MAP 555 : Analog filtering, sampling, reconstruction...

18 Septembre 2015

Today

1. Analog filters
2. Linear systems
3. Sampling
4. Aliasing
5. Practical reconstruction
6. Another approach to sampling

Analog filters

- The tools we have just developed (convolution and the Fourier transform for functions) are going to be used to study analog filters that are governed by a linear differential equation with constant coefficients,

$$\sum_{k=0}^{q} b_k g^{(k)} = \sum_{j=0}^{p} a_j f^{(j)}, \ a_p \cdot b_q \neq 0,$$

  where f is the input and $g = A(f)$ is the output.
- Assumption $f \in \mathcal{S}(\mathbb{R})$. This case is very special. The input has no reason to be so regular, but we will see that this is a step toward more general cases.

input and output are in $\mathcal{S}(\mathbb{R})$

- Assume that $f \in \mathcal{S}(\mathbb{R})$ and look for a solution $g \in \mathcal{S}(\mathbb{R})$. If such a g exists, we can take the Fourier transform of both sides of

$$\sum_{k=0}^{q} b_k g^{(k)} = \sum_{j=0}^{p} a_j f^{(j)}, \;\; a_p \cdot b_q \neq 0, \quad \textbf{(S)}$$

showing that

$$\sum_{k=0}^{q} b_k (2\mathrm{i}\pi\xi)^k \hat{g}(\xi) = \sum_{j=0}^{p} a_j (2\mathrm{i}\pi\xi)^j \hat{f}(\xi) \quad \textbf{(F)}$$

- Consider the two polynomials $P(x) = \sum_{j=0}^{p} a_j x^j$ and $Q(x) = \sum_{k=0}^{q} b_k x^k$ and assume that the rational function $P(x)/Q(x)$ has no poles on the imaginary axis.

input and output are in $\mathcal{S}(\mathbb{R})$

- Then $P(2i\pi\xi)/Q(2i\pi\xi)$ has no poles for real ξ, and **(S)** is equivalent to

$$\hat{g}(\xi) = G(\xi) \quad \text{where} \quad G(\xi) = \frac{P(2i\pi\xi)}{Q(2i\pi\xi)}\hat{f}(\xi)$$

  Note that $G \in \mathcal{S}(\mathbb{R})$.
- This equality completely determines g in $\mathcal{S}(\mathbb{R})$, if it exists, and thus proves the uniqueness of a solution of **(S)** in $\mathcal{S}(\mathbb{R})$.
- $g = \overline{\mathcal{F}}(G)$ is a solution of **(S)** in $\mathcal{S}(\mathbb{R})$. The differential equation has a unique solution without initial conditions being specified. This is because we require the solution g to be in $\mathcal{S}$, which means that g and all of its derivatives vanish at infinity.

Convolution

- Idea : express the solution as a convolution.
- Assumption : $\deg P < \deg Q$. Define the transfer function

$$H(\xi) = \frac{P(2\mathrm{i}\pi\xi)}{Q(2\mathrm{i}\pi\xi)}$$

  is in $L_2(\mathbb{R}) \cap L_\infty(\mathbb{R})$.
- By decomposing this rational function into partial fractions, the impulse response, defined as the inverse Fourier transform of the transfer function

$$h = \overline{\mathcal{F}}(H)$$

  is bounded, rapidly decreasing, continuous except perhaps at the origin.

Simple poles

- The poles of P/Q are assumed to lie off the imaginary axis. There are two cases to consider : P/Q has only simple poles or P/Q has multiple poles.
- Assume first that $P(x)/Q(x)$ has only simple poles. In this case, H can be decomposed in the form

$$H(\xi) = \sum_{k=0}^{q} \frac{\beta_k}{2i\pi\xi - z_k}$$

where $z_1, \ldots, z_q$ are the poles.

Simple poles

- For $a \in \mathbb{C}$, $\mathrm{Re}(a) > 0$, $\epsilon = \pm 1$,

$$\mathrm{e}^{-\epsilon a x} u(\epsilon x) \overset{\mathcal{F}}{\mapsto} \frac{\epsilon}{(\epsilon a + 2\mathrm{i}\pi\xi)}$$

- We conclude that

$$h(t) = \left(\sum_{k \in K-} \beta_k \mathrm{e}^{z_k t} \right) u(t) - \left(\sum_{k \in K_+} \beta_k \mathrm{e}^{z_k t} \right) u(-t) \, .$$

where $t \mapsto u(t)$ is the Heaviside function and

$$K_- = \{k \in \{1, 2, \ldots, q\} | \mathrm{Re}(z_k) < 0\},$$
$$K_+ = \{k \in \{1, 2, \ldots, q\} | \mathrm{Re}(z_k) > 0\}.$$

