MAP 555 : Digital Filters...

9 Octobre 2015

Today

1. Discrete-Time systems
   - Some properties of LTI systems
2. The z-transform
   - Definition
   - Inverse z-transform
   - Property of the z-transform
3. Linear constant-coefficient difference equations

Discrete-time systems

Denote by X and Y the subspaces of input and output signals satisfying the following assumptions :

1. X and Y are linear subspaces of the vector space (over $\mathbb{R}$ or $\mathbb{C}$) of real or complex sequences indexed by $\mathbb{Z}$
2. X and Y closed under translation : if $x \in \mathrm{X}$, then for any $a \in \mathbb{Z}$, $\mathrm{S}_\tau x = \{x[n-\tau], n \in \mathbb{Z}\} \in \mathrm{X}$, for all $\tau \in \mathbb{Z}$ and $x = \{x(n),\ n \in \mathbb{Z}\} \in \mathrm{X}$ (and similarly for Y).

Definition

A discrete-time system $T : \mathrm{X} \to \mathrm{Y}$ is an operator that maps an input sequence $x = \{x(n),\ n \in \mathbb{N}\} \in \mathrm{X}$ into an output sequence $y = \{y(n),\ n \in \mathbb{Z}\}$.

$$y = T(x) \ .$$

Examples

- The Ideal Delay System :

$$y[n] = x[n - n_d], \ -\infty < n < \infty,$$

where n_d is a fixed positive integer called the delay of the system.

- Moving averages $\sum_{k=-\infty}^{\infty} |\psi_k| < \infty$, $\mathsf{X} = \ell^\infty(\mathbb{Z})$, $\mathsf{Y} = \ell^\infty(\mathbb{Z})$

$$y[n] = \sum_{k=-m}^{p} \psi_k x[n-k]$$

- Compressor (or sub-sampling)

$$y[n] = x[Mn] \ , \quad n \in \mathbb{Z}$$

- Interpolator (or up-sampling)

$$y[n] = x[n/M] \quad \text{if } [n]_M = 0 \text{ and } y[n] = 0 \text{ otherwise}$$

- Quadrator

$$y[n] = x^2[n] \ , \quad n \in \mathbb{Z} \ .$$

Linear Systems

Definition (Linear Systems)

A system T on a linear subspace X is linear if

$$T(a_1 x_1 + a_2 x_2) = a_1 T(x_1) + a_2 T(x_2) .$$

- A pure delay system is linear
- The moving average is linear
- The compressor and the interpolator are both linear
- The quadrator is nonlinear !

Time-invariance

Definition (Time invariant)

A system T on $\mathsf{X} \to \mathsf{Y}$ is time-invariant, if for all $\tau \in \mathbb{Z}$,

$$\mathsf{S}_\tau \circ T = T \circ \mathsf{S}_\tau$$

In words, a time-invariant system (often referred to equivalently as a shift-invariant system) is a system for which a time shift or delay of the input sequence causes a corresponding shift in the output sequence.

- The pure delay and the moving-average are linear and time-invariant.
- The quadrator is time-invariant (but not linear)
- The compressor and the interpolator are both linear but not time-invariant.

Causality

Definition (Causal or non-anticipative system)

A system is causal or non-anticipative if for any $n_0 \in \mathbb{Z}$ and $x_1, x_2 \in \mathrm{X}$ satisfying $x_1[n] = x_2[n]$, $n \leq n_0$, we have $y_1[n] = y_2[n]$ for $n \leq n_0$, where $y_1 = T(x_1)$ and $y_2 = T(x_2)$.

- The pure delay is causal.
- The moving average is causal if $\psi_k = 0$ for any $k < 0$.
- The quadrator is non-linear but causal (any memoryless transformation is causal)
- The compressor and the interpolator are both causal

Stability

Definition (BIBO-Stability)

A system is stable in the bounded-input, bounded-output (BIBO) sense if and only if every bounded input sequence produces a bounded output sequence.

