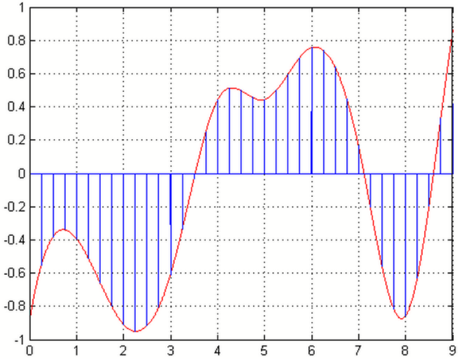
MAP 555 : Discrete Fourier Transform...

25 Septembre 2015

Today

1. Sampling
2. Another approach to sampling
3. Frequency resolution and windowing
4. The Discrete Fourier Transform : definitions and illustration
5. Discrete Fourier Series of a periodic sequence
6. Properties of the DFT of a finite sequence
7. The Fast Fourier Transform

Periodic sampling ?



T the sampling period. $f[n] = f(nT)$ are the samples of the signals. The sampling frequency $1/T$.

Band Limited functions

Definition (Band Limited function)

A function $f \in L_2(\mathbb{R})$ is said to be band limited if there exists $B < \infty$ such that : $\mathcal{F}f(\xi) = 0$ pour $\xi \notin [-B, +B]$. We denote $\mathrm{BL}(B)$ the subspace of $f \in L_2(\mathbb{R})$ of functions satisfying $\mathcal{F}f(\xi) = 0$ for (almost) all $\xi \notin [-B, +B]$.

Periodizing the Fourier transform

- Let $T \leq 1/(2B)$ be the sampling period and $1/T$ be the sampling frequency. Consider the periodic function $\xi \to \mathcal{F}f(\xi)$ given by :

$$F_T(\xi) = \sum_{n\in\mathbb{Z}} [\mathcal{F}f]\left(\xi - \frac{n}{T}\right) \ .$$

- The function $F_T \in L_2[-1/2T, 1/2T]$ can be expanded as a series function

$$F_T(\xi) = \sum_{n\in\mathbb{Z}} c_n(F_T)\mathrm{e}^{-\mathrm{i}2\pi\xi nT},$$

where $\{c_k(F_T)\}$, the Fourier coefficients are given by

$$c_k(F_T) = T\int_{-1/(2T)}^{1/(2T)} F_T(\xi)\mathrm{e}^{+\mathrm{i}2\pi\xi kT}\mathrm{d}\xi \ .$$

Periodizing the Fourier transform

The identity

$$F_T(\xi) = \sum_{n \in \mathbb{Z}} c_n(F_T) \mathrm{e}^{-\mathrm{i}2\pi\xi nT},$$

should be understood as a convergence of the partial sums

$$F_{N,T}(\xi) = \sum_{k=-N}^{N} c_N(F_T) \mathrm{e}^{-\mathrm{i}2\pi\xi nT} ,$$

toward F_T in the topology induced by the norm $\| \cdot \|_2$ in $L_2([-1/(2T), 1/(2T)])$:

$$\lim_{N \to \infty} \int_{-1/(2T)}^{1/(2T)} |F_T(\xi) - F_{N,T}(\xi)|^2 \mathrm{d}\xi = 0 \, .$$

.

Periodizing the Fourier transform

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The Parseval identity implies that $\sum_{n\in\mathbb{Z}} |c_n(F_T)|^2 < \infty$.

A key result

- Since for all $T \leq 1/(2B)$, $[-B, +B] \subset [-1/(2T), 1/(2T)]$, then for all $\xi \in [-1/(2T), 1/(2T)]$,

$$F_T(\xi) = \mathcal{F}f(\xi)$$

- Therefore, the Fourier coefficients $c_k(F_T)$ are given by :

$$c_k(F_T) = T \int_{-B}^{B} \mathcal{F}f(\xi) \mathrm{e}^{+\mathrm{i}2\pi\xi kT} \mathrm{d}\xi = Tf(kT) \; .$$

The Poisson Formula : Key result

If $f \in \mathrm{BL}(B)$ then the discrete-time Fourier transform (DTFT) of the sequence of samples $\{f(nT), n \in \mathbb{Z}\}$

$$T \sum_{n \in \mathbb{Z}} f(nT) \mathrm{e}^{-\mathrm{i}2\pi\xi nT}$$

is equal to the Fourier transform of the function f periodized with a period $1/T$, a relation called the Poisson formula

$$\sum_{n \in \mathbb{Z}} \mathcal{F}f\left(\xi - \frac{n}{T}\right) = T \sum_{n \in \mathbb{Z}} f(nT) \mathrm{e}^{-\mathrm{i}2\pi\xi nT} \ ,$$

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An amazing result : the DTFT is computable only from the knowledge of the samples $\{f(nT), n \in \mathbb{Z}\}$, whereas the knowledge of $\mathcal{F}f$ requires to evaluate the function at all time points

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The DTFT is periodic with period $1/T$. Because of this periodicity, one may restrict the frequency interval to just one-period, $[-1/2T, 1/2T]$

Interpolation formula

Theorem (Nyquist theorem)

If $f \in \mathrm{BL}(B)$ and $T \leq 1/2B$ then

$$f(x) =_{L_2(\mathbb{R})} \sum_{n\in\mathbb{Z}} f(nT) s_T(x - nT) ,$$

where s_T, is the cardinal-sine function

$$s_T(x) = \frac{\sin(\pi x/T)}{\pi x/T} .$$

The condition $T \leq 1/2B$ is the Nyquist condition.

