Homework 6

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1. Let $I_n = \int \sec^n x \, \mathrm{d}x$. Derive the following reduction formula for I_n.

$$I_n = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}.$$

$$\begin{aligned} I_n &= \int \sec^n x \, \mathrm{d}x \\ &= \int \sec^{n-2} x \sec^n x \, \mathrm{d}x \end{aligned}$$

$$\begin{aligned} u = \sec^{n-2} x \implies \quad u' &= (n-2) \sec^{n-3} x \sec x \tan x \\ &= (n-2) \sec^{n-2} x \tan x \end{aligned}$$

$$v' = \sec^2 x \implies \quad v = \tan x$$

$$\begin{aligned} I_n &= \int \sec^{n-2} x \sec^n x \, \mathrm{d}x \\ &= uv - \int vu' \, \mathrm{d}x \\ &= \tan x \sec^{n-2}(x) - (n-2) \int \sec^{n-2} x \tan^2 x \, \mathrm{d}x \\ &= \tan x \sec^{n-2}(x) - (n-2) \int \sec^{n-2} x (\sec^2(x) - 1) \, \mathrm{d}x \\ &= \tan x \sec^{n-2}(x) - (n-2) \int \sec^n(x) - \sec^{n-2} \, \mathrm{d}x \\ &= \tan x \sec^{n-2}(x) - (n-2) \underbrace{\int \sec^n x \, \mathrm{d}x}_{I_n} + (n-2) \underbrace{\int \sec^{n-2} x \, \mathrm{d}x}_{I_{n-2}} \\ &= \tan x \sec^{n-2}(x) - (n-2) I_n + (n-2) I_{n-2} \end{aligned}$$

$$
\begin{aligned}
I_n + (n-2)I_n &= \tan x \sec^{n-2}(x) + (n-2)I_{n-2} \\
I_n(1 + (n-2)) &= \tan x \sec^{n-2}(x) + (n-2)I_{n-2} \\
I_n &= \frac{\tan x \sec^{n-2}(x) + (n-2)I_{n-2}}{1+(n-2)} \\
&= \frac{\tan x \sec^{n-2}(x) + (n-2)I_{n-2}}{n-1} \\
&= \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}
\end{aligned}
$$

2. Use the formula above to find $\int \sec^5 x \, \mathrm{d}x$.

$$
\begin{aligned}
I_5 &= \int \sec^5 \mathrm{d}x \\
&= \frac{\tan x \sec^{5-2} x}{5-1} + \frac{5-2}{5-1} I_{5-2} \\
&= \frac{\tan x \sec^3 x}{4} + \frac{3}{4} I_3 \\
&= \frac{\tan x \sec^3 x}{4} + \frac{3}{4}\left(\frac{\tan x \sec^{3-2} x}{3-1} + \frac{3-2}{3-1} I_{3-2}\right) \\
&= \frac{\tan x \sec^3 x}{4} + \frac{3}{4}\left(\frac{\tan x \sec x}{2} + \frac{1}{2} I_1\right) \\
&= \frac{\tan x \sec^3 x}{4} + \frac{3}{4}\left(\frac{\tan x \sec x}{2} + \frac{1}{2}\int \sec x \, \mathrm{d}x\right) \\
&= \frac{\tan x \sec^3 x}{4} + \frac{3}{4}\left(\frac{\tan x \sec x}{2} + \frac{1}{2}\ln|\tan(x) + \sec(x)|\right) + C \\
&= \frac{\tan x \sec^3 x}{4} + \frac{3}{8}\left(\tan x \sec x + \ln|\tan(x) + \sec(x)|\right) + C \\
&= \frac{1}{8}(2\tan x \sec^3 x + 3(\tan x \sec x + \ln|\tan(x) + \sec(x)|)) + C
\end{aligned}
$$

Book section 3.1

Find the integral by using the simplest method. Not all problems require integration by parts.

(23) $\int \sin(\ln 2x) \, \mathrm{d}x$

$$
\begin{aligned}
u = \ln 2x \quad &\implies \quad \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x \\
&\iff \quad \mathrm{d}x = x\,\mathrm{d}u \\
&\iff \quad e^u = 2x \\
&\iff \quad x = \frac{e^u}{2}
\end{aligned}
$$

$$
\begin{aligned}
\int \sin(\ln 2x)\,\mathrm{d}x &= \int \sin(u)\frac{e^u}{2}\,\mathrm{d}u \\
&= \frac{1}{2}\int e^u \sin u\,\mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
p = \sin u \quad &\implies \quad p' = \cos u \\
q' = e^u \quad &\implies \quad q = e^u
\end{aligned}
$$