- The pure delay is BIBO-stable.
- The moving average is BIBO stable.
- The quadrator is BIBO stable
- The compressor and the interpolator are BIBO stable
- $y[n] = \sum_{k=0}^{n} x[k]$ for $n \geq 0$ and $y[n] = 0$ otherwise is linear, causal, but it is not BIBO-stable (it is not time invariant)

Discrete convolution

- Set $X = Y = \ell^\infty(\mathbb{Z})$.
- Let $\{\psi_k,\ k \in \mathbb{Z}\} \in \ell^1(\mathbb{Z})$. For any $x \in \ell^\infty(\mathbb{Z})$,

$$\sup_{n\in\mathbb{Z}} |y[n]| \le \sum_{k=-\infty}^{\infty} |\psi_k| \sup_{n\in\mathbb{Z}} |x[n]| \ .$$

- For $x \in X$, denote by $y = \psi * x$ the discrete convolution of the sequences $\{\psi_k,\ k \in \mathbb{Z}\}$ and $\{x(k),\ k \in \mathbb{Z}\}$:

$$y[n] = \sum_{k=-\infty}^{\infty} \psi_k x[n-k]$$

- $y = T(x) = \psi * x$ is linear, time-invariant, BIBO stable. It is causal iff $\psi_k = 0$ for $k \le 0$.
- If $x = \delta$ the impulse sequence, then $\psi = T(\delta)$: ψ is the impulse response of the system.

LTI systems and convolutions

- Let T be a linear invariant system over X. Assume that $\delta \in \mathrm{X}$ and denote by $\psi = T(\delta)$.
- For any $x = \{x(n),\ n \in \mathbb{Z}\}$ with finite support, $x[n] = 0$ for $|n| \geq M$, we get

$$x = \sum_{|k| \leq M} x[k] \tau_k \delta$$

- Hence,

$$\begin{aligned} y[n] = T(x)[n] &= \sum_{|k| \leq M} x[k] T(\tau_k \delta)[n] \\ &= \sum_{|k| \leq M} x[k] \tau_k T(\delta)[n] = \sum_{|k| \leq M} x[k] \psi_{n-k} \end{aligned}$$

- Provided that the signal space is equipped with a norm and T is continuous w.r.t. that norm, then it is possible to show that every LTI systems can be represented as a linear convolution.

Cascade connection

- If $\{\alpha_k,\ k \in \mathbb{Z}\} \in L_1(\mathbb{Z})$ and $\{\beta_k,\ k \in \mathbb{Z}\} \in L_1(\mathbb{Z})$, then $\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\alpha_k \beta_{n-k}| < \infty$ and $\alpha * \beta = \{(\alpha * \beta)_n, n \in \mathbb{Z}\}$ where

$$(\alpha * \beta)_n = \sum_{k=-\infty}^{\infty} \alpha_k \beta_{n-k} \in \ell^1(\mathbb{Z}) \ .$$

  In addition, $\alpha * \beta = \beta * \alpha$, the convolution on $\ell^1(\mathbb{Z})$ sequence is commutative.
- For $\alpha \in L_1(\mathbb{Z})$, define the LTI operator F_α on $\ell^\infty(\mathbb{Z})$ by

$$y = \mathsf{F}_\alpha(x) \quad y[n] = \sum_{k \in \mathbb{Z}} \alpha_k x[n-k]$$

- If $\alpha \in \ell^1(\mathbb{Z})$ and $\beta \in \ell^1(\mathbb{Z})$, then

$$\mathsf{F}_\alpha \circ \mathsf{F}_\beta = \mathsf{F}_\beta \circ \mathsf{F}_\alpha = \mathsf{F}_{\alpha * \beta} = \mathsf{F}_{\beta * \alpha} \ .$$

Why z-transform

The z-transform is a generalization of discrete Fourier transform. Why generalize it ?

