3
The z-Transform

3.0 INTRODUCTION

We have seen that the Fourier transform plays a key role in representing and analyzing discrete-time signals and systems. In this chapter, we develop the z-transform representation of a sequence and study how the properties of a sequence are related to the properties of its z-transform. The z-transform for discrete-time signals is the counterpart of the Laplace transform for continuous-time signals, and they each have a similar relationship to the corresponding Fourier transform. One motivation for introducing this generalization is that the Fourier transform does not converge for all sequences and it is useful to have a generalization of the Fourier transform that encompasses a broader class of signals. A second advantage is that in analytical problems the z-transform notation is often more convenient than the Fourier transform notation.

3.1 z-TRANSFORM

The Fourier transform of a sequence $x[n]$ was defined in Chapter 2 as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}. \tag{3.1}$$

The z-transform of a sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}. \tag{3.2}$$

This equation is, in general, an infinite sum or infinite power series, with z being a complex variable. Sometimes it is useful to consider Eq. (3.2) as an operator that transforms a sequence into a function, and we will refer to the *z-transform operator* $\mathcal{Z}\{\cdot\}$, defined as

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z). \tag{3.3}$$

With this interpretation, the z-transform operator is seen to transform the sequence $x[n]$ into the function $X(z)$, where z is a continuous complex variable. The correspondence between a sequence and its z-transform is indicated by the notation

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z). \tag{3.4}$$

The z-transform, as we have defined it in Eq. (3.2), is often referred to as the *two-sided* or *bilateral z-transform,* in contrast to the *one-sided* or *unilateral z-transform,* which is defined as

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}. \tag{3.5}$$

Clearly, the bilateral and unilateral transforms are equivalent only if $x[n] = 0$ for $n < 0$. In this book, we focus on the bilateral transform exclusively.

It is evident from a comparison of Eqs. (3.1) and (3.2) that there is a close relationship between the Fourier transform and the z-transform. In particular, if we replace the complex variable z in Eq. (3.2) with the complex variable $e^{j\omega}$, then the z-transform reduces to the Fourier transform. This is one motivation for the notation $X(e^{j\omega})$ for the Fourier transform; when it exists, the Fourier transform is simply $X(z)$ with $z = e^{j\omega}$. This corresponds to restricting z to have unity magnitude; i.e., for $|z| = 1$, the z-transform corresponds to the Fourier transform. More generally, we can express the complex variable z in polar form as

$$z = re^{j\omega}.$$

With z expressed in this form, Eq. (3.2) becomes

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n},$$

or

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}. \tag{3.6}$$

Equation (3.6) can be interpreted as the Fourier transform of the product of the original sequence $x[n]$ and the exponential sequence r^{-n}. Obviously, for $r = 1$, Eq. (3.6) reduces to the Fourier transform of $x[n]$.

Since the z-transform is a function of a complex variable, it is convenient to describe and interpret it using the complex z-plane. In the z-plane, the contour corresponding to $|z| = 1$ is a circle of unit radius, as illustrated in Figure 3.1. This contour is referred to as the *unit circle*. The z-transform evaluated on the unit circle corresponds to the Fourier transform. Note that ω is the angle between the vector to a point z on the unit circle and the real axis of the complex z-plane. If we evaluate $X(z)$ at points on the

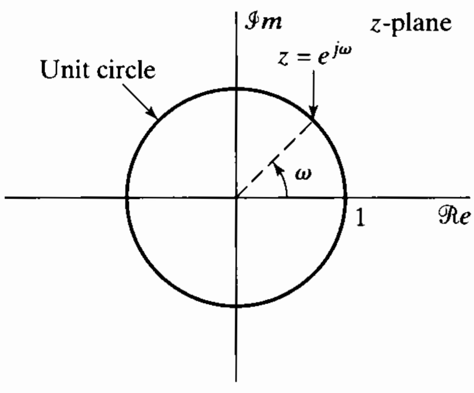

Figure 3.1 The unit circle in the complex z-plane.

unit circle in the z-plane beginning at $z = 1$ (i.e., $\omega = 0$) through $z = j$ (i.e., $\omega = \pi/2$) to $z = -1$ (i.e., $\omega = \pi$), we obtain the Fourier transform for $0 \leq \omega \leq \pi$. Continuing around the unit circle would correspond to examining the Fourier transform from $\omega = \pi$ to $\omega = 2\pi$ or, equivalently, from $\omega = -\pi$ to $\omega = 0$. In Chapter 2, the Fourier transform was displayed on a linear frequency axis. Interpreting the Fourier transform as the z-transform on the unit circle in the z-plane corresponds conceptually to wrapping the linear frequency axis around the unit circle with $\omega = 0$ at $z = 1$ and $\omega = \pi$ at $z = -1$. With this interpretation, the inherent periodicity in frequency of the Fourier transform is captured naturally, since a change of angle of 2π radians in the z-plane corresponds to traversing the unit circle once and returning to exactly the same point.

As we discussed in Chapter 2, the power series representing the Fourier transform does not converge for all sequences; i.e., the infinite sum may not always be finite. Similarly, the z-transform does not converge for all sequences or for all values of z. For any given sequence, the set of values of z for which the z-transform converges is called the *region of convergence,* which we abbreviate ROC. As we stated in Sec. 2.7, if the sequence is absolutely summable, the Fourier transform converges to a continuous function of ω. Applying this criterion to Eq. (3.6) leads to the condition

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty \tag{3.7}$$

for convergence of the z-transform. It should be clear from Eq. (3.7) that, because of the multiplication of the sequence by the real exponential r^{-n}, it is possible for the z-transform to converge even if the Fourier transform does not. For example, the sequence $x[n] = u[n]$ is not absolutely summable, and therefore, the Fourier transform does not converge absolutely. However, $r^{-n}u[n]$ is absolutely summable if $r > 1$. This means that the z-transform for the unit step exists with a region of convergence $|z| > 1$.

Convergence of the power series of Eq. (3.2) depends only on $|z|$, since $|X(z)| < \infty$ if

$$\sum_{n=-\infty}^{\infty} |x[n]|\,|z|^{-n} < \infty, \tag{3.8}$$

i. e., the region of convergence of the power series in Eq. (3.2) consists of all values of z such that the inequality in Eq. (3.8) holds. Thus, if some value of z, say, $z = z_1$, is in the ROC, then all values of z on the circle defined by $|z| = |z_1|$ will also be in the ROC. As one consequence of this, the region of convergence will consist of a ring in the z-plane

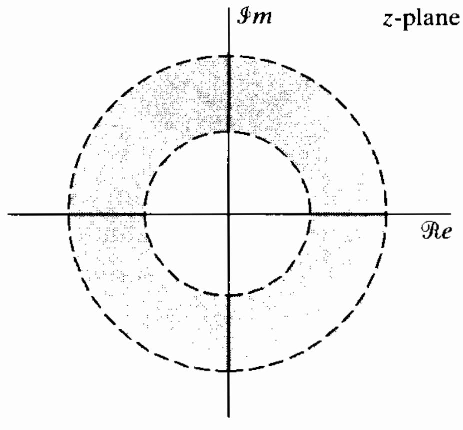

Figure 3.2 The region of convergence (ROC) as a ring in the z-plane. For specific cases, the inner boundary can extend inward to the origin, and the ROC becomes a disc. For other cases, the outer boundary can extend outward to infinity.

centered about the origin. Its outer boundary will be a circle (or the ROC may extend outward to infinity), and its inner boundary will be a circle (or it may extend inward to include the origin). This is illustrated in Figure 3.2. If the ROC includes the unit circle, this of course implies convergence of the z-transform for $|z| = 1$, or equivalently, the Fourier transform of the sequence converges. Conversely, if the ROC does not include the unit circle, the Fourier transform does not converge absolutely.

A power series of the form of Eq. (3.2) is a Laurent series. Therefore, a number of elegant and powerful theorems from the theory of functions of a complex variable can be employed in the study of the z-transform. (See, for example, Churchill and Brown, 1990.) A Laurent series, and therefore the z-transform, represents an analytic function at every point inside the region of convergence; hence, the z-transform and all its derivatives must be continuous functions of z within the region of convergence. This implies that if the region of convergence includes the unit circle, then the Fourier transform and all its derivatives with respect to ω must be continuous functions of ω. Also, from the discussion in Section 2.7, the sequence must be absolutely summable, i.e., a stable sequence.

Uniform convergence of the z-transform requires absolute summability of the exponentially weighted sequence, as stated in Eq. (3.7). Neither of the sequences

$$x_1[n] = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty,$$

and

$$x_2[n] = \cos \omega_0 n, \qquad -\infty < n < \infty,$$

is absolutely summable. Furthermore, neither of these sequences multiplied by r^{-n} would be absolutely summable for any value of r. Thus, these sequences do not have a z-transform that converges absolutely for any z. However, we showed in Section 2.7 that even though sequences such as $x_1[n]$ are not absolutely summable, they do have finite energy, and the Fourier transform converges in the mean-square sense to a discontinuous periodic function. Similarly, the sequence $x_2[n]$ is neither absolutely nor square summable, but a useful Fourier transform for $x_2[n]$ can be defined using impulses. In both cases the Fourier transforms are not continuous, infinitely differentiable functions, so they cannot result from evaluating a z-transform on the unit circle. Thus, in such cases it is not strictly correct to think of the Fourier transform as being the z-transform evaluated on the unit circle, although we nevertheless use the notation $X(e^{j\omega})$ that implies this.

The z-transform is most useful when the infinite sum can be expressed in closed form, i.e., when it can be "summed" and expressed as a simple mathematical formula.

Among the most important and useful z-transforms are those for which $X(z)$ is a rational function inside the region of convergence, i.e.,

$$X(z) = \frac{P(z)}{Q(z)}, \tag{3.9}$$

where $P(z)$ and $Q(z)$ are polynomials in z. The values of z for which $X(z) = 0$ are called the *zeros* of $X(z)$, and the values of z for which $X(z)$ is infinite are referred to as the *poles* of $X(z)$. The poles of $X(z)$ for finite values of z are the roots of the denominator polynomial. In addition, poles may occur at $z = 0$ or $z = \infty$. For rational z-transforms, a number of important relationships exist between the locations of poles of $X(z)$ and the region of convergence of the z-transform. We discuss these more specifically in Section 3.2. First, however, we illustrate the z-transform with several examples.

Example 3.1 Right-Sided Exponential Sequence

Consider the signal $x[n] = a^n u[n]$. Because it is nonzero only for $n \geq 0$, this is an example of a *right-sided* sequence. From Eq. (3.2),

$$X(z) = \sum_{n=-\infty}^{\infty} = a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty.$$

Thus, the region of convergence is the range of values of z for which $|az^{-1}| < 1$ or, equivalently, $|z| > |a|$. Inside the region of convergence, the infinite series converges to

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| > |a|. \tag{3.10}$$

Here we have used the familiar formula for the sum of terms of a geometric series. The z-transform has a region of convergence for any finite value of $|a|$. The Fourier transform of $x[n]$, on the other hand, converges only if $|a| < 1$. For $a = 1$, $x[n]$ is the unit step sequence with z-transform

$$X(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1. \tag{3.11}$$

In Example 3.1, the infinite sum is equal to a rational function of z inside the region of convergence; for most purposes, this rational function is a much more convenient representation than the infinite sum. We will see that any sequence that can be represented as a sum of exponentials can equivalently be represented by a rational z-transform. Such a z-transform is determined to within a constant multiplier by its zeros and its poles. For this example, there is one zero, at $z = 0$, and one pole, at $z = a$. The pole–zero plot and the region of convergence for Example 3.1 are shown in Figure 3.3 where a "$\circ$" denotes the zero and an "$\times$" the pole. For $|a| > 1$, the ROC does not include the unit circle, consistent with the fact that, for these values of a, the Fourier transform of the exponentially growing sequence $a^n u[n]$ does not converge.

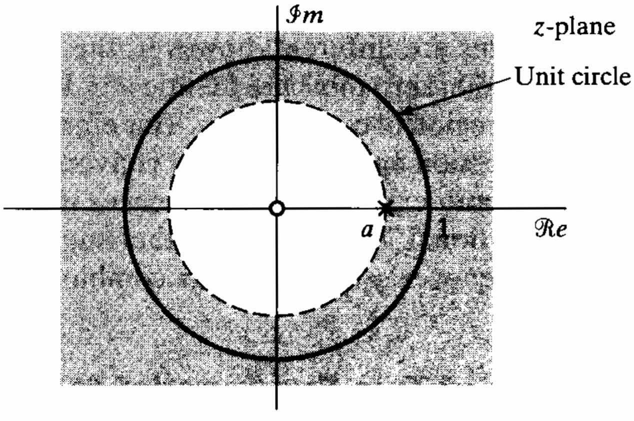

Figure 3.3 Pole–zero plot and region of convergence for Example 3.1.

Example 3.2 Left-Sided Exponential Sequence

Now let $x[n] = -a^n u[-n-1]$. Since the sequence is nonzero only for $n \le -1$, this is a *left-sided* sequence. Then

$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n. \end{aligned} \tag{3.12}$$

If $|a^{-1}z| < 1$ or, equivalently, $|z| < |a|$, the sum in Eq. (3.12) converges, and

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad |z| < |a|. \tag{3.13}$$

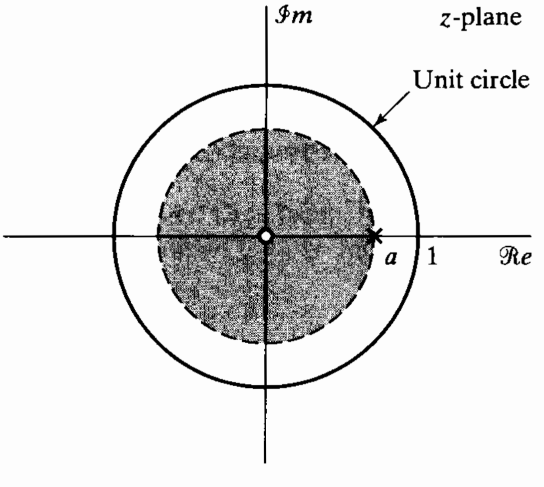

Figure 3.4 Pole–zero plot and region of convergence for Example 3.2.

