MAP 555 : Convolution, Fourier-Plancherel, Filters...

18 Septembre 2015

Today

Convolution

Definition

The convolution of two functions f and g from $\mathbb{R}$ to $\mathbb{C}$ is the function $f * g$, if it exists, defined by

$$f * g(x) = \int_{\mathbb{R}} f(x-t)g(t)\mathrm{d}t = \int_{\mathbb{R}} f(u)g(x-u)\,\mathrm{d}u\,.$$

If no assumption is made about f and g, the convolution is clearly not defined. Take, for example, $f = g \equiv 1$!

Convolution in $L_1(\mathbb{R})$

Theorem

If f and g are in $L_1(\mathbb{R})$, then the following hold

(i) $f * g$ is defined almost everywhere and $f * g$ belongs to $L_1(\mathbb{R})$.

(ii) The convolution is a continuous bilinear operator from $L_1(\mathbb{R}) \times L_1(\mathbb{R})$ to $L_1(\mathbb{R})$ with

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

Since f and g are in $L^1(\mathbb{R})$, Fubini's theorem shows that $(y,z) \mapsto f(y,z) \in L^1(\mathbb{R}^2)$. By making a change of variable

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(y)\, g(z)\, dy\, dz = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t)\, g(t)\, dx\, dt.$$

Fubini: $\quad x \longmapsto \int_{\mathbb{R}} f(x-t)\, g(t)\, dt$ is defined a.e.

② $$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-t)|\, |g(t)|\, dx\, dt = \int_{\mathbb{R}} |g(t)| \left\{ \int_{\mathbb{R}} |f(x-t)|\, dx \right\} dt$$
$$\leq \|f\|_1 \cdot \|g\|_1 .$$

Convolution in $L_p(\mathbb{R})$

Theorem

Assume that $f \in L_p$ an $g \in L_q$ where p and q are conjugates $(p^{-1} + q^{-1} = 1)$. Then the following hold :

(i) $f * g$ is defined everywhere, is continuous and bounded on $\mathbb{R}$.

(ii) $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$.

- We know that $f*g$ is defined a.e. It is bounded:

$$|f*g(x)| = \left|\int f(x-t)\, g(t)\, dt\right| \leqslant \left(\int |f(x-t)|^p dt\right)^{1/p} \left(\int |g(t)|^q\right)^{1/q} < \infty$$

(Hölder)

$$\|f*g\|_\infty \leqslant \|f\|_p \|g\|_q .$$

- Continuity:

$$|f*g(x) - f*g(y)| \leqslant \int |f(x-t) - f(y-t)|\, |g(t)|\, dt$$

$$\leqslant \left(\int |f(x-t) - f(y-t)|^p dt\right)^{1/p} \left(\int |g(t)|^q dt\right)^{1/q} \quad \text{(Hölder)}$$

$$\leqslant \|g\|_q \left(\int |f(s) - f(s+y-x)|^p\, ds\right)^{1/p}.$$

$\rightarrow$ continuity of $\tau_a f$ in $L_p(\mathbb{R})$ $\quad \lim_{a\to 0} \|\tau_a f - f\|_p = 0 \quad f \in L^p(\mathbb{R})$

Classical result from integration theory: The set $C_c^0(\mathbb{R})$ (continuous functions with compact support), is dense in $L_p(\mathbb{R})$. $\forall \varepsilon > 0$, $\exists\, h_\varepsilon \in C_c^0(\mathbb{R})$, $\|f - h_\varepsilon\|_p \leq \varepsilon$.

We have: $\|f - \tau_a f\|_p \leqslant \|f - h_\varepsilon\|_p + \|h_\varepsilon - \tau_a h_\varepsilon\|_p + \|\tau_a h_\varepsilon - \tau_a f\|_p$

$$\leqslant 2\|f - h_\varepsilon\|_p + \|h_\varepsilon - \tau_a h_\varepsilon\|_p \leqslant 2\varepsilon + \|h_\varepsilon - \tau_a h_\varepsilon\|_p .$$

Therefore, it suffices to show that for all $h \in C_c^0(\mathbb{R})$, $\|h - \tau_a h\|_p \to 0$ $a \to 0$.