Multiple poles

- Let $z_1, z_2, \dots, z_l$ the poles and let $m_1, m_2, \dots, m_l$ be their multiplicities.
- Then we can write H as

$$H(\xi) = \sum_{k=1 m}^{l} \sum_{=1}^{m_k} \frac{\beta_{k,m}}{(2i\pi\xi - z_k)^m}.$$

- The impulse response is given by

$$h(t) = \left(\sum_{k \in K-} P_k(t) \mathrm{e}^{z_k t} \right) u(t) - \left(\sum_{k \in K_+} P_k(t) \mathrm{e}^{z_k t} \right) u(-t)$$

where

$$P_k(t) = \sum_{m=1}^{m_k} \beta_{k,m} \frac{t^{m-1}}{(m-1)!}.$$

Convolution

- Since $h = \overline{\mathcal{F}}(H)$ is bounded, rapidly decreasing, continuous except perhaps at the origin **(F)** may be rewritten as

$$\hat{g} = \hat{h} \cdot \hat{f}.$$

- Since $\hat{h} \in L_2(\mathbb{R})$ and $\hat{f} \in \mathcal{S}(\mathbb{R}) \subset L_1(\mathbb{R})$, we have $h * f(t) = \overline{\mathcal{F}}\left(\hat{h}\hat{f}\right)$ implies that

$$g = h * f$$

Generalized solutions

The formula $g = h * f$, obtained when f is in $\mathcal{S}$, makes sense in the following more general cases.

1. If f is in $L_1(\mathbb{R})$, then g is in $L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \cap L_\infty(\mathbb{R})$ and
$$\begin{aligned}\|g\|_1 &\leq \|h\|_1 \|f\|_1,\\ \|g\|_2 &\leq \|h\|_2 \|f\|_1,\\ \|g\|_\infty &\leq \|h\|_\infty \|f\|_1.\end{aligned}$$
2. If f is in $L_2(\mathbb{R})$, then g is in $L_2(\mathbb{R})$, it is bounded and continuous, it tends to 0 at infinity, and
$$\begin{aligned}\|g\|_2 &\leq \|h\|_1 \|f\|_2,\\ \|g\|_\infty &\leq \|h\|_2 \|f\|_2.\end{aligned}$$
3. If f is in $L_\infty(\mathbb{R})$, then g is also bounded and
$$\|g\|_\infty \leq \|h\|_1 \|f\|_\infty.$$

Purely imaginary poles

- What we have done so far does not allow us to treat an equation like
$$g'' + \omega^2 g = f,$$
where $P(x)/Q(x) = 1/(x^2 + \omega^2)$ has two poles are on the imaginary axis.
- In this case h is a sinusoid and the Fourier transform of H (when H is considered to be a function) is no longer defined.
- This problem required the use of the theory of distributions but this degree of sophistication goes far beyond this course.

What happens in $\deg P = \deg Q$

Take for example the equation

$$g'' - \omega^2 g = f''.$$

Again, what we have done so far does not apply. Nevertheless, we can still manage to solve the equation. Changing the unknown function to $g_0 = g - f$ lowers the order of the right-hand side :

$$g_0'' - \omega^2 g_0 = \omega^2 f.$$

Then we have $g_0 = h_0 * f$ and $g = f + h_0 * f$. This is no longer a convolution but it will serve the same purpose.

X the set of input signals and Y the set of output signals.

Assumptions

1. X is a vector spaces (over $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$).
2. X is closed under translation.

Definition (Linearity)

The system $A : \mathrm{X} \to \mathrm{Y}$ is said to be linear if for all $f_1, f_2 \in \mathrm{X}$ and $a_1, a_2 \in \mathbb{K}$

$$A(a_1 f_1 + a_2 f_2) = a_1 A(f_1) + a_2 A(f_2) .$$

- A system A defined $A(f) = h * f$ is linear (assuming that h and f are in proper functions spaces so that $h * f$ is well defined).

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- A system A defined $A(f) = h * f$ is linear (assuming that h and f are in proper functions spaces so that $h * f$ is well defined).
  - For example, if $h \in L_1(\mathbb{R})$ then $A(f) = h * f$ is a linear system from $\mathrm{X} = L_2(\mathbb{R})$ onto $\mathrm{Y} = L_2(\mathbb{R})$.

Stability

Definition

A system $A : \mathrm{X} \to \mathrm{Y}$ is said to be stable if there exists an $M > 0$ such that $\|Af\|_\infty \leq M\|f\|_\infty$ for all $f \in L_\infty(\mathbb{R}) \cap \mathrm{X}$.

