

# A Memorandum on Kan Extensions and Monads

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## Abstract

These notes are meant to remind myself of some facts about Kan extension and monads. The first part is devoted to basic definitions about adjunctions, ends and coends, which are needed to explain the proofs later on. The second part is on monads and Kan extensions.

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# 1 Adjunctions and Monads

Given two functors  $L : \mathcal{D} \rightarrow \mathcal{C}$  and  $R : \mathcal{C} \rightarrow \mathcal{D}$  and *adjunction* is an isomorphism of homsets

$$[\cdot] : \mathcal{C}(LA, B) \cong \mathcal{D}(A, RB) : [\cdot]$$

which is furthermore natural in  $A$  and  $B$ . Here  $[\cdot]$  and  $[\cdot]$  are the functions witnessing the isomorphism. The adjunction is usually depicted as follows

$$\mathcal{C} \xrightleftharpoons[\begin{smallmatrix} \perp \\ R \end{smallmatrix}]{\begin{smallmatrix} L \\ \perp \end{smallmatrix}} \mathcal{D}$$

We say that that  $L$  is left adjoint and  $R$  is right adjoint and it is indicated by  $L \vdash R$ . As a consequence of the isomorphism, for  $f : LA \rightarrow B$  and  $g : A \rightarrow RB$  we have that

$$[f] = g \iff f = [g]$$

and because the isomorphism is natural we can derive the *fusion laws*. For  $a : A' \rightarrow A$ ,  $b : B \rightarrow B'$ ,  $f : LA \rightarrow B$  and  $g : A \rightarrow RB$

$$\begin{aligned} R(b) \cdot [f] &= [b \cdot f] \\ [f] \cdot a &= [f \cdot L(a)] \\ b \cdot [g] &= [R(b) \cdot g] \\ [g] \cdot L(a) &= [g \cdot a] \end{aligned}$$

We can also compute the fusion laws this way.

$$\begin{aligned} R(b) \cdot [f] \cdot a &= [b \cdot f \cdot L(a)] \\ b \cdot [g] \cdot L(a) &= [R(b) \cdot g \cdot a] \end{aligned}$$

This is really all about adjunctions. All the other definitions and constructions are equivalent to this one. Furthermore, this material is very well covered elsewhere [2, 4, 3] so I will not be covering it further.

What is important for the sake of this notes is that an adjunction gives rise to a monad and a comonad where  $RL$  is the monad and  $LR$  is the comonad. The unit and counit of the adjunction are defined as follows

$$\begin{aligned} \eta_A &= [id_{LA}] \\ \epsilon_B &= [id_{RB}] \end{aligned}$$

and are respectively the unit of the monad and the counit of the comonad generated by the adjunction. The join of the monad  $\mu : RLRL \rightarrow RL$  is defined as  $\mu = R\epsilon_L$  and the cojoin  $\delta : LR \rightarrow LRLR$  is defined as  $\delta = L\eta_R$ .

## 2 Limits and Colimits

Given two objects  $X$  and  $Y$  in a category,  $X \times Y$  forms the product of  $X$  and  $Y$ . We can generalise this further. Given a functor  $D : \mathcal{I} \rightarrow \mathcal{C}$  the limit  $\varprojlim D$

is an universal object such that for every  $I \in \mathcal{I}$ , there exists a projection map  $\varprojlim DI \xrightarrow{\pi_I} DI$  such that for every morphism  $DI_1 \xrightarrow{f} DI_2$  we have

$$\pi_{I_2} = f \cdot \pi_{I_1}$$

and furthermore any other object such as this has a unique morphism into the limit commuting with the projections [2, 4].