The pole–zero plot and region of convergence for this example are shown in Figure 3.4. Note that for $|a| < 1$, the sequence $-a^n u[-n-1]$ grows exponentially as $n \to -\infty$, and thus, the Fourier transform does not exist.

Comparing Eqs. (3.10) and (3.13) and Figures 3.3 and 3.4, we see that the sequences and, therefore, the infinite sums are different; however, the algebraic expressions for $X(z)$ and the corresponding pole–zero plots are identical in Examples 3.1 and 3.2. The z-transforms differ only in the region of convergence. This emphasizes the need for specifying both the algebraic expression and the region of convergence for the z-transform of a given sequence. Also, in both examples, the sequences were exponentials and the resulting z-transforms were rational. In fact, as is further suggested by the next example, $X(z)$ will be rational whenever $x[n]$ is a linear combination of real or complex exponentials.

Example 3.3 Sum of Two Exponential Sequences

Consider a signal that is the sum of two real exponentials:

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n]. \tag{3.14}$$

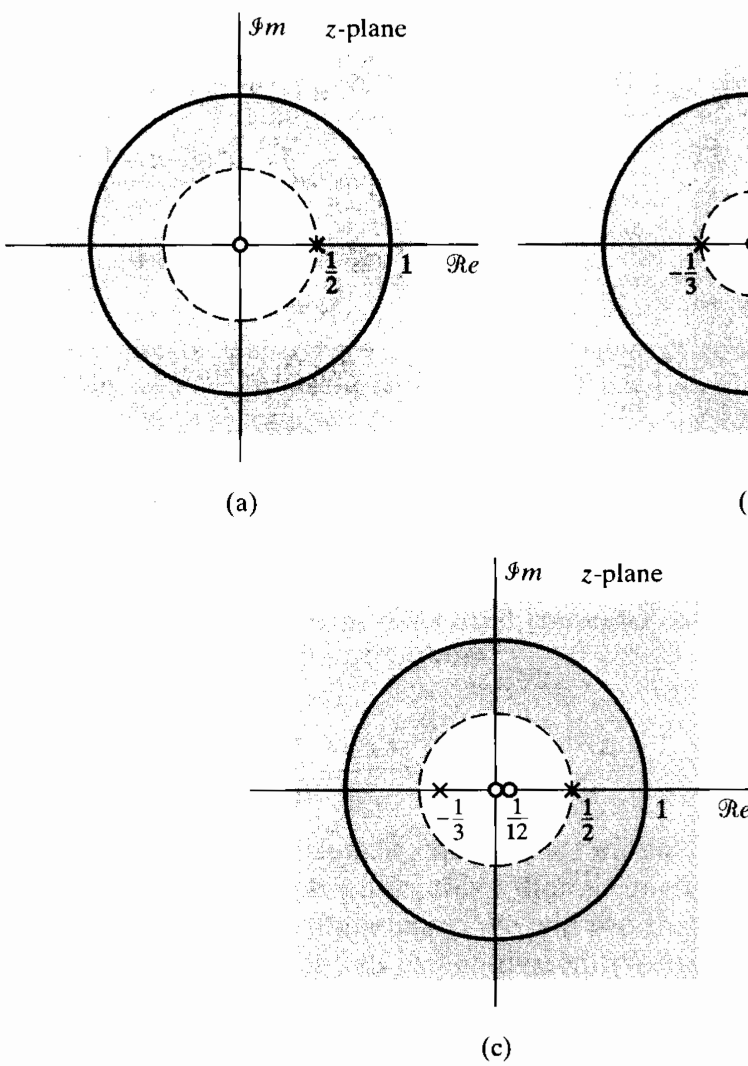

Figure 3.5 Pole–zero plot and region of convergence for the individual terms and the sum of terms in Examples 3.3 and 3.4. (a) $1/(1-\frac{1}{2}z^{-1})$, $|z| > \frac{1}{2}$. (b) $1/(1+\frac{1}{3}z^{-1})$, $|z| > \frac{1}{3}$. (c) $1/(1-\frac{1}{2}z^{-1}) + 1/(1+\frac{1}{3}z^{-1})$, $|z| > \frac{1}{2}$.

The z-transform is then

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} \left\{ \left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n] \right\} z^{-n} \\
&= \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\tfrac{1}{3}\right)^n u[n] z^{-n} \qquad (3.15) \\
&= \sum_{n=0}^{\infty} \left(\tfrac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\tfrac{1}{3} z^{-1}\right)^n \\
&= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} = \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)} \\
&= \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}. \qquad (3.16)
\end{aligned}
$$

For convergence of $X(z)$, both sums in Eq. (3.15) must converge, which requires that both $\left|\frac{1}{2}z^{-1}\right| < 1$ and $\left|\left(-\frac{1}{3}\right)z^{-1}\right| < 1$ or, equivalently, $|z| > \frac{1}{2}$ and $|z| > \frac{1}{3}$. Thus, the region of convergence is the region of overlap, $|z| > \frac{1}{2}$. The pole–zero plot and ROC for the z-transform of each of the individual terms and for the combined signal are shown in Figure 3.5.

In each of the preceding examples, we started with the definition of the sequence and manipulated each of the infinite sums into a form whose sum could be recognized. When the sequence is recognized as a sum of exponential sequences of the form of Examples 3.1 and 3.2, the z-transform can be obtained much more simply using the fact that the z-transform operator is linear. Specifically, from the definition of the z-transform, Eq. (3.2), if $x[n]$ is the sum of two terms, then $X(z)$ will be the sum of the corresponding z-transforms of the individual terms. The ROC will be the intersection of the individual regions of convergence, i.e., the values of z for which both individual sums converge. We have already demonstrated this fact in obtaining Eq. (3.15) in Example 3.3. Example 3.4 shows how the z-transform in Example 3.3 can be obtained in a much more straightforward manner.

Example 3.4 Sum of Two Exponentials (Again)

Again, let $x[n]$ be given by Eq. (3.14). Then using the general result of Example 3.1 with $a = \frac{1}{2}$ and $a = -\frac{1}{3}$, the z-transforms of the two individual terms are easily seen to be

$$\left(\tfrac{1}{2}\right)^n u[n] \stackrel{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| > \tfrac{1}{2}, \tag{3.17}$$

$$\left(-\tfrac{1}{3}\right)^n u[n] \stackrel{z}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \tfrac{1}{3}, \tag{3.18}$$

and, consequently,

$$\left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n] \stackrel{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \tfrac{1}{2}, \tag{3.19}$$

as we had determined in Example 3.3. The pole–zero plot and ROC for the z-transform of each of the individual terms and for the combined signal are shown in Figure 3.5.

All the major points of Examples 3.1–3.4 are summarized in Example 3.5.

Example 3.5 Two-Sided Exponential Sequence

Consider the sequence

$$x[n] = \left(-\tfrac{1}{3}\right)^n u[n] - \left(\tfrac{1}{2}\right)^n u[-n-1]. \tag{3.20}$$

Note that this sequence grows exponentially as $n \to -\infty$. Using the general result of Example 3.1 with $a = -\frac{1}{3}$, we obtain

$$\left(-\tfrac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \tfrac{1}{3},$$

and using the result of Example 3.2 with $a = \frac{1}{2}$ yields

$$-\left(\tfrac{1}{2}\right)^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| < \tfrac{1}{2}.$$

Thus, by the linearity of the z-transform,

$$\begin{aligned} X(z) &= \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad \tfrac{1}{3} < |z|, \quad |z| < \tfrac{1}{2}, \\ &= \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}. \end{aligned} \tag{3.21}$$

In this case, the ROC is the annular region $\frac{1}{3} < |z| < \frac{1}{2}$. Note that the rational function in this example is identical to the rational function in Examples 3.3 and 3.4, but the ROC is different in the three cases. The pole–zero plot and the ROC for this example are shown in Figure 3.6.

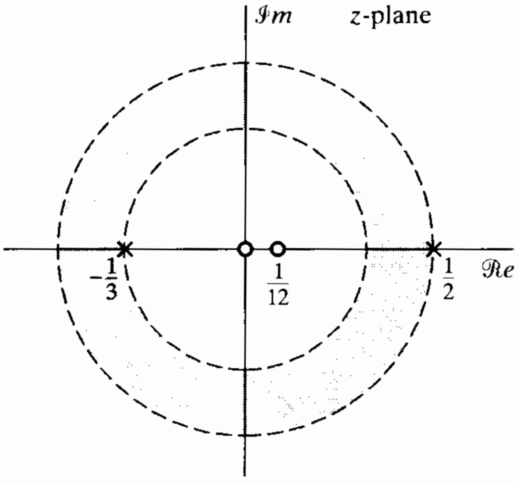

Figure 3.6 Pole–zero plot and region of convergence for Example 3.5.

Note that the ROC does not contain the unit circle, so the sequence in Eq. (3.20) does not have a Fourier transform.

In each of the preceding examples, we expressed the z-transform both as a ratio of polynomials in z and as a ratio of polynomials in z^{-1}. From the form of the definition of the z-transform as given in Eq. (3.2), we see that, for sequences that are zero for

$n < 0$, $X(z)$ involves only negative powers of z. Thus, for this class of signals, it is particularly convenient for $X(z)$ to be expressed in terms of polynomials in z^{-1} rather than z; however, even when $x[n]$ is nonzero for $n < 0$, $X(z)$ can still be expressed in terms of factors of the form $(1 - az^{-1})$. It should be remembered that such a factor introduces both a pole and a zero, as illustrated by the algebraic expressions in the preceding examples.

From these examples, it is easily seen that infinitely long exponential sequences have z-transforms that can be expressed as rational functions of either z or z^{-1}. The case where the sequence has finite length also has a rather simple form. If the sequence is nonzero only in the interval $N_1 \le n \le N_2$, the z-transform

$$X(z) = \sum_{n=N_1}^{N_2} x[n]z^{-n} \tag{3.22}$$

has no problems of convergence, as long as each of the terms $|x[n]z^{-n}|$ is finite. In general, it may not be possible to express the sum of a finite set of terms in a closed form, but in such cases it may be unnecessary. For example, it is easily seen that if $x[n] = \delta[n] + \delta[n-5]$, then $X(z) = 1 + z^{-5}$, which is finite for $|z| > 0$. An example of a case where a finite number of terms can be summed to produce a more compact representation of the z-transform is given in Example 3.6.

Example 3.6 Finite-Length Sequence

Consider the signal

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n \\ &= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}, \end{aligned} \tag{3.23}$$

where we have used the general formula in Eq. (2.56) to sum the finite series. The ROC is determined by the set of values of z for which

$$\sum_{n=0}^{N-1} |az^{-1}|^n < \infty.$$

Since there are only a finite number of nonzero terms, the sum will be finite as long as az^{-1} is finite, which in turn requires only that $|a| < \infty$ and $z \neq 0$. Thus, assuming that $|a|$ is finite, the ROC includes the entire z-plane, with the exception of the origin $(z = 0)$. The pole–zero plot for this example, with $N = 16$ and a real and between zero and unity, is shown in Figure 3.7. Specifically, the N roots of the numerator polynomial are at

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 0, 1, \ldots, N-1. \tag{3.24}$$

(Note that these values satisfy the equation $z^N = a^N$, and when $a = 1$, these complex values are the Nth roots of unity.) The zero at $k = 0$ cancels the pole at $z = a$.

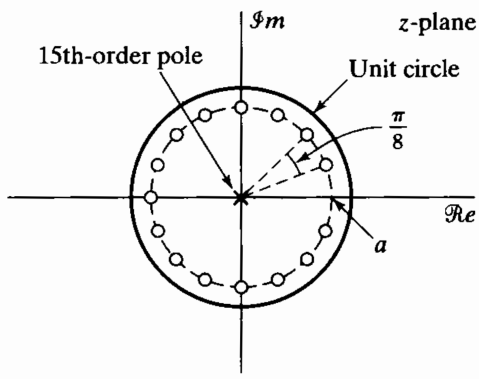

Figure 3.7 Pole–zero plot for Example 3.6 with $N = 16$ and a real such that $0 < a < 1$. The region of convergence in this example consists of all values of z except $z = 0$.

Consequently, there are no poles other than at the origin. The remaining zeros are at

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 1, \ldots, N-1. \tag{3.25}$$

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

The transform pairs corresponding to some of the prêceding examples, as well as a number of other commonly encountered z-transform pairs, are summarized in Table 3.1. We will see that these basic transform pairs are very useful in finding z-transforms given a sequence or, conversely, in finding the sequence corresponding to a given z-transform.

3.2 PROPERTIES OF THE REGION OF CONVERGENCE FOR THE z-TRANSFORM

The examples of the previous section suggest that the properties of the region of convergence depend on the nature of the signal. These properties are summarized next, followed by some discussion and intuitive justification. We assume specifically that the algebraic expression for the z-transform is a rational function and that $x[n]$ has finite amplitude, except possibly at $n = \infty$ or $n = -\infty$.

PROPERTY 1: The ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \le r_R < |z| < r_L \le \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

PROPERTY 4: If $x[n]$ is a *finite-duration sequence,* i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

PROPERTY 5: If $x[n]$ is a *right-sided sequence,* i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

PROPERTY 6: If $x[n]$ is a *left-sided sequence,* i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.

As discussed in Section 3.1, property 1 results from the fact that convergence of Eq. (3.2) for a given $x[n]$ is dependent only on $|z|$, and property 2 is a consequence of the fact that Eq. (3.2) reduces to the Fourier transform when $|z| = 1$. Property 3 follows from the recognition that $X(z)$ is infinite at a pole and therefore, by definition, does not converge.