Why ?

- Sampling theorem can be derived from another perspective, which is in some sense broader.
- It acknowledges the fact that a function can be seen as a superposition if complex exponential (think ! this is the meaning of the Fourier synthesis formula...)
- The approach starts by deriving the sampling theorem first for complex exponential, and then to extends the result to a (much) wider class of signals.

Problem

- Consider the function $s_{\xi_0}(x) = \mathrm{e}^{2\mathrm{i}\pi\xi_0 x}$, $\xi_0 \in \mathbb{R}$. This function is neither in $L_1(\mathbb{R})$ or in $L_2(\mathbb{R})$, and therefore does not fit into the framework presented earlier.
- A natural question is the following : under which conditions on the sampling period T may we retrieve $s(x)$ from the its samples $\{s(nT), n \in \mathbb{Z}\}$.
- Since $s_{\xi_0}(nT) = \mathrm{e}^{2\mathrm{i}\pi\xi_0 nT}$, we have

$$s_{\xi_0}(nT) = s_{\xi_0+k/T}(nT) , \quad \text{for all } k, n \in \mathbb{Z}$$

- If $\xi_0 \in \left]-1/2T, 1/2T\right[$ we can retrieve s_{ξ_0} from its samples without error !

A sampling theorem for complex exponential

Theorem

For all $x \in \mathbb{R}$ *and* $\xi_0 \in \left]-1/2T, 1/2T\right[$,

$$\mathrm{e}^{2\mathrm{i}\pi\xi_0 x} = \sum_{n\in\mathbb{Z}} \mathrm{e}^{2\mathrm{i}\pi\xi_0 nT} s_T(x - nT) \quad s_T(x) = \frac{\sin\left(\pi x/T\right)}{\pi x/T}$$

where the series in the RHS converges uniformly on $[-B, B] \subset \left]-1/2T, 1/2T\right[$

Sampling a finite sum of sinewaves

- If $x \mapsto f(x)$ is a finite superposition of complex sinewaves,

$$f(x) = \sum_{k=1}^{M} \gamma_k \mathrm{e}^{2\mathrm{i}\nu_k x}$$

where $\{\gamma_k\}_{k=1}^{M} \in \mathbb{C}$ and $\{\nu_k\}_{k=1}^{M} \subset]-1/2T, 1/2T[$, then the Nyquist formula holds

$$f(x) = \sum_{k=-\infty}^{\infty} f(kT) s_T(x - kT)$$

Decomposable signals

- This can be extended to any function f which can be written as

$$f(x) = \int_{-B}^{B} \mathrm{e}^{2\mathrm{i}\pi\xi x} \mu(\mathrm{d}x)$$

where μ is a signed measure on $[-B, B]$ as soon as $B < 1/2T$.

From theory to practice

- Even though $T\sum_{k=-\infty}^{\infty} f(kT)\mathrm{e}^{-\mathrm{i}2\pi\xi kT}$ is the closest approximation of $\mathcal{F}f$ that we can achieve digitally, it is still not computable because generally it requires an infinite number of samples.
- To make it computable, we must make a second approximation, keeping only a finite number of samples $\{f(kT), 0 \leq k \leq L-1\}$ (assuming a causal signal).
- In terms of the times samples, the time windowed version is given by

$$T\sum_{k=0}^{L-1} f(kT)\mathrm{e}^{-\mathrm{i}2\pi\xi kT}$$

Normalized frequency

- It is customary to define the normalized frequency as $\lambda = \xi/\xi_s$, where $\xi_s = 1/T$ is the sampling frequency.
- Expressed in terms of the normalized frequency λ, the discrete time Fourier transform is given by

$$\lambda \mapsto T \sum_{k=-\infty}^{\infty} f(kT) \mathrm{e}^{-\mathrm{i}2\pi k\lambda}$$

  which is periodic with period 1. It is customary to represent this function on the interval $[-1/2, 1/2]$.
- It is sometimes more appropriate to work with the normalized pulsation $\omega = 2\pi\lambda$.
- In the sequel we set conventionally $T = 1$

From theory to practice

- The windowed signal may be thought of as an infinite sequence which is zero outside the range of the window and agrees with the original one within the window.
- To express this mathematically, we define the rectangular window of length L :

$$w(n) = \begin{cases} 1 & n \in \{0, \dots, L-1\} \\ 0 & \text{otherwise} \end{cases}$$

- The Discrete time Fourier transform of the windowed signal $\{w(n)f(n), n = 0, \dots, L-1\}$ is given by

$$F_L(\omega) = \sum_{k=0}^{L-1} w(k)f(k)\mathrm{e}^{-\mathrm{i}\omega k} \,.$$

Windowing effect

- In general, the windowing process has two major effects :
  1. First, it reduces the frequency resolution of the computed spectrum, in the sense that the smallest resolvable frequency difference is limited by the length of the data record. This is linked with the uncertainty principle.
  2. Second, it introduces spurious high-frequency components into the spectrum, which are caused by the sharp clipping of the signal at the left and right ends of the rectangular window. This effect is referred to as frequency leakage.
- Both effects can be understood by deriving the precise effect of windowing on the Discrete Tile Fourier Transform.