The limit is right adjoint to the diagonal functor mapping every object to the constant functor and the colimit is the left adjoint to the diagonal functor.

$$\mathcal{C} \begin{array}{c} \xleftarrow{\varprojlim} \\ \perp \\ \xrightarrow{\Delta} \end{array} \mathcal{C}^{\mathcal{I}} \begin{array}{c} \xleftarrow{\Delta} \\ \perp \\ \xrightarrow{\varprojlim} \end{array} \mathcal{C}$$

We right down the isomorphisms

$$\mathcal{C}(\varprojlim_{I \in \mathcal{I}} A(I), B) \cong \mathcal{C}^{\mathcal{I}}(A, \Delta B)$$

$$\mathcal{C}^{\mathcal{I}}(\Delta A, B) \cong \mathcal{C}(A, \varprojlim_{I \in \mathcal{I}} B)$$

## 2.1 Preservation and Creation of (Co)Limits

A functor  $H : \mathcal{C} \rightarrow \mathcal{D}$  is said to *preserve* limits if, given a diagram  $F : \mathcal{I} \rightarrow \mathcal{C}$

$$H \varprojlim_{I \in \mathcal{I}} FI \cong \varprojlim_{I \in \mathcal{I}} HFI$$

In other words,  $H$  preserves limits if the limit of the diagram obtained by composition with  $H$ , namely  $HF$ , corresponds with the limit of  $F$  applied to  $H$ . In particular, such a functor preserves small limits as well. A functor that preserves small (co)limits is called *(co)continuous*.

As a prominent example, the covariant homset functor  $\mathcal{C}(C, -) : \mathcal{C} \rightarrow \mathbf{Set}$  preserve limits

$$\mathcal{C}(C, \varprojlim_{I \in \mathcal{I}} FI) \cong \varprojlim_{I \in \mathcal{I}} \mathcal{C}(C, FI) \quad (1)$$

On the other hand, the contravariant homset functor, which may be written as  $\mathcal{C}(-, C) = \mathcal{C}^{\text{op}}(C, -) : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  carries colimits over to limits in the following sense

$$\mathcal{C}(\varprojlim_{I \in \mathcal{I}} FI, C) \cong \varprojlim_{I \in \mathcal{I}} \mathcal{C}(FI, C) \quad (2)$$

## 2.2 Dependent product and sum

Let us consider the set  $\mathcal{I}$  (or the discrete category  $\mathcal{I}$  with only identities). Then the right adjoint to the diagonal functor is called the *dependent product*  $\prod_{I \in \mathcal{I}} B(I)$  for some functor  $B : \mathcal{I} \rightarrow \mathcal{C}$  and the left adjoint is called the

dependent sum  $\Sigma_{I \in \mathcal{I}} A(I)$  for some functor  $A : \mathcal{I} \rightarrow \mathcal{C}$ .

$$\mathcal{C} \begin{array}{c} \xleftarrow{\Sigma_{I \in \mathcal{I}} (-)_I} \\ \xrightarrow[\Delta]{\perp} \end{array} \mathcal{C}^{\mathcal{I}} \begin{array}{c} \xleftarrow{\Delta} \\ \xrightarrow[\Pi_{I \in \mathcal{I}} (-)_I]{\perp} \end{array} \mathcal{C}$$

The limit preservation (1) and colimit reverse (2) continues to hold for dependent products and sums.

$$\Pi_{I \in \mathcal{I}} \mathcal{C}(X, A(I)) \cong \mathcal{C}(X, \Pi_{I \in \mathcal{I}} A(I)) \quad (3)$$

$$\mathcal{C}(\Sigma_{I \in \mathcal{I}} A(I), X) \cong \Pi_{I \in \mathcal{I}} \mathcal{C}(A(I), X) \quad (4)$$

### 2.2.1 Powers and CoPowers

Now we consider only constant functors and keep  $\mathcal{I}$  as the discrete category. The limits and colimits of these functors are called powers and copowers which can be indicated by  $\Sigma \mathcal{I}.A = \mathcal{I} \bullet A$  and  $\Pi \mathcal{I}.B = B^{\mathcal{I}}$

$$\mathcal{C} \begin{array}{c} \xleftarrow{\Sigma \mathcal{I}.(-)} \\ \xrightarrow[\Delta]{\perp} \end{array} \mathcal{C}^{\mathcal{I}} \begin{array}{c} \xleftarrow{\Delta} \\ \xrightarrow[\Pi \mathcal{I}.(-)]{\perp} \end{array} \mathcal{C}$$