1. The generalized filter A is stable when $\deg P < \deg Q$.
2. It is still stable if $\deg P = \deg Q$...
3. Set $\mathrm{X} = \mathrm{Y} = L_2(\mathbb{R})$ and $h \in L_1\mathbb{R}$. Then $A(f) = h * f$, with $f \in L_2(\mathbb{R})$ is stable.

Causality

Definition

A system $A : \mathrm{X} \to \mathrm{Y}$ is causal if the equality of any two input signals up to time $t = t_0$ implies the equality of the two output signals at least to time t_0,

$$f_1(t) = f_2(t) \text{ for } t \leq t_0 \Rightarrow Af_1(t) = Af_2(t) \text{ for } t < t_0$$

This property is completely natural for a physical system in which the variable is time. It says that the response at time t depends only on what has happened before t.

Causality

Theorem

If $A(f) = h * f$ is a defined with a convolution, then the system is causal if the impulse response vanishes for all $x \leq 0$.

- The system $A(f) = h * f$ with $h(x) = u(x)\mathrm{e}^{-ax}$ $a > 0$ is causal (since $h \in L_1(\mathbb{R})$, we may take here $\mathrm{X} = \mathrm{Y} = L_2(\mathbb{R})$). The support of the impulse reponse is $\mathbb{R}^+$ (it is infinite).
- more generally, if $\deg P < \deg Q$, the generalized filter $A : A(f) = h * f$ is causal if $\mathrm{supp}(h) \subset [0, \infty[$ or equivalently if the poles of P/Q are located to the left of the imaginary axis.

Invariance

Define by τ_a the delay operator : $\tau_a f(x) = f(x-a)$ for all $x \in \mathbb{R}$.

Definition

A system A is invariant if a translation of time in the input leads to the same translation in the output, or equivalently if A and τ_a commute for all $a \in \mathbb{R}$:

$$A \circ \tau_a = \tau_a \circ A$$

- if $\mathrm{X} = \mathrm{Y} = L_2(\mathbb{R})$, $h \in L_1(\mathbb{R})$, then $A(f) = h * f$ is invariant.

Linear filter

Definition

A mapping $A : \mathrm{X} \to \mathrm{Y}$ is a linear filter if

1. A is linear
2. A is invariant.

- $\mathrm{X} = \mathrm{Y} = L_2(\mathbb{R})$, $h \in L_1(\mathbb{R})$, $A(f) = h * f$ is a linear filter.
- If $\mathrm{X} \subset C^1(\mathbb{R})$ is the set of functions f satisfying $f(x) = \int \hat{f}(\xi) \mathrm{e}^{+2\mathrm{i}\pi\xi x} \mathrm{d}\xi$ for all $x \in \mathrm{X}$ for some $(1 + \|\xi|)\hat{f} \in L_1(\mathbb{R})$. The mapping $A(f) = f'$ is a linear filter. Its transfer function is $H(\xi) = 2\mathrm{i}\pi\xi$ but this is not a convolution.

A first order filter

- Consider the first order system $RCg' + g = f$.
- This is rational system with $P(x) = 1$ and $Q(x) = 1 + RC(x)$ $(\deg(P) < \deg(Q))$
- The transfer function is

$$H(\xi) = \frac{P(2\mathrm{i}\pi\xi)}{Q(2\mathrm{i}\pi\xi)} = \frac{1}{1 + 2\mathrm{i}RC\pi\xi}$$

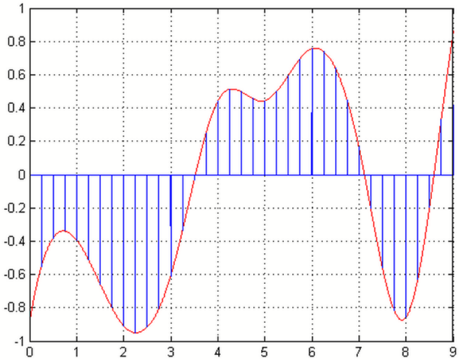
  There is a single pole at $z_1 = -1/RC$. The system is low pass.
- The impulse response is

$$h(t) = \frac{1}{RC}\mathrm{e}^{-t/RC}u(t)\ .$$

- This is a convolutive system, which is causal and stable, which may be defined on $X = L_2(\mathbb{R})$ by

$$g = A(f) = \int h(s)f(t-s)\mathrm{d}s\ .$$

Periodic sampling ?



T the sampling period. $f[n] = f(nT)$ are the samples of the signals. The sampling frequency $1/T$.