Properties 4 through 7 can all be developed more or less directly from the interpretation of the z-transform as the Fourier transform of the original sequence, modified by an exponential weighting. Let us first consider property 4. Figure 3.8 shows a finite-duration sequence and the exponential sequence r^{-n} for $1 < r$ (a decaying exponential) and for $0 < r < 1$ (a growing exponential). Convergence of the z-transform is implied by absolute summability of the sequence $x[n]|z|^{-n}$ or, equivalently, $x[n]r^{-n}$. It should

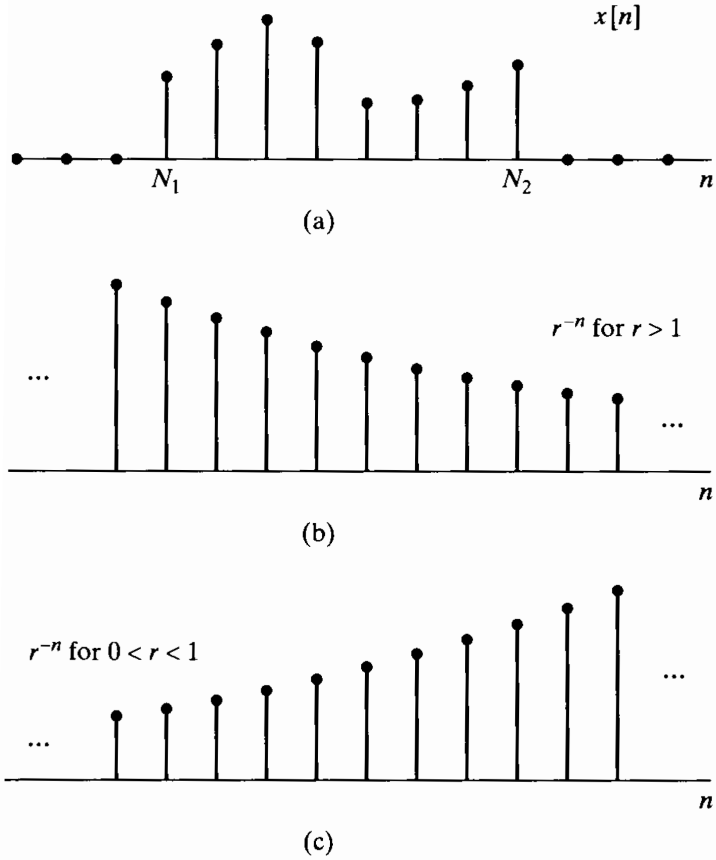

Figure 3.8 Finite-length sequence and weighting sequences implicit in convergence of the z-transform. (a) The finite-length sequence $x[n]$. (b) Weighting sequence r^{-n} for $1 < r$. (c) Weighting sequence r^{-n} for $0 < r < 1$.

be evident from Figure 3.8 that, since $x[n]$ has only a finite number of nonzero values, as long as each of these values is finite, $x[n]$ will be absolutely summable. Furthermore, this will not be affected by the exponential weighting if the weighting sequence has finite amplitude in the interval where $x[n]$ is nonzero, i.e., $N_1 \le n \le N_2$. Therefore, for a finite-duration sequence, $x[n]r^{-n}$ will be absolutely summable for $0 < r < \infty$. The only possible complication arises for $r = 0$ or for $r = \infty$. If $x[n]$ is nonzero for any positive values of n (i.e., if $N_2 > 0$), and if r, or, equivalently, $|z|$, is zero, then $x[n]r^{-n}$ will be infinite for $0 < n \le N_2$. Correspondingly, if $x[n]$ is nonzero for any negative values of n (i.e., if $N_1 < 0$), then $x[n]r^{-n}$ will be infinite for $N_1 \le n < 0$ if r, or, equivalently, $|z|$, is infinite.

Property 5 can be interpreted in a somewhat similar manner. Figure 3.9 illustrates a right-sided sequence and the exponential sequence r^{-n} for two different values of r. A right-sided sequence is zero prior to some value of n, say, N_1. If the circle $|z| = r_0$ is in the ROC, then $x[n]r_0^{-n}$ is absolutely summable, or equivalently, the Fourier transform of $x[n]r_0^{-n}$ converges. Since $x[n]$ is right sided, the sequence $x[n]r_1^{-n}$ will also be absolutely summable if r_1^{-n} decays faster than r_0^{-n}. Specifically, as illustrated in Figure 3.9, this more rapid exponential decay will further attenuate sequence values for positive values of n and cannot cause sequence values for negative values of n to become unbounded, since $x[n]z^{-n} = 0$ for $n < N_1$. Based on this property, we can conclude that, for a right-sided sequence, the ROC extends outward from some circle in the z-plane, concentric with the origin. This circle, in fact, is at the outermost pole in $X(z)$. To see this, assume that the poles occur at $z = d_1, \ldots, d_N$, with d_1 having the smallest magnitude, i.e., corresponding

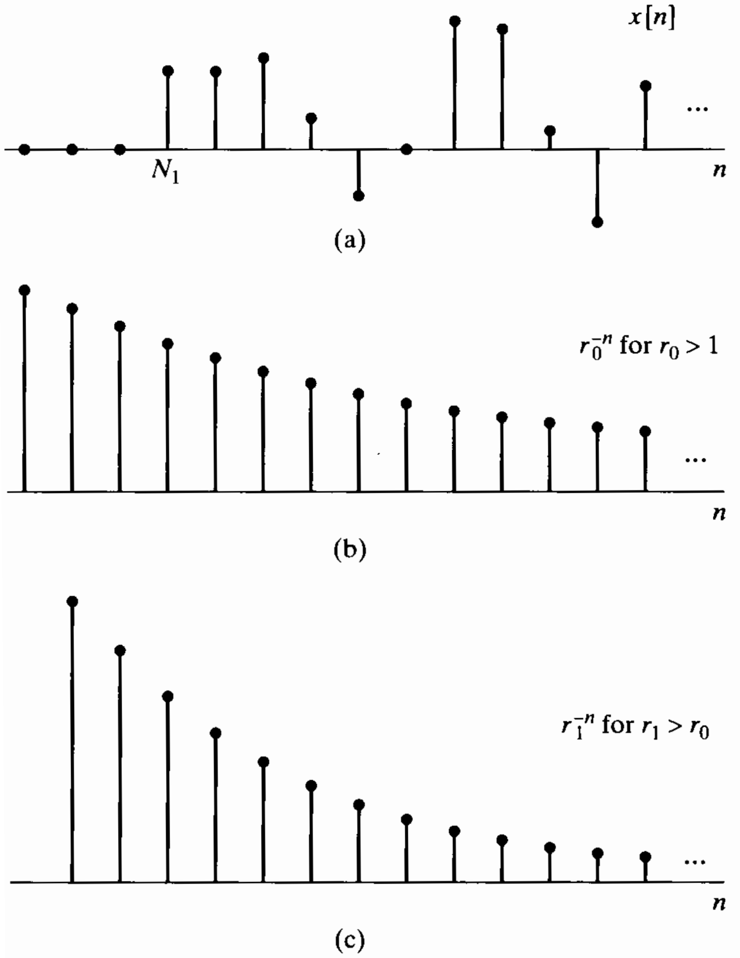

Figure 3.9 Right-sided sequence and weighting sequences implicit in convergence of the z-transform. (a) The right-sided sequence $x[n]$. (b) Weighting sequence r_0^{-n} for $1 < r_0$. (c) Weighting sequence r_1^{-n} for $r_1 > r_0$.

to the innermost pole, and d_N having the largest magnitude, i.e., corresponding to the outermost pole. To simplify the argument, we will assume that all the poles are distinct, although the argument can be easily generalized for multiple-order poles. As we will see in Section 3.3, for $N_1 \leq n$, $x[n]$ will consist of a sum of exponentials of the form

$$x[n] = \sum_{k=1}^{N} A_k (d_k)^n, \qquad n \geq N_1. \tag{3.26}$$

The least rapidly increasing of these exponentials, as n increases, is the one corresponding to the innermost pole, i.e., d_1, and the most slowly decaying (or most rapidly growing) is the one corresponding to the outermost pole, i.e., d_N. Now let us consider $x[n]$ with the exponential weighting r^{-n} applied, i.e.,

$$x[n]r^{-n} = r^{-n} \sum_{k=1}^{N} A_k (d_k)^n, \qquad n \geq N_1, \tag{3.27}$$

$$= \sum_{k=1}^{N} A_k (d_k r^{-1})^n, \qquad n \geq N_1. \tag{3.28}$$

Absolute summability of $x[n]r^{-n}$ requires that each exponential in Eq. (3.28) be absolutely summable; i.e.,

$$\sum_{n=N_1}^{\infty} |d_k r^{-1}|^n < \infty, \qquad k = 1, \ldots, N, \tag{3.29}$$

or equivalently,

$$|r| > |d_k|, \qquad k = 1, \ldots, N. \tag{3.30}$$

Since the outermost pole, d_N, is the one with the largest absolute value,

$$|r| > |d_N|; \tag{3.31}$$

i.e., the ROC is outside the outermost pole, extending to infinity. If $N_1 < 0$, the ROC will not include $|z| = \infty$, since r^{-n} is infinite for r infinite and n negative.

As suggested by the preceding discussion, it is possible to be very precise about property 5 (as well as the associated properties 6 and 7). The essence of the argument, however, is that for a sum of right-sided exponential sequences with an exponential weighting applied, the exponential weighting must be restricted so that all of the exponentially weighted terms decay with increasing n.

For property 6, which is concerned with left-sided sequences, an exactly parallel argument can be carried out. Here, however, $x[n]$ will consist of a sum of exponentials of the same form as Eq. (3.28), but for $n \le N_2$; i.e.,

$$x[n] = \sum_{k=1}^{N} A_k (d_k)^n, \qquad n \le N_2, \tag{3.32}$$

or, with exponential weighting,

$$x[n]r^{-n} = \sum_{k=1}^{N} A_k (d_k r^{-1})^n, \qquad n \le N_2. \tag{3.33}$$

Since $x[n]$ now extends to $-\infty$ along the negative n-axis, r must be restricted so that for each d_k, the exponential sequence $(d_k r^{-1})^n$ decays to zero as n *decreases* toward $-\infty$. Equivalently,

$$|r| < |d_k|, \qquad k = 1, \ldots, N,$$

or, since d_1 has the smallest magnitude,

$$|r| < |d_1|; \tag{3.34}$$

i.e., the ROC is inside the innermost pole. If the left-sided sequence has nonzero values for positive values of n, then the ROC will not include the origin, $z = 0$.

For right-sided sequences, the ROC is dictated by the exponential weighting required to have all exponential terms decay to zero for increasing n; for left-sided sequences, the exponential weighting must be such that all exponential terms decay to zero for decreasing n. For two-sided sequences, the exponential weighting needs to be balanced, since if it decays too fast for increasing n, it may grow too quickly for decreasing n and vice versa. More specifically, for two-sided sequences, some of the poles contribute only for $n > 0$ and the rest only for $n < 0$. The region of convergence is

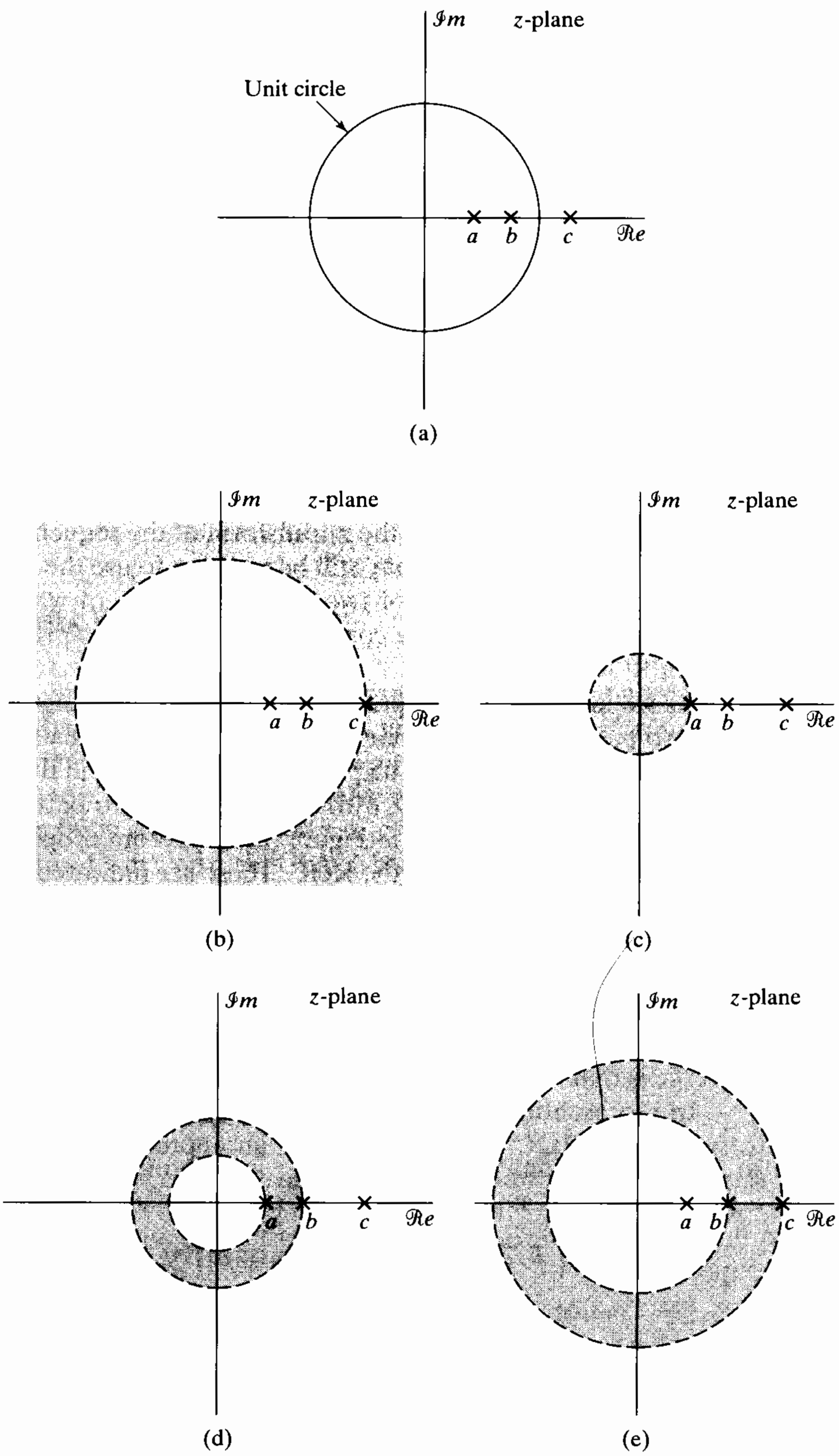

Figure 3.10 Examples of four z- transforms with the same pole–zero locations, illustrating the different possibilities for the region of convergence. Each ROC corresponds to a different sequence: (b) to a right-sided sequence, (c) to a left-sided sequence, (d) to a two-sided sequence, and (e) to a two-sided sequence.

bounded on the inside by the pole with the largest magnitude that contributes for $n > 0$ and on the outside by the pole with the smallest magnitude that contributes for $n < 0$.