Now equations (3) and (4) in turn specialise to powers and copowers

$$\mathcal{C}(X, A)^{\mathcal{I}} \cong \mathcal{C}(X, A^{\mathcal{I}}) \quad (5)$$

$$\mathcal{C}(\mathcal{I} \bullet A, X) \cong \mathcal{C}(A, X)^{\mathcal{I}} \quad (6)$$

As a consequence of (5) and (6) we get that

$$\mathcal{C}(\mathcal{I} \bullet A, B) \cong \mathcal{C}(A, B)^{\mathcal{I}} \cong \mathcal{C}(A, B^{\mathcal{I}})$$

Now since  $\mathcal{I}$  is the discrete category (it has no arrows) it can be regarded as a set! Hence, the set of natural transformation between to constant functors is just the set of functions between the images of these indexed by  $\mathcal{I}$

$$\mathcal{C}^{\mathcal{I}}(\Delta X, \Delta Y) \cong \mathcal{C}(X, Y)^{\mathcal{I}} \cong \mathcal{I} \rightarrow \mathcal{C}(X, Y) \quad (7)$$

Note that, since  $\mathcal{I}$  and  $\mathcal{C}(X, Y)$  are sets, then  $\mathcal{C}(X, Y)^{\mathcal{I}}$  is the exponential object in **Set** and hence it is isomorphic to  $\mathcal{I} \rightarrow \mathcal{C}(X, Y)$  which is the set of functions.

In **Set**,  $\mathcal{I} \bullet A = A \times \mathcal{I}$  and  $B^{\mathcal{I}}$  is the function space  $\mathcal{I} \rightarrow B$ .

## 3 Ends and Coends

Sometimes it useful to talk about limits and colimits of diagrams that have a contravariant component. These are called *ends* and *coends*. Consider a functor  $S : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ . Ends and coends are respectively limits and colimits for  $S$ .

It is easy to prove [4] that whenever  $S$  is “dummy” in the first variable, i.e.  $S$  factors through the second projection as in

$$\mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{\pi_2} \mathcal{C} \xrightarrow{T} \mathcal{D}$$

then the end coincides with the limit of  $T$

$$\int_{C:\mathcal{C}} S(C, C) = \lim_{\overleftarrow{C:\mathcal{C}}} TC$$

Ends behave similarly to universal quantification. For a functor  $S : \mathcal{C}^{\text{op}} \times \mathcal{C} \times \mathcal{D}^{\text{op}} \times \mathcal{D} \rightarrow \mathcal{E}$

$$\int_{C:\mathcal{C}} \int_{D:\mathcal{D}} S(C, C, D, D) \cong \int_{D:\mathcal{D}} \int_{C:\mathcal{D}} S(C, C, D, D) \quad (8)$$

### 3.1 Preservation of Ends

A functor  $H : \mathcal{C} \rightarrow \mathcal{D}$  is said to *preserve the end* of a functor  $S : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$

$$H \int_{C \in \mathcal{C}} S(C, C) = \int_{C \in \mathcal{C}} HS(C, C)$$

In other words, when  $w : e \rightrightarrows S$  is an end of  $S$  and  $Hw : He \rightrightarrows HS$  is an end for  $HS$ .