- Fourier transform does not converge on all sequences. Hence the discussion of stability, causality, etc.. is blurred by convergence problems.
- Bring the power of complex variable theory (poles, zeros, Laurent decomposition)

Definition

- The Fourier transform of a sequence $x \in \ell^2(\mathbb{Z})$ is defined as

$$X(\mathrm{e}^{\mathrm{i}\omega}) =_{\ell^2(\mathbb{Z})} \sum_{n=-\infty}^{\infty} x[n]\mathrm{e}^{-\mathrm{i}\omega n}$$

- The z-transform of the sequence x is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \,,$$

which is, in general, an infinite sum or infinite power series, with z being a complex variable.

Bilateral z-transform

- The z-transform can be seen as an operator which transforms the sequence x into the function $X(z)$, where z is a continuous complex variable.
- The correspondence between a sequence and its z-transform is indicated by the notation

$$x[n] \stackrel{Z}{\leftrightarrow} X(z)$$

- The z-transform $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ is often referred to as the two-sided or bilateral z-transform, in contrast to the one-sided or unilateral z-transform, which is defined as

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

- Clearly, the bilateral and unilateral transforms are equivalent only if $x[n] = 0$ for $n < 0$.
- In this lesson, we focus on the bilateral transform exclusively.

z-transform and TFTD

- It is evident that there is a close relationship between the Fourier transform and the z-transform.
- In particular, if we replace the complex variable z in $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ with the complex variable $\mathrm{e}^{\mathrm{i}\omega}$, then the z-transform reduces to the DTFT.
- This is one motivation for the notation $X(\mathrm{e}^{\mathrm{i}\omega})$ for the Fourier transform ; when it exists, the Fourier transform is simply $X(z)$ with $z = \mathrm{e}^{\mathrm{i}\omega}$ which corresponds to restricting z to have unity magnitude ; i.e., for $|z| = 1$, the z-transform corresponds to the Fourier transform.

Region of convergence

- The z-transform does not converge for all sequences or for all values of z.
- For any given sequence x, the set of values of z for which the z-transform converges is called the region of convergence, which we abbreviate $\mathrm{ROC}(x)$.
- Formally, $z \in \mathrm{ROC}(x)$ if

$$\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty$$

- For example, the sequence $x[n] = u[n]$ is not absolutely summable, and therefore, the Fourier transform does not converge absolutely. However, $\{r^{-n}u[n], n \in \mathbb{Z}\}$ is absolutely summable if $r > 1$. This means that the z-transform for the unit step exists with a region of convergence $|z| > 1$.

Region of convergence

- If some value of z, say, $z = z_1$, is in the $\mathrm{ROC}(x)$, then all values of z on the circle defined by $|z| = |z_1|$ will also be in the $\mathrm{ROC}(x)$.
- As one consequence of this, the region of convergence will consist of a ring in the z-plane centered about the origin.
- Its outer boundary will be a circle (or $\mathrm{ROC}(x)$ may extend outward to infinity), and its inner boundary will be a circle (or it may extend inward to include the origin).
- If $\mathrm{ROC}(x)$ includes the unit circle, this of course implies convergence of the z-transform for $|z| = 1$, or equivalently, the Fourier transform of the sequence converges.
- Conversely, if the $\mathrm{ROC}(x)$ does not include the unit circle, the Fourier transform does not converge absolutely.

The z-transform is not always defined

- Uniform convergence of the z-transform requires absolute summability of the exponentially weighted sequence.
- Neither of the sequences

$$x_1[n] = \frac{\sin \omega_c n}{\pi n}, \qquad n \in \mathbb{Z}$$
$$x_2[n] = \cos \omega_0 n, \qquad n \in \mathbb{Z}$$

  is absolutely summable. Furthermore, neither of these sequences multiplied by r^{-n} would be absolutely summable for any value of r.
- Thus, these sequences do not have a z-transform that converges absolutely for any z.
- However, even though sequences such as $x_1[n]$ is not absolutely summable, $x_1 \in \ell^2(\mathbb{Z})$, and the Fourier transform converges in the mean-square sense to a discontinuous periodic function.