Convolution

- Recall that $f(n)$ is the n-th Fourier coefficient of the π-periodic function $F(\omega)$
$$f(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega')\mathrm{e}^{+\mathrm{i}\omega' n}\mathrm{d}\omega'$$
- Evaluate the DTFT of the windowed sequence $\{w(n)x(n), n \in \mathbb{N}\}$:

$$\sum_{n=-\infty}^{\infty} x(n)w(n)\mathrm{e}^{-\mathrm{i}\omega n} = (2\pi)^{-1}\sum_{n=-\infty}^{\infty} w(n)\int_{-\pi}^{\pi} X(\omega')\mathrm{e}^{-\mathrm{i}(\omega-\omega')n}\mathrm{d}\omega'$$
$$= (2\pi)^{-1}\int_{-\pi}^{\pi} X(\omega')W(\omega-\omega')\mathrm{d}\omega'$$

where $W(\omega)$ is the DTFT of the window

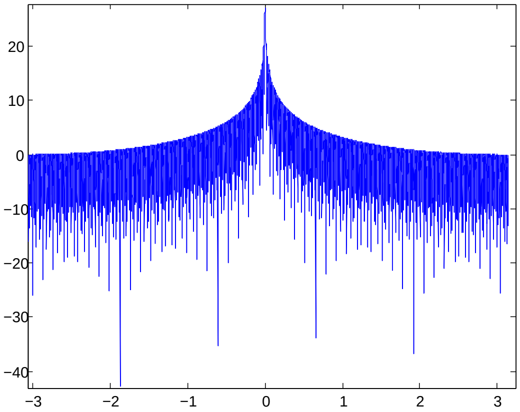
$$W(\omega) = \sum_{k=0}^{L-1} w(n)\mathrm{e}^{-\mathrm{i}\omega n} .$$

Fourier transform of a rectangular window

- The DTFT of the recatngular window is given by

$$\sum_{k=0}^{L-1} \mathrm{e}^{-\mathrm{i}\omega k} = \frac{\sin(\omega L/2)}{\sin(\omega/2)} \mathrm{e}^{-\mathrm{i}\omega(L-1)/2}$$

- The function has a maximum at 0 which is equal to L and is π periodic.


20
10
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3

Mainlobes and sidelobes

- It consists of a main lobe of height L and base width $4\pi/L$ centered at $\omega = 0$, and several smaller sidelobes.
- The sidelobes are between the zeros of $W(\omega)$, which are the zeros of the numerator $\sin(\omega L/2) = 0$, that is, $\omega = 2\pi k/L$, for $k = \pm 1, \pm 2, \ \ldots$ (with $k = 0$ excluded).
- The mainlobe peak at 0 dominates the spectrum, because $w(n)$ is essentially a low-pass signal, except when it cuts off at its endpoints.
- The higher frequency components that have 'leaked" away from 0 and lie under the sidelobes represent the sharp transitions of $w(n)$ at the endpoints.

Mainlobes and sidelobes

- For simplicity, we define the width of the mainlobe to be half the base width, that is, in units of radians per sample :

$$\Delta\omega_w = \frac{2\pi}{L} \quad \text{(rectangular window width)}$$

- In units of Hz, it is defined through $\Delta\omega_w = 2\pi\Delta\xi_w/\xi_s$:

$$\Delta\xi_w = \frac{\xi_s}{L} = \frac{1}{LT} = \frac{1}{T_L}$$

- The mainlobe width $\Delta\xi_w$ determines the frequency resolution limits of the windowed spectrum.
  - As L increases, the height of the mainlobe increases and its width becomes narrower, getting more concentrated around 0.
  - However, the height of the sidelobes also increases, but relative to the mainlobe height, it remains approximately the same and about 13 dB down.

Better analysis windows

- Windows are commonly used for spectral analysis. They have the desirable properties that
  - their Fourier transforms is concentrated around $\omega = 0$,
  - they have simple functional forms that allow them to be computed esasily.
- Examples
  1. Bartlett $w(n) = (2n/L)\mathbb{1}_{\{n \leq L/2\}} + (2 - 2n/L)\mathbb{1}_{\{n > L/2\}}$, for $n = 0, \dots, L-1$.
  2. Hanning $w(n) = 0.5 - 0.5\cos(2\pi n/L)$, for $n = 0, \dots, L-1$.
  3. Hamming $w(n) = 0.54 - 0.46\cos(2\pi n/L)$, for $n = 0, \dots, L-1$.
  4. Blackman $w(n) = 0.42 - 0.5\cos(2\pi n/L) + 0.08\cos(4\pi n/L)$, for $n = 0, \dots, L-1$.

