Homework 7

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Section 3.4

Express the rational function as a sum or difference of two simpler rational expressions.

(183) $\dfrac{x^2+1}{x(x+1)(x+2)}$

$$\frac{x^2+1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}; \quad A, B, C \in \mathbb{R}$$

$$x^2+1 = A(x+1)(x+2) + Bx(x+2) + Cx(x+1)$$

Let $x = 0$.

$$0^2+1 = A(0+1)(0+2)$$
$$1 = 2A$$
$$\therefore A = \frac{1}{2}$$

Let $x = -1$.

$$(-1)^2+1 = B(-1)(-1+2)$$
$$2 = -B$$
$$\therefore B = -2$$

Let $x = -2$.

$$(-2)^2+1 = C(-2)(-2+1)$$
$$5 = 2C$$
$$\therefore C = \frac{5}{2}$$

$$\therefore \frac{x^2+1}{x(x+1)(x+2)} = \frac{1}{2x} - \frac{2}{x+1} + \frac{5}{2(x+2)}$$

(185) $\dfrac{3x+1}{x^2}$

$$\frac{3x+1}{x^2} = \frac{A}{x} + \frac{B}{x^2}; \quad A, B \in \mathbb{R}$$

$$3x + 1 = Ax + B$$

By comparing coefficients, $A = 3, B = 1$.

$$\therefore \frac{3x+1}{x^2} = \frac{3}{x} + \frac{1}{x^2}$$

(187) $\dfrac{2x^4}{x^2 - 2x}$

$$\begin{array}{r|l} & 2x^2 + 4x + 8 \\ \hline x^2 - 2x & 2x^4 \\ & \underline{2x^4 - 4x^3} \\ & \quad 4x^3 \\ & \quad \underline{4x^3 - 8x^2} \\ & \qquad 8x^2 \\ & \qquad \underline{8x^2 - 16x} \\ & \qquad\qquad 16x \end{array}$$

$$\begin{aligned} \therefore \frac{2x^4}{x^2 - 2x} &= 2x^2 + 4x + 8 + \frac{16x}{x^2 - 2x} \\ &= 2x^2 + 4x + 8 + \frac{16x}{x(x-2)} \\ &= 2x^2 + 4x + 8 + \frac{16}{x-2} \end{aligned}$$

(189) $\dfrac{1}{x^2(x-1)}$

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}; \quad A, B, C \in \mathbb{R}$$

$$1 = Ax(x-1) + B(x-1) + Cx^2$$

Let $x = 0$.

$$1 = B(0-1)$$

$$1 = -B$$

$$\therefore B = -1$$

Let $x = 1$.

$$1 = C(1^2)$$

$$\therefore C = 1$$

Let $x = 2$ and substitute $B = -1, C = 1$.

$$1 = A(2)(2-1) - (2-1) + (2^2)$$

$$1 = 2A - 1 + 4$$

$$\therefore A = -1$$

$$\therefore \frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}$$

(192) $\frac{1}{x^4-1} = \frac{1}{(x+1)(x-1)(x^2+1)}$

$$\frac{1}{(x+1)(x-1)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x^2+1}; \quad A, B, C \in \mathbb{R}$$

$$1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + C(x+1)(x-1)$$

Let $x = -1$.

$$1 = A(-1-1)(-1^2+1)$$

$$1 = -4A$$

$$\therefore A = -\frac{1}{4}$$

Let $x = 1$.

$$1 = B(1+1)(1^2+1)$$

$$1 = 4B$$

$$\therefore B = \frac{1}{4}$$

Let $x = 0$ and substitute $A = -\frac{1}{4}, B = \frac{1}{4}$.

$$1 = A(0-1)(0^2+1) + B(0+1)(0^2+1) + C(0+1)(0-1)$$

$$1 = -A + B - C$$

$$1 = \frac{1}{4} + \frac{1}{4} - C$$
$$\frac{1}{2} = -C$$
$$\therefore C = -\frac{1}{2}$$

$$\begin{aligned}\therefore \frac{1}{x^4 - 1} &= \frac{1}{(x+1)(x-1)(x^2+1)} \\ &= -\frac{1}{4(x+1)} + \frac{1}{4(x-1)} - \frac{1}{2(x^2+1)}\end{aligned}$$

Section 3.7

Determine whether the improper integrals converge or diverge. If possible, determine the value of the integrals that converge.

(359) $\int_{-\infty}^{\infty} \frac{1}{x^2+1} \, \mathrm{d}x$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} \, \mathrm{d}x = \int_{-\infty}^{0} \frac{1}{x^2+1} \, \mathrm{d}x + \int_{0}^{\infty} \frac{1}{x^2+1} \, \mathrm{d}x$$

$$\begin{aligned}\int_{-\infty}^{0} \frac{1}{x^2+1} \, \mathrm{d}x &= \lim_{n \to -\infty} \int_{n}^{0} \frac{1}{x^2+1} \, \mathrm{d}x \\ &= \lim_{n \to -\infty} \left[\tan^{-1} x \right]_{n}^{0} \\ &= \lim_{n \to -\infty} -\tan^{-1} n \\ &= \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\int_{0}^{\infty} \frac{1}{x^2+1} \, \mathrm{d}x &= \lim_{n \to \infty} \int_{0}^{n} \frac{1}{x^2+1} \, \mathrm{d}x \\ &= \lim_{n \to \infty} \left[\tan^{-1} x \right]_{0}^{n} \\ &= \lim_{n \to \infty} \tan^{-1} n \\ &= \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\therefore \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x &= \int_{-\infty}^{0} \frac{1}{x^2+1}\,\mathrm{d}x + \int_{0}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x \\ &= \frac{\pi}{2} + \frac{\pi}{2} \\ &= \pi\end{aligned}$$