Property 8 is somewhat more difficult to develop formally, but, at least intuitively, it is strongly suggested by our discussion of properties 4 through 7. Any infinite two-sided sequence can be represented as a sum of a right-sided part (say, for $n \geq 0$) and a left-sided part that includes everything not included in the right-sided part. The right-sided part will have an ROC given by Eq. (3.31), while the ROC of the left-sided part will be given by Eq. (3.34). The ROC of the entire two-sided sequence must be the intersection of these two regions. Thus, if such an intersection exists, it will always be a simply connected annular region of the form

$$r_R < |z| < r_L.$$

There is a possibility of no overlap between the regions of convergence of the right- and left-sided parts; i.e., $r_L < r_R$. An example is the sequence $x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{3}\right)^n u[-n-1]$. In this case, the z-transform of the sequence simply does not exist. If such cases arise, however, it may still be possible to use the z-transform by considering the sequence to be the sum of two sequences, each of which has a z-transform, but the two transforms cannot be combined in algebraic expressions, since they have no common ROC.

As we indicated in comparing Examples 3.1 and 3.2, the algebraic expression or pole–zero pattern does not completely specify the z-transform of a sequence; i.e., the ROC must also be specified. The properties considered in this section limit the possible ROC's that can be associated with a given pole–zero pattern. To illustrate, consider the pole–zero pattern shown in Figure 3.10(a). From properties 1, 3, and 8, there are only four possible choices for the ROC. These are indicated in Figures 3.10(b), (c), (d), and (e), each being associated with a different sequence. Specifically, Figure 3.10(b) corresponds to a right-sided sequence, Figure 3.10(c) to a left-sided sequence, and Figures 3.10(d) and 3.10(e) to two different two-sided sequences. If we assume, as indicated in Figure 3.10(a), that the unit circle falls between the pole at $z = b$ and the pole at $z = c$, then the only one of the four cases for which the Fourier transform would converge is that in Figure 3.10(e).

In representing a sequence through its z-transform, it is sometimes convenient to specify the ROC implicitly through an appropriate time-domain property of the sequence. This is illustrated in Example 3.7.

Example 3.7 Stability, Causality, and the ROC

Consider a system with impulse response $h[n]$ for which the z-transform $H(z)$ has the pole–zero plot shown in Figure 3.11. There are three possible ROC's consistent with properties 1–8 that can be associated with this pole-zero plot. However, if we state in addition that the system is stable (or equivalently, that $h[n]$ is absolutely summable and therefore has a Fourier transform), then the ROC must include the unit circle. Thus, stability of the system and properties 1–8 imply that the ROC is the region $\frac{1}{2} < |z| < 2$. Note that as a consequence, $h[n]$ is two sided, and therefore, the system is not causal.

If we state instead that the system is causal, and therefore that $h[n]$ is right sided, then property 5 would require that the ROC be the region $|z| > 2$. Under this

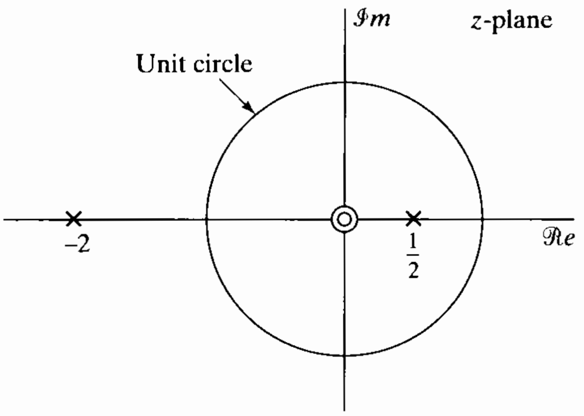

Figure 3.11 Pole–zero plot for the system function in Example 3.7.

condition, the system would not be stable; i.e., for this specific pole–zero plot, there is no ROC that would imply that the system is both stable and causal.

3.3 THE INVERSE z-TRANSFORM

One of the important roles of the z-transform is in the analysis of discrete-time linear systems. Often, this analysis involves finding the z-transform of sequences and, after some manipulation of the algebraic expressions, finding the inverse z-transform. There are a number of formal and informal ways of determining the inverse z-transform from a given algebraic expression and associated region of convergence. There is a formal inverse z-transform expression that is based on the Cauchy integral theorem (Churchill and Brown, 1990). However, for the typical kinds of sequences and z-transforms that we will encounter in the analysis of discrete linear time-invariant systems, less formal procedures are sufficient and preferable. In Sections 3.3.1–3.3.3 we consider some of these procedures, specifically the inspection method, partial fraction expansion, and power series expansion.

3.3.1 Inspection Method

The inspection method consists simply of becoming familiar with, or recognizing "by inspection," certain transform pairs. For example, in Section 3.1, we evaluated the z-transform for sequences of the form $x[n] = a^n u[n]$, where a can be either real or complex. Sequences of this form arise quite frequently, and consequently, it is particularly useful to make direct use of the transform pair

$$a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \qquad |z| > |a|. \tag{3.35}$$

If we need to find the inverse z-transform of

$$X(z) = \left(\frac{1}{1 - \frac{1}{2}z^{-1}}\right), \qquad |z| > \frac{1}{2}, \tag{3.36}$$

and we recall the z-transform pair of Eq. (3.35), we would recognize "by inspection" the associated sequence as $x[n] = \left(\frac{1}{2}\right)^n u[n]$. If the ROC associated with $X(z)$ in Eq. (3.36) had been $|z| < \frac{1}{2}$, we can recall transform pair 6 in Table 3.1 to find by inspection that $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$.

Tables of z-transforms, such as Table 3.1, are invaluable in applying the inspection method. If the table is extensive, it may be possible to express a given z-transform as a sum of terms, each of whose inverse is given in the table. If so, the inverse transform (i.e., the corresponding sequence) can be written from the table.

3.3.2 Partial Fraction Expansion

As already described, inverse z-transforms can be found by inspection if the z-transform expression is recognized or tabulated. Sometimes $X(z)$ may not be given explicitly in an available table, but it may be possible to obtain an alternative expression for $X(z)$ as a sum of simpler terms, each of which is tabulated. This is the case for any rational function, since we can obtain a partial fraction expansion and easily identify the sequences corresponding to the individual terms.

To see how to obtain a partial fraction expansion, let us assume that $X(z)$ is expressed as a ratio of polynomials in z^{-1}; i.e.,

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}. \tag{3.37}$$

Such z-transforms arise frequently in the study of linear time-invariant systems. An equivalent expression is

$$X(z) = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}. \tag{3.38}$$

Equation (3.38) explicitly shows that for such functions, there will be M zeros and N poles at nonzero locations in the z-plane. In addition, there will be either $M-N$ poles at $z=0$ if $M > N$ or $N-M$ zeros at $z=0$ if $N > M$. In other words, z-transforms of the form of Eq. (3.37) always have the same number of poles and zeros in the finite z-plane, and there are no poles or zeros at $z=\infty$. To obtain the partial fraction expansion of $X(z)$ in Eq. (3.37), it is most convenient to note that $X(z)$ could be expressed in the form

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}, \tag{3.39}$$

where the c_k's are the nonzero zeros of $X(z)$ and the d_k's are the nonzero poles of $X(z)$. If $M < N$ and the poles are all first order, then $X(z)$ can be expressed as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}. \tag{3.40}$$

Obviously, the common denominator of the fractions in Eq. (3.40) is the same as the denominator in Eq. (3.39). Multiplying both sides of Eq. (3.40) by $(1 - d_k z^{-1})$ and evaluating for $z = d_k$ shows that the coefficients, A_k, can be found from

$$A_k = (1 - d_k z^{-1}) X(z)\big|_{z=d_k}. \tag{3.41}$$

Example 3.8 Second-Order z-Transform

Consider a sequence $x[n]$ with z-transform

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad |z| > \frac{1}{2}. \tag{3.42}$$

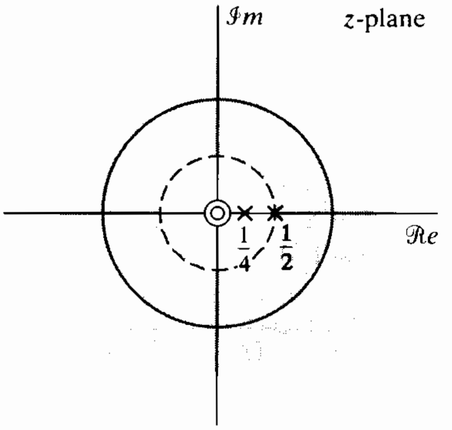

Figure 3.12 Pole-zero plot and ROC for Example 3.8.

The pole–zero plot for $X(z)$ is shown in Figure 3.12. From the region of convergence and property 5, Section 3.2, we see that $x[n]$ is a right-sided sequence. Since the poles are both first order, $X(z)$ can be expressed in the form of Eq. (3.40); i.e.,

$$X(z) = \frac{A_1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1 - \frac{1}{2}z^{-1}\right)}.$$

From Eq. (3.41),

$$A_1 = \left(1 - \frac{1}{4}z^{-1}\right) X(z)\big|_{z=1/4} = -1,$$

$$A_2 = \left(1 - \frac{1}{2}z^{-1}\right) X(z)\big|_{z=1/2} = 2.$$

Therefore,

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)}.$$

Since $x[n]$ is right sided, the ROC for each term extends outward from the outermost pole. From Table 3.1 and the linearity of the z-transform, it then follows that

$$x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n].$$

Clearly, the numerator that would result from adding the terms in Eq. (3.40) would be at most of degree $(N-1)$ in the variable z^{-1}. If $M \geq N$, then a polynomial must be added to the right-hand side of Eq. (3.40), the order of which is $(M-N)$. Thus, for $M \geq N$, the complete partial fraction expansion would have the form

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}. \tag{3.43}$$

If we are given a rational function of the form of Eq. (3.37), with $M \geq N$, the B_r's can be obtained by long division of the numerator by the denominator, with the division process terminating when the remainder is of lower degree than the denominator. The A_k's can still be obtained with Eq. (3.41).

If $X(z)$ has multiple-order poles and $M \geq N$, Eq. (3.43) must be further modified. In particular, if $X(z)$ has a pole of order s at $z = d_i$ and all the other poles are first-order, then Eq. (3.43) becomes

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1, k\neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m}. \tag{3.44}$$

The coefficients A_k and B_r are obtained as before. The coefficients C_m are obtained from the equation

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} [(1 - d_i w)^s X(w^{-1})] \right\}_{w=d_i^{-1}}. \tag{3.45}$$

Equation (3.44) gives the most general form for the partial fraction expansion of a rational z-transform expressed as a function of z^{-1} for the case $M \geq N$ and for d_i a pole of order s. If there are several multiple-order poles, there will be a term like the third sum in Eq. (3.44) for each multiple-order pole. If there are no multiple-order poles, Eq. (3.44) reduces to Eq. (3.43). If the order of the numerator is less than the order of the denominator $(M < N)$, then the polynomial term disappears from Eqs. (3.43) and (3.44).

It should be emphasized that we could have achieved the same results by assuming that the rational z-transform was expressed as a function of z instead of z^{-1}. That is, instead of factors of the form $(1 - az^{-1})$, we could have considered factors of the form $(z - a)$. This would lead to a set of equations similar in form to Eqs. (3.39)–(3.45) that would be convenient for use with a table of z-transforms expressed in terms of z. Since Table 3.1 is expressed in terms of z^{-1}, the development we pursued is more useful.

To see how to find the sequence corresponding to a given rational z-transform, let us suppose that $X(z)$ has only first-order poles, so that Eq. (3.43) is the most general form of the partial fraction expansion. To find $x[n]$, we first note that the z-transform operation is linear, so that the inverse transform of individual terms can be found and then added together to form $x[n]$.

The terms $B_r z^{-r}$ correspond to shifted and scaled impulse sequences, i.e., terms of the form $B_r\delta[n-r]$. The fractional terms correspond to exponential sequences. To decide whether a term

$$\frac{A_k}{1-d_k z^{-1}}$$

corresponds to $(d_k)^n u[n]$ or $-(d_k)^n u[-n-1]$, we must use the properties of the region of convergence that were discussed in Section 3.2. From that discussion, it follows that if $X(z)$ has only simple poles and the ROC is of the form $r_R < |z| < r_L$, then a given pole d_k will correspond to a right-sided exponential $(d_k)^n u[n]$ if $|d_k| < r_R$, and it will correspond to a left-sided exponential if $|d_k| > r_L$. Thus, the region of convergence can be used to sort the poles. Multiple-order poles also are divided into left-sided and right-sided contributions in the same way. The use of the region of convergence in finding inverse z-transforms from the partial fraction expansion is illustrated by the following examples.

