3

Stationary Time Series Models

Limited by a finite number of available observations, we often construct a finite-order parametric model to describe a time series process. In this chapter, we introduce the autoregressive moving average model, which includes the autoregressive model and the moving average model as special cases. This model contains a very broad class of parsimonious time series processes found useful in describing a wide variety of time series. After giving detailed discussions on the characteristics of each process in terms of the autocorrelation and partial autocorrelation functions, we illustrate the results with examples.

3.1 Autoregressive Processes

As mentioned earlier in Section 2.6, in the autoregressive representation of a process, if only a finite number of π weights are nonzero, i.e., $\pi_1 = \phi_1$, $\pi_2 = \phi_2, \ldots, \pi_p = \phi_p$, and $\pi_k = 0$ for $k > p$, then the resulting process is said to be an autoregressive process (model) of order p, which is denoted as AR(p). It is given by

$$\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + \cdots + \phi_p \dot{Z}_{t-p} + a_t \tag{3.1.1}$$

or

$$\phi_p(B)\dot{Z}_t = a_t, \tag{3.1.2}$$

where $\phi_p(B) = (1 - \phi_1 B - \cdots - \phi_p B^p)$, and $\dot{Z}_t = Z_t - \mu$.

Because $\sum_{j=1}^{\infty}|\pi_j| = \sum_{j=1}^{p}|\phi_j| < \infty$, the process is always invertible. To be stationary, the roots of $\phi_p(B) = 0$ must lie outside of the unit circle. The AR processes are useful in describing situations in which the present value of a time series depends on its preceding values plus a random shock. Yule (1927) used an AR process to describe the phenomena of sunspot numbers and the behavior of a simple pendulum. First, let us consider the following simple models.

3.1.1 The First-Order Autoregressive AR(1) Process

For the first-order autoregressive process AR(1), we write

$$(1 - \phi_1 B)\dot{Z}_t = a_t \tag{3.1.3a}$$

or

$$\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + a_t. \tag{3.1.3b}$$

As mentioned above, the process is always invertible. To be stationary, the root of $(1 - \phi_1 B) = 0$ must be outside of the unit circle. That is, for a stationary process, we have $|\phi_1| < 1$. The AR(1) process is sometimes called the Markov process because the distribution of $\dot{Z}_t$ given $\dot{Z}_{t-1}, \dot{Z}_{t-2}, \dot{Z}_{t-3}, \ldots$ is exactly the same as the distribution of $\dot{Z}_t$ given $\dot{Z}_{t-1}$.

ACF of the AR(1) Process The autocovariances are obtained as follows:

$$\begin{aligned} E(\dot{Z}_{t-k}\dot{Z}_t) &= E(\phi_1 \dot{Z}_{t-k}\dot{Z}_{t-1}) + E(\dot{Z}_{t-k}a_t) \\ \gamma_k &= \phi_1 \gamma_{k-1}, \qquad k \geq 1, \end{aligned} \tag{3.1.4}$$

and the autocorrelation function becomes

$$\rho_k = \phi_1 \rho_{k-1} = \phi_1^k, \qquad k \geq 1, \tag{3.1.5}$$

where we use that $\rho_0 = 1$. Hence, when $|\phi_1| < 1$ and the process is stationary, the ACF exponentially decays in one of two forms depending on the sign of ϕ_1. If $0 < \phi_1 < 1$, then all autocorrelations are positive; if $-1 < \phi_1 < 0$, then the sign of the autocorrelations shows an alternating pattern beginning with a negative value. The magnitudes of these autocorrelations decrease exponentially in both cases, as shown in Figure 3.1.

PACF of the AR(1) Process For an AR(1) process, the PACF from (2.3.19) is

$$\phi_{kk} = \begin{cases} \rho_1 = \phi_1, & k = 1, \\ 0, & \text{for } k \geq 2. \end{cases} \tag{3.1.6}$$

Hence, the PACF of the AR(1) process shows a positive or negative spike at lag 1 depending on the sign of ϕ_1 and then cuts off as shown in Figure 3.1.

EXAMPLE 3.1 For illustration, we simulated 250 values from an AR(1) process, $(1 - \phi_1 B)(Z_t - 10) = a_t$, with $\phi_1 = .9$. The white noise series a_t are independent normal $N(0, 1)$ random variables. Figure 3.2 shows the plot of the series. It is relatively smooth.

Table 3.1 and Figure 3.3 show the sample ACF and the sample PACF for the series. Clearly $\hat{\rho}_k$ decreases exponentially and $\hat{\phi}_{kk}$ cuts off after lag 1 because none of the sample

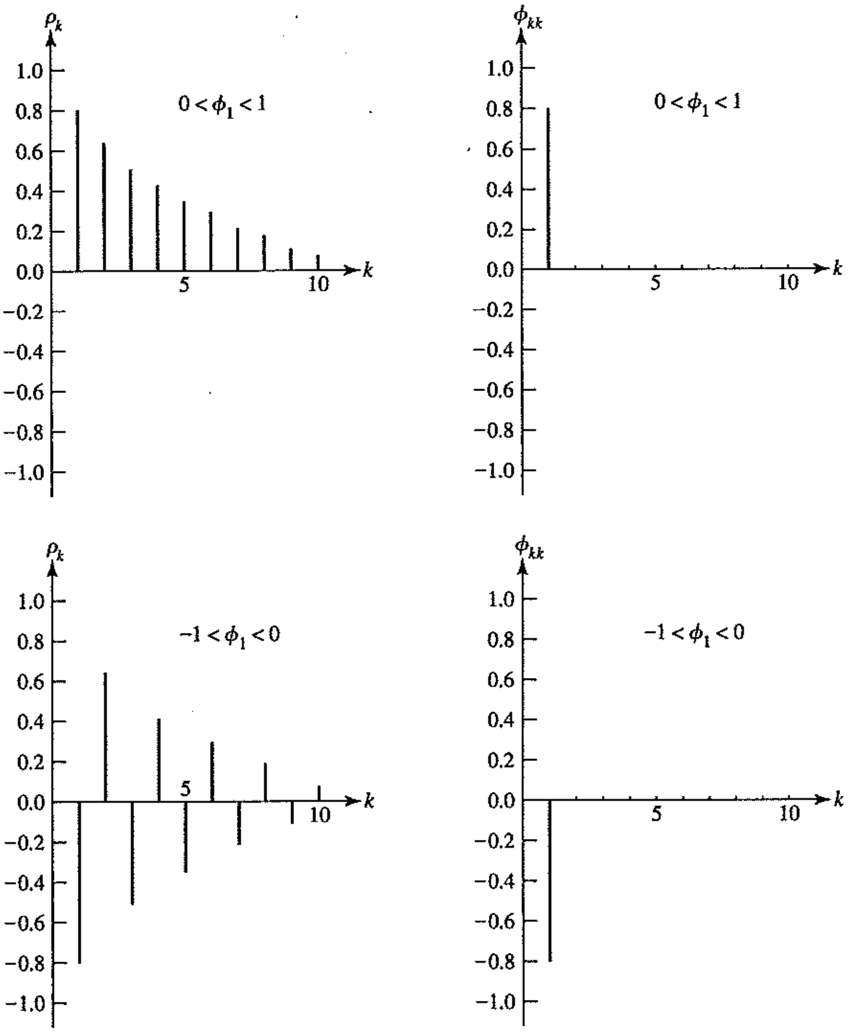

FIGURE 3.1 ACF and PACF of the AR(1) process: $(1 - \phi_1 B)\dot{Z}_t = a_t$.

PACF values is significant beyond that lag and, more important, these insignificant $\hat{\phi}_{kk}$ do not exhibit any pattern. The associated standard error of the sample ACF $\hat{\rho}_k$ is computed by

$$S_{\hat{\rho}_k} \simeq \sqrt{\frac{1}{n}(1 + 2\hat{\rho}_1^2 + \cdots + 2\hat{\rho}_{k-1}^2)}, \tag{3.1.7}$$

and the standard error of the sample PACF $\hat{\phi}_{kk}$ is set to be

$$S_{\hat{\phi}_{kk}} \simeq \sqrt{\frac{1}{n}}, \tag{3.1.8}$$

which are standard outputs used in most time series software.

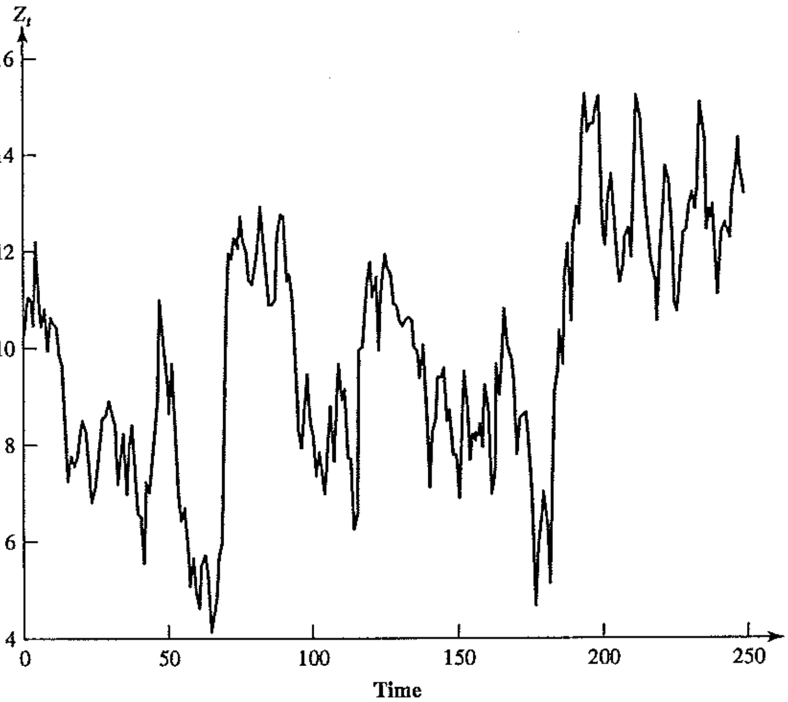

FIGURE 3.2 A simulated AR(1) series, $(1 - .9B)(Z_t - 10) = a_t$.

EXAMPLE 3.2 This example shows a simulation of 250 values from the AR(1) process $(1 - \phi_1 B)(Z_t - 10) = a_t$, with $\phi_1 = -.65$ and a_t being Gaussian $N(0, 1)$ white noise. The series is plotted in Figure 3.4 and is relatively jagged.

The sample ACF and sample PACF of the series are shown in Table 3.2 and Figure 3.5. We see the alternating decreasing pattern beginning with a negative in the sample ACF and the cutoff property of the sample PACF. Because $\hat{\phi}_{11} = \hat{\rho}_1$, $\hat{\phi}_{11}$ is also negative. Also, even though only the first two or three sample autocorrelations are significant, the overall pattern clearly indicates the phenomenon of an AR(1) model with a negative value of ϕ_1.

TABLE 3.1 Sample ACF and sample PACF for a simulated series from $(1 - .9B)(Z_t - 10) = a_t$.

k	1	2	3	4	5	6	7	8	9	10
$\hat{\rho}_k$	.88	.76	.67	.57	.48	.40	.34	.28	.21	.17
St.E.	.06	.10	.12	.14	.14	.15	.16	.16	.16	.16
$\hat{\phi}_{kk}$	.88	.01	−.01	−.11	.02	−.01	.01	−.02	−.06	.05
St.E.	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06

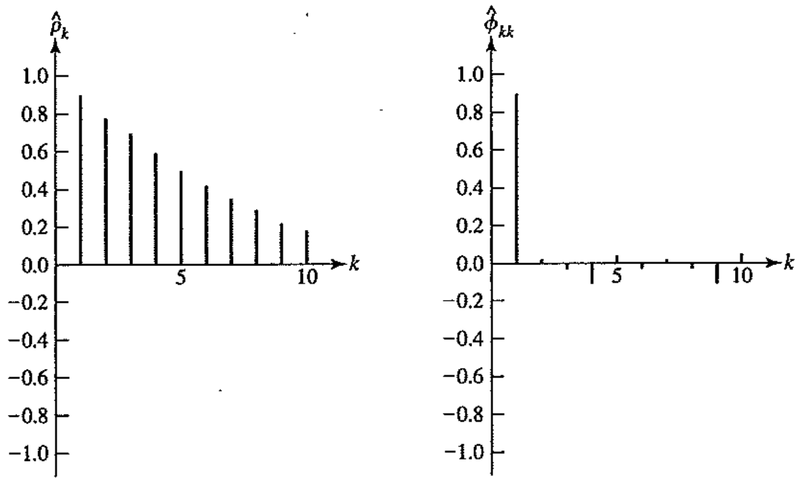

FIGURE 3.3 Sample ACF and sample PACF of a simulated AR(1) series: $(1 - .9B)(Z_t - 10) = a_t$.