Analysis windows

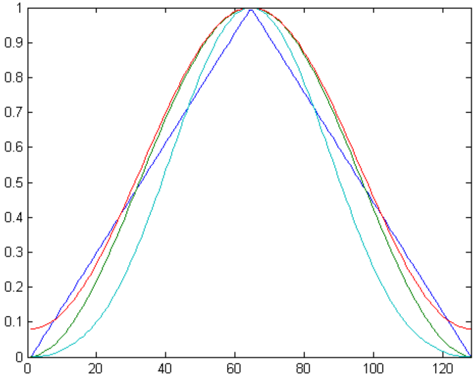


FIGURE – Bartlett, Hanning, Hamming and Blackman windows

Discrete-time Fourier transform of analysis windows

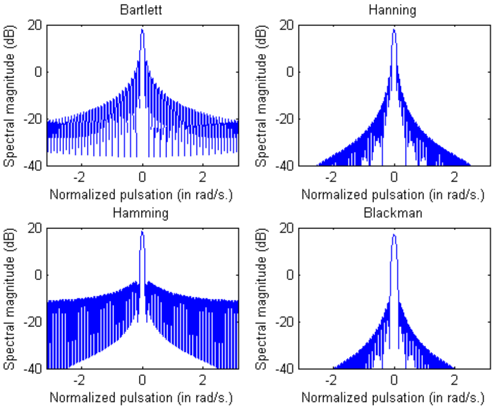


FIGURE – Bartlett, Hanning, Hamming and Blackman windows

Definition

- The N-point Discrete Fourier Transform of a length-L signal is defined to be the DTFT evaluated at N equally spaced frequencies over the full Nyquist interval, $0 \leq \omega \leq 2\pi$. These DFT frequencies are defined in radians per sample as follows :

$$\omega_k = \frac{2\pi k}{N} \quad k = 0, 1, \ \ldots, \ N-1$$

or, in Hz

$$h \ = \frac{kf_s}{N} \quad k = 0, 1, \ \ldots, \ N-1$$

- Thus, the N-point DFT will be, for $k = 0, 1, \ \ldots \ , \ N-1$:

$$F(\omega_k) = \sum_{n=0}^{L-1} f(n) \mathrm{e}^{-\mathrm{i}\omega_k n}$$

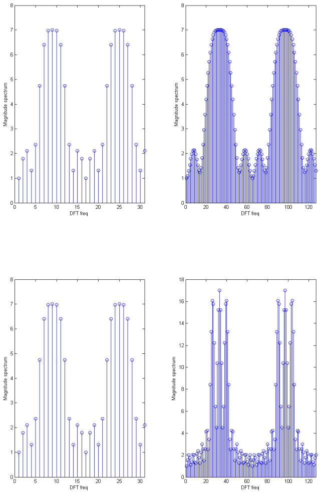
zero-padding

- In principle, the two lengths L and N can be specified independently of each other : L is the number of times samples in the data record and can even be infinite ; N is the number of frequencies at which we choose to evaluate the DTFT.
- Most discussions of the DFT assume that $L = N$. The reason for this will discussed later.
- If $L < N$, we can pad $N - L$ zeros at the end of the data record to make it of length N.
- If $L > N$, we may reduce the data record to length N by wrapping it modulo-N

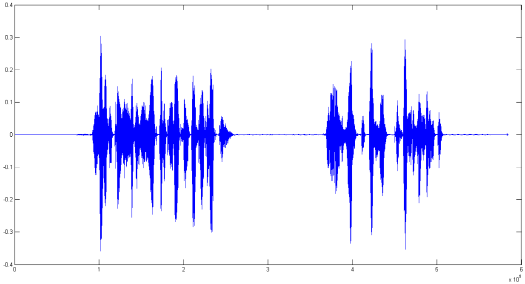
Physical versus computation resolution

- The bin width $2\pi/N$ represents (in normalized frequency) the spacing between the DFT frequencies at which the DFT is computed.
- It should not be confused with the frequency resolution width which is linked with the analysis window (which is roughly proportional to the inverse of the window length $2\pi/L$), which refers to the minimum resolvable frequency separation between sinusoidal components.
- To avoid confusion, we will refer to $2\pi/L$ as the physical resolution and $2\pi/N$ as the computational resolution.

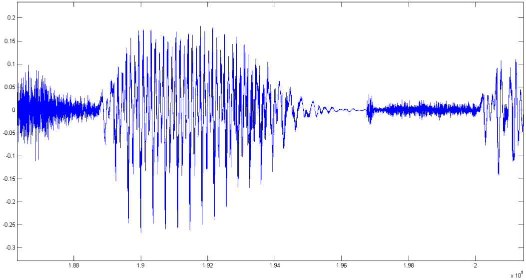
Physical / computational resolution



Speech signal



Speech signal : extract



Speech signal : extract

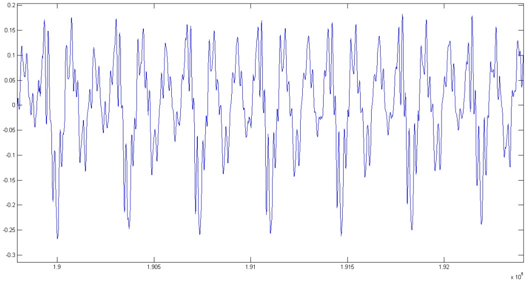


FIGURE – Sampling frequency : 48kHz ; Sampling period : 20.8 μsec ; period : 371 samples ; pitch= 129 Hz

Speech signal : windowing

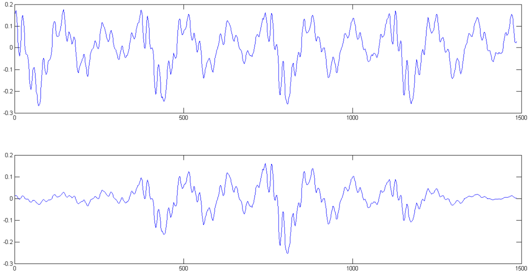


FIGURE – Sampling frequency : 48kHz ; Sampling period : 20.8 μsec ; period : 371 samples ; pitch= 129 Hz