Example 3.9 Inverse by Partial Fractions

To illustrate the case in which the partial fraction expansion has the form of Eq. (3.43), consider a sequence $x[n]$ with z-transform

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = \frac{(1+z^{-1})^2}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})}, \qquad |z| > 1. \tag{3.46}$$

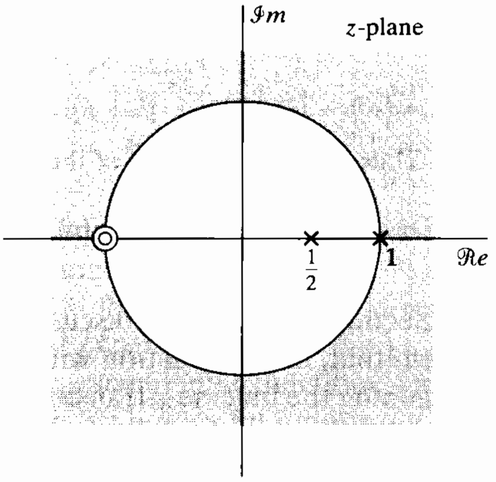

Figure 3.13 Pole–zero plot for the z-transform in Example 3.9.

The pole–zero plot for $X(z)$ is shown in Figure 3.13. From the region of convergence and property 5, Section 3.2, it is clear that $x[n]$ is a right-sided sequence. Since $M = N = 2$ and the poles are all first order, $X(z)$ can be represented as

$$X(z) = B_0 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}.$$

The constant B_0 can be found by long division:

$$\begin{array}{r|l} & \quad 2 \\ \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 & z^{-2}+2z^{-1}+1 \\ & \underline{z^{-2}-3z^{-1}+2} \\ & \quad 5z^{-1}-1 \end{array}$$

Since the remainder after one step of long division is of degree 1 in the variable z^{-1}, it is not necessary to continue to divide. Thus, $X(z)$ can be expressed as

$$X(z) = 2 + \frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}. \tag{3.47}$$

Now the coefficients A_1 and A_2 can be found by applying Eq. (3.41) to Eq. (3.46) or, equivalently, Eq. (3.47). Using Eq. (3.47), we obtain

$$A_1 = \left[\left(2+\frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}\right)\left(1-\frac{1}{2}z^{-1}\right)\right]_{z=1/2} = -9,$$

$$A_2 = \left[\left(2+\frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}\right)\left(1-z^{-1}\right)\right]_{z=1} = 8.$$

Therefore,

$$X(z) = 2 - \frac{9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}}. \tag{3.48}$$

From Table 3.1, we see that since the ROC is $|z| > 1$,

$$2 \overset{z}{\longleftrightarrow} 2\delta[n],$$

$$\frac{1}{1-\frac{1}{2}z^{-1}} \overset{z}{\longleftrightarrow} \left(\tfrac{1}{2}\right)^n u[n],$$

$$\frac{1}{1-z^{-1}} \overset{z}{\longleftrightarrow} u[n].$$

Thus, from the linearity of the z-transform,

$$x[n] = 2\delta[n] - 9\left(\tfrac{1}{2}\right)^n u[n] + 8u[n].$$

In Section 3.4 we will discuss and illustrate a number of properties of the z-transform that, in combination with the partial fraction expansion, provide a means for determining the inverse z-transform from a given rational algebraic expression and associated ROC, even when $X(z)$ is not exactly in the form of Eq. (3.39).

3.3.3 Power Series Expansion

The defining expression for the z-transform is a Laurent series where the sequence values $x[n]$ are the coefficients of z^{-n}. Thus, if the z-transform is given as a power series in the form

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots, \end{aligned} \tag{3.49}$$

we can determine any particular value of the sequence by finding the coefficient of the appropriate power of z^{-1}. We have already used this approach in finding the inverse transform of the polynomial part of the partial fraction expansion when $M \geq N$. This approach is also very useful for finite-length sequences where $X(z)$ may have no simpler form than a polynomial in z^{-1}.

Example 3.10 Finite-Length Sequence

Suppose $X(z)$ is given in the form

$$X(z) = z^2\left(1 - \tfrac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1}). \tag{3.50}$$

Although $X(z)$ is obviously a rational function, its only poles are at $z = 0$, so a partial fraction expansion according to the technique of Section 3.3.2 is not appropriate. However, by multiplying the factors of Eq. (3.50), we can express $X(z)$ as

$$X(z) = z^2 - \tfrac{1}{2}z - 1 + \tfrac{1}{2}z^{-1}.$$

Therefore, by inspection, $x[n]$ is seen to be

$$x[n] = \begin{cases} 1, & n = -2, \\ -\frac{1}{2}, & n = -1, \\ -1, & n = 0, \\ \frac{1}{2}, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently,

$$x[n] = \delta[n+2] - \tfrac{1}{2}\delta[n+1] - \delta[n] + \tfrac{1}{2}\delta[n-1].$$

In finding z-transforms of a sequence, we generally seek to sum the power series of Eq. (3.49) to obtain a simpler mathematical expression, e.g., a rational function. If we wish to use the power series to find the sequence corresponding to a given $X(z)$ expressed in closed form, we must expand $X(z)$ back into a power series. Many power series have been tabulated for transcendental functions such as log, sin, sinh, etc. In some cases such power series can have a useful interpretation as z-transforms, as we illustrate in Example 3.11. For rational z-transforms, a power series expansion can be obtained by long division, as illustrated in Examples 3.12 and 3.13.

Example 3.11 Inverse Transform by Power Series Expansion

Consider the z-transform

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|. \tag{3.51}$$

Using the power series expansion for $\log(1 + x)$, with $|x| < 1$, we obtain

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}a^n z^{-n}}{n}.$$

Therefore,

$$x[n] = \begin{cases} (-1)^{n+1}\dfrac{a^n}{n}, & n \geq 1, \\ 0, & n \leq 0. \end{cases} \tag{3.52}$$

Example 3.12 Power Series Expansion by Long Division

Consider the z-transform

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|. \tag{3.53}$$

Since the region of convergence is the exterior of a circle, the sequence is a right-sided one. Furthermore, since $X(z)$ approaches a finite constant as z approaches infinity, the sequence is causal. Thus, we divide, so as to obtain a series in powers of z^{-1}. Carrying out the long division, we obtain

$$\begin{array}{r|l} & 1 + az^{-1} + a^2z^{-2} + \cdots \\ \hline 1 - az^{-1} & 1 \\ & \underline{1 - az^{-1}} \\ & \quad az^{-1} \\ & \quad \underline{az^{-1} - a^2z^{-2}} \\ & \qquad a^2z^{-2} \quad \cdots \end{array}$$

or

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots.$$

Hence, $x[n] = a^n u[n]$.

Example 3.13 Power Series Expansion for a Left-Sided Sequence

As another example, we can consider the same ratio of polynomials as in Eq. (3.53), but with a different region of convergence:

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| < |a|. \tag{3.54}$$

Because of the region of convergence, the sequence is a left-sided one, and since $X(z)$ at $z = 0$ is finite, the sequence is zero for $n > 0$. Thus, we divide, so as to obtain a series in powers of z as follows:

$$\begin{array}{r|l} & -a^{-1}z - a^{-2}z^{-2} - \cdots \\ \hline -a + z & z \\ & \underline{z - a^{-1}z^2} \\ & \quad a^{-1}z^2 \end{array}$$

Therefore, $x[n] = -a^n u[-n-1]$.

3.4 z-TRANSFORM PROPERTIES

Many of the properties of the z-transform are particularly useful in studying discrete-time signals and systems. For example, these properties are often used in conjunction with the inverse z-transform techniques discussed in Sec. 3.3 to obtain the inverse z-transform of more complicated expressions. In Chapter 5 we will see that the properties also form the basis for transforming linear constant-coefficient difference equations to algebraic equations in terms of the transform variable z, the solution to which can then be obtained using the inverse z-transform. In this section, we consider some of the most frequently used properties. In the following discussion, $X(z)$ denotes the z-transform of $x[n]$, and the ROC of $X(z)$ is indicated by R_x; i.e.,

$$x[n] \overset{z}{\longleftrightarrow} X(z), \qquad \text{ROC} = R_x.$$

As we have seen, R_x represents a set of values of z such that $r_R < |z| < r_L$. For properties that involve two sequences and associated z-transforms, the transform pairs will be denoted as

$$x_1[n] \overset{z}{\longleftrightarrow} X_1(z), \qquad \text{ROC} = R_{x_1},$$

$$x_2[n] \overset{z}{\longleftrightarrow} X_2(z), \qquad \text{ROC} = R_{x_2}.$$

3.4.1 Linearity

The linearity property states that

$$ax_1[n] + bx_2[n] \overset{z}{\longleftrightarrow} aX_1(z) + bX_2(z), \qquad \text{ROC contains } R_{x_1} \cap R_{x_2},$$

and follows directly from the z-transform definition, Eq. (3.2). As indicated, the region of convergence is at least the intersection of the individual regions of convergence. For sequences with rational z-transforms, if the poles of $aX_1(z) + bX_2(z)$ consist of all the poles of $X_1(z)$ and $X_2(z)$ (i.e., if there is no pole–zero cancellation), then the region of convergence will be exactly equal to the overlap of the individual regions of convergence. If the linear combination is such that some zeros are introduced that cancel poles, then the region of convergence may be larger. A simple example of this occurs when $x_1[n]$ and $x_2[n]$ are of infinite duration, but the linear combination is of finite duration. In this case the region of convergence of the linear combination is the entire z-plane, with the possible exception of $z = 0$ or $z = \infty$. An example was given in Example 3.6, where $x[n]$ can be expressed as

$$x[n] = a^n u[n] - a^n u[n - N].$$

Both $a^n u[n]$ and $a^n u[n - N]$ are infinite-extent right-sided sequences, and their z-transforms have a pole at $z = a$. Therefore, their individual regions of convergence would both be $|z| > |a|$. However, as shown in Example 3.6, the pole at $z = a$ is canceled by a zero at $z = a$, and therefore, the ROC extends to the entire z-plane, with the exception of $z = 0$.

We have already exploited the linearity property in our previous discussion of the use of the partial fraction expansion for evaluating the inverse z-transform. With that procedure, $X(z)$ is expanded into a sum of simpler terms, and through linearity, the inverse z-transform is the sum of the inverse transforms of each of these terms.

3.4.2 Time Shifting

According to the time-shifting property,

$$x[n-n_0] \overset{z}{\longleftrightarrow} z^{-n_0}X(z), \qquad \text{ROC} = R_x(\text{except for the possible addition or deletion of } z=0 \text{ or } z=\infty).$$

The quantity n_0 is an integer. If n_0 is positive, the original sequence $x[n]$ is shifted right, and if n_0 is negative, $x[n]$ is shifted left. As in the case of linearity, the ROC can be changed, since the factor z^{-n_0} can alter the number of poles at $z = 0$ or $z = \infty$.

The derivation of this property follows directly from the z-transform expression in Eq. (3.2). Specifically, if $y[n] = x[n - n_0]$, the corresponding z-transform is

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n-n_0]z^{-n}.$$

With the substitution of variables $m = n - n_0$,

$$\begin{aligned} Y(z) &= \sum_{m=-\infty}^{\infty} x[m]z^{-(m+n_0)} \\ &= z^{-n_0}\sum_{m=-\infty}^{\infty} x[m]z^{-m}, \end{aligned}$$

or

$$Y(z) = z^{-n_0}X(z).$$

The time-shifting property is often useful, in conjunction with other properties and procedures, for obtaining the inverse z-transform. We illustrate with an example.

Example 3.14 Shifted Exponential Sequence

Consider the z-transform

$$X(z) = \frac{1}{z-\frac{1}{4}}, \qquad |z| > \tfrac{1}{4}.$$

From the ROC, we identify this as corresponding to a right-sided sequence. We can first rewrite $X(z)$ in the form

$$X(z) = \frac{z^{-1}}{1-\frac{1}{4}z^{-1}}, \qquad |z| > \tfrac{1}{4}. \tag{3.55}$$

This z-transform is of the form of Eq. (3.39) with $M = N = 1$, and its expansion in the form of Eq. (3.43) is

$$X(z) = -4 + \frac{4}{1-\frac{1}{4}z^{-1}}. \tag{3.56}$$

From Eq. (3.56), it follows that $x[n]$ can be expressed as

$$x[n] = -4\delta[n] + 4\left(\tfrac{1}{4}\right)^n u[n]. \tag{3.57}$$

An expression for $x[n]$ can be obtained more directly by applying the time-shifting property. First, $X(z)$ can be written as

$$X(z) = z^{-1}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right), \qquad |z| > \tfrac{1}{4}. \tag{3.58}$$

From the time-shifting property, we recognize the factor z^{-1} in Eq. (3.58) as being associated with a time shift of one sample to the right of the sequence $\left(\frac{1}{4}\right)^n u[n]$; i.e.,

$$x[n] = \left(\tfrac{1}{4}\right)^{n-1} u[n-1]. \tag{3.59}$$

It is easily verified that Eqs. (3.57) and (3.59) are the same for all values of n; i.e., they represent the same sequence.

3.4.3 Multiplication by an Exponential Sequence

The exponential multiplication property is expressed mathematically as

$$z_0^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z/z_0), \qquad \text{ROC} = |z_0| R_x.$$

The notation $\text{ROC} = |z_0|R_x$ denotes that the ROC is R_x scaled by $|z_0|$; i.e., if R_x is the set of values of z such that $r_R < |z| < r_L$, then $|z_0|R_x$ is the set of values of z such that $|z_0|r_R < |z| < |z_0|r_L$.