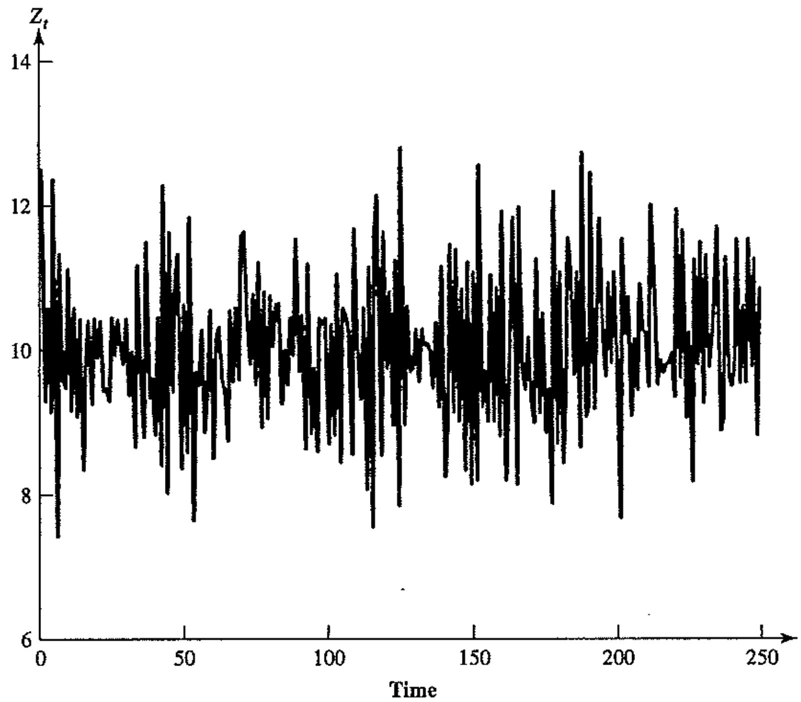

FIGURE 3.4 A simulated AR(1) series $(1 + .65B)(Z_t - 10) = a_t$.

TABLE 3.2 Sample ACF and sample PACF for a simulated series from $(1 + .65B)(Z_t - 10) = a_t$.

k	1	2	3	4	5	6	7	8	9	10
$\hat{\rho}_k$	−.63	.36	−.17	.09	−.07	.06	−.08	.10	−.11	.06
St.E.	.06	.08	.09	.09	.09	.09	.09	.09	.09	.09
$\hat{\phi}_{kk}$	−.63	−.06	.05	.02	−.04	−.01	−.06	.04	−.03	−.05
St.E.	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06

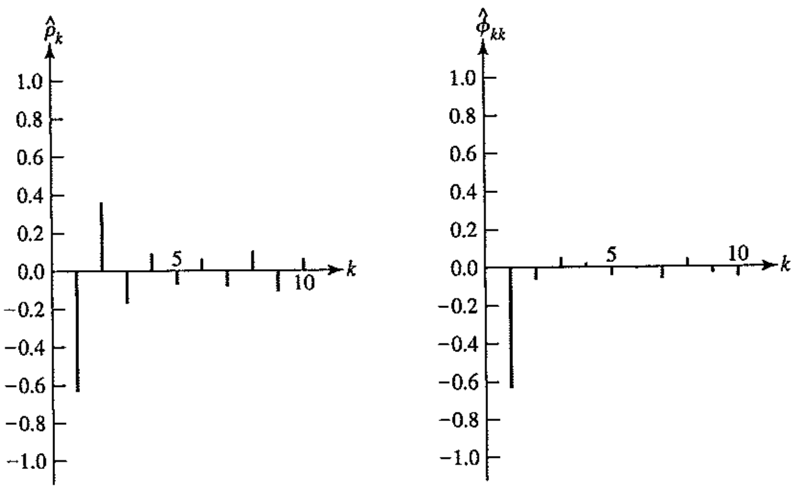

FIGURE 3.5 Sample ACF and sample PACF of a simulated AR(1) series $(1 + .65B)(Z_t - 10) = a_t$.

In discussing stationary autoregressive processes, we have assumed that the zeros of the autoregressive polynomial $\phi_p(B)$ lie outside of the unit circle. In terms of the AR(1) process (3.1.3a or b), it implies that $|\phi_1| < 1$. Thus, when $|\phi_1| \geq 1$, the process is regarded as nonstationary because we have implicitly assumed that the process is expressed as a linear combination of present and past white noise variables. If we also consider a process that is expressed as a linear combination of present and future random shocks, then there exists an AR(1) process with its parameter ϕ_1 greater than 1 in absolute value, which is still stationary in the usual sense of the term as defined in Section 2.1. To see that, consider the process

$$Z_t = \sum_{j=0}^{\infty} (.5)^j a_{t+j}, \tag{3.1.9}$$

where $\{a_t\}$ is a white noise process with mean zero and variance σ_a^2. It is straightforward to verify that the process Z_t in (3.1.9) is indeed stationary in the sense of Section 2.1 with the

ACF, $\rho_k = (.5)^{|k|}$. Now, consider the process (3.1.9) at time $(t - 1)$ and multiply both of its sides by 2, i.e.,

$$\begin{aligned} 2Z_{t-1} &= 2\sum_{j=0}^{\infty}(.5)^j a_{t-1+j} \\ &= 2a_{t-1} + \sum_{j=1}^{\infty}(.5)^{j-1}a_{t-1+j} \\ &= 2a_{t-1} + \sum_{j=0}^{\infty}(.5)^j a_{t+j}. \end{aligned} \tag{3.1.10}$$

Thus, (3.1.9) leads to the following equivalent AR(1) model with $\phi_1 = 2$,

$$Z_t - 2Z_{t-1} = b_t, \tag{3.1.11}$$

where $b_t = -2a_{t-1}$. Note, however, that although the b_t in (3.1.11) is a white noise process with mean zero, its variance becomes $4\sigma_a^2$, which is four times larger than the variance of a_t in the following AR(1) model with the same ACF, $\rho_k = (.5)^{|k|}$,

$$Z_t - .5Z_{t-1} = a_t, \tag{3.1.12}$$

which can be written as a linear combination of present and past random shocks, i.e., $Z_t = \sum_{j=0}^{\infty}(.5)^j a_{t-j}$.

In summary, although a process with an ACF of the form $\phi^{|k|}$, where $|\phi| < 1$, can be written either as

$$Z_t - \phi Z_{t-1} = a_t \tag{3.1.13}$$

or

$$Z_t - \phi^{-1}Z_{t-1} = b_t, \tag{3.1.14}$$

where both a_t and b_t are zero mean white noise processes, the variance of b_t in (3.1.14) is larger than the variance of a_t in (3.1.13) by a factor of ϕ^{-2}. Thus, for practical purposes, we will choose the representation (3.1.13). That is, in terms of a stationary AR(1) process, we always refer to the case in which the parameter value is less than 1 in absolute value.

3.1.2 The Second-Order Autoregressive AR(2) Process

For the second-order autoregressive AR(2) process, we have

$$(1 - \phi_1 B - \phi_2 B^2)\dot{Z}_t = a_t \tag{3.1.15a}$$

or

$$\dot{Z}_t = \phi_1\dot{Z}_{t-1} + \phi_2\dot{Z}_{t-2} + a_t. \tag{3.1.15b}$$

The AR(2) process, as a finite autoregressive model, is always invertible. To be stationary, the roots of $\phi(B) = (1 - \phi_1 B - \phi_2 B^2) = 0$ must lie outside of the unit circle. For example, the process $(1 - 1.5B + .56B^2)Z_t = a_t$ is stationary because $(1 - 1.5B + .56B^2) = (1 - .7B)(1 - .8B) = 0$ gives $B = 1/.7$ and $B = 1/.8$ as the two roots, which are larger than 1 in absolute value. Yet $(1 - .2B - .8B^2)\dot{Z}_t = a_t$ is not stationary because one of the roots of $(1 - .2B - .8B^2) = 0$ is $B = 1$, which is not outside of the unit circle.

The stationarity condition of the AR(2) model can also be expressed in terms of its parameter values. Let B_1 and B_2 be the roots of $(1 - \phi_1 B - \phi_2 B^2) = 0$ or, equivalently, of $\phi_2 B^2 + \phi_1 B - 1 = 0$. We have

$$B_1 = \frac{-\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2}$$

and

$$B_2 = \frac{-\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2}.$$

Now,

$$\frac{1}{B_1} = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

and

$$\frac{1}{B_2} = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}.$$

The required condition $|B_i| > 1$ implies that $|1/B_i| < 1$ for $i = 1$ and 2. Hence,

$$\left|\frac{1}{B_1}\cdot\frac{1}{B_2}\right| = |\phi_2| < 1$$

and

$$|\phi_1| = \left|\frac{1}{B_1} + \frac{1}{B_2}\right| < 2.$$

Thus, we have the following necessary condition for stationarity regardless of whether the roots are real or complex:

$$\begin{cases} -1 < \phi_2 < 1, \\ -2 < \phi_1 < 2. \end{cases} \tag{3.1.16}$$

For real roots, we need $\phi_1^2 + 4\phi_2 \geq 0$, which implies that

$$-1 < \frac{1}{B_2} = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} \leq \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} = \frac{1}{B_1} < 1,$$

or, equivalently,

$$\begin{cases} \phi_2 + \phi_1 < 1, \\ \phi_2 - \phi_1 < 1. \end{cases} \tag{3.1.17}$$

For complex roots, we have $\phi_2 < 0$ and $\phi_1^2 + 4\phi_2 < 0$. Thus, in terms of the parameter values, the stationarity condition of the AR(2) model is given by the following triangular region in Figure 3.6 satisfying

$$\begin{cases} \phi_2 + \phi_1 < 1, \\ \phi_2 - \phi_1 < 1, \\ -1 < \phi_2 < 1. \end{cases} \tag{3.1.18}$$

ACF of the AR(2) Process We obtain the autocovariances by multiplying Z_{t-k} on both sides of (3.1.15b) and taking the expectation

$$\begin{aligned} E(\dot{Z}_{t-k}\dot{Z}_t) &= \phi_1 E(\dot{Z}_{t-k}\dot{Z}_{t-1}) + \phi_2 E(\dot{Z}_{t-k}\dot{Z}_{t-2}) + E(\dot{Z}_{t-k}a_t) \\ \gamma_k &= \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2}, \qquad k \geq 1. \end{aligned}$$

Hence, the autocorrelation function becomes

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}, \qquad k \geq 1. \tag{3.1.19}$$

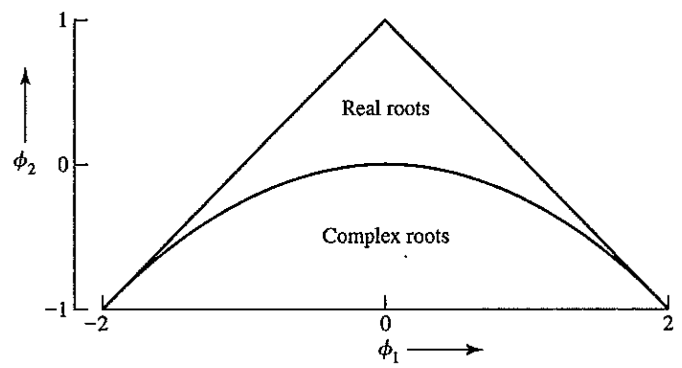

FIGURE 3.6 Stationary regions for the AR(2) model.