Rational models

- Among the most important and useful z-transforms are those for which $X(z)$ is a rational function inside the region of convergence, i.e.,

$$X(z) = \frac{P(z)}{Q(z)}$$

  where $P(z)$ and $Q(z)$ are polynomials in z.
- The values of z for which $X(z) = 0$ are called the zeros of $X(z)$, and the values of z for which $X(z)$ is infinite are referred to as the poles of $X(z)$.
- The poles of $X(z)$ for finite values of z are the roots of the denominator polynomial. In addition, poles may occur at $z = 0$ or $z = \infty$.
- For rational z-transforms, a number of important relationships exist between the locations of poles of $X(z)$ and the region of convergence of the z-transform. (wait...)

Right-sided exponential sequence

- Consider the signal $x[n] = a^n u[n]$. Because it is nonzero only for $n \geq 0$, this is an example of a right-sided sequence.

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

- For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty.$$

Thus, the region of convergence is the range of values of z for which $|az^{-1}| < 1$ or, equivalently, $|z| > |a|$.

- Inside the region of convergence, the infinite series converges to

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a|.$$

Right-sided exponential sequence

- The z-transform has a region of convergence for any finite value of $|a|$. The Fourier transform of $x[n]$, on the other hand, converges only if $|a| < 1$.
- The infinite sum is equal to a rational function of z inside the region of convergence ; for most purposes, this rational function is a much more convenient representation than the infinite sum.
- We will see that any sequence that can be represented as a sum of exponentials can equivalently be represented by a rational z-transform.
- Such a z-transform is determined to within a constant multiplier by its zeros and its poles.

Right-sided exponential sequence

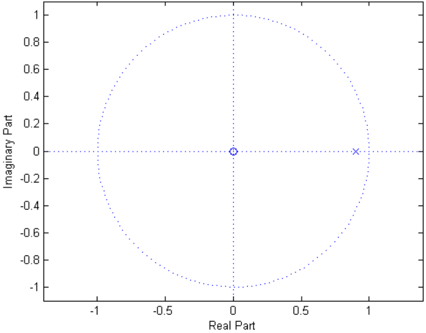

FIGURE – For right-sided exponential sequence, there is one zero, at $z = 0$, and one pole, at $z = a$.

Left-sided exponential sequence

- Now let $x[n] = -a^n u[-n-1]$. Since the sequence is nonzero only for $n \leq -1$, this is a left-sided sequence. Then

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

- If $|a^{-1}z| < 1$ or, equivalently, $|z| < |a|$, the sum converges, and

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \ |z| < |a|.$$

Always specify the ROC !

- We see that the right-sided and left-sided exponential sequences are different and, therefore, the infinite sums are different ;
- however, the algebraic expressions for $X(z)$ and the corresponding pole-zero plots are identical !
- The z-transforms differ only in the region of convergence.
- This emphasizes the need for specifying both the algebraic expression and the region of convergence for the z-transform of a given sequence.

Some common z-transform

1. $\delta[n]$	1	$\mathbb{C}$
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	$\mathbb{C} \setminus \{0\}$ if $m > 0$ or $\mathbb{C} \setminus \infty$ if $m < 0$
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $[\cos \omega_0 n]u[n]$	$\frac{1-[\cos \omega_0]z^{-1}}{1-[2\cos \omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $[\sin \omega_0 n]u[n]$	$\frac{[\sin \omega_0]z^{-1}}{1-[2\cos \omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\frac{1-[r\cos \omega_0]z^{-1}}{1-[2r\cos \omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\frac{[r\sin \omega_0]z^{-1}}{1-[2r\cos \omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$

Properties of the z-transform

- Property 1 : The ROC is a ring or disk in the z-plane centered at the origin ; i.e.,
$$0 \leq r_R < |z| < r_L \leq \infty.$$
- Property 2 : The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.
- Property 3 : The ROC cannot contain any poles.
- Property 4 : If $x[n]$ is a finite-duration sequence, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

Properties of the z-transform

- Property 5 : If $x[n]$ is a right-sided sequence, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the outermost (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.
- Property 6 : If $x[n]$ is a left-sided sequence, i.e.. a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the innermost (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.
- Property 7 : A two-sided sequence is an infinite-duration sequence that is neither right sided nor left sided. if $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.
- Property 8 : The ROC must be a connected region.