$$
\begin{aligned}
\int e^u \sin u\,\mathrm{d}u &= pq - \int qp'\,\mathrm{d}u \\
&= e^u \sin u - \int e^u \cos u\,\mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
r = \cos u \quad &\implies \quad r' = -\sin u \\
s' = e^u \quad &\implies \quad s = e^u
\end{aligned}
$$

$$
\begin{aligned}
\int e^u \cos u\,\mathrm{d}u &= rs - \int sr'\,\mathrm{d}u \\
&= e^u \cos u + \int e^u \sin u\,\mathrm{d}u
\end{aligned}
$$

$$\begin{aligned}
\therefore \int e^u \sin u \,\mathrm{d}u &= e^u \sin u - \int e^u \cos \mathrm{d}u \\
&= e^u \sin u - e^u \cos u - \int e^u \sin u \,\mathrm{d}u
\end{aligned}$$

$$2\int e^u \sin u \,\mathrm{d}u = e^u \sin u - e^u \cos u + C$$

$$\int e^u \sin u \,\mathrm{d}u = \frac{1}{2}(e^u \sin u - e^u \cos u) + C$$

$$\begin{aligned}
\therefore \int \sin(\ln 2x)\,\mathrm{d}x &= \frac{1}{2}\int e^u \sin u \,\mathrm{d}u \\
&= \frac{1}{4}(e^u \sin u - e^u \cos u) + C \\
&= \frac{1}{4}(2x \sin(\ln 2x) - 2x \cos(\ln 2x)) + C \\
&= \frac{1}{2}x(\sin(\ln 2x) - \cos(\ln 2x)) + C
\end{aligned}$$

(25) $\int (\ln x)^2 \,\mathrm{d}x$

$$u = \ln^2 x \quad \Longrightarrow \quad u' = \frac{2\ln x}{x}$$

$$v' = 1 \quad \Longrightarrow \quad v = x$$

$$\begin{aligned}
\int (\ln x)^2 \,\mathrm{d}x &= uv - \int vu' \,\mathrm{d}x \\
&= x\ln^2 x - \int \frac{2\cancel{x}\ln x}{\cancel{x}}\,\mathrm{d}x \\
&= x\ln^2 x - 2\int \ln x \,\mathrm{d}x
\end{aligned}$$

$$p = \ln x \quad \Longrightarrow \quad p' = \frac{1}{x}$$

$$q' = 1 \quad \Longrightarrow \quad q = x$$

$$
\begin{aligned}
\int \ln x \, \mathrm{d}x &= pq - \int qp' \, \mathrm{d}x \\
&= x \ln x - \int \frac{x}{x} \, \mathrm{d}x \\
&= x \ln x - x + C
\end{aligned}
$$

$$
\begin{aligned}
\int (\ln x)^2 \, \mathrm{d}x &= x \ln^2 x - 2 \int \ln x \, \mathrm{d}x \\
&= x \ln^2(x) - 2(x \ln(x) - x) + C \\
&= x \ln^2(x) - 2x \ln(x) + 2x + C \\
&= x(\ln^2(x) - 2 \ln(x) + 2) + C
\end{aligned}
$$

(28) $\int \sin^{-1} x \, \mathrm{d}x$

$$
\begin{aligned}
u = \sin^{-1} x \quad &\implies \quad u' = \frac{1}{\sqrt{1-x^2}} \\
v' = 1 \quad &\implies \quad v = x
\end{aligned}
$$

$$
\begin{aligned}
\int \sin^{-1} \mathrm{d}x &= uv - \int vu' \, \mathrm{d}x \\
&= x \sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}} \, \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
p = 1 - x^2 \quad &\implies \quad \mathrm{d}p = -2x \, \mathrm{d}x \\
&\iff \quad \mathrm{d}x = -\frac{1}{2x} \, \mathrm{d}p
\end{aligned}
$$

$$\begin{aligned}
\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x &= \int \frac{\cancel{x}}{\sqrt{p}}\left(-\frac{1}{2\cancel{x}}\,\mathrm{d}p\right) \\
&= -\frac{1}{2}\int \frac{1}{\sqrt{p}}\,\mathrm{d}p \\
&= -\frac{1}{2}\int p^{-1/2}\,\mathrm{d}p \\
&= -\frac{1}{2}\cdot\frac{p^{1/2}}{1/2}+C \\
&= -\sqrt{p}+C \\
&= -\sqrt{1-x^2}+C
\end{aligned}$$

$$\begin{aligned}
\therefore \int \sin^{-1}\,\mathrm{d}x &= x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x \\
&= x\sin^{-1}(x) - (-\sqrt{1-x^2}) + C \\
&= \sqrt{1-x^2} + x\sin^{-1}(x) + C
\end{aligned}$$

