Homework 8

Mos Kullathon

921425216

1. Find the following limits

a. $\lim_{n\to\infty} \frac{(-1)^{n^3}}{n}$

By squeeze theorem:

$$\lim_{n\to\infty} -\frac{1}{n} \leq \lim_{n\to\infty} \frac{(-1)^{n^3}}{n} \leq \lim_{n\to\infty} \frac{1}{n}$$
$$0 \leq \lim_{n\to\infty} \frac{(-1)^{n^3}}{n} \leq 0$$
$$\therefore \lim_{n\to\infty} \frac{(-1)^{n^3}}{n} = 0.$$

b. $\lim_{n\to\infty} \frac{n^2+1}{n+100}$

$$\deg(n^2+1) > \deg(n+100)$$
$$\therefore \lim_{n\to\infty} \frac{n^2+1}{n+100} = \infty$$

c. $\lim_{n\to\infty} (\frac{1}{5e})^n$

$$\lim_{n\to\infty} \left(\frac{1}{5e}\right)^n = \lim_{n\to\infty} \frac{1^n}{5e^n} = \frac{\lim\limits_{n\to\infty} 1^n}{\lim\limits_{n\to\infty} 5e^n} = 0$$

d. $\lim_{n\to\infty} \frac{n^{100}}{e^{0.01n}}$

The polynomial n^{100} grows slower than the exponential $e^{0.01n}$.

$$\therefore \lim_{n\to\infty} \frac{n^{100}}{e^{0.01n}} = 0$$

e. $\lim_{n\to\infty} (1+\frac{1}{n})(2+\frac{\cos n}{3n^2})$

$$
\begin{aligned}
&\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(2+\frac{\cos n}{3n^2}\right)\\
&=\underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)}_{1}\lim_{n\to\infty}\left(2+\frac{\cos n}{3n^2}\right)\\
&=\lim_{n\to\infty}2+\lim_{n\to\infty}\frac{\cos n}{3n^2}
\end{aligned}
$$

By squeeze theorem:

$$
\begin{gathered}
\lim_{n\to\infty}-\frac{1}{3n^2}\leq\lim_{n\to\infty}\frac{\cos n}{3n^2}\leq\lim_{n\to\infty}\frac{1}{3n^2}\\
0\leq\lim_{n\to\infty}\frac{\cos n}{3n^2}\leq 0\\
\therefore\lim_{n\to\infty}\frac{\cos n}{3n^2}=0.
\end{gathered}
$$

As such,

$$
\begin{aligned}
&\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(2+\frac{\cos n}{3n^2}\right)\\
&=\lim_{n\to\infty}2+0\\
&=2.
\end{aligned}
$$

f. $\lim_{n\to\infty}\frac{\ln\sqrt{n+1}}{n^2}$

$$
\begin{aligned}
\lim_{n\to\infty}\frac{\ln\sqrt{n+1}}{n^2}&=\lim_{n\to\infty}\frac{\frac{1}{2}\ln(n+1)}{n^2}\\
&=\lim_{n\to\infty}\frac{\frac{1}{2(n+1)}}{2n}\\
&=\lim_{n\to\infty}\frac{1}{4n(n+1)}\\
&=0
\end{aligned}
$$

g. $\lim_{n\to\infty}\frac{n!}{n^n}$

n^n grows faster than $n!$.

$$\therefore\lim_{n\to\infty}\frac{n!}{n^n}=0$$

h. $\lim_{n\to\infty}2^{1/n}$

$$\lim_{n\to\infty} 2^{1/n} = 2^{\lim_{n\to\infty} 1/n} = 2^0 = 1$$

i. $\lim_{n\to\infty} 2^{n/(n^2+1)}$

$$\lim_{n\to\infty} 2^{n/(n^2+1)} = 2^{\lim_{n\to\infty} n/(n^2+1)} = 2^0 = 1$$

j. $\lim_{n\to\infty} n^{1/n}$

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n} = e^{\lim_{n\to\infty} \frac{\ln n}{n}} = e^0 = 1$$

k. $\lim_{n\to\infty} (-1)^n \tan \frac{n}{n^5-1}$

By squeeze theorem:

$$\lim_{n\to\infty} -1 \cdot \tan \frac{n}{n^5-1} \leq \lim_{n\to\infty} (-1)^n \tan \frac{n}{n^5-1} \leq \lim_{n\to\infty} 1 \cdot \tan \frac{n}{n^5-1}$$
$$-1(0) \leq \lim_{n\to\infty} (-1)^n \tan \frac{n}{n^5-1} \leq 1(0)$$
$$0 \leq \lim_{n\to\infty} (-1)^n \tan \frac{n}{n^5-1} \leq 0$$
$$\therefore \lim_{n\to\infty} (-1)^n \tan \frac{n}{n^5-1} = 0.$$