Laurent Decomposition

- If a function is analytic over an annular domain $r < |z| < R$, then the Laurent series expansion shows that $X(z)$ can be decomposed as

$$X(z) = \sum_{n=-\infty}^{\infty} a_n z^{-n} \quad a_n = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} X(z) z^{n-1} \mathrm{d}\xi$$

  where γ is any rectifiable path containing no self-intersections in the annulus.
- The series is absolutely converging in any compact domain of the $\{r < |z| < R\}$

Laurent series

- A Laurent series (therefore the z-transform) is analytic over $0 \leq r < |z| < R \leq \infty$; hence, the z-transform and all its derivatives must be infinitely differentiable functions of z within the region of convergence.
- This implies that if the region of convergence includes the unit circle, then the Fourier transform and all its derivatives with respect to ω must be continuous functions of ω.
- Also, the sequence must be absolutely summable, i.e., a stable sequence.

Inverse z-transform

- Beware !... Always specify the ROC on which you are willing to invert... there are has many possible inverse than the number of possible ROCs !
- When $X(z) = P(z)/Q(z)$ is a rational the easiest way to proceed is to use partial fraction expansion. Assuming that $\deg P < \deg Q = N$, assuming simple poles write

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$

where $N = \deg Q$ and $p_1, \ldots, p_N$ are the poles (the zeros of $Q(z)$) and $A_k = \lim_{z \to p_k} (z - p_k) X(z)$ is the residue at pole p_k. We assume that $|p_1| < |p_2| < \cdots < |p_N|$

- Assume first that we are willing to invert the z-transform on $|z| > R$, where $R > \max_{1 \leq k \leq N}(|p_k|)$. Then

$$x[n] = \sum_{k=1}^{N} p_k^n u[n]$$

Inverse z-transform

- Assume now that we are inverting on $|p_\ell| < |z| < |p_{\ell+1}|$ for some $\ell \in \{1, \ldots, N-1\}$. We need to consider the two possible expansions for each factor in the decomposition

$$\frac{1}{1 - p_k z^{-1}} = \sum_{n=0}^{\infty} p_k^n z^{-n} \qquad |p_k| < |z| \quad \text{causal}$$

$$\frac{1}{1 - p_k z^{-1}} = - \sum_{n=-\infty}^{-1} p_k^n z^{-n} \qquad |p_k| > |z| \quad \text{anti-causal} .$$

- Hence, the inverse z-transform is

$$x[n] = \sum_{k=1}^{\ell} p_k^n u[n] - \sum_{k=\ell+1}^{N} p_k^n u[-n-1]$$

Other techniques : power series expansion

The function $X(z) = \log(1 + az^{-1})$ is analytic for $|z| > |a|$. A power series expansion show that, converging on the $|z| > |a|$ shows that

$$\log(1 + az^{-1}) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{a^n}{n} z^{-n} , \quad |z| > |a|$$

showing that

$$x[n] = (-1)^{n+1} \frac{a^n}{n} u[n-1]$$

Inverse z-transform : contour integration

- If $X(z)$ is analytic on $0 \leq r < |z| < R \leq \infty$, then for any closed rectifiable curve γ inside the ROC and any $n \in \mathbb{N}$

$$x[n] = \frac{1}{2\pi\mathrm{i}} \oint_{\gamma} X(z) z^{n-1} \mathrm{d}z$$

- Any contour give the same integral
- Best to use with the Cauchy residue theorem.

Residue theorem

Suppose U is a simply connected open subset of the complex plane, and $a_1, \ldots, a_n$ are finitely many points of U and f is a function which is defined and holomorphic on $U \setminus \{a_1, \ldots, a_n\}$. If γ is a closed rectifiable curve in U which does not meet any of the a_k, then

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi \mathrm{i} \sum_{k=1}^{n} \mathrm{I}(\gamma, a_k)\,\mathrm{Res}(f, a_k).$$

If γ is a positively oriented simple closed curve, $I(\gamma, a_k) = 1$ if a_k is in the interior of γ, and 0 if not, so

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi \mathrm{i} \sum \mathrm{Res}(f, a_k)$$

with the sum over those k for which a_k is inside γ.