Band Limited functions

Definition (Band Limited function)

A function $f \in L_2(\mathbb{R})$ is said to be band limited if there exists $B < \infty$ such that : $\mathcal{F}f(\xi) = 0$ pour $\xi \notin [-B, +B]$. We denote $\mathrm{BL}(B)$ the subspace of $f \in L_2(\mathbb{R})$ of functions satisfying $\mathcal{F}f(\xi) = 0$ for (almost) all $\xi \notin [-B, +B]$.

Band-Limited functions

- The Fourier transform defines an isomporphism on $L_2(\mathbb{R})$. The inverse is $\overline{\mathcal{F}}$.
- Let $f \in \mathrm{BL}(B)$. Since $\mathcal{F}f$ is compactly supported and belongs $L_2(\mathbb{R})$, $\mathcal{F}f \in L_1(\mathbb{R})$.
- Therefore $x \mapsto \overline{\mathcal{F}} \circ \mathcal{F}f(x)$ is continuous and since $\overline{\mathcal{F}} \circ \mathcal{F}f = f$ a.e., every function in $\mathrm{BL}(B)$ has a continuous version (and even a C^∞ version).

Periodizing the Fourier transform

Let $T \leq 1/(2B)$ be the sampling period and $1/T$ be the sampling frequency. Consider the periodic function $\xi \to \mathcal{F}f(\xi)$ given by :

$$F_T(\xi) = \sum_{n\in\mathbb{Z}} [\mathcal{F}f]\left(\xi - \frac{n}{T}\right) .$$

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$\xi \mapsto F_T(\xi)$ is periodic with period $1/T$

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$$F_T(\xi) = \sum_{n \in \mathbb{Z}} [\mathcal{F}f] \left(\xi - \frac{n}{T}\right) .$$

The function $F_T \in L_2[-1/2T, 1/2T]$ can be expanded as a series function

$$F_T(\xi) = \sum_{n \in \mathbb{Z}} c_n(F_T) \mathrm{e}^{-\mathrm{i}2\pi\xi nT},$$

where $\{c_k(F_T)\}$, the Fourier coefficients are given by

$$c_k(F_T) = T \int_{-1/(2T)}^{1/(2T)} F_T(\xi) \mathrm{e}^{+\mathrm{i}2\pi\xi kT} \mathrm{d}\xi .$$

Periodizing the Fourier transform

The identity

$$F_T(\xi) = \sum_{n\in\mathbb{Z}} c_n(F_T)\mathrm{e}^{-\mathrm{i}2\pi\xi nT},$$

should be understood as a convergence of the partial sums

$$F_{N,T}(\xi) = \sum_{k=-N}^{N} c_N(F_T)\mathrm{e}^{-\mathrm{i}2\pi\xi nT} \ ,$$

toward F_T in the topology induced by the norm $\|\cdot\|_2$ in $L_2([-1/(2T), 1/(2T)])$:

$$\lim_{N\to\infty} \int_{-1/(2T)}^{1/(2T)} |F_T(\xi) - F_{N,T}(\xi)|^2 \mathrm{d}\xi = 0 \ .$$

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The Parseval identity implies that $\sum_{n\in\mathbb{Z}} |c_n(F_T)|^2 < \infty$.

A key result

- Since for all $T \leq 1/(2B)$, $[-B, +B] \subset [-1/(2T), 1/(2T)]$, then for all $\xi \in [-1/(2T), 1/(2T)]$,

$$F_T(\xi) = \mathcal{F}f(\xi)$$

- Therefore, the Fourier coefficients $c_k(F_T)$ are given by :

$$c_k(F_T) = T \int_{-B}^{B} \mathcal{F}f(\xi) \mathrm{e}^{+\mathrm{i}2\pi\xi kT} \mathrm{d}\xi = Tf(kT) \, .$$

The Poisson Formula : Key result

If $f \in \mathrm{BL}(B)$ then the discrete-time Fourier transform (DTFT) of the sequence of samples $\{f(nT), n \in \mathbb{Z}\}$

$$T \sum_{n \in \mathbb{Z}} f(nT) \mathrm{e}^{-\mathrm{i}2\pi\xi nT}$$

is equal to the Fourier transform of the function f periodized with a period $1/T$, a relation called the Poisson formula

$$\sum_{n \in \mathbb{Z}} \mathcal{F}f\left(\xi - \frac{n}{T}\right) = T \sum_{n \in \mathbb{Z}} f(nT) \mathrm{e}^{-\mathrm{i}2\pi\xi nT} ,$$

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An amazing result : the DTFT is computable only from the knowledge of the samples $\{x(nT), n \in \mathbb{Z}\}$, whereas the knowledge of $\mathcal{F}f$ requires to evaluate the function at all time points