2. Evaluate the following infinite series

(i) $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$

The common ratio r is

$$r = \frac{\left(\frac{2}{5}\right)^2}{\frac{2}{5}} = \frac{\left(\frac{2}{5}\right)^3}{\left(\frac{2}{5}\right)^2} = \frac{\left(\frac{2}{5}\right)^n}{\left(\frac{2}{5}\right)^{n-1}} = \frac{2}{5}.$$

Since $|r| < 1$, the series converges.

Where the initial term $a = \frac{2}{5}$, using the geometric sum formula yields:

$$\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n = \frac{\frac{2}{5}}{1-\frac{2}{5}}$$
$$= \frac{2}{3}$$

(ii) $\sum_{n=2}^{\infty} (-1)^n \frac{5}{7^n}$

The common ratio r is

$$r = \frac{\frac{5(-1)^3}{7^3}}{\frac{5(-1)^2}{7^2}} = \frac{\frac{5(-1)^4}{7^4}}{\frac{5(-1)^3}{7^3}} = \frac{\frac{5(-1)^n}{7^n}}{\frac{5(-1)^{n-1}}{7^{n-1}}} = -\frac{1}{7}.$$

Since $|r| < 1$, the series converges.

Where the initial term $a = \frac{5(-1)^2}{7^2} = \frac{5}{49}$, using the geometric sum formula yields:

$$\begin{aligned}\sum_{n=2}^{\infty} (-1)^n \frac{5}{7^n} &= \frac{\frac{5}{49}}{1+\frac{1}{7}} \\ &= \frac{5}{56}\end{aligned}$$

(iii) $4 + \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots$

$$\begin{aligned}&4 + \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots \\ &= 4 + \sum_{n=1}^{\infty} \frac{3}{10^n}\end{aligned}$$

For $\displaystyle\sum_{n=1}^{\infty} \frac{3}{10^n}$:

The common ratio r is

$$r = \frac{\frac{3}{10^2}}{\frac{3}{10}} = \frac{\frac{3}{10^3}}{\frac{3}{10^2}} = \frac{\frac{3}{10^n}}{\frac{3}{10^{n-1}}} = \frac{1}{10}$$

Since $|r| < 1$, the series converges.

Where the initial term $a = \frac{3}{10}$, using the geometric sum formula yields:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{3}{10^n} &= \frac{\frac{3}{10}}{1-\frac{1}{10}} \\ &= \frac{1}{3}.\end{aligned}$$

$$\begin{aligned}\therefore 4 + \sum_{n=1}^{\infty} \frac{3}{10^n} &= 4 + \frac{1}{3} \\ &= \frac{13}{3}\end{aligned}$$

(iv) $\sum_{n=1}^{\infty}(\ln 2)^n, \quad \sum_{n=1}^{\infty}(\ln 3)^n$ (Be careful of the convergence)

For $\displaystyle\sum_{n=1}^{\infty}(\ln 2)^n$:

The common ratio r is

$$r = \frac{\ln^2 2}{\ln 2} = \frac{\ln^3 2}{\ln^2 2} = \frac{\ln^n 2}{\ln^{n-1} 2} = \ln 2 \approx 0.69.$$

Since $|r| < 1$, the series converges.

Where the initial term $a = \ln 2$, using the geometric sum formula yields:

$$\sum_{n=1}^{\infty}(\ln 2)^n = \frac{\ln 2}{1 - \ln 2} \approx 2.26.$$

For $\displaystyle\sum_{n=1}^{\infty}(\ln 3)^n$:

The common ratio r is

$$r = \frac{\ln^2 3}{\ln 2} = \frac{\ln^3 3}{\ln^2 3} = \frac{\ln^n 3}{\ln^{n-1} 3} = \ln 3 \approx 1.10.$$

Since $|r| > 1$, the series diverges.