If $h \in C_c^0(\mathbb{R})$ is uniformly continuous, $\|h - \tau_a h\|_\infty \to 0$ $a \to 0$. The proof follows.

Theorem

If $f \in L_1(\mathbb{R})$ and $g \in L_2(\mathbb{R})$, then the following hold :

(i) $f * g(x)$ *exists almost everywhere.*

(ii) $f * g$ *belongs to* $L_2(\mathbb{R})$ *and*

$$\|f * g\|_2 \leq \|f\|_1 \|g\|_2.$$

remark : can be generalized to the convolution $L_p(\mathbb{R}) * L_q(\mathbb{R})$ with $p^{-1} + q^{-1} - 1 = r^{-1}$, where $p,\ q,\ r$ are ≥ 1. For $f \in L_p(\mathbb{R})$ and $g \in$ '$\mathbb{R}$, $f * g$ is in $L_r(\mathbb{R})$.

(i) Write $|f(u)\,g(x-u)| = \left(|f(u)|\,|g(x-u)|^2\right)^{1/2}\left(|f(u)|\right)^{1/2}$.

Since $f \in L_1(\mathbb{R})$ and $|g|^2 \in L_1(\mathbb{R}) \Rightarrow u \mapsto |f(u)|\,|g(x-u)|^2$

is integrable a.e. (Fubini: $\iint |f(u)|\,|g(x-u)|^2\,du\,dx = \|f\|_1 \|g\|_2^2 < \infty$).

$$\int |f(u)\,g(x-u)|\,du \leq \underbrace{\left(\int |f(u)|\,|g(x-u)|^2\,du\right)^{1/2}}_{<\infty \text{ a.e.}} \left(\int |f(u)|\,du\right)^{1/2} \quad \text{(Cauchy-Schwarz).}$$

$\Rightarrow$ $f * g(x)$ is defined a.e.

(ii)
$$|f*g(x)|^2 \leq \|f\|_1 \int |f(u)|\,|g(x-u)|^2\,du$$

Fubini-Tonelli

$$\iint |f(u)|\,|g(x-u)|^2\,du\,dx = \int |f(u)|\,du \int |g(x-u)|^2\,dx = \|f\|_1 \|g\|_2^2.$$

$$\Rightarrow \int |f*g(x)|^2\,dx \leq \|f\|_1\,\|f\|_1\,\|g\|_2^2$$

Derivation

Theorem

Let f be in $L_1(\mathbb{R})$ and let g be in $C^p(\mathbb{R})$. Assume that $g^{(k)}$ is bounded for $k = 0, 1, \ \ldots, \ p$. Then,

(i) $f * g \in C^p(\mathbb{R})$,

(ii) $(f * g)^{(k)} = f * g^{(k)}$ *for* $k = 1, 2, \ p$.

$(x,t) \mapsto f(t)\, g(x-t) = h(x,t)$. is p- times differentiable.

$\frac{\partial}{\partial x^k} h(x,t) = f(t)\, g^{(k)}(x-t)$ for all $k \in [\![0, p]\!]$.

$\left| \frac{\partial}{\partial x^k} h(x,t) \right| \leqslant \|g^{(k)}\|_\infty \cdot |f(t)|$ for all $x \in \mathbb{R}$

Therefore: $x \mapsto \int h(x,t)\, dt$ is p - tines continuously differentiable and:

$$\frac{\partial}{\partial x^k} \int h(x,t)\, dt = \int \frac{\partial}{\partial x^k} h(x,t)\, dt = \int f(t)\, g^{(k)}(x-t)\, dt.$$

Regularization

Definition (Regularizing sequence (mollifier))

A sequence of functions ρ_n in $\mathcal{D}(\mathbb{R})$ is called a regularizing sequence if it satisfies the following conditions :

(i) $\rho_n(x) \geq 0$ for all $x \in \mathbb{R}$,

(ii) $\int_{\mathbb{R}} \rho_n(x) \mathrm{d}x = 1$,

(iii) The support of ρ_n is included in $[-\epsilon_n, \epsilon_n]$ for some $\epsilon_n > 0$, and $\lim_{n\to\infty} \epsilon_n = 0$.