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$$\sum_{n \in \mathbb{Z}} \mathcal{F}f\left(\xi - \frac{n}{T}\right) = T \sum_{n \in \mathbb{Z}} f(nT) \mathrm{e}^{-\mathrm{i}2\pi\xi nT} ,$$

the DTFT is periodic with period $1/T$, which is the sampling frequency. Because of this periodicity, one may restrict the frequency interval to just one-period, $[-1/2T, 1/2T]$

Interpolation formula

- Multiply the Poisson formula

$$\sum_{n\in\mathbb{Z}} \mathcal{F}f\left(\xi - \frac{n}{T}\right) = T\sum_{n\in\mathbb{Z}} f(nT)\mathrm{e}^{-\mathrm{i}2\pi\xi nT} \ ,$$

by the indicator function $\mathbb{1}_{[-1/(2T),1/(2T)]}(\xi)$

- Use the identity

$$\mathbb{1}_{[-1/(2T),1/(2T)]}(\xi)\mathcal{F}f(\xi) = \mathbb{1}_{[-1/(2T),1/(2T)]}(\xi)F_T(\xi)$$

Interpolation formula

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Interpolation formula

For all $\xi \in [-1/2T, +1/2T]$,

$$\mathcal{F}f(\xi) = T \sum_{n \in \mathbb{Z}} f(nT) \mathbb{1}_{[-1/(2T), 1/(2T)]}(\xi) \mathrm{e}^{-\mathrm{i}2\pi\xi nT} \; .$$

which should be understood as

$$\lim_{N\to\infty} \int \left| \mathcal{F}f(\xi) - T \sum_{n=-N}^{N} f(nT) \mathbb{1}_{[-\frac{1}{2T}, \frac{1}{2T}]}(\xi) \mathrm{e}^{-\mathrm{i}2\pi\xi nT} \right|^2 \mathrm{d}\xi = 0 \; .$$

Since $\overline{\mathcal{F}}$ is an isometry and $\overline{\mathcal{F}}\mathcal{F}f = f$ a.e.

$$\lim_{N\to\infty} \int \left| f(t) - T \sum_{n=-N}^{N} f(nT) \overline{\mathcal{F}} \left(\mathbb{1}_{[-\frac{1}{2T}, \frac{1}{2T}]}(\xi) \mathrm{e}^{+\mathrm{i}2\pi\xi nT} \right) (t) \right|^2 \mathrm{d}t = 0 \; .$$

Interpolation formula

Lemma

$$\overline{\mathcal{F}}\left(\xi \to \mathbb{1}_{[-1/(2T),1/(2T)]}(\xi)\mathrm{e}^{+\mathrm{i}2\pi\xi nT}\right)(x) = \frac{\sin\left(\frac{\pi}{T}(x-nT)\right)}{\pi(x-nT)}$$

Interpolation formula

Theorem (Nyquist theorem)

If $f \in \mathrm{BL}(B)$ *and* $T \leq 1/2B$ *then*

$$f(x) =_{L_2(\mathbb{R})} \sum_{n \in \mathbb{Z}} f(nT) s_T(x - nT) \ ,$$

where s_T*, is the cardinal-sine function*

$$s_T(x) = \frac{\sin(\pi x/T)}{\pi x/T} \ .$$

If in addition,

$$\sum_{k \in \mathbb{Z}} |f(kT)| < \infty,$$

the series converges uniformly to f*.*

What happens if the Nyquist condition is violated ?

- Assume that $f \in \mathrm{BL}(B)$ but $T \geq 1/(2B)$.
- It still makes sense to consider the periodization of the Fourier transform
$$F_T(\xi) = \sum_{n \in \mathbb{Z}} [\mathcal{F}f] \left(\xi - \frac{n}{T} \right) \ .$$
- but now the different spectral replicas now overlap, a phenomenon called aliasing.

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- but now the different spectral replicas now overlap, a phenomenon called aliasing.

Aliasing

- The periodized spectrum $\xi \mapsto F_T(\xi)$ still belongs to $L^2(\mathbb{R})$.
- We may still develop this function as a Fourier series :

$$F_T(\xi) = \sum_{n\in\mathbb{Z}} c_k(T)\mathrm{e}^{-\mathrm{i}2\pi\xi nT},$$

where $c_k(T)$ are the Fourier coefficients.

- Key result ! We still have

$$c_k(T) = Tf(kT)$$

the Fourier coefficients of the periodized spectrum are the samples of the functions !

Aliasing...