Specifically, when $k = 1$ and 2

$$\rho_1 = \phi_1 + \phi_2\rho_1$$
$$\rho_2 = \phi_1\rho_1 + \phi_2,$$

which implies that

$$\rho_1 = \frac{\phi_1}{1 - \phi_2} \tag{3.1.20}$$

$$\rho_2 = \frac{\phi_1^2}{1 - \phi_2} + \phi_2 = \frac{\phi_1^2 + \phi_2 - \phi_2^2}{1 - \phi_2}, \tag{3.1.21}$$

and ρ_k for $k \geq 3$ is calculated recursively through (3.1.19).

The pattern of the ACF is governed by the difference equation given by (3.1.19), namely $(1 - \phi_1 B - \phi_2 B^2)\,\rho_k = 0$. Using Theorem 2.7.1, we obtain

$$\rho_k = \begin{cases} b_1\left[\dfrac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}\right]^k + b_2\left[\dfrac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}\right]^k, & \text{if } \phi_1^2 + 4\phi_2 \neq 0, \\ (b_1 + b_2 k)\left[\dfrac{\phi_1}{2}\right]^k, & \text{if } \phi_1^2 + 4\phi_2 = 0, \end{cases} \tag{3.1.22}$$

where the constants b_1 and b_2 can be solved using the initial conditions given in (3.1.20) and (3.1.21). Thus, the ACF will be an exponential decay if the roots of $(1 - \phi_1 B - \phi_2 B^2) = 0$ are real and a damped sine wave if the roots of $(1 - \phi_1 B - \phi_2 B^2) = 0$ are complex.

The AR(2) process was originally used by G. U. Yule in 1921 to describe the behavior of a simple pendulum. Hence, the process is also sometimes called the Yule process.

PACF of the AR(2) Process For the AR(2) process, because

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}$$

for $k \geq 1$ as shown in (3.1.19), we have, from (2.3.19),

$$\phi_{11} = \rho_1 = \frac{\phi_1}{1 - \phi_2} \tag{3.1.23a}$$

$$\phi_{22} = \frac{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix}} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}$$

$$= \frac{\left(\frac{\phi_1^2 + \phi_2 - \phi_2^2}{1 - \phi_2}\right) - \left(\frac{\phi_1}{1 - \phi_2}\right)^2}{1 - \left(\frac{\phi_1}{1 - \phi_2}\right)^2} \tag{3.1.23b}$$

$$= \frac{\phi_2\left[(1 - \phi_2)^2 - \phi_1^2\right]}{(1 - \phi_2)^2 - \phi_1^2} = \phi_2$$

$$\phi_{33} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 1 & \rho_1 & \phi_1 + \phi_2\rho_1 \\ \rho_1 & 1 & \phi_1\rho_1 + \phi_2 \\ \rho_2 & \rho_1 & \phi_1\rho_2 + \phi_2\rho_1 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix}} = 0 \tag{3.1.23c}$$

because the last column of the numerator is a linear combination of the first two columns. Similarly, we can show that $\phi_{kk} = 0$ for $k \geq 3$. Hence, the PACF of an AR(2) process cuts off after lag 2. Figure 3.7 illustrates the PACF and corresponding ACF for a few selected AR(2) processes.

EXAMPLE 3.3 Table 3.3 and Figure 3.8 show the sample ACF and the sample PACF for a series of 250 values simulated from the AR(2) process $(1 + .5B - .3B^2)Z_t = a_t$, with the a_t being Gaussian $N(0, 1)$ white noise. The oscillating pattern of the ACF is similar to that of an AR(1) model with a negative parameter value. The rate of the decreasing of the autocorrelations, however, rejects the possibility of being an AR(1) model. That $\hat{\phi}_{kk}$ cuts off after lag 2, on the other hand, indicates an AR(2) model.

EXAMPLE 3.4 To consider an AR(2) model with the associated polynomial having complex roots, we simulated a series of 250 values from $(1 - B + .5B^2)Z_t = a_t$, with the a_t being Gaussian $N(0, 1)$ white noise. Table 3.4 and Figure 3.9 show the sample ACF and the sample PACF of this series. The sample ACF exhibits a damped sine wave, and the sample PACF cuts off after lag 2. Both give a fairly clear indication of an AR(2) model.

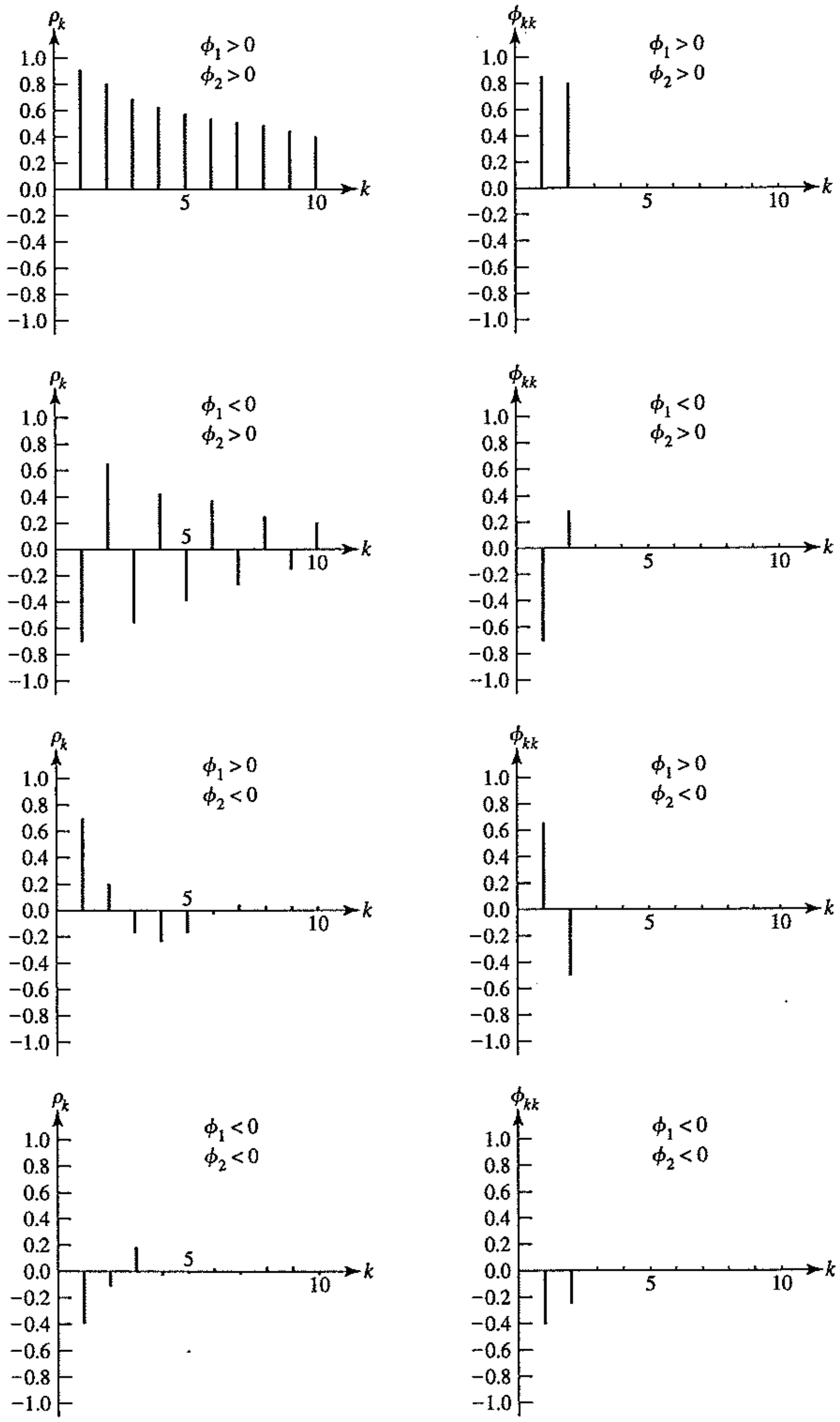

FIGURE 3.7 ACF and PACF of AR(2) process: $(1 - \phi_1 B - \phi_2 B^2)\dot{Z}_t = a_t$.

TABLE 3.3 Sample ACF and sample PACF for a simulated series from $(1 + .5B - .3B^2)Z_t = a_t$.

k	1	2	3	4	5	6	7	8	9	10
$\hat{\rho}_k$	−.70	.62	−.48	.41	−.37	.32	−.30	.27	−.25	.20
St.E.	.06	.09	.11	.11	.12	.12	.13	.13	.13	.13
$\hat{\phi}_{kk}$	−.70	.26	.05	.03	−.08	.00	−.04	.03	−.01	−.05
St.E.	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06

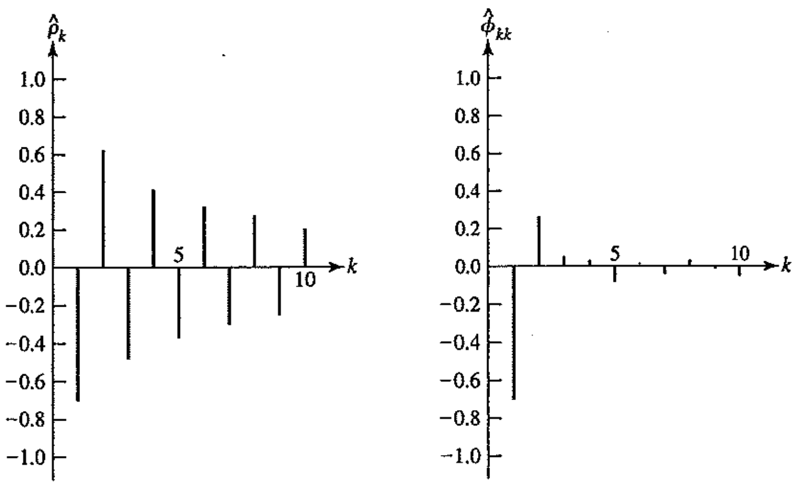

FIGURE 3.8 Sample ACF and sample PACF of a simulated AR(2) series: $(1 + .5B - .3B^2)Z_t = a_t$.