Definition

If $f \in L_1(\mathbb{R})$, the functions $f * \rho_n$ are called regularizations of f.

Key property : $f * \rho_n$ is in $C^\infty(\mathbb{R})$

An example

Set

$$\rho(x) = \begin{cases} c^{-1}\mathrm{e}^{-1/(1-x)} & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \leq 1 \end{cases}$$

with $c \ = \int_{-1}^{1} \mathrm{e}^{-1/(1-x)} \mathrm{d}x$

The sequence $\rho_n(x) = n\rho(nx)$ is a regularizing sequence.

In practice, regularizing sequences are used without defining them explicitly.

Density of $\mathcal{D}(\mathbb{R})$ in $L^1(\mathbb{R})$

Theorem (Density of $\mathcal{D}(\mathbb{R})$ in $L^1(\mathbb{R})$)

Let f be a function in $L_p(\mathbb{R})$, $1 \leq p < \infty$. For $\epsilon > 0$ there exists g_ϵ in $\mathcal{D}(\mathbb{R})$ such that $\|f - g_\epsilon\|_p \leq \epsilon$.

Density: $\forall \varepsilon > 0$, $\exists f_\varepsilon \in C_c^0(\mathbb{R})$ such that $\|f - f_\varepsilon\|_p \leq \varepsilon$.

Assume that $\mathrm{supp}(f_\varepsilon) \subseteq [a,b]$. Denote by $g_n = f_\varepsilon * \rho_n$, where $(\rho_n)_{n\geq 0}$ is a sequence of regularizer.

(1) $\mathrm{supp}(g_n) \subseteq [a-1, b+1]$ for all sufficiently large n. In addition, $g_n \in C_\infty$.

(2) $\|f_\varepsilon - g_n\|_p = \left\{\int |f_\varepsilon(x) - g_n(x)|^p dx\right\}^{1/p} \leq (b-a+2) \sup\limits_{x \in [a-1, b+1]} |f_\varepsilon(x) - g_n(x)|$.

(3) $f_\varepsilon(x) - g_n(x) = \int \{f_\varepsilon(x) - f_\varepsilon(x-t)\} \rho_n(t)\, dt$ (since $\int \rho_n(t) dt = 1$).

(4) $|f_\varepsilon(x) - g_n(x)| \leq \sup\limits_{|t| \leq \varepsilon_n} |f_\varepsilon(x) - f_\varepsilon(x-t)|$

(5) f_ε is continuous and is compactly supported; it is uniformly continuous and $\|f_\varepsilon - g_n\|_\infty \xrightarrow[n\to\infty]{} 0$

Choose n sufficiently large. $\|f_\varepsilon - g_n\|_p \leq \varepsilon$.

We get $\|f - g_n\|_1 \leq \|f - f_\varepsilon\|_1 + \|f_\varepsilon - g_n\|_1 \leq 2\varepsilon$ ✓.

The convolution $\mathcal{S}(\mathbb{R}) * \mathcal{S}(\mathbb{R})$

Theorem

Assume that f and g are in $\mathcal{S}(\mathbb{R})$. Then the following hold :

(i) $f * g$ *is in* $\mathcal{S}(\mathbb{R})$.

(ii) The convolution is a continuous operator from $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ *to* $\mathcal{S}(\mathbb{R})$.

(i) $f \in \mathcal{S}(\mathbb{R}), g \in \mathcal{S}(\mathbb{R}) \Rightarrow f * g \in C^\infty(\mathbb{R})$ ($f \in \mathcal{S}(\mathbb{R}) \subseteq L_1(\mathbb{R})$ and for all k, $g \in C^k(\mathbb{R})$, $g^{(k)} \in L^\infty(\mathbb{R})$).