The Fourier transform of the Nyquist interpolation

$$\tilde{f}(x) = \sum_{n\in\mathbb{Z}} f(nT)s_T(x - nT) \; ,$$

is the $\mathrm{BL}(1/2T)$ function whose Fourier transform is equal to

$$\mathbb{1}_{[-1/2T,1/2T]}(\xi) \sum_{n=-\infty}^{\infty} \mathcal{F}f(\xi - n/T)$$

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Beware ! It is essential to use a lowpass antialiasing prefilter to bandlimit the input signal to within the Nyquist interval so that the resulting replicas after sampling will not overlap

An illustration

- This example illustrates the effect of sampling a non bandlimited signal and the degree to which the portion of the Fourier transform (spectrum) within the Nyquist interval approximates the original spectrum.
- Consider the exponentially decaying signal and its spectrum

$$f(x) = \mathrm{e}^{-ax}u(x) , \quad \mathcal{F}f(\xi) = \frac{1}{a + 2\pi\mathrm{i}\xi} .$$

- The discrete time Fourier transform of the sampled signal can be obtained in two different ways...

Calculation of the DFT

$$\hat{F}_T(\xi) = T \sum_{n=-\infty}^{\infty} f(nT) \mathrm{e}^{-\mathrm{i}2\pi n\xi T}$$

$$= T \sum_{n=0}^{\infty} \mathrm{e}^{-anT} \mathrm{e}^{-\mathrm{i}2\pi n\xi T} = \frac{T}{1 - \mathrm{e}^{-aT} \mathrm{e}^{-\mathrm{i}2\pi\xi T}}$$

On the other hand, by the Poisson formula (which is still valid) the DTFT is the equal to the periodized spectrum

$$\hat{F}_T(\xi) = \sum_{m=-\infty}^{\infty} \frac{1}{a + 2\pi\mathrm{i}(\xi - n\xi_s)}$$

where $\xi_s = 1/T$ is the sampling frequency.

A not so-obvious identity

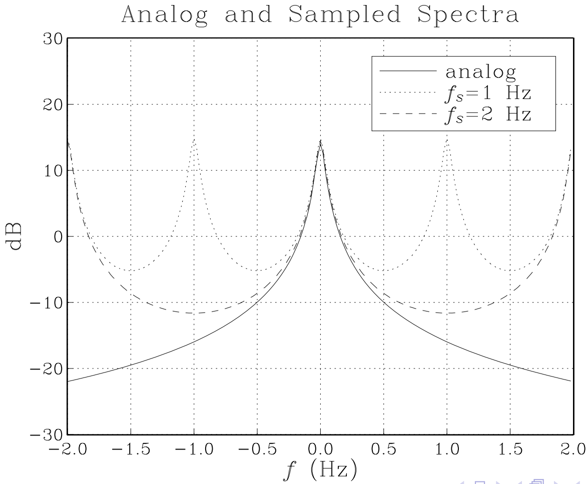
- Combining the two expressions of the DTFT, we obtain the not-so-obvious identity

$$\sum_{m=-\infty}^{\infty} \frac{1}{a + 2\pi \mathrm{i}(\xi - n\xi_s)} = \frac{T}{1 - \mathrm{e}^{-aT}\mathrm{e}^{-\mathrm{i}2\pi\xi T}}$$

- A sanity check... When $T \to 0$ (the sampling frequency goes to ∞), then

$$\frac{T}{1 - \mathrm{e}^{-aT}\mathrm{e}^{-\mathrm{i}2\pi\xi T}} \to \frac{1}{a + 2\pi\mathrm{i}\xi}$$

Aliasing : the exponential case



Ideal antialiasing filter

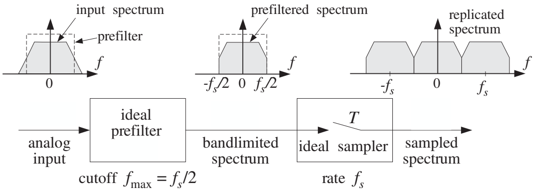


FIGURE – An ideal analog lowpass prefilter removes all the frequency components of the analog input that lie beyond the Nyquist frequency $\xi_s/2$

Practical antialiasing filter

- Antialiasing filter used in practice are not ideal and do not completely remove all the frequency components outside the Nyquist interval. Thus, some aliasing will always take place.
- By proper design, the prefilters may be made as good as desired and the amount of aliasing reduced to tolerable levels.
- A practical anti-aliasing filter is a low pass filter with passband usually taken to be the frequency-range of interest for the application at hand and must be entirely within the Nyquist interval.