For example, as it was the case for the limits and colimits, the homset functor preserves ends in the following way

$$\mathcal{C}(X, \int_{C:\mathcal{C}} S(C, C)) = \int_{C:\mathcal{C}} (\mathcal{C}(X, S(C, C))) \quad (9)$$

and reserves ends into coends

$$\mathcal{C}(\int_{C:\mathcal{C}} S(C, C), X) = \int_{C:\mathcal{C}} (\mathcal{C}(S(C, C), X)) \quad (10)$$

### 3.2 Natural transformations and Ends

Natural transformations are examples of ends. Given two functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$ , the end of the homset functor  $\mathcal{D}^{\mathcal{C}}(F-, G-)$  is the set of natural transformations from  $F$  to  $G$

$$\text{Nat}(F, G) = \int_{C:\mathcal{C}} \mathcal{D}(FC, GC) \quad (11)$$

## 4 The Yoneda Lemma

Assume a locally small category  $\mathcal{C}$ . For all covariant functors  $F : \mathcal{C} \rightarrow \mathbf{Set}$ ,

$$FC \cong \mathcal{C}(C, -) \dot{\rightarrow} F \quad (12)$$

The functions witnessing the isomorphism are given by  $f(x, g) = F(g)(x)$  and its inverse  $f^{-1}(i) = i_C(id_C)$ .

This result holds for contravariant functors as well. Say  $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  is our contravariant functor then

$$FC \cong \mathcal{C}^{\text{op}}(C, -) \dashrightarrow F \cong \mathcal{C}(-, C) \dashrightarrow F$$

The homset functor  $\mathcal{C}(-, -) : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$  is also called the Yoneda embedding,  $y$ . The fact that  $y$  is an embedding means that the  $y$  is a fully faithful functor and therefore it preserves (because it is a functor) and reflects (because it is fully faithful) isomorphisms

$$y_C \cong y_D \iff C \cong D \tag{13}$$

## 5 Kan Extensions

Consider a reindexing functor  $J : \mathcal{C} \rightarrow \mathcal{D}$  and define the functor  $- \circ J : \mathcal{E}^{\mathcal{D}} \rightarrow \mathcal{E}^{\mathcal{C}}$  also named  $\text{App}_J$ .

We want to find out whether  $\text{App}_J$  has a left and right adjoint. We call its left and right adjoint  $\text{Lan}_J$  and  $\text{Ran}_J$  respectively.

$$\mathcal{E}^{\mathcal{C}} \begin{array}{c} \xleftarrow{\text{App}_J} \\ \perp \\ \xrightarrow{\text{Ran}_J} \end{array} \mathcal{E}^{\mathcal{D}} \begin{array}{c} \xleftarrow{\text{Lan}_J} \\ \perp \\ \xrightarrow{\text{App}_J} \end{array} \mathcal{E}^{\mathcal{C}}$$

Now because  $\text{Lan}_J$  is the left adjoint and  $\text{Ran}_J$  to be the right one we would expect the following to be a natural isomorphisms

$$\mathcal{E}^{\mathcal{D}}(\text{Lan}_J H, G) \cong \mathcal{E}^{\mathcal{C}}(H, \text{App}_J G) \quad \mathcal{E}^{\mathcal{C}}(\text{App}_J H, G) \cong \mathcal{E}^{\mathcal{D}}(H, \text{Ran}_J G)$$

In the MacLane [4] he is defining the right Kan extension and the proving it is the right adjoint. Here we take a different approach. By using the Yoneda lemma we derive the right adjoint which is unique up to isomorphism so it must be the right Kan extension. This proof is taken from Hinze's work on generic

programming [3]. Here I have made it a bit more precise.