This property is easily shown simply by substituting $z_0^n x[n]$ into Eq. (3.2). As a consequence of the exponential multiplication property, all the pole–zero locations are scaled by a factor z_0, since, if $X(z)$ has a pole at $z = z_1$, then $X(z_0^{-1}z)$ will have a pole at $z = z_0 z_1$. If z_0 is a positive real number, the scaling can be interpreted as a shrinking or expanding of the z-plane; i.e., the pole and zero locations change along radial lines in the z-plane. If z_0 is complex with unity magnitude, so that $z_0 = e^{j\omega_0}$, the scaling corresponds to a rotation in the z-plane by an angle of ω_0; i.e., the pole and zero locations change in position along circles centered at the origin. This in turn can be interpreted as a frequency shift or translation, associated with the modulation in the time domain by the complex exponential sequence $e^{j\omega_0 n}$. That is, if the Fourier transform exists, this property has the form

$$e^{j\omega_0 n}x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}).$$

Example 3.15 Exponential Multiplication

Starting with the transform pair

$$u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \qquad |z| > 1, \tag{3.60}$$

we can use the exponential multiplication property to determine the z-transform of

$$x[n] = r^n \cos(\omega_0 n)u[n]. \tag{3.61}$$

First, $x[n]$ is expressed as

$$x[n] = \tfrac{1}{2}(re^{j\omega_0})^n u[n] + \tfrac{1}{2}(re^{-j\omega_0})^n u[n].$$

Then, using Eq. (3.60) and the exponential multiplication property, we see that

$$\tfrac{1}{2}(re^{j\omega_0})^n u[n] \overset{z}{\longleftrightarrow} \frac{\frac{1}{2}}{1 - re^{j\omega_0} z^{-1}}, \qquad |z| > r,$$

$$\tfrac{1}{2}(re^{-j\omega_0})^n u[n] \overset{z}{\longleftrightarrow} \frac{\frac{1}{2}}{1 - re^{-j\omega_0} z^{-1}}, \qquad |z| > r.$$

From the linearity property, it follows that

$$\begin{aligned} X(z) &= \frac{\frac{1}{2}}{1 - re^{j\omega_0} z^{-1}} + \frac{\frac{1}{2}}{1 - re^{-j\omega_0} z^{-1}}, \qquad |z| > r \\ &= \frac{(1 - r\cos\omega_0 z^{-1})}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}, \qquad |z| > r. \end{aligned} \tag{3.62}$$

3.4.4 Differentiation of $X(z)$

The differentiation property states that

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \qquad \text{ROC} = R_x.$$

This property is verified by differentiating the z-transform expression of Eq. (3.2); i.e.,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n},$$

$$\begin{aligned} -z\frac{dX(z)}{dz} &= -z \sum_{n=-\infty}^{\infty} (-n) x[n] z^{-n-1} \\ &= \sum_{n=-\infty}^{\infty} n x[n] z^{-n} = \mathcal{Z}\{nx[n]\}. \end{aligned}$$

We illustrate the use of the differentiation property with two examples.

Example 3.16 Inverse of Non-Rational z-Transform

In this example, we use the differentiation property together with the time-shifting property to determine the inverse z-transform considered in Example 3.11. With

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|,$$

we first differentiate to obtain a rational expression:

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1 + az^{-1}}.$$

From the differentiation property,

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1 + az^{-1}}, \qquad |z| > |a|. \tag{3.63}$$

The inverse transform of Eq. (3.63) can be obtained by the combined use of the z-transform pair of Example 3.1, the linearity property, and the time-shifting property. Specifically, we can express $nx[n]$ as

$$nx[n] = a(-a)^{n-1}u[n-1].$$

Therefore,

$$x[n] = (-1)^{n+1}\frac{a^n}{n}u[n-1] \overset{z}{\longleftrightarrow} \log(1 + az^{-1}), \qquad |z| > |a|.$$

Example 3.17 Second-Order Pole

As another example of the use of the differentiation property, let us determine the z-transform of the sequence

$$x[n] = na^n u[n] = n(a^n u[n]).$$

From the z-transform pair of Example 3.1 and the differentiation property, it follows that

$$\begin{aligned} X(z) &= -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right), \qquad |z| > |a| \\ &= \frac{az^{-1}}{(1-az^{-1})^2}, \qquad |z| > |a|. \end{aligned}$$

Therefore,

$$na^n u[n] \overset{z}{\longleftrightarrow} \frac{az^{-1}}{(1-az^{-1})^2}, \qquad |z| > |a|.$$

3.4.5 Conjugation of a Complex Sequence

The conjugation property is expressed as

$$x^*[n] \overset{z}{\longleftrightarrow} X^*(z^*), \qquad \text{ROC} = R_x.$$

This property follows in a straightforward manner from the definition of the z-transform, the details of which are left as an exercise (Problem 3.51).

3.4.6 Time Reversal

By the time-reversal property,

$$x^*[-n] \overset{z}{\longleftrightarrow} X^*(1/z^*), \qquad \text{ROC} = \frac{1}{R_x}.$$

The notation $\text{ROC} = 1/R_x$ implies that R_x is inverted; i.e., if R_x is the set of values of z such that $r_R < |z| < r_L$, then the ROC is the set of values of z such that $1/r_L < |z| < 1/r_R$. Thus, if z_0 is in the ROC for $x[n]$, then $1/z_0^*$ is in the ROC for the z-transform of $x^*[-n]$. If the sequence $x[n]$ is real or we do not conjugate a complex sequence, the result becomes

$$x[-n] \overset{z}{\longleftrightarrow} X(1/z), \qquad \text{ROC} = \frac{1}{R_x}.$$

As with the conjugation property, the time-reversal property follows easily from the definition of the z-transform, and the details are left as an exercise (Problem 3.51).

Example 3.18 Time-Reversed Exponential Sequence

As an example of the use of the property of time reversal, consider the sequence

$$x[n] = a^{-n}u[-n],$$

which is a time-reversed version of $a^n u[n]$. From the time-reversal property, it follows that

$$X(z) = \frac{1}{1-az} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}}, \qquad |z| < |a^{-1}|.$$

3.4.7 Convolution of Sequences

According to the convolution property,

$$x_1[n] * x_2[n] \overset{z}{\longleftrightarrow} X_1(z)X_2(z), \qquad \text{ROC contains } R_{x_1} \cap R_{x_2}.$$

To derive this property formally, we consider

$$y[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k],$$

so that

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right\} z^{-n}. \end{aligned}$$

If we interchange the order of summation,

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}.$$

Changing the index of summation in the second sum from n to $m = n - k$, we obtain

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1[k] \left\{ \sum_{m=-\infty}^{\infty} x_2[m]z^{-m} \right\} z^{-k}.$$

Thus, for values of z inside the regions of convergence of both $X_1(z)$ and $X_2(z)$, we can write

$$Y(z) = X_1(z)X_2(z),$$

where the region of convergence includes the intersection of the regions of convergence of $X_1(z)$ and $X_2(z)$. If a pole that borders on the region of convergence of one of the z-transforms is canceled by a zero of the other, then the region of convergence of $Y(z)$ may be larger. As we develop and exploit it in Chapter 5, the convolution property plays a particularly important role in the analysis of LTI systems. Specifically, as a consequence of this property, the z-transform of the output of an LTI system is the product of the z-transform of the input and the z-transform of the system impulse response. The z-transform of the impulse response of an LTI system is typically referred to as the system function.

Example 3.19 Evaluating a Convolution Using the z-Transform

Let $x_1[n] = a^n u[n]$ and $x_2[n] = u[n]$. The corresponding z-transforms are

$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|,$$

and

$$X_2(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}, \qquad |z| > 1.$$

If $|a| < 1$, the z-transform of the convolution of $x_1[n]$ with $x_2[n]$ is then

$$Y(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})} = \frac{z^2}{(z-a)(z-1)}, \qquad |z| > 1. \tag{3.64}$$

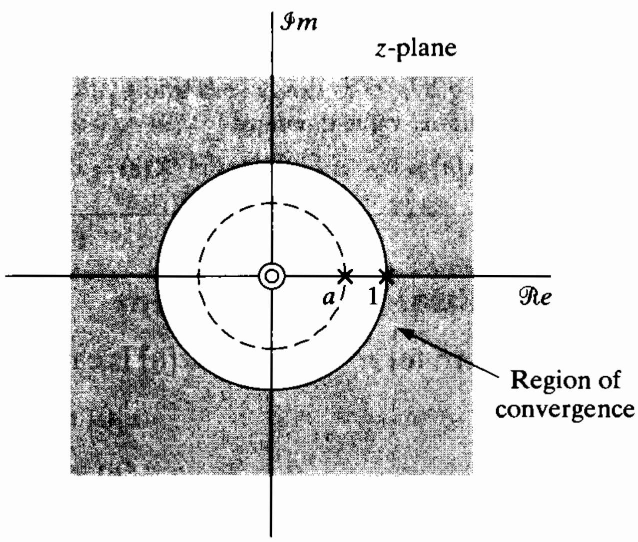

Figure 3.14 Pole–zero plot for the z-transform of the convolution of the sequences $u[n]$ and $a^n u[n]$.

The poles and zeros of $Y(z)$ are plotted in Figure 3.14, and the region of convergence is seen to be the overlap region. The sequence $y[n]$ can be obtained by determining the inverse z-transform. Expanding $Y(z)$ in Eq. (3.64) in a partial fraction expansion, we get

$$Y(z) = \frac{1}{1-a}\left(\frac{1}{1 - z^{-1}} - \frac{a}{1 - az^{-1}}\right), \qquad |z| > 1.$$

Therefore,

$$y[n] = \frac{1}{1-a}(u[n] - a^{n+1}u[n]).$$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Section Reference	Sequence	Transform	ROC
	$x[n]$	$X(z)$	R_x
	$x_1[n]$	$X_1(z)$	R_{x_1}
	$x_2[n]$	$X_2(z)$	R_{x_2}
3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
3.4.2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
	$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
	$\mathcal{J}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
3.4.8	Initial-value theorem:		
	$x[n] = 0, \quad n < 0$ $\quad \lim_{z\to\infty} X(z) = x[0]$		

3.4.8 Initial-Value Theorem

If $x[n]$ is zero for $n < 0$ (i.e., if $x[n]$ is causal), then

$$x[0] = \lim_{z\to\infty} X(z).$$

This theorem is shown by considering the limit of each term in the series of Eq. (3.2). (See Problem 3.56.)

3.4.9 Summary of Some z-Transform Properties

We have presented and discussed a number of the theorems and properties of z-transforms, many of which are useful in manipulating z-transforms. These properties and a number of others are summarized for convenient reference in Table 3.2.

3.5 SUMMARY

In this chapter, we have defined the z-transform of a sequence and shown that it is a generalization of the Fourier transform. The discussion focused on the properties of the z-transform and techniques for obtaining the z-transform of a sequence and vice versa. Specifically, we showed that the defining power series of the z-transform may converge when the Fourier transform does not. We explored in detail the dependence of the shape of the region of convergence on the properties of the sequence. A full understanding of the properties of the region of convergence is essential for successful use of the z-transform. This is particularly true in developing techniques for finding the sequence that corresponds to a given z-transform, i.e., finding inverse z-transforms. Much of the discussion focused on z-transforms that are rational functions in their region of convergence. For such functions, we described a technique of inverse transformation based on the partial fraction expansion of $X(z)$. We also discussed other techniques for inverse transformation, such as the use of tabulated power series expansions and long division.

An important part of the chapter was a discussion of some of the many properties of the z-transform that make it useful in analyzing discrete-time signals and systems. A variety of examples demonstrated how these properties can be used to find direct and inverse z-transforms.

PROBLEMS

Basic Problems with Answers

3.1. Determine the z-transform, including the region of convergence, for each of the following sequences:

(a) $\left(\frac{1}{2}\right)^n u[n]$

(b) $-\left(\frac{1}{2}\right)^n u[-n-1]$

(c) $\left(\frac{1}{2}\right)^n u[-n]$

(d) $\delta[n]$

(e) $\delta[n-1]$

(f) $\delta[n+1]$

(g) $\left(\frac{1}{2}\right)^n (u[n]-u[n-10])$

3.2. Determine the z-transform of the sequence

$$x[n] = \begin{cases} n, & 0 \le n \le N-1, \\ N, & N \le n. \end{cases}$$

3.3. Determine the z-transform of each of the following sequences. Include with your answer the region of convergence in the z-plane and a sketch of the pole–zero plot. Express all sums in closed form; α can be complex.

(a) $x_a[n] = \alpha^{|n|}, \qquad 0 < |\alpha| < 1.$

(b) $x_b[n] = \begin{cases} 1, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$

(c) $x_c[n] = \begin{cases} n, & 0 \le n \le N, \\ 2N-n, & N+1 \le n \le 2N, \\ 0, & \text{otherwise.} \end{cases}$

Hint: Note that $x_b[n]$ is a rectangular sequence and $x_c[n]$ is a triangular sequence. First express $x_c[n]$ in terms of $x_b[n]$.

3.4. Consider the z-transform $X(z)$ whose pole–zero plot is as shown in Figure P3.4-1.

(a) Determine the region of convergence of $X(z)$ if it is known that the Fourier transform exists. For this case, determine whether the corresponding sequence $x[n]$ is right sided, left sided, or two sided.

(b) How many possible two-sided sequences have the pole–zero plot shown in Figure P3.4-1?

(c) Is it possible for the pole–zero plot in Figure P3.4-1 to be associated with a sequence that is both stable and causal? If so, give the appropriate region of convergence.

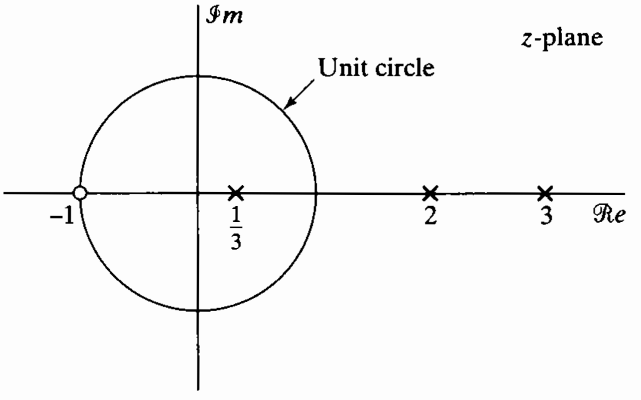

Figure P3.4-1

3.5. Determine the sequence $x[n]$ with z-transform

$$X(z) = (1 + 2z)(1 + 3z^{-1})(1 - z^{-1}).$$

3.6. Following are several z-transforms. For each, determine the inverse z-transform using both methods—partial fraction expansion and power series expansion—discussed in Section 3.3. In addition, indicate in each case whether the Fourier transform exists.

