r) + \sqrt{r^2 - y^2}\right)^2 - \left((R-r) - \sqrt{r^2 - y^2}\right)^2 \\
&= (A+B)^2 - (A-B)^2 \\
&= (\cancel{A^2 + B^2} + 2AB) - (\cancel{A^2 + B^2} - 2AB) \\
&= 2AB - (-2AB) \\
&= 4AB \\
&= 4(R-r)\sqrt{r^2 - y^2}.
\end{aligned}$$

And so, we have that:

$$\begin{aligned}
V &= \pi \int_{-r}^{r} \left((R-r) + \sqrt{r^2 - y^2}\right)^2 - \left((R-r) - \sqrt{r^2 - y^2}\right)^2 \mathrm{d}y \\
&= \pi \int_{-r}^{r} 4(R-r)\sqrt{r^2 - y^2}\,\mathrm{d}y \\
&= 4\pi(R-r) \int_{-r}^{r} \sqrt{r^2 - y^2}\,\mathrm{d}y.
\end{aligned}$$

(b) Compute the integral and find the volume of the donut. (Substitute $y = r\sin\theta$)

With the integrand being in form $\sqrt{r^2 - y^2}$, we can use trigonometic substitution.

Let $y = r\sin\theta$.

$$\begin{aligned}
y = r\sin\theta \qquad &\implies \qquad \mathrm{d}y = r\cos\theta\,\mathrm{d}\theta \\
y = r\sin\theta = -r \qquad &\implies \qquad \theta = \sin^{-1}(-1) = -\frac{\pi}{2} \\
y = r\sin\theta = r \qquad &\implies \qquad \theta = \sin^{-1}(1) = \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
\therefore \int_{-r}^{r} \sqrt{r^2 - y^2}\,\mathrm{d}y &= \int_{-\pi/2}^{\pi/2} \sqrt{r^2 - r^2 \sin^2\theta} \cdot r\cos\theta\,\mathrm{d}\theta \\
&= \int_{-\pi/2}^{\pi/2} \sqrt{r^2(1-\sin^2\theta)} \cdot r\cos\theta\,\mathrm{d}\theta \\
&= \int_{-\pi/2}^{\pi/2} \sqrt{r^2\cos^2\theta} \cdot r\cos\theta\,\mathrm{d}\theta \\
&= \int_{-\pi/2}^{\pi/2} r\cos\theta \cdot r\cos\theta\,\mathrm{d}\theta \\
&= \int_{-\pi/2}^{\pi/2} r^2\cos^2\theta\,\mathrm{d}\theta \\
&= r^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\,\mathrm{d}\theta \\
&= r^2 \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2\theta}{2}\,\mathrm{d}\theta \\
&= \frac{r^2}{2} \int_{-\pi/2}^{\pi/2} 1+\cos 2\theta\,\mathrm{d}\theta \\
&= \frac{r^2}{2} \left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2} \\
&= \frac{r^2}{2} \left(\frac{\pi}{2} + \cancel{\frac{\sin\pi}{2}} + \frac{\pi}{2} - \cancel{\frac{\sin(-\pi)}{2}}\right) \\
&= \frac{\pi r^2}{2}
\end{aligned}$$

Finally, we substitute the result of this integral into our expression for volume.

$$\begin{aligned}
V &= 4\pi(R-r)\int_{-r}^{r} \sqrt{r^2 - y^2}\,\mathrm{d}y \\
&= 4\pi(R-r)\frac{\pi r^2}{2} \\
&= 2\pi^2 r^2 (R-r)
\end{aligned}$$

2. Find the volume of the solid of revolution by rotating the region bounded by $y = 2x^2, y = 8$ and the y-axis.

> **Question open to interpretation**
> Axis of revolution not specified. Assuming revolution around the y-axis.

$$y = 2x^2 \implies x = \pm\sqrt{\frac{y}{2}}$$

Since the region is bounded by the y-axis, we only consider the positive component.

$$x = \sqrt{\frac{y}{2}}$$

And at $x = 0$, $y = 0$. So, the integral for the volume V is

$$\begin{aligned} V &= \pi \int_0^8 \left(\sqrt{\frac{y}{2}}\right)^2 \mathrm{d}y \\ &= \pi \int_0^8 \frac{y}{2} \mathrm{d}y \\ &= \pi \left[\frac{y^2}{4}\right]_0^8 \\ &= 16\pi. \end{aligned}$$

Long Question 1

In Figure 1, the region R is enclosed by the parabola $y = 4 - (x-3)^2$ and the line segment AC where A, C are the points $(1,0)$ and $(5,0)$ respectively. B is the vertec ofthe parabola.

(a) (i) Write down the coordinates of B.

$$\begin{aligned} y = 4 - (x-3)^2 &\implies & \frac{\mathrm{d}y}{\mathrm{d}x} = -2(x-3) \\ -2(x-3) = 0 &\implies & x = 3 \\ x = 3 &\implies & y = 4 - (3-3)^2 = 4 \end{aligned}$$

$$\therefore B = (3, 4)$$

(ii) Write down the equation of the curve AB and BC as a function of $x = f(y)$.

$$\begin{aligned} y &= 4 - (x-3)^2 \\ y - 4 &= -(x-3)^2 \end{aligned}$$

$$-y+4=(x-3)^2$$
$$\pm\sqrt{-y+4}=x-3$$
$$x=3\pm\sqrt{-y+4}$$

$$\therefore f_{AB}(y)=3-\sqrt{4-y}$$

$$f_{BC}(y)=3+\sqrt{4-y}$$

A jelly ring in the shape of solid of revolution of the region R is revolved the y-axis as shown in Figure 2. The jelly ring contains two layers and we let h be the height of the lower layer.