Speech signal : physical resolution

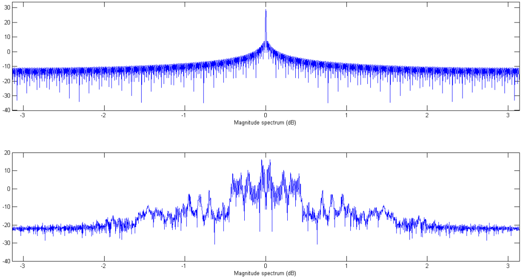


FIGURE – Sampling frequency : 48kHz ; Sampling period : 20.8 μsec ; period : 371 samples ; pitch= 129 Hz

Speech signal : physical resolution

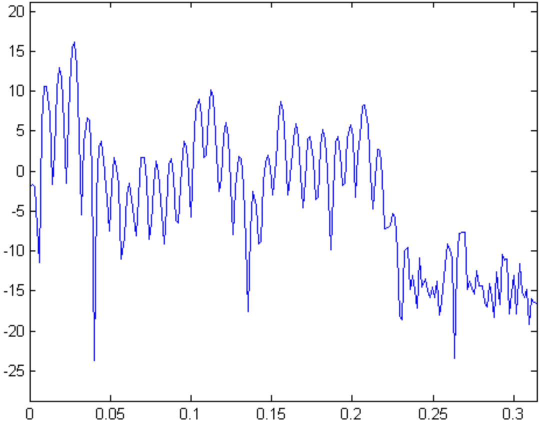


FIGURE – Sampling frequency : 48kHz ; Sampling period : 20.8 μsec ; period :

Discrete periodic sequences

- Consider a sequence $\{\tilde{x}(n), n \in \mathbb{Z}\}$ that is periodic with period N, i.e. for all n and r,
$$\tilde{x}(n) = \tilde{x}(n + rN)\ .$$
- As with continuous-time periodic signals, such a sequence can be represented by a Fourier series corresponding to a sum of harmonically related complex exponentials,
$$\tilde{x}(n) = \frac{1}{N} \sum_{k \in \mathbb{Z}} \tilde{X}(k) \mathrm{e}^{\mathrm{i}(2\pi/N)kn}$$
- Denoting $e_k : n \mapsto e_k(n) = \mathrm{e}^{\mathrm{i}(2\pi/N)nk}$ these complex exponentials, the periodic discrete time signal $\tilde{x}$ may be decomposed as
$$\tilde{x} = \frac{1}{N} \sum_{k \in \mathbb{Z}} \tilde{X}(k) e_k$$

Discrete periodic sequences

- The Fourier series representation of a continuous-time periodic signal generally requires infinitely many harmonically related complex exponentials.
- The Fourier series of any discrete time-signal with period N requires only N harmonically related complex exponentials since

$$e_{k+\ell N}(n) = e_k(n) \quad \text{for all } k, \ell \in \mathbb{Z} \,.$$

- Consequently the set of N periodic complex exponentials $\{e_0, e_1, \ldots, e_{N-1}\}$ defines all the distinct periodic exponentials and

$$\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) e_k(n) \,, \quad e_k : n \mapsto e_k(n) = \mathrm{e}^{\mathrm{i}(2\pi/N)nk}$$

Orthogonality

- To obtain the sequence of Fourier series coefficients $\tilde{X}(k)$ from the periodic sequence $\tilde{x}$, we exploit the orthogonality of the set of complex exponentials $\{e_j\}_{j=0}^{N-1}$ Note indeed that

$$\sum_{n=0}^{N-1} \tilde{x}(n)\mathrm{e}^{-\mathrm{i}(2\pi/N)rn} = \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k)\mathrm{e}^{\mathrm{i}(2\pi/N)(k-r)n}$$

- After exchanging the order of summations, we obtain

$$\sum_{n=0}^{N-1} \tilde{x}(n)\mathrm{e}^{-\mathrm{i}(2\pi/N)rn} = \sum_{k=0}^{N-1} \tilde{X}(k) \left[\frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{\mathrm{i}(2\pi/N)(k-r)n} \right]$$

Orthogonality

- The following identity expresses the orthogonality of the complex exponentials :

$$\frac{1}{N}\sum_{n=1}^{N-1} \mathrm{e}^{\mathrm{i}(2\pi/N)(k-r)n} = \begin{cases} 1, k-r = mN, & m \text{ an integer} \\ 0 & \text{otherwise} \end{cases}$$

- Conclusion

$$\sum_{n=0}^{N-1} \tilde{x}[n]\mathrm{e}^{-\mathrm{i}(2\pi/N)rn} = \tilde{X}[r].$$

DFT coefficients properties

- the sequence $\tilde{X}[k]$ is periodic with period N, i.e. for any integer k :

$$\begin{aligned}\tilde{X}[k+N] &= \sum_{n=0}^{N-1} \tilde{x}(n) \mathrm{e}^{-\mathrm{i}(2\pi/N)(k+N)n} \\ &= \left(\sum_{n=0}^{N-1} \tilde{x}(n) \mathrm{e}^{-\mathrm{i}(2\pi/N)kn} \right) \mathrm{e}^{-\mathrm{i}2\pi n} = \tilde{X}[k]\end{aligned}$$

- An advantage of interpreting the Fourier series coefficients $\tilde{X}[k]$ as a periodic sequence is that there is then a duality between the time and frequency domains for the Fourier series representation of periodic sequences.

