Exercice 1. Given f and g in $L_2(\mathbb{R})$, show that

(i) $f * g(t) = \overline{\mathcal{F}}\left(\hat{f} \cdot \hat{g}\right)(t)$ for all t in $\mathbb{R}$.

(ii) $\widehat{f \cdot g} = \hat{f} * \hat{g}$ for all t in $\mathbb{R}$.

(iii) Compute $f * f$ when $f(t) = \sin(2\pi\lambda t)/\pi t$.

(iv) Compute $g * g$ when $g(x) = \mathrm{e}^{-\pi x^2}$.

We establish the result using the density of $\mathcal{S}(\mathbb{R})$ in $L_2(\mathbb{R})$. Thus let f_n and g_n be two sequences in $\mathcal{S}(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|f - f_n\|_2 = 0 \quad \text{and} \quad \lim_{n\to\infty} \|g - g_n\|_2 = 0.$$

We see that $f_n * g_n = \overline{\mathcal{F}}\left(\hat{f}_n \cdot \hat{g}_n\right)$ by taking the inverse Fourier transform of both sides. On the other hand,

$$\begin{aligned}\|\hat{f} \cdot \hat{g} - \hat{f}_n \cdot \hat{g}_n\|_1 &\leq \|\hat{f} - \hat{f}_n\|_2 \|\hat{g}\|_2 + \|\hat{f}_n\|_2 \|\hat{g} - \hat{g}_n\|_2 \\ &= \|f - h\|_2 \|g\|_2 + \|f_n\|_2 \|g - g_n\|_2\end{aligned}$$

and hence $\lim_{n\to\infty} \|\hat{f} \cdot \hat{g} - \hat{f}_n \cdot \hat{g}_n\|_1 = 0$. By applying the Riemann-Lebesgue theorem to the inverse Fourier transform, we see that $\overline{\mathcal{F}}\left(\hat{f}_n \cdot \hat{g}_n\right)$ tends to $\overline{\mathcal{F}}\left(\hat{f} \cdot \hat{g}\right)$ uniformly on $\mathbb{R}$. The last step is to determine the limit of $f_n * g_n$. We have

$$\|f * g - f_n * g_n\|_\infty \leq \|f - f_n\|_2 \|g\|_2 + \|f_n\|_2 \|g - g_n\|_2;$$

thus $f_n * g_n$ converges uniformly to $f * g$, which is continuous.

Exercice 2. Consider the generalized filter determined by

$$g'' + 2ag' + bg = f$$

given $a,\ b \in \mathbb{R}$.

(a) Determine the regions of the $(a,\ b)$-plane where the poles of Q are not on the imaginary axis.

(b) Determine the regions corresponding to a causal filter.

(c) Show that the filter is unstable if $b = 0$.