Residue theorem

At a simple pole c, the residue of f is given by :

$$\mathrm{Res}(f, c) = \lim_{z \to c} (z - c) f(z).$$

More generally, if c is a pole of order n, then the residue of f around $z = c$ can be found by the formula :

$$\mathrm{Res}(f, c) = \frac{1}{(n-1)!} \lim_{z \to c} \frac{d^{n-1}}{dz^{n-1}} \left((z - c)^n f(z) \right).$$

Linearity

$$y[n] = a_1 x_1[n] + a_2 x_2[n] \overset{Z}{\leftrightarrow} a_1 X_1(z) + a_2 X_2(z) \ ,$$
$$\mathrm{ROC}(y) \supset \mathrm{ROC}(x_1) \cap \mathrm{ROC}(x_2) \ .$$

- For sequences with rational z-transforms, if the poles of $a_1 X_1(z) + a_2 X_2(z)$ consist of all the poles of $X_1(z)$ and $X_2(z)$ (i.e., if there is no pole-zero cancellation), then the region of convergence will be exactly equal to the overlap of the individual regions of convergence.
- If the linear combination is such that some zeros are introduced that cancel poles, then the region of convergence may be larger.

Time-shifting and Differentiation

$$x[n-n_0] \overset{Z}{\leftrightarrow} z^{-n_0} X(z) \quad \mathrm{ROC}(\mathsf{S}_{n_0}\, x) = \mathrm{ROC}(x)$$

(except for the possible addition or deletion of $z = 0$ or $z = \infty$).

The ROC can be changed, since the factor z^{-n_0} can alter the number of poles at $z = 0$ or $z = \infty$.

$$y[n] = nx[n] \overset{Z}{\leftrightarrow} -z\frac{\mathrm{d}X(z)}{\mathrm{d}z} \ , \quad \mathrm{ROC}(y) = \mathrm{ROC}(x) \ .$$

Convolution of sequences

$$y[n] = x_1 * x_2[n] \overset{Z}{\leftrightarrow} X_1(z)X_2(z) \; , \mathrm{ROC}(y) \supset \mathrm{ROC}(x_1) \cap \mathrm{ROC}(x_2) \; .$$

- If $\mathrm{ROC}(x_1) \cap \mathrm{ROC}(x_2) = \emptyset$, then the convolution on the RHS is not defined.
- the convolution property plays a particularly important role in the analysis of LTI systems.
- Specifically, as a consequence of this property, the z-transform of the output of an LTI system is the product of the z-transform of the input and the z-transform of the system impulse response called the transfer function.

Evaluating a convolution with the z-transform

Let $x_1[n] = a^n u[n]$ and $x_2[n] = u[n]$. The corresponding z-transforms are

$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}}, \ |z| > |a|,$$

and

$$X_2(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}, \ |z| > 1.$$

If $|a| < 1$, the z-transform of the convolution of $x_1[n]$ with $x_{2[n]}$ is then

$$Y(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})} = \frac{z^2}{(z - a)(z - 1)}, \ |z| > 1.$$

The sequence $y[n]$ can be obtained by determining the inverse z-transform. Expanding $Y(z)$ on $|z| > 1$ in a partial fraction expansion, we get

$$Y(z) = \frac{1}{1 - a}\left\{\frac{1}{1 - z^{-1}} - \frac{a}{1 - az^{-1}}\right\}, \quad |z| > 1,$$

showing that

$$y[n] = \frac{1}{1 - a}\left(u[n] - a^{n+1}u[n]\right).$$

constant-coefficient difference equation

- Consider the class of systems whose input and output satisfy a linear constant-coefficient difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

- Applying the z-transform to both sides and using the linearity property and the time-shifting property, we obtain

$$\sum_{k=0}^{N} a_k z^{-k} \mathrm{Y}(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) \ ,$$

or equivalently,

$$\left(\sum_{k=0}^{N} a_k z^{-k} \right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k} \right) X(z) \ .$$

System function

- The system function has the algebraic form

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- $H(z)$ is a ratio of polynomials in z^{-1}, It is often convenient to express $H(z)$ in factored form as

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})} .$$

- Each of the factors $(1 - c_k z^{-1})$ in the numerator contributes a zero at $z = c_k$ and a pole at $z = 0$.
- Similarly, each of the factors $(1 - d_k z^{-1})$ in the denominator contributes a zero at $z = 0$ and a pole at $z = d_k$.