(ii)
$$x^p (f*g)^{(q)}(x) = x^p \int f(t)\, g^{(q)}(x-t)\, dt$$
$$= \int f(t)\, (t + x - t)^p\, g^{(q)}(x-t)\, dt$$
$$= \sum_{j=0}^{p} \binom{p}{j} \int t^j f(t)\, (x-t)^{p-j}\, g^{(q)}(x-t)\, dt.$$

Since $\sup_u |u|^{p-j} g^{(q)}(u) < \infty$ $\quad |t^j f(t) (x-t)^{p-j} g^{(q)}(x-t)| \leq C_{p,j} |t^j f(t)|$

$\hookrightarrow$ dominated convergence theorem.

$$\lim_{|x| \to \infty} x^p (f*g)^{(q)}(x) = \sum_{j=0}^{p} \binom{p}{j} \int t^j f(t) \lim_{|x| \to \infty} (x-t)^{p-j} g^{(q)}(x-t)\, dt = 0.$$

density of $\mathcal{S}(\mathbb{R})$ in $L_2(\mathbb{R})$

Theorem

$\mathcal{S}(\mathbb{R})$ is a dense linear subspace of $L_2(\mathbb{R})$.

Démonstration.

$\mathcal{S}(\mathbb{R}) \subset L_2(\mathbb{R})$ = trivial

$\mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ is dense in $L_2(\mathbb{R})$.

$\square$

The Plancherel-Parseval inequality

Theorem

For $f, g \in \mathcal{S}$,

$$\int \hat{f}(\xi)\bar{\hat{g}}(\xi)\mathrm{d}\xi = \int f(x)\bar{g}(x)\mathrm{d}x$$

$$\int |\hat{f}(\xi)|^2\mathrm{d}\xi = \int |f(x)|^2\mathrm{d}x \ .$$

The Hahn-Banach theorem (elementary version)

Theorem

Let E and F be two normed vector spaces. Assume that F is complete and that G is a dense linear subspace of E. If A is a continuous linear operator from G to F, then there exists has a unique continuous linear extension of A, denoted by $\tilde{A}$, from E to F. Furthermore, the norm of $\overline{A}$ is equal to the norm of A.

The Fourier transform in $L_2(\mathbb{R})$

Theorem

The Fourier transform $\mathcal{F}$ and its inverse $\overline{\mathcal{F}}$ extend uniquely to isometries on $L_2(\mathbb{R})$. Using the same notation for these extensions, we have the following results : for all f and g in $L_2(\mathbb{R})$:

(i) $\mathcal{F} \circ \overline{\mathcal{F}} f = \overline{\mathcal{F}} \mathcal{F} f = f$ *a.e*

(ii) $\int_{\mathbb{R}} f(x)\overline{g}(x)\mathrm{d}x = \int_{\mathbb{R}} \mathcal{F}(f)(\xi)\overline{\mathcal{F}}(g)(\xi)\mathrm{d}\xi.$

(iii) $\|f\|_2 = \|\mathcal{F}(f)\|_2.$

Let $(\phi_n)_{n \geq 0}$ a sequence of functions in $\mathcal{S}(\mathbb{R})$ $\|f - \phi_n\|_2 \to 0$ $n \to \infty$.

$$\|f - \mathcal{F}\overline{\mathcal{F}} f\|_2 \leq \|f - \phi_n\|_2 + \|\phi_n - \mathcal{F}\overline{\mathcal{F}}\phi_n\|_2 + \|\mathcal{F}\overline{\mathcal{F}}\phi_n - \mathcal{F}\overline{\mathcal{F}} f\|_2$$

We have: $\phi_n = \mathcal{F}\overline{\mathcal{F}}\phi_n$. for all $n \in \mathbb{N}$

Since $\mathcal{F}$ is an isometry $\|\mathcal{F}\overline{\mathcal{F}}\phi_n - \mathcal{F}\overline{\mathcal{F}} f\|_2 = \|\phi_n - f\|_2$

Therefore: $\|f - \mathcal{F}\overline{\mathcal{F}} f\|_2 = 0 \Rightarrow f = \mathcal{F}\overline{\mathcal{F}} f$ a.e.