3.1.3 The General pth-Order Autoregressive AR(p) Process

The pth-order autoregressive process AR(p) is

$$(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p)\dot{Z}_t = a_t \tag{3.1.24a}$$

or

$$\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + \phi_2 \dot{Z}_{t-2} + \cdots + \phi_p \dot{Z}_{t-p} + a_t \tag{3.1.24b}$$

ACF of the General AR(p) Process To find the autocovariance function, we multiply Z_{t-k} on both sides of (3.1.24b)

$$\dot{Z}_{t-k}\dot{Z}_t = \phi_1 \dot{Z}_{t-k}\dot{Z}_{t-1} + \cdots + \phi_p \dot{Z}_{t-k}\dot{Z}_{t-p} + \dot{Z}_{t-k}a_t$$

and take the expected value

$$\gamma_k = \phi_1 \gamma_{k-1} + \cdots + \phi_p \gamma_{k-p}, \qquad k > 0, \tag{3.1.25}$$

TABLE 3.4 Sample ACF and sample PACF for a simulated series from $(1 - B + .5B^2)Z_t = a_t$.

k						$\hat{\rho}_k$						
1–12	.67	.20	−.13	−.26	−.22	−.09	.02	.08	.06	.00	−.10	−.17
St.E.	.06	.09	.09	.09	.09	.09	.09	.09	.10	.10	.10	.10
12–24	−.13	−.04	.07	.13	.10	.03	−.05	−.07	−.09	−.13	−.12	−.09
St.E.	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10

k						$\hat{\phi}_{kk}$						
1–12	.67	−.45	−.04	−.08	.05	−.01	.03	−.01	−.04	−.01	−.13	−.03
St.E.	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06
12–24	.06	−.04	.09	−.02	−.04	.01	−.02	.03	−.12	−.07	−.03	−.03
St.E.	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06

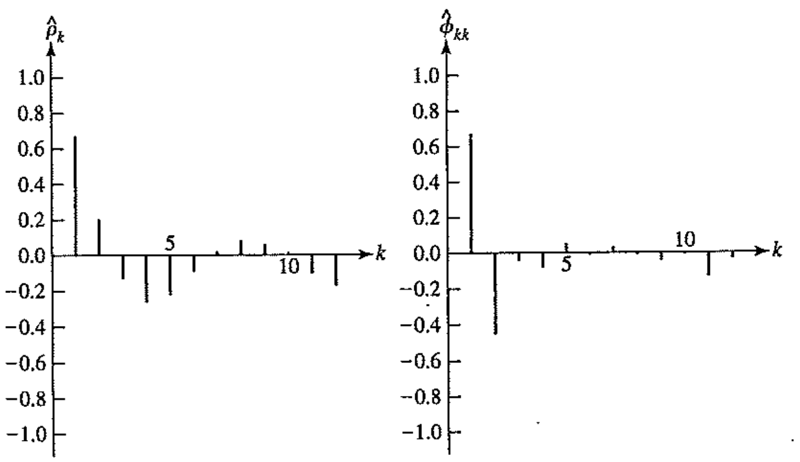

FIGURE 3.9 Sample ACF and sample PACF of a simulated AR(2) series: $(1 - B + .5B^2)Z_t = a_t$.

where we recall that $E(a_t Z_{t-k}) = 0$ for $k > 0$. Hence, we have the following recursive relationship for the autocorrelation function:

$$\rho_k = \phi_1 \rho_{k-1} + \cdots + \phi_p \rho_{k-p}, \qquad k > 0. \tag{3.1.26}$$

From (3.1.26) we see that the ACF, ρ_k, is determined by the difference equation $\phi_p(B)\rho_k = (1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p)\rho_k = 0$ for $k > 0$. Now, we can write

$$\phi_p(B) = \prod_{i=1}^{m} (1 - G_i B)^{d_i},$$

where $\sum_{i=1}^{m} d_i = p$, and G_i^{-1} $(i = 1, 2, \ldots, m)$ are the roots of multiplicity d_i of $\phi_p(B) = 0$. Using the difference equation result in Theorem 2.7.1, we have

$$\rho_k = \sum_{i=1}^{m} \sum_{j=0}^{d_i - 1} b_{ij} k^j G_i^k. \tag{3.1.27}$$

If $d_i = 1$ for all i, then G_i^{-1} are all distinct and the above reduces to

$$\rho_k = \sum_{i=1}^{p} b_i G_i^k, \qquad k > 0. \tag{3.1.28}$$

For a stationary process, $|G_i^{-1}| > 1$ and $|G_i| < 1$. Hence, the ACF ρ_k tails off as a mixture of exponential decays or damped sine waves depending on the roots of $\phi_p(B) = 0$. Damped sine waves appear if some of the roots are complex.

PACF of the General AR(p) Process Because $\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \cdots + \phi_p \rho_{k-p}$ for $k > 0$, we can easily see that when $k > p$ the last column of the matrix in the numerator of ϕ_{kk} in (2.3.19) can be written as a linear combination of previous columns of the same matrix. Hence, the PACF ϕ_{kk} will vanish after lag p. This property is useful in identifying an AR model as a generating process for a time series discussed in Chapter 6.

3.2 Moving Average Processes

In the moving average representation of a process, if only a finite number of ψ weights are nonzero, i.e., $\psi_1 = -\theta_1$, $\psi_2 = -\theta_2, \ldots, \psi_q = -\theta_q$, and $\psi_k = 0$ for $k > q$, then the resulting process is said to be a moving average process or model of order q and is denoted as MA(q). It is given by

$$\dot{Z}_t = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} \tag{3.2.1a}$$

or

$$\dot{Z}_t = \theta(B) a_t, \tag{3.2.1b}$$

where

$$\theta(B) = (1 - \theta_1 B - \cdots - \theta_q B^q).$$

Because $1 + \theta_1^2 + \cdots + \theta_q^2 < \infty$, a finite moving average process is always stationary. This moving average process is invertible if the roots of $\theta(B) = 0$ lie outside of the unit circle. Moving average processes are useful in describing phenomena in which events produce an immediate effect that only lasts for short periods of time. The process arose as a result of the study by Slutzky (1927) on the effect of the moving average of random events. To discuss other properties of the MA(q) process, let us first consider the following simpler cases.

3.2.1 The First-Order Moving Average MA(1) Process

When $\theta(B) = (1 - \theta_1 B)$, we have the first-order moving average MA(1) process

$$\begin{aligned}\dot{Z}_t &= a_t - \theta_1 a_{t-1} \\ &= (1 - \theta_1 B)a_t, \end{aligned} \tag{3.2.2}$$

where $\{a_t\}$ is a zero mean white noise process with constant variance σ_a^2. The mean of $\{\dot{Z}_t\}$ is $E(\dot{Z}_t) = 0$, and hence $E(Z_t) = \mu$.

ACF of the MA(1) Process The autocovariance generating function of a MA(1) process is, using (2.6.9),

$$\gamma(B) = \sigma_a^2(1 - \theta_1 B)(1 - \theta_1 B^{-1}) = \sigma_a^2\{-\theta_1 B^{-1} + (1 + \theta_1^2) - \theta_1 B\}.$$

Hence, the autocovariances of the process are

$$\gamma_k = \begin{cases} (1 + \theta_1^2)\sigma_a^2, & k = 0, \\ -\theta_1\sigma_a^2, & k = 1, \\ 0, & k > 1. \end{cases} \tag{3.2.3}$$

The autocorrelation function becomes

$$\rho_k = \begin{cases} \dfrac{-\theta_1}{1 + \theta_1^2}, & k = 1, \\ 0, & k > 1, \end{cases} \tag{3.2.4}$$

which cuts off after lag 1, as shown in Figure 3.10.

Because $1 + \theta_1^2$ is always bounded, the MA(1) process is always stationary. For the process to be invertible, however, the root of $(1 - \theta_1 B) = 0$ must lie outside the unit circle. Because the root $B = 1/\theta_1$, we require $|\theta_1| < 1$ for an invertible MA(1) process.

Two remarks are in order.

1. Both the process $\dot{Z}_t = (1 - .4B)a_t$ and the process $\dot{Z}_t = (1 - 2.5B)a_t$ have the same autocorrelation function

$$\rho_k = \begin{cases} \dfrac{-1}{2.9}, & k = 1, \\ 0, & k > 1. \end{cases}$$

   In fact, more generally, for any θ_1, $\dot{Z}_t = (1 - \theta_1 B)a_t$ and $\dot{Z}_t = (1 - 1/\theta_1 B)a_t$ have the same autocorrelations. If the root of $(1 - \theta_1 B)$ lies outside the unit circle, however, then the root of $(1 - 1/\theta_1 B) = 0$ lies inside the unit circle, and vice versa. In other words, among the two processes that produce the same autocorrelations, one and only one is invertible. Thus, for uniqueness and meaningful implication for forecasting discussed in Chapter 5, we restrict ourselves to an invertible process in the model selections.
2. From (3.2.4), it is easy to see that $2|\rho_k| < 1$. Hence, for an MA(1) process, $|\rho_k| < .5$.

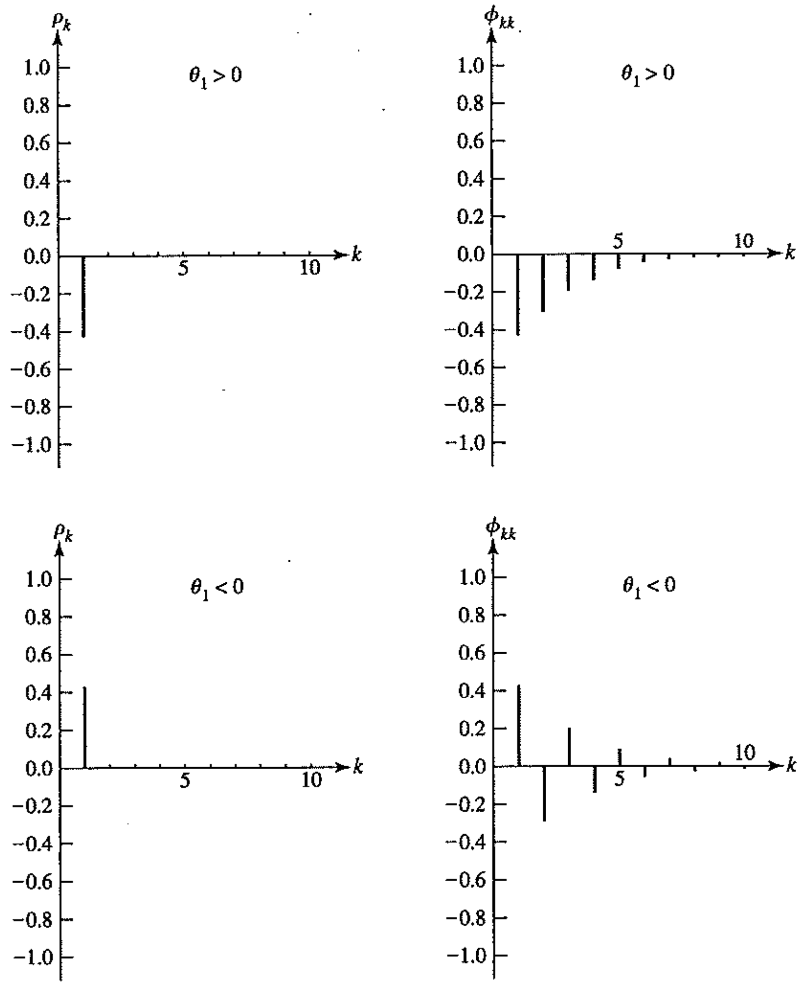

FIGURE 3.10 ACF and PACF of MA(1) processes: $\dot{Z}_t = (1 - \theta_1 B)a_t$.

PACF of the MA(1) Process Using (2.3.19) and (3.2.4), the PACF of an MA(1) process can be easily seen to be

$$\phi_{11} = \rho_1 = \frac{-\theta_1}{1 + \theta_1^2} = \frac{-\theta_1(1 - \theta_1^2)}{1 - \theta_1^4}$$

$$\phi_{22} = \frac{-\rho_1^2}{1 - \rho_1^2} = \frac{-\theta_1^2}{1 + \theta_1^2 + \theta_1^4} = \frac{-\theta_1^2(1 - \theta_1^2)}{1 - \theta_1^6}$$

$$\phi_{33} = \frac{\rho_1^3}{1 - 2\rho_1^2} = \frac{-\theta_1^3}{1 + \theta_1^2 + \theta_1^4 + \theta_1^6} = \frac{-\theta_1^3(1 - \theta_1^2)}{(1 - \theta_1^8)}.$$

In general,

$$\phi_{kk} = \frac{-\theta_1^k(1 - \theta_1^2)}{1 - \theta_1^{2(k+1)}}, \qquad \text{for } k \geq 1. \tag{3.2.5}$$

Contrary to its ACF, which cuts off after lag 1, the PACF of an MA(1) model tails off exponentially in one of two forms depending on the sign of θ_1 (hence on the sign of ρ_1). If alternating in sign, then it begins with a positive value; otherwise, it decays on the negative side, as shown in Figure 3.10. We also note that $|\phi_{kk}| < \frac{1}{2}$.

EXAMPLE 3.5 The sample ACF and sample PACF are calculated from a series of 250 values simulated from the MA(1) model $Z_t = (1 - .5B)a_t$, using a_t as Gaussian $N(0, 1)$ white noise. They are shown in Table 3.5 and plotted in Figure 3.11. Statistically, only one autocorrelation $\hat{\rho}_1$ and two partial autocorrelations $\hat{\phi}_{11}$ and $\hat{\phi}_{22}$ are significant. From the overall pattern, however, $\hat{\rho}_k$ clearly cuts off after lag 1 and $\hat{\phi}_{kk}$ tails off, which indicate a clear MA(1) phenomenon.