Practical antialiasing filter

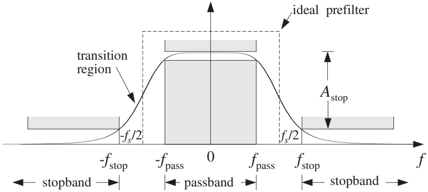


FIGURE – Practical antialiasing lowpass filter

Practical antialiasing filter

- The prefilter must be essentially flat over his passband in order not to distort the frequencies of interest.
- Even if it is not completely flat over the passband, it can be equalized digitally at a subsequent processing stage by a digital filter.
- The stopband frequency ξ_{stop} of the prefilter and the minimum stop band attenuation must be chosen appropriately to minimize aliasing effects.
- The stop band frequency should be chosen so that

$$\xi_{\mathrm{stop}} = \xi_{\mathrm{s}} - \xi_{\mathrm{pass}} \ .$$

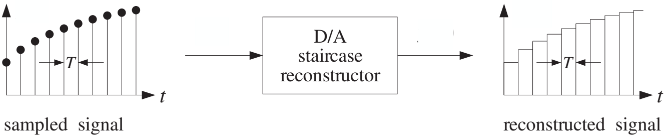
Analog reconstructors

- The ideal interpolation is not physically achievable (the support of s_T is the whole real-line and is non causal).
- Any reasonable way of filling the gaps between samples will result in some sort of reconstruction.
- A typical reconstruction formula is

$$f_a(x) = \sum_{n=-\infty}^{\infty} f(nT)h_T(x - nT)$$

  where h_T has a finite support.
- The simplest reconstruction formula is the sample and hold or staircase in which $h_T(x) = \mathbb{1}_{[0,T]}(x)$.

Staircase reconstructor



Practical reconstructor

- Assume that $h_T \in L_1(\mathbb{R}) \times L_2(\mathbb{R})$. Then

$$\mathcal{F}h_T(\xi) = \hat{h}_T(\xi) = \int_{-\infty}^{\infty} h_T(x)\mathrm{e}^{-\mathrm{i}2\pi\xi x}\mathrm{d}x$$

Note that $\hat{h}_T \in L_2(\mathbb{R}) \cap L_\infty(\mathbb{R})$.

- For any $N \in \mathbb{N}$,

$$\overline{\mathcal{F}}\left(\sum_{n=-N}^{N} f(nT)h_T(x-nT)\right)(\xi) = \hat{h}_T(\xi)\sum_{n=-N}^{N} f(nT)\mathrm{e}^{-\mathrm{i}2\pi\xi nT}$$
$$= T^{-1}\hat{h}_T(\xi)F_{N,T}(\xi)$$

Practical reconstruction

- $\lim_{N\to\infty} \|F_{N,T} - F_T\|_2 = 0$ where F_T is the T-periodized spectrum.
- Therefore, in $L_2(\mathbb{R})$,

$$\overline{\mathcal{F}}\left(\sum_{n=-N}^{N} f(nT)h_T(x-nT)\right)(\xi) = T^{-1}\hat{h}_T(\xi)\sum_{m=-\infty}^{\infty}\mathcal{F}f(\xi - m/T)$$

Practical reconstruction

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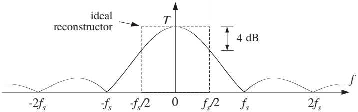
- Practical interpolators do not completely eliminate the replicated spectral images as the ideal reconstructor does.

Staircase reconstruction

For the staircase reconstructor, $h_T(x) = \mathbb{1}_{[0,T]}(x)$ and

$$\hat{h}_T(\xi) = T\frac{\sin(\pi\xi T)}{\pi\xi T}\mathrm{e}^{-\mathrm{i}\pi\xi T} .$$



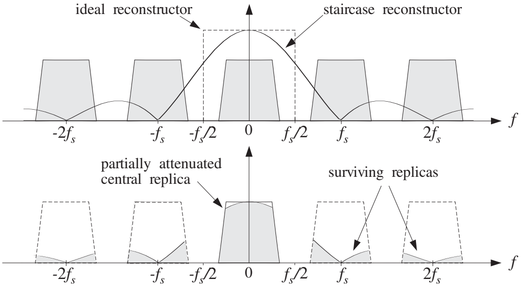
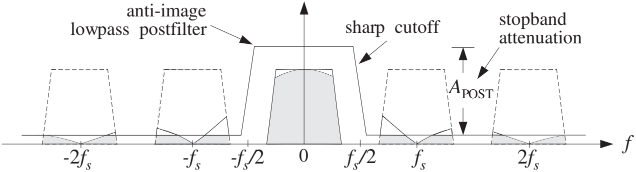


FIGURE – The staircase reconstructor does not completely eliminate the replicated spectral images as the ideal reconstructor does

Anti-image postfilters

- The surviving spectral replicas may be removed by an additional lowpass filter, called the anti-image postfilter, whose cutoff frequency in the Nyquist frequency $\xi_s/2$.
- In the frequency domain, the combined effect of the staircase reconstructor followed by the anti-image postfilter is to remove the spectral replica as much as possible.
- The reason for using this two-stage reconstruction procedure is the simplicity pf implementation of the staircase reconstruction. A typical D/A converter will act as such a reconstructor.