(36) $\int x\sec^2 x\,\mathrm{d}x$

$$\begin{aligned}
u = x \quad &\Longrightarrow \quad u' = 1 \\
v' = \sec^2 x \quad &\Longrightarrow \quad v = \tan x
\end{aligned}$$

$$\begin{aligned}
\int x\sec^2 x\,\mathrm{d}x &= uv - \int vu'\,\mathrm{d}x \\
&= x\tan x - \int \tan x\,\mathrm{d}x \\
&= x\tan x - \int \frac{\sin x}{\cos x}\,\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
p = \cos x \quad &\Longrightarrow \quad \mathrm{d}p = -\sin x\,\mathrm{d}x \\
&\Longleftrightarrow \quad \mathrm{d}x = \frac{\mathrm{d}p}{-\sin x}
\end{aligned}$$

$$\int \frac{\sin x}{\cos x}\,\mathrm{d}x = \int \frac{\cancel{\sin x}}{p} \cdot \frac{\mathrm{d}p}{-\cancel{\sin x}}$$

$$= -\int \frac{1}{p}\,\mathrm{d}p$$

$$= -\ln(p) + C$$

$$= -\ln(\cos x) + C$$

$$\therefore \int x \sec^2 x\,\mathrm{d}x = x\tan x - \int \frac{\sin x}{\cos x}\,\mathrm{d}x$$

$$= x\tan x - (-\ln(\cos x)) + C$$

$$= x\tan x + \ln(\cos x) + C$$

Derive the following formulas using the technique of integration by parts. Assume that n is a positive integer. These formulas are called reduction formulas because the exponent in the x term has been reduced by one in each case. The second integral is simpler than the original integral.

(49) $\int x^n \cos x\,\mathrm{d}x = x^n \sin x - n\int x^{n-1}\sin x\,\mathrm{d}x$

$$u = x^n \implies u' = nx^{n-1}$$

$$v' = \cos x \implies v = \sin x$$

$$\int x^n \cos x\,\mathrm{d}x = uv - \int vu'\,\mathrm{d}x$$

$$= x^n \sin x - \int nx^{n-1}\sin x\,\mathrm{d}x$$

$$= x^n \sin x - n\int x^{n-1}\sin x\,\mathrm{d}x$$

(65) Find the area of the region enclosed by the curve $y = x\cos x$ and the *x*-axis for $\frac{11\pi}{2} \leq x \leq \frac{13\pi}{2}$. (Express the answer in exact form.)

Since $x\cos x \geq 0$ for all $x \in [\frac{11\pi}{2}, \frac{13\pi}{2}]$, the area A between the curve $y = x\cos x$ and the x-axis is

$$\int_{11\pi/2}^{13\pi/2} x\cos x\,\mathrm{d}x.$$

$$u = x \implies u' = 1$$

$$v' = \cos x \implies v = \sin x$$

$$\begin{aligned}
\int_{11\pi/2}^{13\pi/2} x\cos x\,\mathrm{d}x &= \Big[x\sin x\Big]_{11\pi/2}^{13\pi/2} - \int_{11\pi/2}^{13\pi/2} \sin x\,\mathrm{d}x \\
&= \Big[x\sin x\Big]_{11\pi/2}^{13\pi/2} - \Big[-\cos x\Big]_{11\pi/2}^{13\pi/2} \\
&= \frac{13\pi}{2} - \frac{11\pi}{2}(-1) - 0 \\
&= 12\pi
\end{aligned}$$

(66) Find the volume of the solid generated by revolving the region bounded by the curve $y = \ln x$, the x-axis, and the vertical line $x = e^2$ about the *x*-axis. (Express the answer in exact form.)

Since $y = \ln x$ is increasing and $\ln x = 0 \implies x = 1$, and the region is bounded by $x = e^2$, we have that the volume V of the solid generated by revolving the region around the x-axis is

$$\begin{aligned}
V &= \pi\int_1^{e^2} \ln^2(x) - 0^2\,\mathrm{d}x \\
&= \pi\int_1^{e^2} \ln^2(x)\,\mathrm{d}x
\end{aligned}$$

From (25), we found $\int \ln^2 x\,\mathrm{d}x$.

$$\int \ln^2 x\,\mathrm{d}x = x(\ln^2(x) - 2\ln(x) + 2) + C$$

Therefore:

$$
\begin{aligned}
V &= \pi \int_1^{e^2} \ln^2 x \, \mathrm{d}x \\
&= \pi \left[x(\ln^2(x) - 2\ln(x) + 2) \right]_1^{e^2} \\
&= \pi(e^2(4 - 4 + 2) - (2)) \\
&= \pi(2e^2 - 2) \\
&= 2\pi(e^2 - 1).
\end{aligned}
$$