TABLE 3.5 Sample ACF and sample PACF for a simulated series from $Z_t = (1 - .5B)a_t$.

k	1	2	3	4	5	6	7	8	9	10
$\hat{\rho}_k$	−.44	.00	.02	−.03	−.01	−.05	.04	−.03	−.03	.02
St.E.	.06	.07	.07	.07	.07	.07	.07	.07	.08	.08
$\hat{\phi}_{kk}$	−.44	−.24	−.11	−.08	−.07	−.12	−.06	−.07	−.10	−.08
St.E.	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06

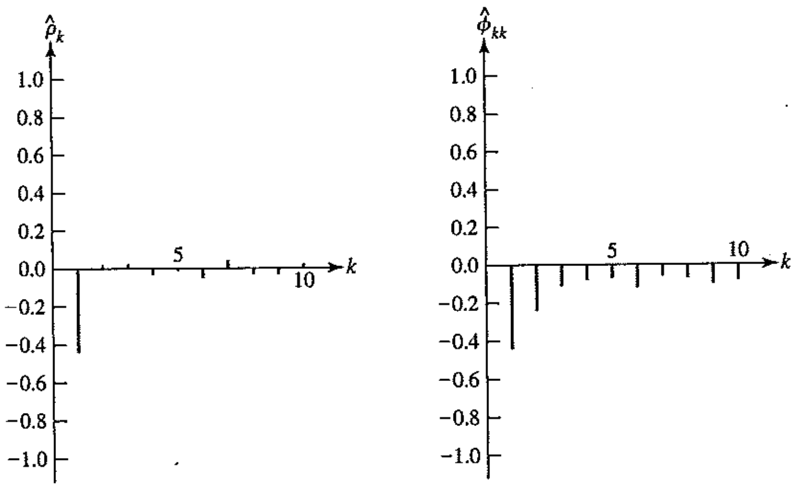

FIGURE 3.11 Sample ACF and sample PACF of a simulated MA(1) series: $Z_t = (1 - .5B)a_t$.

3.2.2 The Second-Order Moving Average MA(2) Process

When $\theta(B) = (1 - \theta_1 B - \theta_2 B^2)$, we have the second-order moving average process

$$\dot{Z}_t = (1 - \theta_1 B - \theta_2 B^2)a_t, \tag{3.2.6}$$

where $\{a_t\}$ is a zero mean white noise process. As a finite-order moving average model, the MA(2) process is always stationary. For invertibility, the roots of $(1 - \theta_1 B - \theta_2 B^2) = 0$ must lie outside of the unit circle. Hence,

$$\begin{cases} \theta_2 + \theta_1 < 1 \\ \theta_2 - \theta_1 < 1 \\ -1 < \theta_2 < 1, \end{cases} \tag{3.2.7}$$

which is parallel to the stationary condition of the AR(2) model, as shown in (3.1.18).

ACF of the MA(2) Process The autocovariance generating function via (2.6.9) is

$$\begin{aligned} \gamma(B) &= \sigma_a^2(1 - \theta_1 B - \theta_2 B^2)(1 - \theta_1 B^{-1} - \theta_2 B^{-2}) \\ &= \sigma_a^2\{-\theta_2 B^{-2} - \theta_1(1 - \theta_2)B^{-1} + (1 + \theta_1^2 + \theta_2^2) - \theta_1(1 - \theta_2)B - \theta_2 B^2\}. \end{aligned}$$

Hence, the autocovariances of the MA(2) model are

$$\begin{aligned} \gamma_0 &= (1 + \theta_1^2 + \theta_2^2)\sigma_a^2, \\ \gamma_1 &= -\theta_1(1 - \theta_2)\sigma_a^2, \\ \gamma_2 &= -\theta_2\sigma_a^2, \end{aligned}$$

and

$$\gamma_k = 0, \qquad k > 2.$$

The autocorrelation function is

$$\rho_k = \begin{cases} \dfrac{-\theta_1(1 - \theta_2)}{1 + \theta_1^2 + \theta_2^2}, & k = 1, \\ \dfrac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}, & k = 2, \\ 0, & k > 2, \end{cases} \tag{3.2.8}$$

which cuts off after lag 2.

PACF of the MA(2) Process From (2.3.19), using that $\rho_k = 0$ for $k \geq 3$, we obtain

$$\phi_{11} = \rho_1$$

$$\phi_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}$$

$$\phi_{33} = \frac{\rho_1^3 - \rho_1\rho_2(2 - \rho_2)}{1 - \rho_2^2 - 2\rho_1^2(1 - \rho_2)}$$

$$\vdots$$

The MA(2) process contains the MA(1) process as a special case. Hence, the PACF tails off as an exponential decay or a damped sine wave depending on the signs and magnitudes of θ_1 and θ_2 or, equivalently, the roots of $(1 - \theta_1 B - \theta_2 B^2) = 0$. The PACF will be damped sine wave if the roots of $(1 - \theta_1 B - \theta_2 B^2) = 0$ are complex. They are shown in Figure 3.12 together with the corresponding ACF.

EXAMPLE 3.6 A series of 250 values is simulated from the MA(2) process $Z_t = (1 - .65B - .24B^2)a_t$ with a Gaussian $N(0, 1)$ white noise series a_t. The sample ACF and sample PACF are in Table 3.6 and plotted in Figure 3.13. We see that $\hat{\rho}_k$ clearly cuts off after lag 2 and $\hat{\phi}_{kk}$ tails off as expected for an MA(2) process.

3.2.3 The General qth-Order Moving Average MA(q) Process

The general qth-order moving average process is

$$\dot{Z}_t = (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q)a_t. \tag{3.2.9}$$

For this general MA(q) process, the variance is

$$\gamma_0 = \sigma_a^2 \sum_{j=0}^{q} \theta_j^2, \tag{3.2.10}$$

where $\theta_0 = 1$, and the other autocovariances are

$$\gamma_k = \begin{cases} \sigma_a^2(-\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{q-k}\theta_q), & k = 1, 2, \ldots, q, \\ 0, & k > q. \end{cases} \tag{3.2.11}$$

Hence, the autocorrelation function becomes

$$\rho_k = \begin{cases} \dfrac{-\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \cdots + \theta_q^2}, & k = 1, 2, \ldots, q, \\ 0, & k > q. \end{cases} \tag{3.2.12}$$

The autocorrelation function of an MA(q) process cuts off after lag q. This important property enables us to identify whether a given time series is generated by a moving average process.

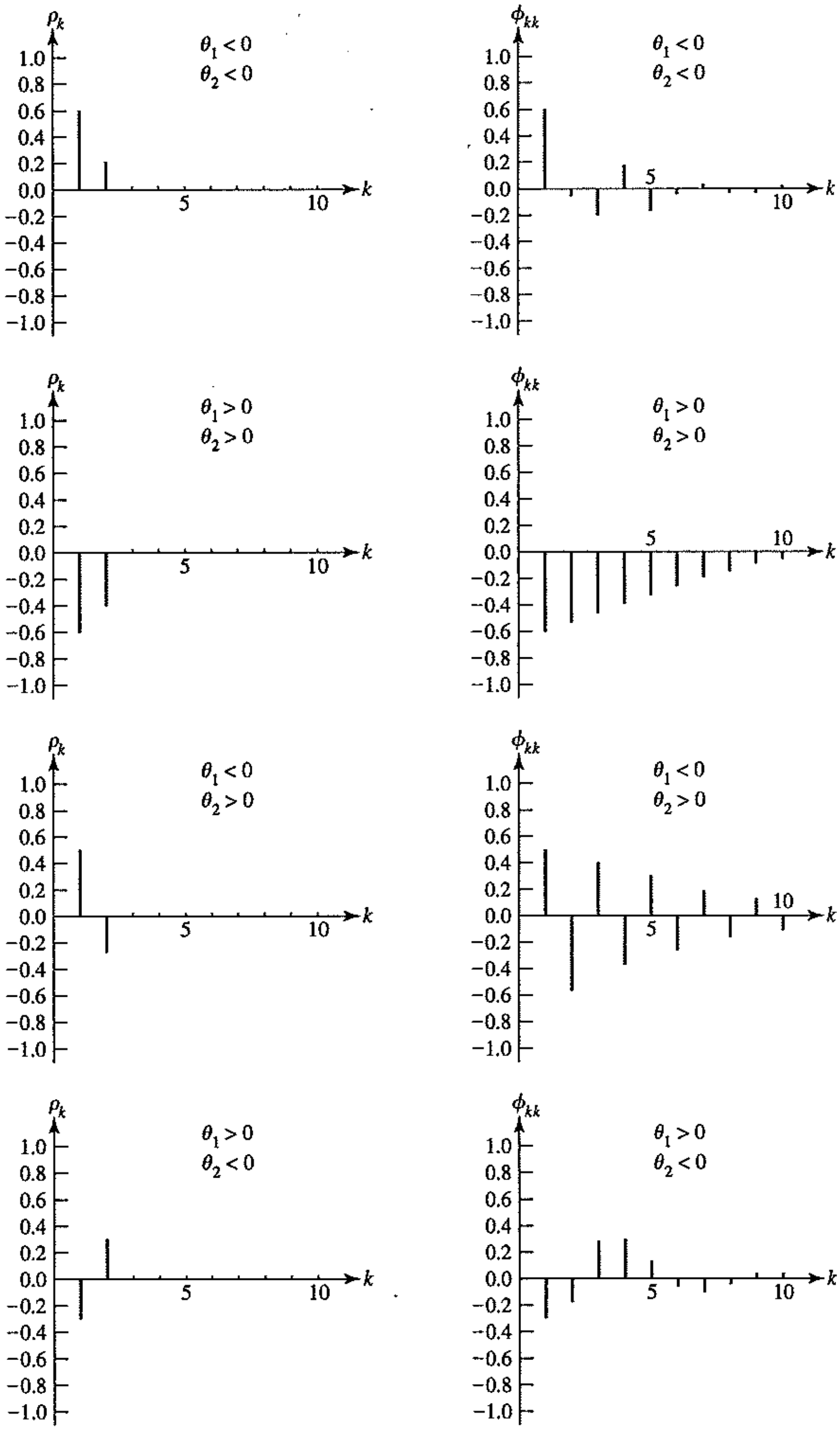

FIGURE 3.12 ACF and PACF of MA(2) processes: $Z_t = (1 - \theta_1 B - \theta_2 B^2)a_t$.

TABLE 3.6 Sample ACF and sample PACF for a simulated MA(2) series from $Z_t = (1 - .65B - .24B^2)a_t$.

k	1	2	3	4	5	6	7	8	9	10
$\hat{\rho}_k$	−.35	−.17	.09	−.06	.01	−.01	−.04	.07	−.07	.09
St.E.	.06	.07	.07	.07	.07	.07	.07	.07	.07	.07
$\hat{\phi}_{kk}$	−.35	−.34	−.15	−.18	−.11	−.12	−.14	−.05	−.14	.00
St.E.	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06

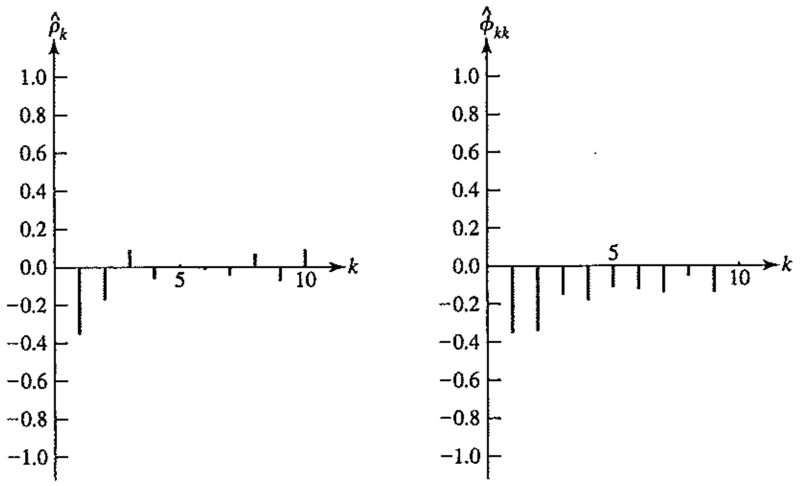

FIGURE 3.13 Sample ACF and sample PACF of a simulated MA(2) series: $Z_t = (1 - .65B - .24B^2)a_t$.