(a) $X(z) = \dfrac{1}{1 + \frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}$

(b) $X(z) = \dfrac{1}{1 + \frac{1}{2}z^{-1}}, \qquad |z| < \frac{1}{2}$

(c) $X(z) = \dfrac{1 - \frac{1}{2}z^{-1}}{1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}, \qquad |z| > \frac{1}{2}$

(d) $X(z) = \dfrac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}}, \qquad |z| > \frac{1}{2}$

(e) $X(z) = \dfrac{1 - az^{-1}}{z^{-1} - a}, \qquad |z| > |1/a|$

3.7. The input to a causal linear time-invariant system is

$$x[n] = u[-n-1] + \left(\tfrac{1}{2}\right)^n u[n].$$

The z-transform of the output of this system is

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + z^{-1}\right)}.$$

(a) Determine $H(z)$, the z-transform of the system impulse response. Be sure to specify the region of convergence.

(b) What is the region of convergence for $Y(z)$?

(c) Determine $y[n]$.

3.8. The system function of a causal linear time-invariant system is

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{3}{4}z^{-1}}.$$

The input to this system is

$$x[n] = \left(\tfrac{1}{3}\right)^n u[n] + u[-n-1].$$

(a) Find the impulse response of the system, $h[n]$.
(b) Find the output $y[n]$.
(c) Is the system stable? That is, is $h[n]$ absolutely summable?

3.9. A causal LTI system has impulse response $h[n]$, for which the z-transform is

$$H(z) = \frac{1 + z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{4}z^{-1}\right)}.$$

(a) What is the region of convergence of $H(z)$?
(b) Is the system stable? Explain.
(c) Find the z-transform $X(z)$ of an input $x[n]$ that will produce the output

$$y[n] = -\tfrac{1}{3}\left(-\tfrac{1}{4}\right)^n u[n] - \tfrac{4}{3}(2)^n u[-n-1].$$

(d) Find the impulse response $h[n]$ of the system.

3.10. Without explicitly solving for $X(z)$, find the region of convergence of the z-transform of each of the following sequences, and determine whether the Fourier transform converges:

(a) $x[n] = \left[\left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n\right] u[n-10]$

(b) $x[n] = \begin{cases} 1, & -10 \le n \le 10, \\ 0, & \text{otherwise,} \end{cases}$

(c) $x[n] = 2^n u[-n]$

(d) $x[n] = \left[\left(\frac{1}{4}\right)^{n+4} - (e^{j\pi/3})^n\right] u[n-1]$

(e) $x[n] = u[n+10] - u[n+5]$

(f) $x[n] = \left(\frac{1}{2}\right)^{n-1} u[n] + (2+3j)^{n-2} u[-n-1]$

3.11. Following are four z-transforms. Determine which ones *could* be the z-transform of a *causal* sequence. Do not evaluate the inverse transform. You should be able to give the answer by inspection. Clearly state your reasons in each case.

(a) $\dfrac{(1 - z^{-1})^2}{\left(1 - \frac{1}{2}z^{-1}\right)}$

(b) $\dfrac{(z-1)^2}{\left(z - \frac{1}{2}\right)}$

(c) $\dfrac{\left(z - \frac{1}{4}\right)^5}{\left(z - \frac{1}{2}\right)^6}$

(d) $\dfrac{\left(z - \frac{1}{4}\right)^6}{\left(z - \frac{1}{2}\right)^5}$

3.12. Sketch the pole–zero plot for each of the following z-transforms and shade the region of convergence:

(a) $X_1(z) = \dfrac{1 - \frac{1}{2}z^{-1}}{1 + 2z^{-1}}$, ROC: $|z| < 2$

(b) $X_2(z) = \dfrac{1 - \frac{1}{3}z^{-1}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{2}{3}z^{-1}\right)}$, $x_2[n]$ causal

(c) $X_3(z) = \dfrac{1 + z^{-1} - 2z^{-2}}{1 - \frac{13}{6}z^{-1} + z^{-2}}$, $x_3[n]$ absolutely summable.

3.13. A causal sequence $g[n]$ has the z-transform

$$G(z) = \sin(z^{-1})(1 + 3z^{-2} + 2z^{-4}).$$

Find $g[11]$.

3.14. If $H(z) = \dfrac{1}{1 - \frac{1}{4}z^{-2}}$ and $h[n] = A_1\alpha_1^n u[n] + A_2\alpha_2^n u[n]$, determine the values of A_1, A_2, α_1, and α_2.

3.15. If $H(z) = \dfrac{1 - \frac{1}{1024}z^{-10}}{1 - \frac{1}{2}z^{-1}}$ for $|z| > 0$, is the corresponding LTI system causal? Justify your answer.

3.16. When the input to an LTI system is

$$x[n] = \left(\frac{1}{3}\right)^n u[n] + (2)^n u[-n-1],$$

the corresponding output is

$$y[n] = 5\left(\frac{1}{3}\right)^n u[n] - 5\left(\frac{2}{3}\right)^n u[n].$$

(a) Find the system function $H(z)$ of the system. Plot the pole(s) and zero(s) of $H(z)$ and indicate the region of convergence.
(b) Find the impulse response $h[n]$ of the system.
(c) Write a difference equation that is satisfied by the given input and output.
(d) Is the system stable? Is it causal?

3.17. Consider an LTI system with input $x[n]$ and output $y[n]$ that satisfies the difference equation

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n] - x[n-1].$$

Determine all possible values for the system's impulse response $h[n]$ at $n = 0$.

3.18. A causal LTI system has the system function

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{\left(1 + \frac{1}{2}z^{-1}\right)(1 - z^{-1})}.$$

(a) Find the impulse response of the system, $h[n]$.
(b) Find the output of this system, $y[n]$, for the input

$$x[n] = e^{j(\pi/2)n}.$$

3.19. For each of the following pairs of input z-transform $X(z)$ and system function $H(z)$, determine the region of convergence for the output z-transform $Y(z)$:

(a)

$$X(z) = \frac{1}{1 + \frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}$$

$$H(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}, \qquad |z| > \frac{1}{4}$$

(b)

$$X(z) = \frac{1}{1 - 2z^{-1}}, \qquad |z| < 2$$

$$H(z) = \frac{1}{1 - \frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3}$$

(c)

$$X(z) = \frac{1}{\left(1 - \frac{1}{5}z^{-1}\right)\left(1 + 3z^{-1}\right)}, \qquad \frac{1}{5} < |z| < 3$$

$$H(z) = \frac{1 + 3z^{-1}}{1 + \frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3}$$

3.20. For each of the following pairs of input and output z-transforms $X(z)$ and $Y(z)$, determine the region of convergence for the system function $H(z)$:

(a)

$$X(z) = \frac{1}{1 - \frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4}$$

$$Y(z) = \frac{1}{1 + \frac{2}{3}z^{-1}}, \qquad |z| > \frac{2}{3}$$

(b)

$$X(z) = \frac{1}{1 + \frac{1}{3}z^{-1}}, \qquad |z| < \frac{1}{3}$$

$$Y(z) = \frac{1}{\left(1 - \frac{1}{6}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right)}, \qquad \frac{1}{6} < |z| < \frac{1}{3}$$

Basic Problems

3.21. Consider a linear time-invariant system with impulse response

$$h[n] = \begin{cases} a^n, & n \geq 0, \\ 0, & n < 0, \end{cases}$$

and input

$$x[n] = \begin{cases} 1, & 0 \leq n \leq (N-1), \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the output $y[n]$ by explicitly evaluating the discrete convolution of $x[n]$ and $h[n]$.

(b) Determine the output $y[n]$ by computing the inverse z-transform of the product of the z-transforms of $x[n]$ and $h[n]$.

3.22. Consider an LTI system that is stable and for which $H(z)$, the z-transform of the impulse response, is given by

$$H(z) = \frac{3}{1 + \frac{1}{3}z^{-1}}.$$

Suppose $x[n]$, the input to the system, is a unit step sequence.

(a) Find the output $y[n]$ by evaluating the discrete convolution of $x[n]$ and $h[n]$.

(b) Find the output $y[n]$ by computing the inverse z-transform of $Y(z)$.

3.23. An LTI system is characterized by the system function

$$H(z) = \frac{\left(1 - \frac{1}{2}z^{-2}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}, \qquad |z| > \frac{1}{2}.$$

(a) Determine the impulse response of the system.

(b) Determine the difference equation relating the system input $x[n]$ and the system output $y[n]$.

3.24. Sketch each of the following sequences and determine their z-transforms, including the region of convergence:

(a) $\displaystyle\sum_{k=-\infty}^{\infty} \delta[n - 4k]$

(b) $\displaystyle\frac{1}{2}\left[e^{j\pi n} + \cos\left(\frac{\pi}{2}n\right) + \sin\left(\frac{\pi}{2} + 2\pi n\right)\right] u[n]$

3.25. Consider a right-sided sequence $x[n]$ with z-transform

$$X(z) = \frac{1}{(1 - az^{-1})(1 - bz^{-1})} = \frac{z^2}{(z - a)(z - b)}.$$

In Section 3.3 we considered the determination of $x[n]$ by carrying out a partial fraction expansion, with $X(z)$ considered as a ratio of polynomials in z^{-1}. Carry out a partial fraction expansion of $X(z)$, considered as a ratio of polynomials in z, and determine $x[n]$ from this expansion.

Advanced Problems

3.26. Determine the inverse z-transform of each of the following. In Parts (a)–(c), use the methods specified. In Part (d), use any method you prefer.

(a) Long division:

$$X(z) = \frac{1 - \frac{1}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}}, \qquad x[n] \text{ a right-sided sequence}$$

(b) Partial fraction:

$$X(z) = \frac{3}{z - \frac{1}{4} - \frac{1}{8}z^{-1}}, \qquad x[n] \text{ stable}$$

(c) Power series:

$$X(z) = \ln(1 - 4z), \qquad |z| < \frac{1}{4}$$

(d) $X(z) = \dfrac{1}{1 - \frac{1}{3}z^{-3}}, \qquad |z| > (3)^{-1/3}$

3.27. Using any method, determine the inverse z-transform for each of the following:

(a) $X(z) = \dfrac{1}{\left(1+\frac{1}{2}z^{-1}\right)^2(1-2z^{-1})(1-3z^{-1})}$, stable sequence

(b) $X(z) = e^{z^{-1}}$

(c) $X(z) = \dfrac{z^3-2z}{z-2}$, left-sided sequence

3.28. Determine the inverse z-transform of each of the following. You should find the z-transform properties in Section 3.4 helpful.

(a) $X(z) = \dfrac{3z^{-3}}{\left(1-\frac{1}{4}z^{-1}\right)^2}$, $x[n]$ left sided

(b) $X(z) = \sin(z)$, ROC includes $|z| = 1$

(c) $X(z) = \dfrac{z^7-2}{1-z^{-7}}$, $|z| > 1$

3.29. Determine a sequence $x[n]$ whose z-transform is $X(z) = e^z + e^{1/z}$, $z \neq 0$.

3.30. Determine the inverse z-transform of

$$X(z) = \log 2\left(\tfrac{1}{2} - z\right), \qquad |z| < \tfrac{1}{2},$$

by

(a) using the power series

$$\log(1-x) = -\sum_{i=1}^{\infty} \frac{x^i}{i}, \qquad |x| < 1;$$

(b) first differentiating $X(z)$ and then using the derivative to recover $x[n]$.

3.31. For each of the following sequences, determine the z-transform and region of convergence, and sketch the pole–zero diagram:

(a) $x[n] = a^n u[n] + b^n u[n] + c^n u[-n-1]$, $|a| < |b| < |c|$

(b) $x[n] = n^2 a^n u[n]$

(c) $x[n] = e^{n^4}\left[\cos\left(\frac{\pi}{12}n\right)\right]u[n] - e^{n^4}\left[\cos\left(\frac{\pi}{12}n\right)\right]u[n-1]$

3.32. The pole–zero diagram in Figure P3.32-1 corresponds to the z-transform $X(z)$ of a causal sequence $x[n]$. Sketch the pole–zero diagram of $Y(z)$, where $y[n] = x[-n+3]$. Also, specify the region of convergence for $Y(z)$.

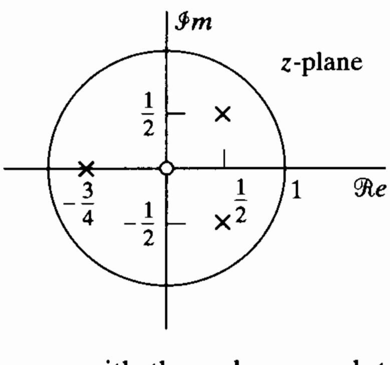

Figure P3.32-1

3.33. Let $x[n]$ be the sequence with the pole–zero plot shown in Figure P3.33-1. Sketch the pole–zero plot for:

(a) $y[n] = \left(\frac{1}{2}\right)^n x[n]$

(b) $w[n] = \cos\left(\frac{\pi n}{2}\right)x[n]$

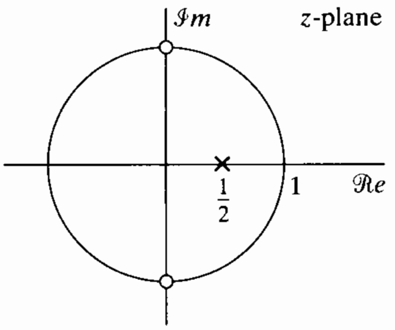

Figure P3.33-1

3.34. Consider an LTI system that is stable and for which $H(z)$, the z-transform of the impulse response, is given by

$$H(z) = \frac{3 - 7z^{-1} + 5z^{-2}}{1 - \frac{5}{2}z^{-1} + z^{-2}}.$$

Suppose $x[n]$, the input to the system, is a unit step sequence.