Analysis-Synthesis

$$\tilde{X}(k) = \sum_{n=0}^{N-1} x(n) \mathrm{e}^{-\mathrm{i}(2\pi/N)kn} \quad \tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) \mathrm{e}^{\mathrm{i}(2\pi/N)kn}$$

are an analysis-synthesis pair and will be referred to as the discrete Fourier series (DFS) representation of a periodic sequence.

For convenience in notation, these equations are often written in terms of the complex quantity $W_N = \mathrm{e}^{-\mathrm{i}(2\pi/N)}$ and the DFS analysis-synthesis pair is expressed as follows :

Analysis equation : $\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$

Synthesis equation : $\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$

Linearity

Consider two periodic sequences $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$, both with period N, such that

$$\tilde{x}_1[n] \overset{\text{DFS}}{\leftrightarrow} \tilde{X}_1[k]$$
$$\tilde{x}_2[n] \overset{\text{DFS}}{\leftrightarrow} \tilde{X}_2[k]$$

Then

$$a\tilde{x}_1[n] + b\tilde{x}_2[n] \overset{\text{DFS}}{\leftrightarrow} a\tilde{X}_1[k] + b\tilde{X}_2[k].$$

Shift of a sequence

- If a periodic sequence $\tilde{x}[n]$ has Fourier coefficients $\tilde{X}[k]$, then $\tilde{x}[n-m]$ is a shifted version of $\tilde{x}[n]$, and

$$\tilde{x}[n-m] \overset{\text{DFS}}{\leftrightarrow} W_N^{km}\tilde{X}[k].$$

- Any shift that is greater than or equal to the period (i.e. $m \geq N$) cannot be distinguished in the time domain from a shorter shift m_1 such that $m = m_1 + m_2 N$, where m_1 and m_2 are integers and $0 \leq m_1 \leq N-1$.
- Because the sequence of Fourier series coefficients of a periodic sequence is a periodic sequence, a similar result applies to a shift in the Fourier coefficients by an integer ℓ. Specifically,

$$W_N^{-n\ell}\tilde{x}[n] \overset{\text{DFS}}{\leftrightarrow} \tilde{X}[k-\ell]$$

Periodic convolution

- Let $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ be two periodic sequences, each with period N and with discrete Fourier series coefficients denoted by $\tilde{X}_1[k]$ and $\tilde{X}_2[k]$, respectively.
- If we form the product

$$\tilde{X}_3[k] = \tilde{X}_1[k]\tilde{X}_2[k],$$

then the periodic sequence $\tilde{x}_3[n]$ with Fourier series coefficients $\tilde{X}_3[k]$ is

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m].$$

Periodic convolution

- This result is at first sight not surprising, since our previous experience with transforms suggests that multiplication of frequency-domain functions corresponds to convolution of time-domain functions
- Equation
$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m].$$
looks very much like a convolution sum : it involves the summation of values of the product of $\tilde{x}_1[m]$ with $\tilde{x}_2[n-m]$, which is a time-reversed and time-shifted version of $\tilde{x}_2[m]$, just as in aperiodic discrete convolution.
- Beware ! the sequences $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{x}_3$ are all periodic with period N, and the summation is over only one period.
- A convolution of this form is referred to as a periodic convolution.

Periodizing a finite sequence

- We begin by considering a finite-length sequence $x[n]$ of length N samples such that $x[n] = 0$ outside the range $0 \leq n \leq N-1$.
- In many instances, we will want to assume that a sequence has length N even if its length is $L \leq N$. In such cases, we simply recognize that the last $(N-L)$ samples are zero.
- To each finite-length sequence of length N, we can always associate a periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN].$$

- The finite-length sequence $x[n]$ can be recovered from $\tilde{x}[n]$ through

$$x[n] = \begin{cases} \tilde{x}[n] & 0 \leq n \leq N-1, \\ 0 & \text{otherwise.} \end{cases}$$

Periodizing a finite sequence

- The relation

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

  can alternatively be written as

$$\tilde{x}[n] = x[(n \text{ modulo } N)].$$

- For convenience, we will use the notation $((n))_N$ to denote $(n \text{ modulo } N)$; with this notation,

$$\tilde{x}[n] = x[((n))_N].$$

- Note that this identity is valid only when $x[n]$ has length less than or equal to N.

DFT coefficients

- The sequence of discrete Fourier series coefficients $\tilde{X}[k]$ of the periodic sequence $\tilde{x}[n]$ is itself a periodic sequence with period N.
- To maintain a duality between the time and frequency domains, we will choose the Fourier coefficients that we associate with a finite-duration sequence to be a finite-duration sequence corresponding to one period of $\tilde{X}[k]$.
- This finite-duration sequence, $X[k]$, will be referred to as the discrete Fourier transform (DFT). Thus, the DFT, $X[k]$, is related to the DFS coefficients, $\tilde{X}[k]$, by

$$X[k] = \begin{cases} \tilde{X}[k] & 0 \leq k \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{X}[k] = X[(\ k \ \mathrm{modulo}\ N)\ ] \ = X[((k))_N].$$

DFT Analysis-Synthesis formula

Analysis equation : $X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] W_N^{kn} & 0 \leq k \leq N-1 \\ 0 & \text{otherwise} \end{cases}$

Synthesis equation : $x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$

The relationship between $x[n]$ and $X[k]$ implied by the Analysis and Synthesis equations is denoted

$$x[n] \overset{\mathrm{DFT}}{\leftrightarrow} X[k]$$

Linearity

- If $x_1[n]$ and $x_2[n]$ two length-N sequences are linearly combined, i.e.

$$x_3[n] = a_1 x_1[n] + a_2 x_2[n]$$

  then the DFT of $x_3[n]$ is

$$X_3[k] = a_1 X_1[k] + a_2 X_2[k]$$

- This result remains true if $x_1[n]$ is a N_1-length sequence and $x_2[n]$ is a N_2 length sequence by setting $N \geq max(N_1, N_2)$ are zero-padding the sequences.