From the discussion of MA(1) and MA(2) processes, we can easily see that the partial autocorrelation function of the general MA(q) process tails off as a mixture of exponential decays and/or damped sine waves depending on the nature of the roots of $(1 - \theta_1 B - \cdots - \theta_q B^q) = 0$. The PACF will contain damped sine waves if some of the roots are complex.

3.3 The Dual Relationship Between AR(p) and MA(q) Processes

For a given stationary AR(p) process,

$$\phi_p(B)\dot{Z}_t = a_t, \tag{3.3.1}$$

where $\phi_p(B) = (1 - \phi_1 B - \cdots - \phi_p B^p)$, we can write

$$\dot{Z}_t = \frac{1}{\phi_p(B)} a_t = \psi(B) a_t, \tag{3.3.2}$$

with $\psi(B) = (1 + \psi_1 B + \psi_2 B^2 + \cdots)$ such that

$$\phi_p(B)\psi(B) = 1. \tag{3.3.3}$$

The ψ weights can be derived by equating the coefficients of B^j on both sides of (3.3.3). For example, we can write the AR(2) process as

$$\dot{Z}_t = \frac{1}{(1 - \phi_1 B - \phi_2 B^2)} a_t = (1 + \psi_1 B + \psi_2 B^2 + \cdots)a_t, \tag{3.3.4}$$

which implies that

$$(1 - \phi_1 B - \phi_2 B^2)(1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots) = 1,$$

i.e.,

$$\begin{aligned} 1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots& \\ - \phi_1 B - \psi_1\phi_1 B^2 - \psi_2\phi_1 B^3 - \cdots& \\ - \phi_2 B^2 - \psi_1\phi_2 B^3 - \cdots &= 1. \end{aligned}$$

Thus, we obtain the ψ_j's as follows:

$$\begin{aligned} &B^1: & \psi_1 - \phi_1 = 0 &\longrightarrow \psi_1 = \phi_1 \\ &B^2: & \psi_2 - \psi_1\phi_1 - \phi_2 = 0 &\longrightarrow \psi_2 = \psi_1\phi_1 + \phi_2 = \phi_1^2 + \phi_2 \\ &B^3: & \psi_3 - \psi_2\phi_1 - \psi_1\phi_2 = 0 &\longrightarrow \psi_3 = \psi_2\phi_1 + \psi_1\phi_2 \\ &\vdots & & \end{aligned}$$

Actually, for $j \geq 2$, we have

$$\psi_j = \psi_{j-1}\phi_1 + \psi_{j-2}\phi_2, \tag{3.3.5}$$

where $\psi_0 = 1$. In a special case when $\phi_2 = 0$, we have $\psi_j = \phi_1^j$ for $j \geq 0$. Therefore,

$$\dot{Z}_t = \frac{1}{(1 - \phi_1 B)} a_t = (1 + \phi_1 B + \phi_1^2 B^2 + \cdots)a_t. \tag{3.3.6}$$

This equation implies that a finite-order stationary AR process is equivalent to an infinite-order MA process.

Given a general invertible MA(q) process,

$$\dot{Z}_t = \theta_q(B)a_t \tag{3.3.7}$$

with $\theta_q(B) = (1 - \theta_1 B - \cdots - \theta_q B^q)$, we can rewrite it as

$$\pi(B)\dot{Z}_t = \frac{1}{\theta_q(B)}\dot{Z}_t = a_t, \tag{3.3.8}$$

where

$$\begin{aligned}\pi(B) &= 1 - \pi_1 B - \pi_2 B^2 - \cdots \\ &= \frac{1}{\theta_q(B)}.\end{aligned} \tag{3.3.9}$$

For example, we can write the MA(2) process as

$$(1 - \pi_1 B - \pi_2 B^2 - \pi_3 B^3 - \cdots)\dot{Z}_t = \frac{1}{(1 - \theta_1 B - \theta_2 B^2)}\dot{Z}_t = a_t, \tag{3.3.10}$$

where

$$(1 - \theta_1 B - \theta_2 B^2)(1 - \pi_1 B - \pi_2 B^2 - \pi_3 B^3 - \cdots) = 1,$$

or

$$\begin{aligned}1 - \pi_1 B - \pi_2 B^2 - \pi_3 B^3 - \cdots& \\ - \theta_1 B + \pi_1\theta_1 B^2 + \pi_2\theta_1 B^3 + \cdots& \\ - \theta_2 B^2 + \pi_1\theta_2 B^3 + \cdots &= 1.\end{aligned}$$

Thus, the π weights can be derived by equating the coefficients of B^j as follows:

$$\begin{aligned}
B^1&: \quad -\pi_1 - \theta_1 = 0 \longrightarrow \pi_1 = -\theta_1 \\
B^2&: \quad -\pi_2 + \pi_1\theta_1 - \theta_2 = 0 \longrightarrow \pi_2 = \pi_1\theta_1 - \theta_2 = -\theta_1^2 - \theta_2 \\
B^3&: \quad -\pi_3 + \pi_2\theta_1 + \pi_1\theta_2 = 0 \longrightarrow \pi_3 = \pi_2\theta_1 + \pi_1\theta_2 \\
&\vdots
\end{aligned}$$

In general,

$$\pi_j = \pi_{j-1}\theta_1 + \pi_{j-2}\theta_2, \quad \text{for } j \geq 3. \tag{3.3.11}$$

When $\theta_2 = 0$ and the process becomes the MA(1) process, we have $\pi_j = -\theta_1^j$ for $j \geq 1$, and

$$(1 + \theta_1 B + \theta_1^2 B^2 + \cdots)\dot{Z}_t = \frac{1}{(1 - \theta_1 B)}\dot{Z}_t = a_t. \tag{3.3.12}$$

Thus, in terms of the AR representation, a finite-order invertible MA process is equivalent to an infinite-order AR process.

In summary, a finite-order stationary AR(p) process corresponds to an infinite-order MA process, and a finite-order invertible MA(q) process corresponds to an infinite-order AR process. This dual relationship between the AR(p) and the MA(q) processes also exists in the autocorrelation and partial autocorrelation functions. The AR(p) process has its autocorrelations tailing off and partial autocorrelations cutting off, but the MA(q) process has its autocorrelations cutting off and partial autocorrelations tailing off.

3.4 Autoregressive Moving Average ARMA(p, q) Processes

A natural extension of the pure autoregressive and the pure moving average processes is the mixed autoregressive moving average process, which includes the autoregressive and moving average processes as special cases. The process contains a large class of parsimonious time series models that are useful in describing a wide variety of time series encountered in practice.

3.4.1 The General Mixed ARMA(p, q) Process

As we have shown, a stationary and invertible process can be represented either in a moving average form or in an autoregressive form. A problem with either representation, though, is that it may contain too many parameters, even for a finite-order moving average and a finite-order autoregressive model because a high-order model is often needed for good approximation. In general, a large number of parameters reduces efficiency in estimation. Thus, in model building, it may be necessary to include both autoregressive and moving average terms in a model, which leads to the following useful mixed autoregressive moving average (ARMA) process:

$$\phi_p(B)\dot{Z}_t = \theta_q(B)a_t, \tag{3.4.1}$$

where

$$\phi_p(B) = 1 - \phi_1 B - \cdots - \phi_p B^p,$$

and

$$\theta_q(B) = 1 - \theta_1 B - \cdots - \theta_q B^q.$$

For the process to be invertible, we require that the roots of $\theta_q(B) = 0$ lie outside the unit circle. To be stationary, we require that the roots of $\phi_p(B) = 0$ lie outside the unit circle. Also, we assume that $\phi_p(B) = 0$ and $\theta_q(B) = 0$ share no common roots. Henceforth, we refer to this process as an ARMA(p, q) process or model, in which p and q are used to indicate the orders of the associated autoregressive and moving average polynomials, respectively.

The stationary and invertible ARMA process can be written in a pure autoregressive representation discussed in Section 2.6, i.e.,

$$\pi(B)\dot{Z}_t = a_t, \tag{3.4.2}$$

where

$$\pi(B) = \frac{\phi_p(B)}{\theta_q(B)} = (1 - \pi_1 B - \pi_2 B^2 - \cdots). \tag{3.4.3}$$

This process can also be written as a pure moving average representation,

$$\dot{Z}_t = \psi(B)a_t, \tag{3.4.4}$$

where

$$\psi(B) = \frac{\theta_q(B)}{\phi_p(B)} = (1 + \psi_1 B + \psi_2 B^2 + \cdots). \tag{3.4.5}$$

ACF of the ARMA(p, q) Process To derive the autocovariance function, we rewrite (3.4.1) as

$$\dot{Z}_t = \phi_1\dot{Z}_{t-1} + \cdots + \phi_p\dot{Z}_{t-p} + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

and multiply by $\dot{Z}_{t-k}$ on both sides

$$\dot{Z}_{t-k}\dot{Z}_t = \phi_1\dot{Z}_{t-k}\dot{Z}_{t-1} + \cdots + \phi_p\dot{Z}_{t-k}\dot{Z}_{t-p} + \dot{Z}_{t-k}a_t - \theta_1\dot{Z}_{t-k}a_{t-1} - \cdots - \theta_q\dot{Z}_{t-k}a_{t-q}.$$

We now take the expected value to obtain

$$\gamma_k = \phi_1\gamma_{k-1} + \cdots + \phi_p\gamma_{k-p} + E(\dot{Z}_{t-k}a_t) - \theta_1 E(\dot{Z}_{t-k}a_{t-1}) - \cdots - \theta_q E(\dot{Z}_{t-k}a_{t-q}).$$

Because

$$E(\dot{Z}_{t-k}a_{t-i}) = 0 \qquad \text{for } k > i,$$

we have

$$\gamma_k = \phi_1\gamma_{k-1} + \cdots + \phi_p\gamma_{k-p}, \qquad k \geq (q + 1), \tag{3.4.6}$$

and hence,

$$\rho_k = \phi_1\rho_{k-1} + \cdots + \phi_p\rho_{k-p}, \qquad k \geq (q + 1). \tag{3.4.7}$$

Equation (3.4.7) satisfies the pth-order homogeneous difference equation as shown in (3.1.26) for the AR(p) process. Therefore, the autocorrelation function of an ARMA(p, q) model tails off after lag q just like an AR(p) process, which depends only on the autoregressive parameters in the model. The first q autocorrelations $\rho_q, \rho_{q-1}, \ldots, \rho_1$, however, depend on both autoregressive and moving average parameters in the model and serve as initial values for the pattern. This distinction is useful in model identification.

PACF of the ARMA(p, q) Process Because the ARMA process contains the MA process as a special case, its PACF will also be a mixture of exponential decays or damped sine waves depending on the roots of $\phi_p(B) = 0$ and $\theta_q(B) = 0$.

3.4.2 The ARMA(1, 1) Process

$$(1 - \phi_1 B)\dot{Z}_t = (1 - \theta_1 B)a_t, \tag{3.4.8a}$$

or

$$\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + a_t - \theta_1 a_{t-1}. \tag{3.4.8b}$$

For stationarity, we assume that $|\phi_1| < 1$, and for invertibility, we require that $|\theta_1| < 1$. When $\phi_1 = 0$, (3.4.8a) is reduced to an MA(1) process, and when $\theta_1 = 0$, it is reduced to an AR(1) process. Thus, we can regard the AR(1) and MA(1) processes as special cases of the ARMA(1, 1) process.