Stability and Causality

- To transform the difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad \text{into} \quad Y(z) = \frac{B(z)}{A(z)} X(z) = H(z)X(z)$$

  we made no further assumption about stability or causality.
- Correspondingly, from the difference equation, we can obtain the algebraic expression for the system function, but not the region of convergence. The region of convergence of $H(z)$ is not determined from the derivation leading , since all that is required for $Y(z) = H(z)X(z)$ is that $X(z)$ and $Y(z)$ have overlapping regions of convergence !
- Warning ! the difference equation does not uniquely specify the impulse response of a linear time-invariant system !

Stability and Causality

- For the system function $H(z) = B(z)/A(z)$, there are a number of choices for the region of convergence.
- For a given ratio of polynomials, each possible choice for the region of convergence will lead to a different impulse response, but they will all correspond to the same difference equation.
- However, if we assume that the system is causal, it follows that $h[n]$ must be a right-sided sequence, and therefore, the region of convergence of $H(z)$ must be outside the outermost pole.
- Alternatively, if we assume that the system is stable, then, the impulse response must be absolutely summable, i.e.,

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty.$$

Determining the ROC

- Consider the LTI system with input and output related through the difference equation

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n].$$

- The system function $H(z)$ is given by

$$H(z) = \frac{1}{1 - \frac{5}{2}z^{-1} + z^{-2}} = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}.$$

- The system has a zero (of order 2) at 0 and two poles at $1/2$ and 2. There are three possible choices for the ROC.
  1. If the system is assumed to be causal, then the ROC is outside the outermost pole, i.e., $|z| > 2$. In this case the system will not be stable, since the ROC does not include the unit circle.
  2. If we assume that the system is stable, then the ROC will be $1/2 < |z| < 2$. The system will not be causal.
  3. For the third possible choice of ROC, $|z| < 1/2$, the system will be neither stable nor causal.

Stable and causal LTI

- Causality and stability are not necessarily compatible requirements.
- In order for a linear time-invariant system whose input and output satisfy a difference equation to be both causal and stable, the ROC of the corresponding system function must be outside the outermost pole and include the unit circle. Clearly, this requires that all the poles of the system function be inside the unit circle.

A first order IIR filter

- Consider a system whose input and output satisfy the difference equation

$$y[n] - ay[n-1] \; = x[n]$$

- The system function is

$$H(z) = \frac{1}{1 - az^{-1}}.$$

  and has a single pole at a.
- There are two possible ROC $|z| > |a|$ and $|z| < |a|$.
  1. If the system is assumed to be causal, then the ROC should be outside the outermost pole $|z| > |a|$.
  2. if the system is assumed to be stable, the unit circle should be included in $|z| > |a|$, which implies that $|a| < 1$.
- For a causal and stable solution ($|a| < 1$), the associated impulse response is

$$h[n] = a^n u[n]$$

  The impulse response has infinitely many no zero terms ; this is an IIR LTI system.

Frequency response for rational filters

If a stable linear time-invariant system has a rational system function (i.e., if its input and output satisfy a difference equation), then its frequency response (the system function evaluated on the unit circle) has the form

$$H(\mathrm{e}^{\mathrm{i}\omega}) = \frac{\sum_{k=0}^{M} b_k \mathrm{e}^{-\mathrm{i}\omega k}}{\sum_{k=0}^{N} a_k \mathrm{e}^{-\mathrm{i}\omega k}}$$

That is, $H(\mathrm{e}^{\mathrm{i}\omega})$ is a ratio of polynomials in the variable $\mathrm{e}^{-\mathrm{i}\omega}$.