Circular shift

- Let $x[n]$ a a finite-duration sequence of length N and m be integer. Then :

$$x[n] \overset{\mathrm{DFT}}{\leftrightarrow} X[k]$$
$$x[((n-m))_N] \overset{\mathrm{DFT}}{\leftrightarrow} W_N^{mk} X[k]$$

- Beware : multiplying the DFT coefficients $X[k]$ by W_N^{mk} does not correspond to a linear shift.

We associate to $x[n]$ the periodic sequence $\tilde{x}[n] = x[((n))_N]$.

On a :

$$x[n] \overset{DFT}{\longleftrightarrow} X[k]$$

$$\tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k] \quad \text{où} \quad \tilde{X}[k] = X[((k))_N].$$

We know that : $\tilde{x}[n-m] \overset{DFS}{\longleftrightarrow} W_N^{mk}\,\tilde{X}[k]$.

Since, for $k \in [\![0, N-1]\!]$ $W_N^{mk} X[k] = W_N^{mk}\tilde{X}[k]$

$$\tilde{x}[((n-m))_N] \overset{DFT}{\longleftrightarrow} W_N^{mk} X[k].$$

Direct proof

$$\frac{1}{N}\sum_{k=0}^{N-1} W_N^{mk} X[k]\, W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{k(m-n)}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{k((m-n))_N}$$

synthesis formula.

$$= x[((m-n))_N]$$

Circular convolution

- Consider two finite duration sequences $x_1[n]$ and $x_2[n]$, both of length N and assume that $x_1[n] \overset{\mathrm{DFT}}{\leftrightarrow} X_1[k]$ and $x_2[n] \overset{\mathrm{DFT}}{\leftrightarrow} X_2[k]$.
- Set $X_3[k] = X_1[k]X_2[k]$ and

$$\begin{aligned} x_3[n] &= \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N] \\ &= \sum_{m=0}^{N-1} x_2[m]x_1[((n-m))_N] \end{aligned}$$

- This operation is referred to a circular convolution.

Direct proof:

$$x_3[m] = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]\, X_2[k]\, W_N^{-km}$$

Circular shift:

$$x_2[((n-m))_N] \overset{DFT}{\longleftrightarrow} X_2[k]\, W_N^{-km}.$$

Hence: $$X_2[k]\, W_N^{-km} = \sum_{n=0}^{N-1} x_2[((n-m))_N]\, W_N^{kn}$$ (Analysis formula)

Substitute: $$x_3[m] = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]\left(\sum_{n=0}^{N-1} x_2[((n-m))_N]\, W_N^{kn}\right)$$

$$= \sum_{n=0}^{N-1} x_2[((n-m))_N]\, \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]\, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x_1(n)\, x_2[((n-m))_N].$$

Some history

- As far as we can tell, Gauss was the first to propose the techniques that we now call the fast Fourier transform (FFT) for calculating the coefficients in a trigonometric expansion of an asteroid's orbit in 1805.
- However, it was the seminal paper by Cooley and Tukey [link] in 1965 that caught the attention of the science and engineering community and, in a way, founded the discipline of digital signal processing (DSP).
- The impact of the Cooley-Tukey FFT was enormous. Problems could be solved quickly that were not even considered a few years earlier. A flurry of research expanded the theory and developed excellent practical programs as well as opening new applications.

Fast algorithm ?

- The development of fast algorithms usually consists of using special properties of the algorithm of interest to remove redundant or unnecessary operations of a direct implementation.
- Because of the periodicity, symmetries, and orthogonality of the basis functions and the special relationship with convolution, the discrete Fourier transform (DFT) has enormous capacity for improvement of its arithmetic efficiency.

Complexity

- The DFT of a finite-length sequence of length N is

$$X[k] = \sum_{k=0}^{N-1} x[n] W_N^{nk} \ , \quad k = 0, 1, \ldots, N-1.$$

- Since $x[n]$ may be complex, N complex multiplications and $(N-1)$ complex additions are required to compute each DFT coefficients if we use directly the above expression as a formula for computation.
- To compute all N values therefore requires a total of N^2 complex multiplications and $N(N-1)$ additions.