(b) (i) Show that the volume of the lower layer of the jelly ring is

$$8\pi(8-(4-h)^{3/2})$$

Similar to question one, revolving the right-hand side ($f_{BC}(y)$) will produce the volume of the entire jelly. Revolving the left-hand side ($f_{AB}(y)$) will produce the volume of the hole.

As such, the volume of the lower portion of the jelly ring V_{lower} would be the volume produced by revolving the outside minus the inside from 0 to height h.

$$\begin{aligned} V_{\text{lower}} &= \pi\int_0^h f_{BC}(y)^2 - f_{AB}(y)^2 \,\mathrm{d}y \\ &= \pi\int_0^h (3+\sqrt{4-y})^2-(3-\sqrt{4-y})^2 \,\mathrm{d}y \end{aligned}$$

Let $A=3$ and $B=\sqrt{4-y}$. Then,

$$(3+\sqrt{4-y})^2-(3-\sqrt{4-y})^2$$

$$=(A+B)^2-(A-B)^2$$

$$=4AB$$

$$=12\sqrt{4-y}.$$

And so, we have that:

$$
\begin{aligned}
V_{\text{lower}} &= \pi \int_0^h (3+\sqrt{4-y})^2 - (3-\sqrt{4-y})^2 \, \mathrm{d}y \\
&= \pi \int_0^h 12\sqrt{4-y} \, \mathrm{d}y \\
&= 12\pi \int_0^h \sqrt{4-y} \, \mathrm{d}y \\
&= 12\pi \int_0^h (4-y)^{1/2} \, \mathrm{d}y \\
&= 12\pi \left[-\frac{2(4-y)^{3/2}}{3} \right]_0^h \\
&= 12\pi \left(-\frac{2(4-h)^{3/2}}{3} + \frac{2(4-0)^{3/2}}{3} \right) \\
&= 12\pi \left(-\frac{2(4-h)^{3/2}}{3} + \frac{16}{3} \right) \\
&= 12\pi \left(\frac{1}{3} \right) (-2(4-h)^{3/2} + 16) \\
&= 12\pi \left(\frac{1}{3} \right) (2)(-(4-h)^{3/2} + 8) \\
&= 8\pi(-(4-h)^{3/2} + 8) \\
&= 8\pi(8 - (4-h)^{3/2})
\end{aligned}
$$

which matches the provided expression for volume.

(ii) If the upper and the lower layers have the same volume, find the value of h. (You leave your answer of h in three decimal places).

Since the integral found for the lower portion is valid for the entire jelly ring, we simply need to change its bound to reflect the upper portion.

$$
\begin{aligned}
V_{\text{upper}} &= \pi \int_h^4 12\sqrt{4-y}\,\mathrm{d}y \\
&= 12\pi \int_h^4 \sqrt{4-y}\,\mathrm{d}y \\
&= 12\pi \left[-\frac{2(4-y)^{3/2}}{3}\right]_h^4 \\
&= 12\pi \left(\cancel{-\frac{2(4-4)^{3/2}}{3}} + \frac{2(4-h)^{3/2}}{3}\right) \\
&= \frac{24\pi(4-h)^{3/2}}{3} \\
&= 8\pi(4-h)^{3/2}
\end{aligned}
$$

Since $V_{\text{upper}} = V_{\text{lower}}$, we can then solve for h.

$$
\begin{aligned}
V_{\text{upper}} &= V_{\text{lower}} \\
\cancel{8\pi}(4-h)^{3/2} &= \cancel{8\pi}(8-(4-h)^{3/2}) \\
\sqrt{(4-h)^3} &= 8 - \sqrt{(4-h)^3} \\
2\sqrt{(4-h)^3} &= 8 \\
\sqrt{(4-h)^3} &= 4 \\
(4-h)^3 &= 16 \\
4-h &= \sqrt[3]{16} \\
h &= -2\sqrt[3]{2} + 4 \\
&\approx 1.480
\end{aligned}
$$

(c) If milk is poured into the center of the jelly ring until it is fully filled. Find the volume of the milk required.

Revolving $f_{AB}(y)$ around the y-axis produces the volume of the hole. So, its volume is also the volume of the milk.

$$\begin{aligned}
V_{\text{milk}} &= \pi \int_0^4 f_{AB}(y)^2 \, \mathrm{d}y \\
&= \pi \int_0^4 (3 - \sqrt{4-y})^2 \, \mathrm{d}y \\
&= \pi \int_0^4 (9 + (4-y) - 6\sqrt{4-y}) \, \mathrm{d}y \\
&= \pi \int_0^4 \left(13 - y - 6(4-y)^{1/2}\right) \mathrm{d}y \\
&= \pi \left[13y - \frac{y^2}{2} + \frac{12(4-y)^{3/2}}{3}\right]_0^4 \\
&= \pi \left[13y - \frac{y^2}{2} + 4(4-y)^{3/2}\right]_0^4 \\
&= \pi \left(13(4) - \frac{16}{2} - 4(4)^{3/2}\right) \\
&= 12\pi
\end{aligned}$$