In terms of a pure autoregressive representation, we write

$$\pi(B)\dot{Z}_t = a_t,$$

where

$$\pi(B) = (1 - \pi_1 B - \pi_2 B^2 - \cdots) = \frac{(1 - \phi_1 B)}{(1 - \theta_1 B)},$$

i.e.,

$$(1 - \theta_1 B)(1 - \pi_1 B - \pi_2 B^2 - \pi_3 B^3 - \cdots) = (1 - \phi_1 B)$$

or

$$\left[1 - (\pi_1 + \theta_1)B - (\pi_2 - \pi_1\theta_1)B^2 - (\pi_3 - \pi_2\theta_1)B^3 - \cdots\right] = (1 - \phi_1 B).$$

By equating coefficients of B^j on both sides of the above equation, we get

$$\pi_j = \theta_1^{j-1}(\phi_1 - \theta_1), \qquad \text{for } j \geq 1. \tag{3.4.9}$$

We can write the ARMA(1, 1) process in a pure moving average representation as

$$Z_t = \psi(B)a_t = \frac{(1 - \theta_1 B)}{(1 - \phi_1 B)} a_t.$$

We note that

$$(1 - \phi_1 B)(1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots) = (1 - \theta_1 B),$$

i.e.,

$$\left[1 + (\psi_1 - \phi_1)B + (\psi_2 - \psi_1\phi_1)B^2 + \cdots\right] = (1 - \theta_1 B).$$

Hence,

$$\psi_j = \phi_1^{j-1}(\phi_1 - \theta_1), \qquad \text{for } j \geq 1. \tag{3.4.10}$$

ACF of the ARMA(1, 1) Process To obtain the autocovariance for $\{Z_t\}$, we multiply Z_{t-k} on both sides of (3.4.8b),

$$\dot{Z}_{t-k}\dot{Z}_t = \phi_1\dot{Z}_{t-k}\dot{Z}_{t-1} + \dot{Z}_{t-k}a_t - \theta_1\dot{Z}_{t-k}a_{t-1},$$

and take the expected value to obtain

$$\gamma_k = \phi_1\gamma_{k-1} + E(\dot{Z}_{t-k}a_t) - \theta_1 E(\dot{Z}_{t-k}a_{t-1}). \tag{3.4.11}$$

More specifically, when $k = 0$,

$$\gamma_0 = \phi_1\gamma_1 + E(\dot{Z}_t a_t) - \theta_1 E(\dot{Z}_t a_{t-1}).$$

Recall that $E(\dot{Z}_t a_t) = \sigma_a^2$. For the term $E(\dot{Z}_t a_{t-1})$, we note that

$$\begin{aligned} E(\dot{Z}_t a_{t-1}) &= \phi_1 E(\dot{Z}_{t-1}a_{t-1}) + E(a_t a_{t-1}) - \theta_1 E(a_{t-1}^2) \\ &= (\phi_1 - \theta_1)\sigma_a^2. \end{aligned}$$

Hence,

$$\gamma_0 = \phi_1\gamma_1 + \sigma_a^2 - \theta_1(\phi_1 - \theta_1)\sigma_a^2. \tag{3.4.12}$$

When $k = 1$, we have from (3.4.11)

$$\gamma_1 = \phi_1\gamma_0 - \theta_1\sigma_a^2. \tag{3.4.13}$$

Substituting (3.4.13) in (3.4.12), we have

$$\gamma_0 = \phi_1^2\gamma_0 - \phi_1\theta_1\sigma_a^2 + \sigma_a^2 - \phi_1\theta_1\sigma_a^2 + \theta_1^2\sigma_a^2,$$

i.e.,

$$\gamma_0 = \frac{(1 + \theta_1^2 - 2\phi_1\theta_1)}{(1 - \phi_1^2)}\sigma_a^2.$$

Thus,

$$\begin{aligned}\gamma_1 &= \phi_1\gamma_0 - \theta_1\sigma_a^2 \\ &= \frac{\phi_1(1 + \theta_1^2 - 2\phi_1\theta_1)}{(1 - \phi_1^2)}\sigma_a^2 - \theta_1\sigma_a^2 \\ &= \frac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)}{(1 - \phi_1^2)}\sigma_a^2.\end{aligned}$$

For $k \geq 2$, we have from (3.4.11)

$$\gamma_k = \phi_1\gamma_{k-1}, \qquad k \geq 2.$$

Hence, the ARMA(1, 1) model has the following autocorrelation function:

$$\rho_k = \begin{cases} 1 & k = 0, \\ \dfrac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)}{1 + \theta_1^2 - 2\phi_1\theta_1}, & k = 1, \\ \phi_1\rho_{k-1}, & k \geq 2. \end{cases} \tag{3.4.14}$$

Note that the autocorrelation function of an ARMA(1, 1) model combines characteristics of both AR(1) and MA(1) processes. The moving average parameter θ_1 enters into the calculation of ρ_1. Beyond ρ_1, the autocorrelation function of an ARMA(1, 1) model follows the same pattern as the autocorrelation function of an AR(1) process.

PACF of the ARMA(1, 1) Process The general form of the PACF of a mixed model is complicated and is not needed. It suffices to note that, because the ARMA(1, 1) process contains the MA(1) process as a special case, the PACF of the ARMA(1, 1) process also tails off exponentially like the ACF, with its shape depending on the signs and magnitudes of ϕ_1 and θ_1. Thus, that both ACF and PACF tail off indicates a mixed ARMA model. Some of the ACF and PACF patterns for the ARMA(1, 1) model are shown in Figure 3.14. By examining Figure 3.14, it can be seen that due to the combined effect of both ϕ_1 and θ_1, the PACF of the ARMA(1, 1) process contains many more different shapes than the PACF of the MA(1) process, which consists of only two possibilities.

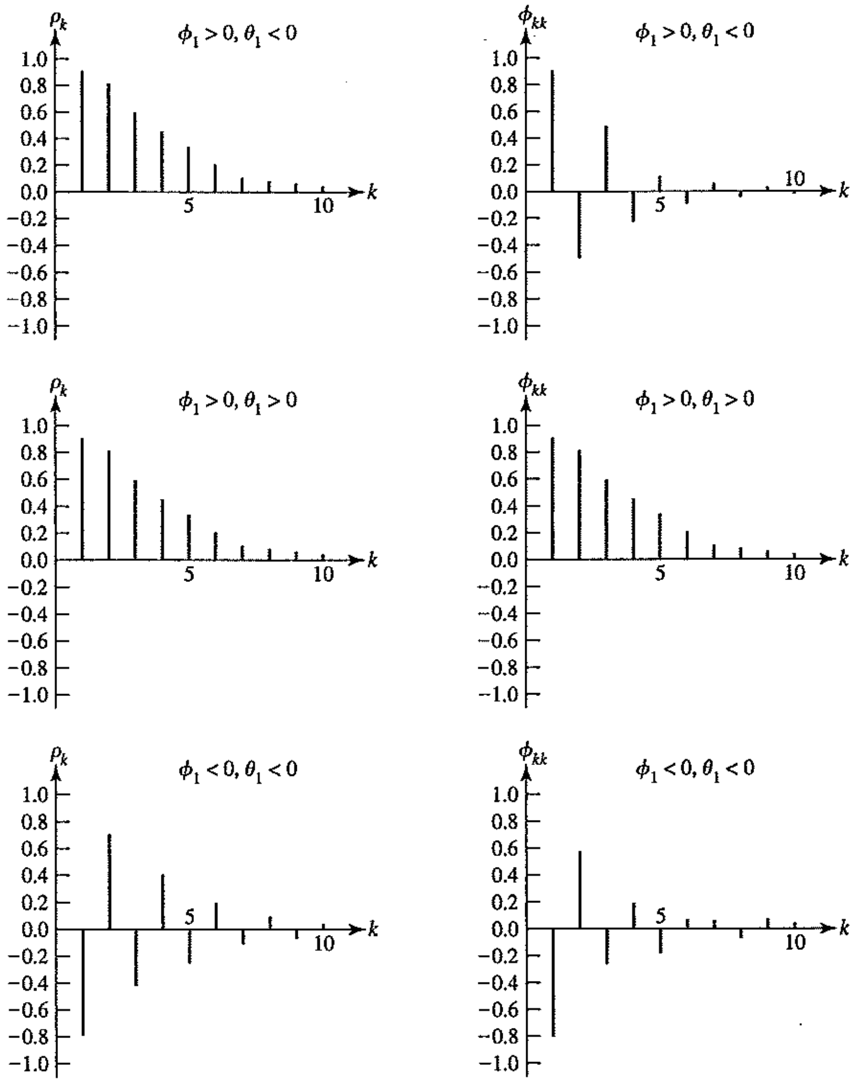

FIGURE 3.14 ACF and PACF of ARMA(1, 1) model $(1 - \phi_1 B)\dot{Z}_t = (1 - \theta_1 B)a_t$.

EXAMPLE 3.7 A series of 250 values is simulated from the ARMA(1, 1) process $(1 - .9B)Z_t = (1 - .5B)a_t$, with the a_t being a Gaussian $N(0, 1)$ white noise series. The sample ACF and sample PACF are shown in Table 3.7 and also plotted in Figure 3.15. That both $\hat{\rho}_k$ and $\hat{\phi}_{kk}$ tail off indicates a mixed ARMA model. To decide the proper orders of p and q in a mixed model is a much more difficult and challenging task, sometimes requiring considerable experience and skill. Some helpful methods are discussed in Chapter 6 on model identification. For now, it is sufficient to identify tentatively from the sample ACF and sample PACF whether the phenomenon is a pure AR, pure MA, or mixed ARMA model. Solely based on the sample PACF as shown in Table 3.7, without looking at the sample ACF, we know that the phenomenon

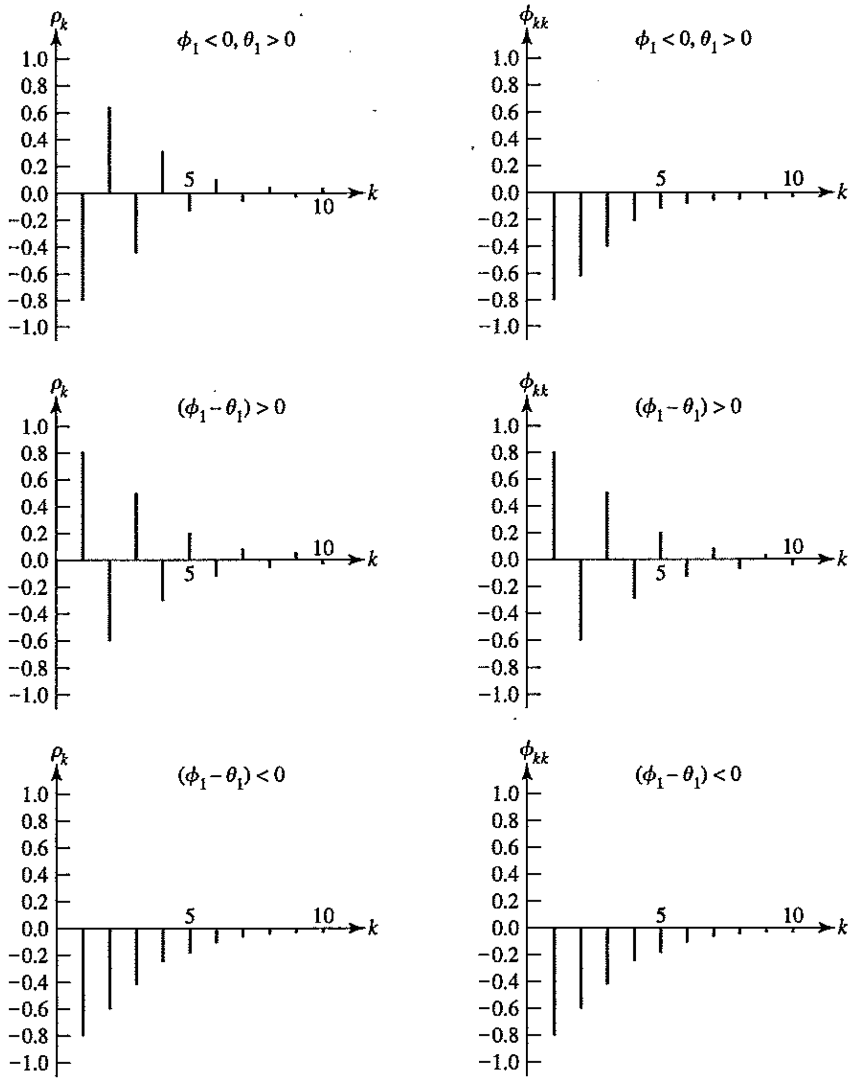

FIGURE 3.14 (*Continued*)

cannot be an MA process because the MA process cannot have a positively exponentially decaying PACF.