$$\begin{aligned}
\mathcal{E}^{\mathcal{C}}(\text{App}_J A, B) &= \{ \text{Homsets in the exponential category are natural transformations} \} \\
&\quad \text{Nat}(\text{App}_J A, B) \\
&\cong \{ \text{Natural transformations are ends (11)} \} \\
&\quad \int_{X:\mathcal{C}} \mathcal{C}(AJX, BX) \\
&\cong \{ \text{Yoneda with } \mathcal{C}(A-, BX) \} \\
&\quad \int_{X:\mathcal{C}} \text{Nat}(\mathcal{D}(-, JX), \mathcal{C}(A-, BX)) \\
&\cong \{ \text{by (11)} \} \\
&\quad \int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{D}(Y, JX) \rightarrow \mathcal{C}(AY, BX) \\
&\cong \{ \text{by (7)} \} \\
&\quad \int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{C}(AY, BX)^{\mathcal{D}(Y, JX)} \\
&\cong \{ \text{by (5)} \} \\
&\quad \int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{C}(AY, BX^{\mathcal{D}(Y, JX)}) \\
&\cong \{ \text{by (8)} \} \\
&\quad \int_{Y:\mathcal{D}} \int_{X:\mathcal{C}} \mathcal{C}(AY, BX^{\mathcal{D}(Y, JX)}) \\
&\cong \{ \text{Homsets preserve ends (9)} \} \\
&\quad \int_{Y:\mathcal{D}} \mathcal{C}(AY, \int_{X:\mathcal{C}} BX^{\mathcal{D}(Y, JX)}) \\
&\cong \{ \text{Natural transformation are ends (11)} \} \\
&\quad \text{Nat}(A-, \int_{X:\mathcal{C}} BX^{\mathcal{D}(-, JX)}) \\
&\cong \{ \text{Homsets in the exponential category are natural transformations} \} \\
&\quad \mathcal{E}^{\mathcal{D}}(A-, \int_{X:\mathcal{C}} BX^{\mathcal{D}(-, JX)})
\end{aligned}$$

For all functors  $J : \mathcal{C} \rightarrow \mathcal{D}$ ,  $A : \mathcal{C} \rightarrow \mathcal{E}$  and  $B : \mathcal{D} \rightarrow \mathcal{E}$

$$\text{Ran}_J AY = \int_{X \in \mathcal{C}} \Pi_{\mathcal{D}(Y, JX)} AX$$

We now compute the left Kan extension. I could not find this proof is not

in [3, 4].

$$\begin{aligned}
\mathcal{E}^{\mathcal{C}}(A, \text{App}_J B) &= A \dot{\rightarrow} \text{App}_J B \\
&\cong \{ \text{Natural transformations are ends (11)} \} \\
&\int_{X:\mathcal{C}} \mathcal{C}(AX, BJX) \\
&\cong \{ \text{by Yoneda with } \mathcal{C}(AX, B-) \} \\
&\int_{X:\mathcal{C}} \mathcal{D}(JX, -) \dot{\rightarrow} \mathcal{C}(AX, B-) \\
&\cong \{ \text{Natural transformations are ends(11)} \} \\
&\int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{D}(JX, Y) \rightarrow \mathcal{C}(AX, BY) \\
&\cong \{ \text{The set functions space is a power (7)} \} \\
&\int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{C}(AX, BY)^{\mathcal{D}(JX, Y)} \\
&\cong \{ \text{Homsets revert powers into copowers (6)} \} \\
&\int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{C}(\mathcal{D}(JX, Y) \bullet AX, BY) \\
&\cong \{ \text{Switching over ends (8)} \} \\
&\int_{Y:\mathcal{D}} \int_{X:\mathcal{C}} \mathcal{C}(\mathcal{D}(JX, Y) \bullet AX, BY) \\
&\cong \{ \text{Homsets reverse ends into coends (10)} \} \\
&\int_{Y:\mathcal{D}} \mathcal{C}(\int_{X:\mathcal{C}} \mathcal{D}(JX, Y) \bullet AX, BY) \\
&\cong \{ \text{Natural transformations are ends (11)} \} \\
&\text{Nat}(\int_{X:\mathcal{C}} \mathcal{D}(JX, -) \bullet AX, B) \\
&= \mathcal{E}^{\mathcal{D}}(\int_{X:\mathcal{C}} \mathcal{D}(JX, -) \bullet AX, B)
\end{aligned}$$