Decimation-in-time FFT algorithms

- The principle of the decimation-in-time is most conveniently illustrated by the special case $N = 2^\nu$. Since N is an even-integer, we can consider computing $X[k]$ by separating $x[n]$ into two $N/2$-point sequences consisting of the even-numbered points in $x[n]$ and the odd-numbered points :
- With $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$ and separating $x[n]$ into its even- and odd-numbered points, we obtain

$$X[k] = \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{(2r+1)k}$$
$$= \sum_{r=0}^{(N/2)-1} x[2r] (W_N^2)^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] (W_N^2)^{rk}$$

Decimation-in-time FFT algorithms

- But $W_N^2 = W_{N/2}$, since

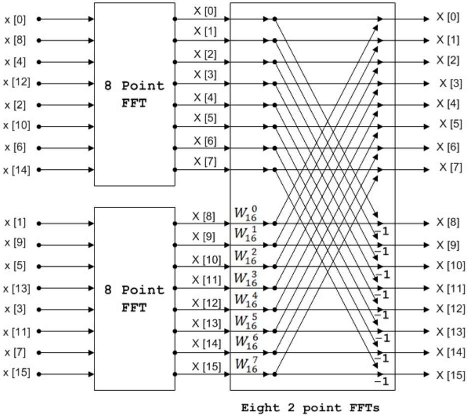
$$W_N^2 = \mathrm{e}^{-2\mathrm{i}(2\pi/N)} = \mathrm{e}^{-\mathrm{i}2\pi/(N/2)} = W_{N/2}.$$

- Consequently, $X[k]$ can be rewritten as

$$\begin{aligned} X[k] &= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_{N/2}^{rk} \\ &= G[k] + W_N^k H[k], \quad k = 0, 1, \cdots, N-1. \end{aligned}$$

- Each of the sums above is recognized as an $(N/2)$-point DFT, the first sum being the $(N/2)$-point DFT of the even-numbered points of the original sequence and the second being the $(N/2)$-point DFT of the odd-numbered points of the original sequence.
- Although the index k ranges over N values, $k = 0, 1, \ldots, N-1$, each of the sums must be computed only for k between 0 and $(N/2) - 1$, since $G[k]$ and $H[k]$ are each periodic in k with period $N/2$.

Divide and Conquer : step 1



Complexity

- Previously we saw that, for direct computation without exploiting symmetry, N^2 complex multiplications and additions were required.
- By comparison, our novel algorithm requires the computation of two $(N/2)$-point DFTs, which in turn requires $2(N/2)^2$ complex multiplications and approximately $2(N/2)^2$ complex additions if we do the $(N/2)$-point DFTs by the direct method.
- Then the two $(N/2)$-point DFTs must be combined, requiring N complex multiplications, corresponding to multiplying the second sum by W_N^k, and N complex additions, corresponding to adding the product obtained to the first sum.
- Consequently, the computation of $X[k]$ for all values of k requires at most $N + 2(N/2)^2$ or $N + N^2/2$ complex multiplications and complex additions.
- It is easy to verify that for $N > 2$, the total $N + N^2/2$ will be less than N^2

One step further

- The first step corresponds to breaking the original N-point computation into two $(N/2)$-point DFT computations.
- If $N/2$ is even, as it is when N is equal to a power of 2, then we can consider computing each of the $(N/2)$-point DFTs by breaking each of the sums in that equation into two $(N/4)$-point DFTs, which would then be combined to yield the $(N/2)$-point DFTs.
- Thus, $G[k]$ in the previous step would be represented as

$$G[k] = \sum_{r=0}^{(N/2)-1} g[r] W_{N/2}^{rk} = \sum_{\ell=0}^{(N/4)-1} g[2\ell] W_{N/2}^{2\ell k} + \sum_{\ell=0}^{(N/4)-1} g[2\ell+1] W_{N/2}^{(2\ell+1)k}$$

or

$$G[k] = \sum_{\ell=0}^{(N/4)-1} g[2\ell] W_{N/4}^{\ell k} + W_{N/2}^{k} \sum_{\ell=0}^{(N/4)-1} g[2\ell+1] W_{N/4}^{\ell k}.$$

One step further

- Similarly, $H[k]$ would be represented as

$$H[k] = \sum_{\ell=0}^{(N/4)-1} h[2\ell] W_{N/4}^{\ell k} + W_{N/2}^{k} \sum_{\ell=0}^{(N/4)-1} h[2\ell+1] W_{N/4}^{\ell k}.$$

- Consequently, the $(N/2)$-point DFT $G[k]$ can be obtained by combining the $(N/4)$-point DFTs of the sequences $g[2\ell]$ and $g[2\ell+1]$.
- Similarly, the $(N/2)$-point DFT $H[k]$ can be obtained by combining the $(N/4)$-point DFTs of the sequences $h[2\ell]$ and $h[2\ell+1]$.

Divide and Conquer !

- For the more general case, but with N still a power of 2, we would proceed by decomposing the $(N/4)$-point transforms into $(N/8)$-point transforms and continue until we were left with only 2-point transforms.
- This requires $\nu = \log_2 N$ stages of computation.
- Previously, we found that in the original decomposition of an N-point transform into two $(N/2)$-point transforms, the number of complex multiplications and additions required was $N + 2(N/2)^2$
- When the $(N/2)$-point transforms are decomposed into $(N/4)$-point transforms, the factor of $(N/2)^2$ is replaced by $N/2 + 2(N/4)^2$, so the overall computation then requires $N + N + 4(N/4)^2$ complex multiplications and additions.
- If $N = 2^\nu$, this can be done at most $\nu = \log_2 N$ times, so that after carrying out this decomposition as many times as possible, the number of complex multiplications and additions is equal to $N\nu = N \log_2 N$.

Divide and Conquer : step 2