The exchange formula

Theorem

If f and g are in $L_2(\mathbb{R})$, $\mathcal{F}(f) \cdot g$ and $f \cdot \mathcal{F}(g)$ are in $L_1(\mathbb{R})$, and

$$\int_{\mathbb{R}} \mathcal{F}(f)(t) g(t) \mathrm{d}t = \int_{\mathbb{R}} f(u) \mathcal{F}(g)(u) \mathrm{d}u.$$

(i): $f \in L^2(\mathbb{R}) \Rightarrow \mathcal{F}f \in L^2(\mathbb{R})$, $g \in L^2(\mathbb{R})$ $\Bigg\} \Rightarrow g \cdot \mathcal{F}f \in L^2(\mathbb{R})$ idem for $f \cdot \mathcal{F}g$.

(ii) $(f_n) \subseteq \mathcal{S}(\mathbb{R})$, $\|f - f_n\|_2 \xrightarrow{n\to\infty} 0$ $\qquad (g_n) \subseteq \mathcal{S}(\mathbb{R})$, $\|g - g_n\|_2 \xrightarrow{n\to\infty} 0$.

(iii)
$$\int \mathcal{F}f(t)\, g(t)\, dt - \int \mathcal{F}f_n(t)\, g_n(t)\, dt = \int \{\mathcal{F}f(t) - \mathcal{F}f_n(t)\}\, g(t)\, dt + \int \mathcal{F}f_n(t)\, \{g(t) - g_n(t)\}\, dt.$$

$$\left| \int \mathcal{F}f(t)\, g(t)\, dt - \int \mathcal{F}f_n(t)\, g_n(t)\, dt \right| \leq \|\mathcal{F}f - \mathcal{F}f_n\|_2 \|g\|_2 + \|\mathcal{F}f_n\|_2 \|g - g_n\|_2$$
$$= \|f - f_n\|_2 \|g\|_2 + \|f_n\|_2 \|g - g_n\|_2.$$

Extension

Theorem

The Fourier transform defined on $L_1(\mathbb{R})$ and the one obtained by extension to $L_2(\mathbb{R})$ coincide on $L_1(\mathbb{R}) \cap L^2(\mathbb{R})$. If $f \in L_2(\mathbb{R})$, then $\mathcal{F}(f)$ is the limit in $L_2(\mathbb{R})$ of the sequence $\hat{f}_n$ defined by

$$\hat{f}_n(\xi) = \int_{-n}^{n} \mathrm{e}^{-2\mathrm{i}\pi\xi x} f(x)\, \mathrm{d}x.$$

We will continue to denote the Fourier transform by $\hat{f}$ or $\mathcal{F}(f)$. The meaning of these notations is now clear, depending on whether $f \in L_1(\mathbb{R})$ or $f \in L_2(\mathbb{R})$.

Examples

If $f \in L_2(\mathbb{R})$ then $\mathcal{F} \circ \mathcal{F} f = f_\sigma$, a.e.

For $a \in \mathbb{C}$, $\mathrm{Re}(a) > 0$,

$$\frac{1}{a + \mathrm{i}2\pi x} \mapsto \mathrm{e}^{a\xi} u(-\xi)$$

If $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ then $\mathcal{F}\left(\hat{f}\right) = f_\sigma$, a.e.

the cardinal sine function

$$\frac{\sin(x)}{x} \mapsto \pi \mathbb{1}_{[-(2\pi)^{-1},(2\pi)^{-1}]}(\xi) \; .$$

Inverse Fourier transform in $L_1(\mathbb{R})$

We saw that $\overline{\mathcal{F}}\hat{f}(t) = f(t)$ at every point t where f is continuous when f and $\hat{f}$ are both in $L_1(\mathbb{R})$. In particular, if $f \in \mathcal{S}(\mathbb{R})$, then $\overline{\mathcal{F}}\hat{f}(t) = f(t)$ for all $t \in \mathbb{R}$.

Theorem

If f and $\hat{f}$ are $L_1(\mathbb{R})$, then $\overline{\mathcal{F}}\hat{f} = f$ a.e.

Convolution and Fourier transform in $L_1(\mathbb{R})$

Theorem

Given f and g in $L_1(\mathbb{R})$, we have

(i) $\widehat{f * g}(\xi) = \hat{f}(\xi) \cdot \hat{g}(\xi)$ *for all* $\xi \in \mathbb{R}$.