Anti-image postfilters



Why ?

- Sampling theorem can be derived from another perspective, which is in some sense broader.
- It acknowledges the fact that a function can be seen as a superposition if complex exponential (think ! this is the meaning of the Fourier synthesis formula...)
- The approach starts by deriving the sampling theorem first for complex exponential, and then to extends the result to a (much) wider class of signals.

Problem

- Consider the function $s_{\xi_0}(x) = \mathrm{e}^{2\mathrm{i}\pi\xi_0 x}$, $\xi_0 \in \mathbb{R}$. This function is neither in $L_1(\mathbb{R})$ or in $L_2(\mathbb{R})$, and therefore does not fit into the framework presented earlier.
- A natural question is the following : under which conditions on the sampling period T may we retrieve $s(x)$ from the its samples $\{s(nT), n \in \mathbb{Z}\}$.
- Since $s_{\xi_0}(nT) = \mathrm{e}^{2\mathrm{i}\pi\xi_0 nT}$, we have

$$s_{\xi_0}(nT) = s_{\xi_0+k/T}(nT)\ , \quad \text{for all } k, n \in \mathbb{Z}$$

- If $\xi_0 \in \left]-1/2T, 1/2T\right[$ we can retrieve s_{ξ_0} from its samples without error !

A sampling theorem for complex exponential

Theorem

For all $x \in \mathbb{R}$ *and* $\xi_0 \in \left]-1/2T, 1/2T\right[$,

$$\mathrm{e}^{2\mathrm{i}\pi\xi_0 x} = \sum_{n\in\mathbb{Z}} \mathrm{e}^{2\mathrm{i}\pi\xi_0 nT} s_T(x - nT) \quad s_T(x) = \frac{\sin\left(\pi x/T\right)}{\pi x/T}$$

where the series in the RHS converges uniformly on $[-B, B] \subset \left]-1/2T, 1/2T\right[$

If f is a-periodic

$$f(y) = \sum_{k \in \mathbb{Z}} c_k(f)\, e^{+2i\pi \frac{k}{a} y} \qquad c_k(f) = \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(y) \cdot e^{-i2\pi \frac{k}{a} y}\, dy.$$

We apply this result to $g_x(\xi) = e^{+2i\pi \xi x}$. $\quad \xi \in \left]-\frac{1}{2T}, +\frac{1}{2T}\right[$;

Then,
$$g_x(\xi) = \sum_{k \in \mathbb{Z}} c_k(g_x)\, e^{+i2\pi \xi k T} \qquad \left(\text{here, } a \leftarrow \frac{1}{T},\ y \leftarrow \xi\right)$$

and
$$c_k(g_x) = T \int_{-\frac{1}{2T}}^{+\frac{1}{2T}} g_x(\xi)\, e^{-i2\pi \xi k T}\, d\xi$$

$$= T \int_{-\frac{1}{2T}}^{+\frac{1}{2T}} e^{+i2\pi \xi x}\, e^{-i2\pi \xi k T}\, d\xi$$

$$= \frac{1}{i2\pi(x - kT)} \left[e^{+i2\pi \xi (x - kT)} \right]_{-1/2T}^{+\frac{1}{2T}} = \frac{\sin\left(\frac{\pi}{T}(x - kT)\right)}{\frac{\pi}{T}(x - kT)}$$

Sampling a decomposable signal

- If $x \mapsto f(x)$ is a finite superposition of complex sinewaves,

$$f(x) = \sum_{k=1}^{M} \gamma_k \mathrm{e}^{2\mathrm{i}\nu_k x}$$

where $\{\gamma_k\}_{k=1}^{M} \in \mathbb{C}$ and $\{\nu_k\}_{k=1}^{M} \subset]-1/2T, 1/2T[$, then the Nyquist formula holds

$$f(x) = \sum_{k=-\infty}^{\infty} f(kT) s_T(x - kT)$$

- This can be extended to any function f which can be written as

$$f(x) = \int_{-B}^{B} \mathrm{e}^{2\mathrm{i}\pi\xi x} \mu(\mathrm{d}x)$$

where μ is a measure on $[-B, B]$ as soon as $B < 1/2T$.