(362) $\int_0^{\infty} e^{-x}\,\mathrm{d}x$

$$\begin{aligned}\int_0^{\infty} e^{-x}\,\mathrm{d}x &= \lim_{n\to\infty} \int_0^{n} e^{-x}\,\mathrm{d}x \\ &= \lim_{n\to\infty} \Big[-e^{-x}\Big]_0^n \\ &= \lim_{n\to\infty} -e^{-n} - (-e^0) \\ &= \lim_{n\to\infty} -\frac{1}{e^n} + \lim_{n\to\infty} 1 \\ &= 0 + 1 \\ &= 1\end{aligned}$$

(365) $\int_0^1 \frac{\mathrm{d}x}{\sqrt[3]{x}}$

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{\sqrt[3]{x}} &= \lim_{n\to 0} \int_n^1 x^{-\frac{1}{3}}\,\mathrm{d}x \\ &= \lim_{n\to 0} \left[\frac{x^{\frac{2}{3}}}{2/3}\right]_n^1 \\ &= \lim_{n\to 0} \left[\frac{3x^{2/3}}{2}\right]_n^1 \\ &= \lim_{n\to 0} \frac{3(1)^{2/3}}{2} - \frac{3n^{2/3}}{2} \\ &= \frac{3}{2}\end{aligned}$$

Evaluate the integrals. If the integral diverges, answer "diverges."

(374) $\int_1^{\infty} \frac{\mathrm{d}x}{x^e}$

$$
\begin{aligned}
\int_1^\infty \frac{\mathrm{d}x}{x^e} &= \lim_{n\to\infty} \int_1^n x^{-e}\,\mathrm{d}x \\
&= \lim_{n\to\infty} \left[\frac{x^{1-e}}{1-e}\right]_1^n \\
&= \lim_{n\to\infty} \frac{n^{1-e}}{1-e} - \frac{1^{1-e}}{1-e} \\
&= \lim_{n\to\infty} \frac{1}{1-e(n^{e-1})} - \frac{1^{1-e}}{1-e} \\
&= 0 - \frac{1}{1-e} \\
&= \frac{1}{e-1}
\end{aligned}
$$

(375) $\int_0^1 \frac{\mathrm{d}x}{x^\pi}$

$$
\begin{aligned}
\int_0^1 \frac{\mathrm{d}x}{x^\pi} &= \lim_{n\to 0} \int_n^1 x^{-\pi}\,\mathrm{d}x \\
&= \lim_{n\to 0} \left[\frac{x^{1-\pi}}{1-\pi}\right]_n^1 \\
&= \lim_{n\to 0} \frac{1}{1-\pi} - \frac{0^{1-\pi}}{1-\pi} \\
&= \infty
\end{aligned}
$$

Therefore, $\int_0^1 \frac{\mathrm{d}x}{x^\pi}$ diverges.

(382) $\int_0^\infty xe^{-x}\,\mathrm{d}x$

$$
\int_0^\infty xe^{-x}\,\mathrm{d}x = \lim_{n\to\infty} \int_0^n xe^{-x}\,\mathrm{d}x
$$

$$
u = x \implies u' = 1
$$

$$
v' = e^{-x} \implies v = -e^{-x}
$$

$$\begin{aligned}\int_0^n xe^{-x}\,\mathrm{d}x &= \Big[uv\Big]_0^n - \int_0^n vu'\,\mathrm{d}x \\ &= \Big[-xe^{-x}\Big]_0^n + \int_0^n e^{-x}\,\mathrm{d}x \\ &= -ne^{-n} + \Big[-e^{-x}\Big]_0^n \\ &= -ne^{-n} - e^{-n} + 1 \\ &= -\frac{n}{e^n} - \frac{1}{e^n} + 1\end{aligned}$$

By L'Hôpital's:

$$\lim_{n\to\infty} -\frac{n}{e^n} = \lim_{n\to\infty} -\frac{1}{e^n} = 0.$$

$$\begin{aligned}\therefore \int_0^\infty xe^{-x}\,\mathrm{d}x &= \lim_{n\to\infty} -\frac{n}{e^n} - \frac{1}{e^n} + 1 \\ &= -0 - 0 + 1 \\ &= 1\end{aligned}$$

(394) Find the area of the region in the first quadrant between the curve $y = e^{-6x}$ and the *x*-axis.

$$\begin{aligned}A &= \int_0^\infty e^{-6x}\,\mathrm{d}x \\ &= \lim_{n\to\infty} \int_0^n e^{-6x}\,\mathrm{d}x \\ &= \lim_{n\to\infty} \left[-\frac{e^{-6x}}{6}\right]_0^n \\ &= \lim_{n\to\infty} -\frac{e^{-6n}}{6} + \frac{1}{6} \\ &= -0 + \frac{1}{6} \\ &= \frac{1}{6}\end{aligned}$$