(ii) If in addition $\hat{f}$ *and* $\hat{g}$ *are in* $L_1(\mathbb{R})$, *then* $\widehat{f \cdot g}(\xi) = \hat{f} * \hat{g}(\xi)$ *for all* $\xi \in \mathbb{R}$.

If $f \in L_1(\mathbb{R})$, $g \in L_1(\mathbb{R}) \Rightarrow f * g \in L_1(\mathbb{R})$ (see first slide).

$$\int e^{-i2\pi\xi x} f*g(x)\,dx = \int e^{-i2\pi\xi x}\left[\int f(x-t)\,g(t)\,dt\right]dx$$

Since $\iint |e^{-i2\pi\xi x}||f(x-t)||g(t)|\,dx\,dt < \infty$, we may apply the Fubini theorem $\rightarrow$

$$\int e^{-i2\pi\xi x} f*g(x)\,dx = \iint e^{-i2\pi\xi(x-t)} f(x-t)\, e^{-i2\pi\xi t} g(t)\,dt =$$

$$= \int e^{-i2\pi\xi s} f(s)\,ds \int e^{-i2\pi\xi t} g(t)\,dt = \hat{f}(\xi)\hat{g}(\xi).$$

(ii) if $\hat{f}, \hat{g} \in L_1(\mathbb{R})$, $\quad \hat{f} * \hat{g} \in L_1(\mathbb{R})$

$$\overline{\mathcal{F}}(\hat{f} * \hat{g}) = \overline{\mathcal{F}}(\hat{f})\,\overline{\mathcal{F}}(\hat{g}) = f \cdot g$$

f and g are bounded (since $f = \overline{\mathcal{F}}(\hat{f})$ with $\hat{f} \in L_1(\mathbb{R})$)

$\Rightarrow f \cdot g \in L_1(\mathbb{R})$ $\quad \int |f \cdot g(x)|\,dx \leq \|f\|_\infty \|g\|_1 < \infty.$

Convolution and Fourier transform in $L_1(\mathbb{R}) * L_2(\mathbb{R})$

Theorem

If $f \in L_2(\mathbb{R})$ and $g \in L_1(\mathbb{R})$, then $\hat{f} \cdot \hat{g}$ is in $L_2(\mathbb{R})$ and $f * g = \overline{\mathcal{F}}\left(\hat{f} \cdot \hat{g}\right)$, with equality in $L_2(\mathbb{R})$.

Analog filters

The tools we have just developed (convolution and the Fourier transform for functions) are going to be used to study analog filters that are governed by a linear differential equation with constant coefficients,

$$\sum_{k=0}^{q} b_k g^{(k)} = \sum_{j=0}^{p} a_j f^{(j)}, \; a_p \cdot b_q \neq 0,$$

where f is the input and $g = A(f)$ is the output.

Assumption $f \in \mathcal{S}(\mathbb{R})$. This case is very special. The input has no reason to be so regular, but we will see that this is a step toward more general cases.

input and output are in $\mathcal{S}(\mathbb{R})$

Assume that $f \in \mathcal{S}(\mathbb{R})$ and look for a solution $g \in \mathcal{S}(\mathbb{R})$. If such a g exists, we can take the Fourier transform of both sides of

$$\sum_{k=0}^{q} b_k g^{(k)} = \sum_{j=0}^{p} a_j f^{(j)}, \;\; a_p \cdot b_q \neq 0, \qquad \textbf{(S)}$$

showing that

$$\sum_{k=0}^{q} b_k (2\mathrm{i}\pi\xi)^k \hat{g}(\xi) = \sum_{j=0}^{p} a_j (2\mathrm{i}\pi\xi)^j \hat{f}(\xi) \qquad \textbf{(F)}$$

Consider the two polynomials $P(x) = \sum_{j=0}^{p} a_j x^j$ and $Q(x) = \sum_{k=0}^{q} b_k x^k$ and assume that the rational function $P(x)/Q(x)$ has no poles on the imaginary axis.

input and output are in $\mathcal{S}(\mathbb{R})$

Then $P(2\mathrm{i}\pi\xi)/Q(2\mathrm{i}\pi\xi)$ has no poles for real ξ, and **(S)** is equivalent to

$$\hat{g}(\xi) = G(\xi) \quad \text{where} \quad G(\xi) = \frac{P(2\mathrm{i}\pi\xi)}{Q(2\mathrm{i}\pi\xi)}\hat{f}(\xi)$$

Note that $G \in \mathcal{S}(\mathbb{R})$.