Now by looking at what we have got we define the left Kan extension as a functor parametrised by  $A$

$$\text{Lan}_J AY = \int^{X \in \mathcal{C}} \mathcal{D}(JX, Y) \bullet AX$$



## 5.1 Yoneda revisited

### 5.1.1 Yoneda

Given a functor  $F : \mathcal{C} \rightarrow \mathbf{Set}$ , the Yoneda lemma says that  $F$  is isomorphic to the right Kan extension over the identity functor of  $F$

$$F \cong \mathrm{Ran}_{\mathrm{Id}} F \quad (14)$$

We compute as follows

$$\begin{aligned} FC &\cong \mathrm{Nat}(\mathcal{C}(C, -), F) \\ &\int_{X:\mathcal{C}} \mathbf{Set}(\mathcal{C}(C, X), FX) \\ &\int_{X:\mathcal{C}} \mathcal{C}(C, X) \rightarrow FX \\ &\int_{X:\mathcal{C}} FX^{\mathcal{C}(C, X)} \\ &\mathrm{Ran}_{\mathrm{Id}} FC \end{aligned}$$

Another way of proving this is by using the fact that  $\mathrm{App}_{\mathrm{Id}}$  is left adjoint to  $\mathrm{Ran}_{\mathrm{Id}}$ .

$$\mathcal{E}^{\mathcal{C}}(\mathrm{App}_{\mathrm{Id}} G, F) = \mathcal{E}^{\mathcal{C}}(G, F) \cong \mathcal{E}^{\mathcal{C}}(G, \mathrm{Ran}_{\mathrm{Id}} F)$$

Thus, since  $\mathcal{E}^{\mathcal{C}}(-, F) \cong \mathcal{E}^{\mathcal{C}}(-, \mathrm{Ran}_{\mathrm{Id}} F)$  by Yoneda (13) we have  $F \cong \mathrm{Ran}_{\mathrm{Id}} F$ .

### 5.1.2 CoYoneda

The coYoneda lemma states that

$$F \cong \mathrm{Lan}_{\mathrm{Id}} F \quad (15)$$

This can be proven by instantiating the left Kan extension with the identity functor obtaining the adjunction  $\mathrm{Lan}_{\mathrm{Id}} \dashv \mathrm{App}_{\mathrm{Id}}$ . From the adjunction we know that

$$\mathcal{E}^{\mathcal{C}}(\mathrm{Lan}_{\mathrm{Id}} F, G) \cong \mathcal{E}^{\mathcal{C}}(F, \mathrm{App}_{\mathrm{Id}} G) = \mathcal{E}^{\mathcal{C}}(F, G \circ \mathrm{Id}) = \mathcal{E}^{\mathcal{C}}(F, G)$$

is a natural in  $F$  and  $G$ . Because this isomorphism is natural we know that  $\mathcal{E}^{\mathcal{C}}(\mathrm{Lan}_{\mathrm{Id}} F, -) \cong \mathcal{E}^{\mathcal{C}}(F, -)$  which by (13) implies  $\mathrm{Lan}_{\mathrm{Id}} F \cong F$  since  $\mathcal{E}^{\mathcal{C}}(-, -)$  is the Yoneda embedding.

## 5.2 Left and Right Shifts

As a particular case  $D : 1 \rightarrow \mathcal{D}$ . So  $\mathrm{App}_D H = H \circ D = HD(*)$

$$\mathcal{E} \xrightleftharpoons[\mathrm{Rsh}_D]{-D} \mathcal{E}^{\mathcal{D}} \xrightleftharpoons[-D]{\mathrm{Lsh}_D} \mathcal{E}$$

$$\mathcal{E}^{\mathcal{D}}(\text{Lsh}_J H, G) \cong \mathcal{E}(H, GD)$$

$$\mathcal{E}(HD, G) \cong \mathcal{E}^{\mathcal{D}}(H, \text{Rsh}_J G)$$

$$\text{Lsh}_D AX = \Sigma_{\mathcal{D}(D, X)} A$$

$$\text{Ran}_D AX = \Pi_{\mathcal{D}(X, D)} A$$

## 6 Monads from Kan Extensions

### 6.1 The Codensity Monad

The codensity monad is just the right Kan extension of  $J$  along  $J$

$$\text{Cod } JX = \text{Ran}_J JX = \int_{Y:\mathcal{C}} JY^{\mathcal{D}(X, JY)}$$