(a) Find the output $y[n]$ by evaluating the discrete convolution of $x[n]$ and $h[n]$.

(b) Find the output $y[n]$ by computing the inverse z-transform of $Y(z)$.

3.35. Determine the unit step response of the causal system for which the z-transform of the impulse response is

$$H(z) = \frac{1 - z^3}{1 - z^4}.$$

3.36. If the input $x[n]$ to an LTI system is $x[n] = u[n]$, the output is

$$y[n] = \left(\tfrac{1}{2}\right)^{n-1}u[n+1].$$

(a) Find $H(z)$, the z-transform of the system impulse response, and plot its pole–zero diagram.

(b) Find the impulse response $h[n]$.

(c) Is the system stable?

(d) Is the system causal?

3.37. Consider a sequence $x[n]$ for which the z-transform is

$$X(z) = \frac{\frac{1}{3}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{4}}{1 - 2z^{-1}}$$

and for which the region of convergence includes the unit circle. Determine $x[0]$ using the initial-value theorem.

3.38. Consider a stable linear time-invariant system. The z-transform of the impulse response is

$$H(z) = \frac{z^{-1} + z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right)}.$$

Suppose $x[n]$, the input to the system, is $2u[n]$. Determine $y[n]$ at $n = 1$.

3.39. Suppose the z-transform of $x[n]$ is

$$X(z) = \frac{z^{10}}{\left(z - \frac{1}{2}\right)\left(z - \frac{3}{2}\right)^{10}\left(z + \frac{3}{2}\right)^2\left(z + \frac{5}{2}\right)\left(z + \frac{7}{2}\right)}.$$

It is also known that $x[n]$ is a stable sequence.

(a) Determine the region of convergence of $X(z)$.

(b) Determine $x[n]$ at $n = -8$.

3.40. In Figure P3.40-1, $H(z)$ is the system function of a causal LTI system.

(a) Using z-transforms of the signals shown in the figure, obtain an expression for $W(z)$ in the form

$$W(z) = H_1(z)X(z) + H_2(z)E(z),$$

where both $H_1(z)$ and $H_2(z)$ are expressed in terms of $H(z)$.

(b) For the special case $H(z) = z^{-1}/(1 - z^{-1})$, determine $H_1(z)$ and $H_2(z)$.

(c) Is the system $H(z)$ stable? Are the systems $H_1(z)$ and $H_2(z)$ stable?

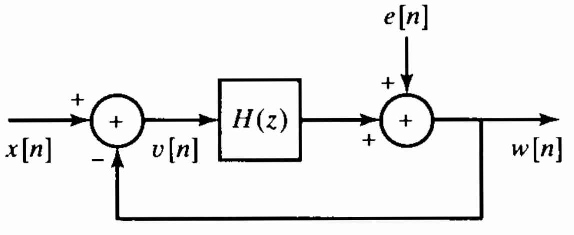

Figure P3.40-1

3.41. In Figure P3.41-1, $h[n]$ is the impulse response of the LTI system within the inner box. The input to system $h[n]$ is $v[n]$, and the output is $w[n]$. The z-transform of $h[n]$, $H(z)$, exists in the following region of convergence:

$$0 < r_{\min} < |z| < r_{\max} < \infty.$$

(a) Can the LTI system with impulse response $h[n]$ be BIBO stable? If so, determine inequality constraints on $r_{\min}$ and $r_{\max}$ such that it is stable. If not, briefly explain why.

(b) Is the overall system (in the large box, with input $x[n]$ and output $y[n]$) LTI? If so, find its impulse response $g[n]$. If not, briefly explain why.

(c) Can the overall system be BIBO stable? If so, determine inequality constraints relating α, $r_{\min}$, and $r_{\max}$ such that it is stable. If not, briefly explain why.

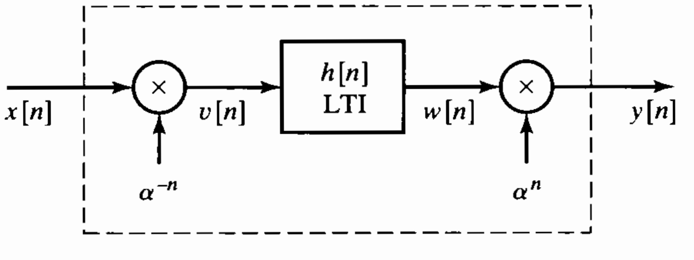

Figure P3.41-1

3.42. A causal and stable LTI system $\mathcal{S}$ has its input $x[n]$ and output $y[n]$ related by the linear constant-coefficient difference equation

$$y[n] + \sum_{k=1}^{10} \alpha_k y[n-k] = x[n] + \beta x[n-1].$$

Let the impulse response of $\mathcal{S}$ be the sequence $h[n]$.

(a) Show that $h[0]$ must be nonzero.

(b) Show that α_1 can be determined from the knowledge of $h[0]$ and $h[1]$.

(c) If $h[n] = (0.9)^n \cos(\pi n/4)$ for $0 \le n \le 10$, sketch the pole–zero plot for the system function of $\mathcal{S}$, and indicate the region of convergence.

3.43. When the input to an LTI system is

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1],$$

the output is

$$y[n] = 6\left(\frac{1}{2}\right)^n u[n] - 6\left(\frac{3}{4}\right)^n u[n].$$

(a) Find the system function $H(z)$ of the system. Plot the poles and zeros of $H(z)$, and indicate the region of convergence.
(b) Find the impulse response $h[n]$ of the system.
(c) Write the difference equation that characterizes the system.
(d) Is the system stable? Is it causal?

3.44. When the input to a causal LTI system is

$$x[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}2^n u[-n-1],$$

the z-transform of the output is

$$Y(z) = \frac{1+z^{-1}}{\left(1-z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}.$$

(a) Find the z-transform of $x[n]$.
(b) What is the region of convergence of $Y(z)$?
(c) Find the impulse response of the system.
(d) Is the system stable?

3.45. Let $x[n]$ be a discrete-time signal with $x[n] = 0$ for $n \leq 0$ and z-transform $X(z)$. Furthermore, given $x[n]$, let the discrete-time signal $y[n]$ be defined by

$$y[n] = \begin{cases} \frac{1}{n}x[n], & n > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Compute $Y(z)$ in terms of $X(z)$.
(b) Using the result of Part (a), find the z-transform of

$$w[n] = \frac{1}{n + \delta[n]}u[n-1].$$

3.46. The signal $y[n]$ is the output of an LTI system with impulse response $h[n]$ for a given input $x[n]$. Throughout the problem, assume that $y[n]$ is stable and has a z-transform $Y(z)$ with the pole–zero diagram shown in Figure P3.46-1. The signal $x[n]$ is stable and has the pole–zero diagram shown in Figure P3.46-2.

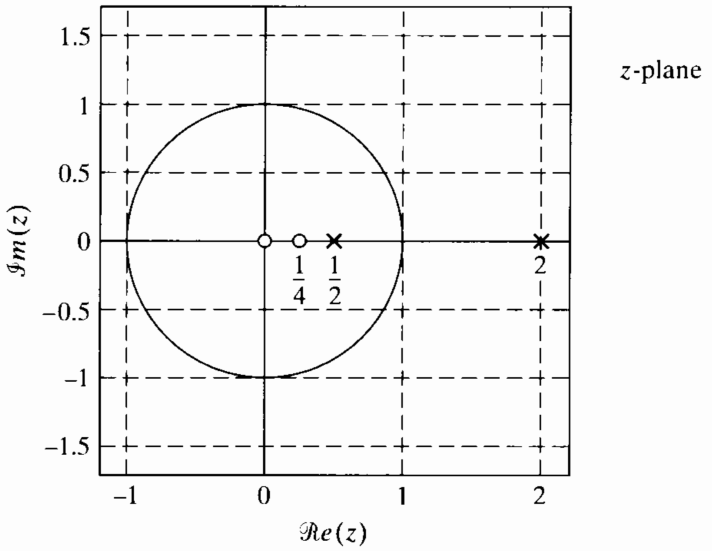

Figure P3.46-1

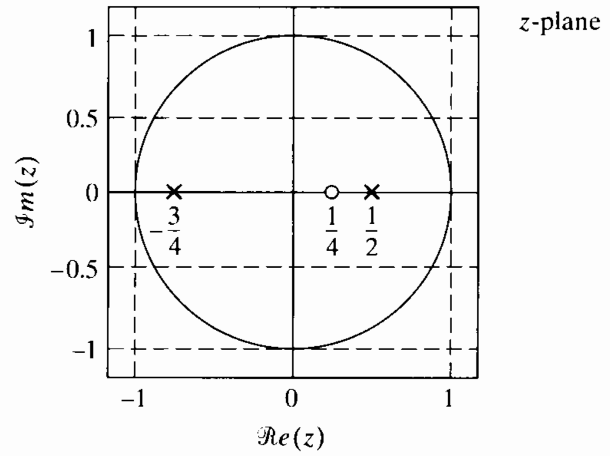

Figure P3.46-2

(a) What is the region of convergence, $Y(z)$?
(b) Is $y[n]$ left sided, right sided, or two sided?
(c) What is the ROC of $X(z)$?
(d) Is $x[n]$ a causal sequence? That is, does $x[n] = 0$ for $n < 0$?
(e) What is $x[0]$?
(f) Draw the pole–zero plot of $H(z)$, and specify its ROC.
(g) Is $h[n]$ anticausal? That is, does $h[n] = 0$ for $n > 0$?

Extension Problems

3.47. Let $x[n]$ denote a causal sequence; i.e., $x[n] = 0, n < 0$. Furthermore, assume that $x[0] \neq 0$.
(a) Show that there are no poles or zeros of $X(z)$ at $z = \infty$, i.e., that $\lim_{z\to\infty} X(z)$ is nonzero and finite.
(b) Show that the number of poles in the finite z-plane equals the number of zeros in the finite z-plane. (The finite z-plane excludes $z = \infty$.)

3.48. Consider a sequence with z-transform $X(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials in z. If the sequence is absolutely summable and if all the roots of $Q(z)$ are inside the unit circle, is the sequence necessarily causal? If your answer is yes, clearly explain. If your answer is no, give a counterexample.

3.49. Let $x[n]$ be a causal stable sequence with z-transform $X(z)$. The *complex cepstrum* $\hat{x}[n]$ is defined as the inverse transform of the logarithm of $X(z)$; i.e.,

$$\hat{X}(z) = \log X(z) \stackrel{z}{\longleftrightarrow} \hat{x}[n],$$

where the ROC of $\hat{X}(z)$ includes the unit circle. (Strictly speaking, taking the logarithm of a complex number requires some careful considerations. Furthermore, the logarithm of a valid z-transform may not be a valid z-transform. For now, we assume that this operation is valid.)

Determine the complex cepstrum for the sequence

$$x[n] = \delta[n] + a\delta[n-N], \qquad \text{where } |a| < 1.$$

3.50. Assume that $x[n]$ is real and even; i.e., $x[n] = x[-n]$. Further, assume that z_0 is a zero of $X(z)$; i.e., $X(z_0) = 0$.

(a) Show that $1/z_0$ is also a zero of $X(z)$.

(b) Are there other zeros of $X(z)$ implied by the information given?

3.51. Using the definition of the z-transform in Eq. (3.2), show that if $X(z)$ is the z-transform of $x[n] = x_R[n] + jx_I[n]$, then

(a) $x^*[n] \stackrel{z}{\longleftrightarrow} X^*(z^*)$

(b) $x[-n] \stackrel{z}{\longleftrightarrow} X(1/z)$

(c) $x_R[n] \stackrel{z}{\longleftrightarrow} \frac{1}{2}[X(z) + X^*(z^*)]$

(d) $x_I[n] \stackrel{z}{\longleftrightarrow} \dfrac{1}{2j}[X(z) - X^*(z^*)]$

3.52. Consider a *real* sequence $x[n]$ that has all the poles and zeros of its z-transform inside the unit circle. Determine, in terms of $x[n]$, a *real* sequence $x_1[n]$ not equal to $x[n]$, but for which $x_1[0] = x[0]$, $|x_1[n]| = |x[n]|$, and the z-transform of $x_1[n]$ has all its poles and zeros inside the unit circle.

3.53. A real finite-duration sequence whose z-transform has no zeros at conjugate reciprocal pair locations and no zeros on the unit circle is uniquely specified to within a positive scale factor by its Fourier transform phase (Hayes et al., 1980).

An example of zeros at conjugate reciprocal pair locations is $z = a$ and $(a^*)^{-1}$. Even though we can generate sequences that do not satisfy the preceding set of conditions, almost any sequence of practical interest satisfies the conditions and therefore is uniquely specified to within a positive scale factor by the phase of its Fourier transform.

Consider a sequence $x[n]$ that is real, that is zero outside $0 \le n \le N-1$, and whose z-transform has no zeros at conjugate reciprocal pair locations and no zeros on the unit circle. We wish to develop an algorithm that reconstructs $cx[n]$ from $\sphericalangle X(e^{j\omega})$, the Fourier transform phase of $x[n]$, where c is a positive scale factor.

(a) Specify a set of $(N-1)$ linear equations, the solution to which will provide the recovery of $x[n]$ to within a positive or negative scale factor from $\tan\{\sphericalangle X(e^{j\omega})\}$. You do not have to prove that the set of $(N-1)$ linear equations has a unique solution. Further, show that if we know $\sphericalangle X(e^{j\omega})$ rather than just $\tan\{\sphericalangle X(e^{j\omega})\}$, the sign of the scale factor can also be determined.

(b) Suppose

$$x[n] = \begin{cases} 0, & n < 0, \\ 1, & n = 0, \\ 2, & n = 1, \\ 3, & n = 2, \\ 0, & n \ge 3. \end{cases}$$