This equality completely determines g in $\mathcal{S}(\mathbb{R})$, if it exists, and thus proves the uniqueness of a solution of **(S)** in $\mathcal{S}(\mathbb{R})$.

$g = \overline{\mathcal{F}}(G)$ is a solution of **(S)** in $\mathcal{S}(\mathbb{R})$. The differential equation has a unique solution without initial conditions being specified. This is because we require the solution g to be in $\mathcal{S}$, which means that g and all of its derivatives vanish at infinity.

Convolution

Idea : express the solution as a convolution.

Assumption : $\deg P < \deg Q$. Define the transfer function

$$H(\xi) = \frac{P(2i\pi\xi)}{Q(2i\pi\xi)}$$

is in $L_2(\mathbb{R}) \cap L_\infty(\mathbb{R})$.

By decomposing this rational function into partial fractions, the impulse response, defined as the inverse Fourier transform of the transfer function

$$h = \overline{\mathcal{F}}(H)$$

is bounded, rapidly decreasing, continuous except perhaps at the origin.

Simple poles

The poles of P/Q are assumed to lie off the imaginary axis. There are two cases to consider : P/Q has only simple poles or P/Q has multiple poles.

Assume first that $P(x)/Q(x)$ has only simple poles. In this case, H can be decomposed in the form

$$H(\xi) = \sum_{k=0}^{q} \frac{\beta_k}{2i\pi\xi - z_k}$$

where $z_1, \ \ldots, \ z_q$ are the poles.

Simple poles

For $a \in \mathbb{C}$, $\mathrm{Re}(a) > 0$, $\epsilon = \pm 1$,

$$\mathrm{e}^{-\epsilon a x} u(\epsilon x) \overset{\mathcal{F}}{\mapsto} \frac{\epsilon}{(\epsilon a + 2\mathrm{i}\pi\xi)}$$

We conclude that

$$h(t) = \left(\sum_{k \in K-} \beta_k \mathrm{e}^{z_k t} \right) u(t) - \left(\sum_{k \in K_+} \beta_k \mathrm{e}^{z_k t} \right) u(-t) .$$

where $t \mapsto u(t)$ is the Heaviside function and

$$K_- = \{k \in \{1, 2, \ldots, q\} | \mathrm{Re}(z_k) < 0\},$$
$$K_+ = \{k \in \{1, 2, \ldots, q\} | \mathrm{Re}(z_k) > 0\}.$$

Multiple poles

Let $z_1, z_2, \ldots, z_l$ the poles and let $m_1, m_2, \ldots, m_l$ be their multiplicities.

Then we can write H as

$$H(\xi) = \sum_{k=1}^{l} \sum_{m=1}^{m_k} \frac{\beta_{k,m}}{(2i\pi\xi - z_k)^m}.$$

The impulse response is given by

$$h(t) = \left(\sum_{k \in K_-} P_k(t) \mathrm{e}^{z_k t} \right) u(t) - \left(\sum_{k \in K_+} P_k(t) \mathrm{e}^{z_k t} \right) u(-t)$$

where

$$P_k(t) = \sum_{m=1}^{m_k} \beta_{k,m} \frac{t^{m-1}}{(m-1)!}.$$

Convolution

Since $h = \overline{\mathcal{F}}(H)$ is bounded, rapidly decreasing, continuous except perhaps at the origin **(F)** may be rewritten as

$$\hat{g} = \hat{h} \cdot \hat{f}.$$

Since $\hat{h} \in L_2(\mathbb{R})$ and $\hat{f} \in \mathcal{S}(\mathbb{R}) \subset L_1(\mathbb{R})$, we have $h * f(t) = \overline{\mathcal{F}}\left(\hat{h}\hat{f}\right)$ implies that

$$g = h * f$$

Generalized solutions

The formula $g = h * f$, obtained when f is in $\mathcal{S}$, makes sense in the following more general cases.