### 6.2 The Codensity Transformation

If  $L \dashv R$  then both  $L \circ - \vdash R \circ -$  and  $- \circ R \vdash - \circ L$  are adjunctions.

$$\text{If } \mathcal{L} \xrightleftharpoons[\mathcal{R}]{\mathcal{L}} \mathcal{R} \text{ then } \mathcal{E}^{\mathcal{L}} \xrightleftharpoons[\mathcal{E}^{\mathcal{R}}]{\mathcal{L}} \mathcal{E}^{\mathcal{R}}$$

Because of this fact there is a natural isomorphism

$$\mathcal{E}^{\mathcal{L}}(F \circ R, G) \cong \mathcal{E}^{\mathcal{R}}(F, G \circ L)$$

Now, since  $F \circ R$  is  $\text{App}_R F$  then  $\mathcal{E}^{\mathcal{L}}(F \circ R, G) \cong \mathcal{E}^{\mathcal{R}}(F, \text{Ran}_R G)$ . But then we know also that

$$\mathcal{E}^{\mathcal{L}}(F, G \circ L) \cong \mathcal{E}^{\mathcal{R}}(F, \text{Ran}_R G)$$

Since the Yoneda embedding  $\mathcal{E}^{\mathcal{R}}(-, -) : \mathcal{C} \rightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$  is fully faithful, by (13) then  $G \circ L \cong \text{Ran}_R G$ . (But this is also the proof that adjoints are unique up-to isomorphism). Now, if  $G = R$  then we get that the monad

$$R \circ L \cong \text{Ran}_R R \tag{16}$$

## 7 On Free Monads

Given an endofunctor  $F : \mathcal{C} \rightarrow \mathcal{C}$  the free monad  $M$  over  $F$  is the monad freely generated by the constructors  $\eta : \text{Id} \rightarrow M$  and  $\text{op} : FM \rightarrow M$ . The fact that is a monad implies it must have a join operation as well  $\mu : MM \rightarrow M$ . This monad  $M$  is also indicated by  $F^*$ .

When it exists, the free monad is the least solution to the equation

$$F^*A \cong A + FF^*A$$

It can be shown that this is the free  $F$ -algebra in the sense that it is the monad arising from a free-forgetful adjunction

$$F\text{-Alg} \begin{array}{c} \xleftarrow{\text{Free}} \\ \xrightarrow[U]{\perp} \end{array} \mathcal{C} \begin{array}{c} \curvearrowright \\ F^* \end{array}$$

where  $F^* = U\text{Free}$ .

The ceiling and floor witness the natural isomorphism on both  $A$  and  $B$

$$[\cdot] : F\text{-Alg}(\text{Free } A, B) \cong \mathcal{C}(A, UB) : [\cdot]$$

As usual, the unit of the monad is given by  $\eta = [id_{\text{Free } A}]$  and the multiplication is derived by using the counit  $\mu = U\epsilon_{\text{Free}}$ . Now the map  $FF^*A \rightarrow F^*A$  is the  $F$ -algebra given by  $\text{Free } A$ . To see this let  $F^*A$  to be  $U\text{Free } A$  (it's just a name). Say that  $\text{Free}$  takes an object  $A : \mathcal{C}$  and sends it to an  $F$ -algebra,  $\text{Free } A = (X, \text{alg} : FX \rightarrow X)$  for some  $X$  and some algebras  $\text{alg}$ . But that means that  $U\text{Free } A = X$  which implies  $X$  is  $F^*A$  and that  $\text{alg}$  is the map  $\text{op} : FF^*A \rightarrow F^*A$  we were looking for.

## 7.1 Free Monads and The Right Kan Extension

From the previous section we know that every monad arising from an adjunction  $L \dashv R$  (hence every monad!) is isomorphic to the right Kan extension of  $R$  along  $R$ , denoted by  $\text{Ran}_R R$  and also known as the *codensity monad*.