EXAMPLE 3.8 The sample ACF and PACF are calculated for a series of 250 values as shown in Table 3.8 and plotted in Figure 3.16. None is statistically significant from 0, which would indicate a white noise phenomenon. In fact, the series is the simulation result from the ARMA(1, 1) process, $(1 - \phi_1 B)Z_t = (1 - \theta_1 B)a_t$ with $\phi_1 = .6$ and $\theta_1 = .5$. The sample ACF and sample PACF are both small because the AR polynomial $(1 - .6B)$ and the MA polynomial $(1 - .5B)$ almost cancel each other out. Recall from (3.4.14) that the ACF of the

TABLE 3.7 Sample ACF and sample PACF for a simulated ARMA(1, 1) series from $(1 - .9B)Z_t = (1 - .5B)a_t$.

k	1	2	3	4	5	6	7	8	9	10
$\hat{\rho}_k$	.57	.50	.47	.35	.31	.25	.21	.18	.10	.12
St.E.	.06	.08	.09	.10	.11	.11	.11	.11	.11	.11
$\hat{\phi}_{kk}$	.57	.26	.18	−.03	.01	−.01	.01	.01	−.08	.05
St.E.	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06

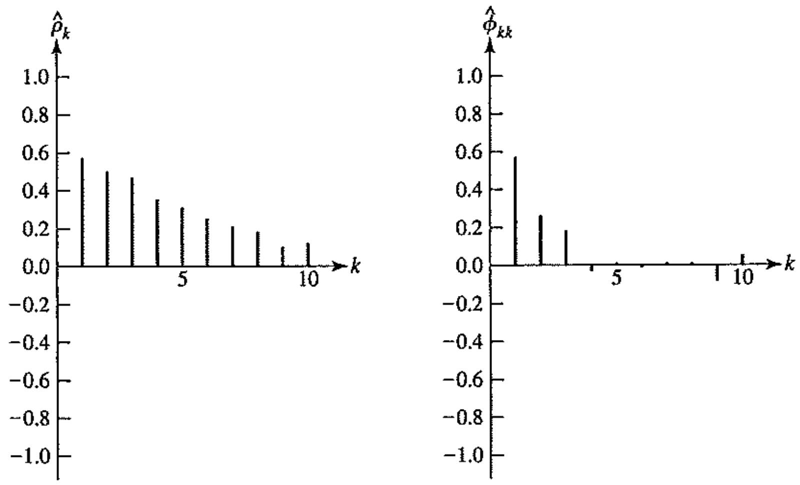

FIGURE 3.15 Sample ACF and sample PACF of a simulated ARMA(1, 1) series: $(1 - .9B)Z_t = (1 - .5B)a_t$.

ARMA(1, 1) process is $\rho_k = \phi_1^{k-1}(\phi_1 - \theta_1)(1 - \phi_1\theta_1)/(1 + \theta_1^2 - 2\phi_1\theta_1)$ for $k \geq 1$, which is approximately equal to zero when $\phi_1 \simeq \theta_1$. Thus, the sample phenomenon of a white noise series implies that the underlying model is either a random noise process or an ARMA process with its AR and MA polynomials being nearly equal. The assumption of no common roots between $\phi_p(B) = 0$ and $\theta_q(B) = 0$ in the mixed model is needed to avoid this confusion.

Before closing this chapter, we note that the ARMA(p, q) model in (3.4.1), i.e.,

$$(1 - \phi_1 B - \cdots - \phi_p B^p)(Z_t - \mu) = (1 - \theta_1 B - \cdots - \theta_q B^q)a_t,$$

can also be written as

$$(1 - \phi_1 B - \cdots - \phi_p B^p)Z_t = \theta_0 + (1 - \theta_1 B - \cdots - \theta_q B^q)a_t, \tag{3.4.15}$$

TABLE 3.8 Sample ACF and sample PACF for a simulated series of the ARMA(1, 1) process: $(1 - .6B)Z_t = (1 - .5B)a_t$.

k	1	2	3	4	5	6	7	8	9	10
$\hat{\rho}_k$	.10	.05	.09	.00	−.02	.02	−.02	.04	−.04	.01
St.E.	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06
$\hat{\phi}_{kk}$	.10	.04	.08	−.02	−.02	.01	−.02	.05	−.05	.02
St.E.	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06

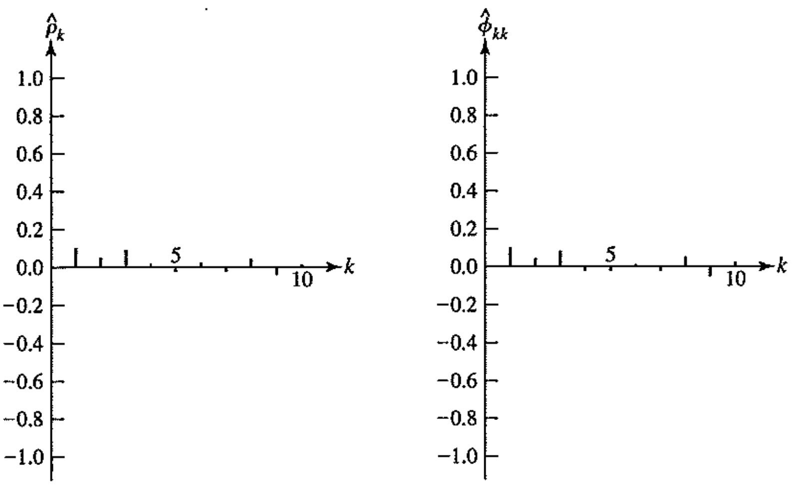

FIGURE 3.16 Sample ACF and sample PACF of a simulated ARMA(1, 1) series: $(1 - .6B)Z_t = (1 - .5B)a_t$.

where

$$\begin{aligned}\theta_0 &= (1 - \phi_1 B - \cdots - \phi_p B^p)\mu \\ &= (1 - \phi_1 - \cdots - \phi_p)\mu. \end{aligned} \tag{3.4.16}$$

In terms of this form, the AR(p) model becomes

$$(1 - \phi_1 B - \cdots - \phi_p B^p)Z_t = \theta_0 + a_t \tag{3.4.17}$$

and the MA(q) model becomes

$$Z_t = \theta_0 + (1 - \theta_1 B - \cdots - \theta_q B^q)a_t. \tag{3.4.18}$$

It is clear that in the MA(q) process, $\theta_0 = \mu$.

EXERCISES

3.1 Find the ACF and PACF and plot the ACF ρ_k for $k = 0, 1, 2, 3, 4,$ and 5 for each of the following models where the a_t is a Gaussian white noise process.

(a) $Z_t - .5Z_{t-1} = a_t,$

(b) $Z_t + .98Z_{t-1} = a_t,$

(c) $Z_t - 1.3Z_{t-1} + .4Z_{t-2} = a_t,$

(d) $Z_t - 1.2Z_{t-1} + .8Z_{t-2} = a_t.$

3.2 Consider the following AR(2) models:

(i) $Z_t - .6Z_{t-1} - .3Z_{t-2} = a_t,$

(ii) $Z_t - .8Z_{t-1} + .5Z_{t-2} = a_t.$

(a) Find the general expression for ρ_k.

(b) Plot the ρ_k, for $k = 0, 1, 2, \ldots, 10$.

(c) Calculate σ_Z^2 by assuming that $\sigma_a^2 = 1$.

3.3 Simulate a series of 100 observations from each of the models with $\sigma_a^2 = 1$ in Exercise 3.1. For each case, plot the simulated series, and calculate and study its sample ACF $\hat{\rho}_k$ and PACF $\hat{\phi}_{kk}$ for $k = 0, 1, \ldots, 20$.

3.4 (a) Show that the ACF ρ_k for the AR(1) process satisfies the difference equation

$$\rho_k - \phi_1\rho_{k-1} = 0, \qquad \text{for } k \geq 1.$$

(b) Find the general expression for ρ_k.

3.5 Consider the AR(2) process $Z_t = Z_{t-1} - .25Z_{t-2} + a_t$.

(a) Calculate ρ_1.

(b) Use ρ_0, ρ_1 as starting values and the difference equation to obtain the general expression for ρ_k.

(c) Calculate the values ρ_k for $k = 1, 2, \ldots, 10$.

3.6 (a) Find the range of α such that the AR(2) process

$$Z_t = Z_{t-1} + \alpha Z_{t-2} + a_t$$

is stationary.

(b) Find the ACF for the model in part (a) with $\alpha = -\frac{1}{2}$.

3.7 Show that if an AR(2) process is stationary, then

$$\rho_1^2 < \frac{\rho_2 + 1}{2}.$$

3.8 Consider the MA(2) process $Z_t = (1 - 1.2B + .5B^2)a_t$.

(a) Find the ACF using the definition of autocovariance function.

(b) Find the ACF using the autocovariance generating function.

(c) Find the PACF ϕ_{kk} for the process.

3.9 Find an invertible process which has the following ACF:

$$\rho_0 = 1, \qquad \rho_1 = .25, \qquad \text{and} \qquad \rho_k = 0 \qquad \text{for } k \geq 2.$$

3.10 (a) Find a process that has the following autocovariance function:

$$\gamma_0 = 10, \qquad \gamma_1 = 0, \qquad \gamma_2 = -4, \qquad \gamma_k = 0, \qquad \text{for } |k| > 2.$$

(b) Examine stationarity and invertibility for the process obtained in part (a).

3.11 Consider the MA(2) process $Z_t = a_t - .1a_{t-1} + .21a_{t-2}$.

(a) Is the model stationary? Why?
(b) Is the model invertible? Why?
(c) Find the ACF for the above process.

3.12 Simulate a series of 100 observations from the model with $\sigma_a^2 = 1$ in Exercise 3.8. Plot the simulated series and calculate and study its sample ACF, $\hat{\rho}_k$, and PACF, $\hat{\phi}_{kk}$ for k = 0, 1, . . . , 20.

3.13 One can calculate the PACF using either (2.3.19) or (2.5.25). Illustrate the computation of ϕ_{11}, ϕ_{22}, and ϕ_{33} for the MA(1) process using both procedures.

3.14 Consider each of the following models:

(i) $(1 - B)Z_t = (1 - 1.5B)a_t$,
(ii) $(1 - .8B)Z_t = (1 - .5B)a_t$,
(iii) $(1 - 1.1B + .8B^2)Z_t = (1 - 1.7B + .72B^2)a_t$,
(iv) $(1 - .6B)Z_t = (1 - 1.2B + .2B^2)a_t$

(a) Verify whether it is stationary, or invertible, or both.
(b) Express the model in an MA representation if it exists.
(c) Express the model in an AR representation if it exists.

3.15 Consider each of the following processes:

(i) $(1 - .6B)Z_t = (1 - .9B)a_t$,
(ii) $(1 - 1.4B + .6B^2)Z_t = (1 - .8B)a_t$

(a) Find the ACF ρ_k.
(b) Find the PACF ϕ_{kk} for $k = 1, 2, 3$.
(c) Find the autocovariance generating function.

3.16 Simulate a series of 100 observations from each of the models with $\sigma_a^2 = 1$ in Exercise 3.15. For each simulated series, plot the series, calculate, and study its sample ACF $\hat{\rho}_k$ and PACF $\hat{\phi}_{kk}$ for $k = 0, 1, \ldots, 20$.