(i) If f is in $L_1(\mathbb{R})$, then g is in $L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \cap L_\infty(\mathbb{R})$ and

$$\begin{aligned} \|g\|_1 &\leq \|h\|_1 \|f\|_1, \\ \|g\|_2 &\leq \|h\|_2 \|f\|_1, \\ \|g\|_\infty &\leq \|h\|_\infty \|f\|_1. \end{aligned}$$

(ii) If f is in $L_2(\mathbb{R})$, then g is in $L_2(\mathbb{R})$, it is bounded and continuous, it tends to 0 at infinity, and

$$\begin{aligned} \|g\|_2 &\leq \|h\|_1 \|f\|_2, \\ \|g\|_\infty &\leq \|h\|_2 \|f\|_2. \end{aligned}$$

(iii) If f is in $L_\infty(\mathbb{R})$, then g is also bounded and

$$\|g\|_\infty \leq \|h\|_1 \|f\|_\infty.$$

Purely imaginary poles

What we have done so far does not allow us to treat an equation like

$$g'' + \omega^2 g = f,$$

where $P(x)/Q(x) = 1/(x^2 + \omega^2)$ has two poles are on the imaginary axis.

In this case h is a sinusoid and the Fourier transform of H (when H is considered to be a function) is no longer defined.

This problem required the use of the theory of distributions but this degree of sophistication goes far beyond this course.

What happens in $\deg P = \deg Q$

Take for example the equation

$$g'' - \omega^2 g = f''.$$

Again, what we have done so far does not apply. Nevertheless, we can still manage to solve the equation. Changing the unknown function to $g_0 = g - f$ lowers the order of the right-hand side :

$$g_0'' - \omega^2 g_0 = \omega^2 f.$$

Then we have $g_0 = h_0 * f$ and $g = f + h_0 * f$. This is no longer a convolution but it will serve the same purpose.

Denote by X the set of input signals and Y the set of output signals which are assumed to be vector spaces (over $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$).

Definition (Linearity)

The system $A : \mathrm{X} \to \mathrm{Y}$ is said to be linear if for all $x_1, x_2 \in \mathrm{X}$ and $\lambda_1, \lambda_2 \in \mathbb{K}$

$$A(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 A(x_1) + \lambda_2 A(x_2) \ .$$

A system A defined by a convolution $A(f) = h * f$ is linear.

Stability

Definition

A system $A : \mathrm{X} \to \mathrm{Y}$ is said to be stable if there exists an $M > 0$ such that $\|Af\|_\infty \leq M\|f\|_\infty$ for all $f \in L_\infty(\mathbb{R}) \cap \mathrm{X}$.

(i) The generalized filter A is stable when $\deg P < \deg Q$.

(ii) It is still stable if $\deg P = \deg Q$...

Causality

Definition

A system $A : \mathrm{X} \to \mathrm{Y}$ is causal if the equality of any two input signals up to time $t = t_0$ implies the equality of the two output signals at least to time t_0,

$$x_1(t) = x_2(t) \text{ for } t \leq t_0 \Rightarrow Ax_1(t) = Ax_2(t) \text{ for } t < t_0$$

This property is completely natural for a physical system in which the variable is time. It says that the response at time t depends only on what has happened before t.

Invariance

Define by τ_a the delay operator : $\tau_a x(t) = x(t-a)$ for all $t \in \mathbb{R}$.

Definition

A system A is invariant if a translation of time in the input leads to the same translation in the output, or equivalently if A and τ_a commute for all $a \in \mathbb{R}$:

$$A \circ \tau_a = \tau_a \circ A$$

Causality

(i) When A is linear and invariant, the causality condition becomes the following : For all $t_0 \in \mathbb{R}$,

$$f(t) = 0 \text{ for } t < t_0 \Rightarrow Af(t) = 0 \text{ for } t < t_0.$$

(ii) Assume that $\deg P \leq \deg Q$. The generalized filter $A : A(f) = h * f$ is causal if $\mathrm{supp}(h) \subset [0, \infty[$ or equivalently if the poles of P/Q are located to the left of the imaginary axis.