Since the free monad is a monad, also the free monad can be transformed using the codensity transformation. Since the algebraically free monad factors as  $F^* = U\text{Free}$  then by (16)

$$F^* \cong \text{Ran}_U U$$

By unfolding the definitions we get that

$$F^*A \cong \int_{Z : F\text{-Alg}} UZ^{\mathcal{C}(A, UZ)} \quad (17)$$

However, there is another way to transform the free monad. Using Yoneda (14)

$$F^*A \cong A + FF^*A \cong A + (\text{Ran}_{\text{Id}} F)F^*A \cong A + \int_{X : \mathcal{C}} FX^{\mathcal{C}(F^*A, X)} \quad (18)$$

## 7.2 Free Monads and the Left Kan Extension

Using CoYoneda (15)

$$F^*A \cong A + F \circ F^* \cong A + (\text{Lan}_{\text{Id}} F)F^*A \cong A + \int^{X : \text{Set}} \mathcal{C}(X, F^*A) \bullet FX \quad (19)$$

In **Set**,

$$F^*A \cong A + \int_{X:\mathcal{C}} FX \times (X \rightarrow F^*A)$$

### 7.3 Algebras for the Left Kan extension

Every  $G$ -algebra is isomorphic to the algebras for the left Kan extension on  $G$ .

$$\begin{aligned} \mathcal{D}(\text{Lan}_{\text{Id}}GA, A) &\cong \mathcal{D}^{\mathcal{D}}(\text{Lan}_{\text{Id}}G, \text{Rsh}_A A) && \{ \text{ by } -A \dashv \text{Rsh}_{\text{Id}} \} \\ &\cong \mathcal{D}^{\mathcal{D}}(G, \text{Rsh}_A A \circ \text{Id}) && \{ \text{ by } \text{Lan}_{\text{Id}} \dashv \text{App}_{\text{Id}} \} \\ &\cong \mathcal{D}^{\mathcal{D}}(G, \text{Rsh}_A A) \\ &\cong \mathcal{D}(GA, A) && \{ \text{ by } -A \dashv \text{Rsh}_{\text{Id}} \} \end{aligned}$$

This proof is worth of reminding, but a simpler way to do it is to use Yoneda (14) directly

$$\mathcal{D}(\text{Lan}_{\text{Id}}GA, A) \cong \mathcal{D}(GA, A)$$

### 7.4 The Freest Monad

Given a functor  $J : \mathcal{C} \rightarrow \mathcal{D}$  and an endofunctor  $F : \mathcal{C} \rightarrow \mathcal{D}$  the freest monad is defined as follows

$$F_J^{\text{st}} A \cong JA + \text{Lan}_J F(F_J^{\text{st}} A)$$

Note that  $F$  is not an endofunctor and so the freest monad is in fact a *relative monad* [1]. The free monad over an endofunctor  $F : \mathcal{C} \rightarrow \mathcal{C}$  is derivable by setting  $J$  to the identity functor

$$F^*A = F_{\text{Id}}^{\text{st}} A \cong \text{Id}A + \text{Lan}_{\text{Id}} F(F^{\text{st}} A) \cong A + F(F^{\text{st}} A)$$

The last step is of course the coYoneda lemma (15). I am not sure yet if the freest monad is derivable from the free monad.

## References

- [1] Thorsten Altenkirch, James Chapman, and Tarmo Uustalu. Monads need not be endofunctors. *Log. Methods Comput. Sci.*, 11(1), 2015.
- [2] Steve Awodey. *Category Theory*. Oxford University Press, Inc., USA, 2nd edition, 2010.
- [3] Ralf Hinze. Generic programming with adjunctions. In Jeremy Gibbons, editor, *Generic and Indexed Programming - International Spring School, SSGIP 2010, Oxford, UK, March 22-26, 2010, Revised Lectures*, volume 7470 of *Lecture Notes in Computer Science*, pages 47–129. Springer, 2010.
- [4] Saunders MacLane. *Categories for the Working Mathematician*. Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.