A Memorandum on Kan Extensions and Monads

Marco Paviotti

June 8, 2023

Abstract

These notes are meant to remind myself of some facts about Kan extension and monads. The first part is devoted to basic definitions about adjunctions, ends and coends, which are needed to explain the proofs later on. The second part is on monads and Kan extensions.

Contents

1 Adjunctions and Monads

Given two functors $L : \mathcal{D} \to \mathcal{C}$ and $R : \mathcal{C} \to \mathcal{D}$ and *adjunction* is an isomorphism of homsets

$$\lfloor \cdot \rfloor : \mathcal{C}(LA, B) \cong \mathcal{D}(A, RB) : \lceil \cdot \rceil$$

which is furthermore natural in A and B. Here $\lfloor \dot{} \rfloor$ and $\lceil \dot{} \rceil$ are the functions witnessing the isomorphism. The adjunction is usually depicted as follows

$$\begin{tikzcd} \mathcal{C} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "R"'] & \mathcal{D} \arrow[l, bend right, "L"'] \end{tikzcd}$$

We say that that L is left adjoint and R is right adjoint and it is indicated by $L \vdash R$. As a consequence of the isomorphism, for $f : LA \to B$ and $g : A \to RB$ we have that

$$\lfloor f \rfloor = g \iff f = \lceil g \rceil$$

and because the isomorphism is natural we can derive the *fusion laws*. For $a : A' \to A$, $b : B \to B'$, $f : LA \to B$ and $g : A \to RB$

$$\begin{aligned} R(b) \cdot \lfloor f \rfloor &= \lfloor b \cdot f \rfloor \\ \lfloor f \rfloor \cdot a &= \lfloor f \cdot L(a) \rfloor \\ b \cdot \lceil g \rceil &= \lceil R(b) \cdot g \rceil \\ \lceil g \rceil \cdot L(a) &= \lceil g \cdot a \rceil \end{aligned}$$

We can also compute the fusion laws this way.

$$\begin{aligned} R(b) \cdot \lfloor f \rfloor \cdot a &= \lfloor b \cdot f \cdot L(a) \rfloor \\ b \cdot \lceil g \rceil \cdot L(a) &= \lceil R(b) \cdot g \cdot a \rceil \end{aligned}$$

This is really all about adjunctions. All the other definitions and constructions are equivalent to this one. I reference here some material for further reading [2, 4, 3].

What is important for the sake of this notes is that every adjunction gives rise to a monad and a comonad where RL is the monad and LR is the comonad. The unit and counit of the adjunction are defined as follows

$$\begin{aligned} \eta_A &= \lfloor id_{LA} \rfloor \\ \epsilon_B &= \lceil id_{RB} \rceil \end{aligned}$$

and are respectively the unit of the monad and the counit of the comonad generated by the adjunction. The join of the monad $\mu : RLRL \to RL$ is defined as $\mu = R\epsilon_L$ and the cojoin $\delta : LR \to LRLR$ is defined as $\delta = L\eta_R$.

2 Limits and Colimits

Given two objects X and Y in a category, $X \times Y$ forms the product of X and Y. We can generalise this further. Given a functor $D : \mathcal{I} \to \mathcal{C}$ the limit $\varprojlim D$ is an universal object such that for every $I \in \mathcal{I}$, there exists a projection map $\varprojlim DI \xrightarrow{\pi_I} DI$ such that for every morphism $DI_1 \xrightarrow{f} DI_2$ we have

$$\pi_{I_2} = f \cdot \pi_{I_1}$$

and furthermore any other object such as this has a unique morphism into the limit commuting with the projections [2, 4].

The limit is right adjoint to the diagonal functor mapping every object to the constant functor and the colimit is the left adjoint to the diagonal functor.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift right, "\Delta"'] \arrow[r, phantom, "\perp"] & \mathcal{C}^{\mathcal{I}} \arrow[l, shift right, "\varinjlim"'] \arrow[r, shift right, "\varprojlim"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "\Delta"']
\end{tikzcd}$$

We right down the isomorphisms

$$\mathcal{C}(\varinjlim_{I \in \mathcal{I}} A(I), B) \cong \mathcal{C}^{\mathcal{I}}(A, \Delta B)$$

$$\mathcal{C}^{\mathcal{I}}(\Delta A, B) \cong \mathcal{C}(A, \varprojlim_{I \in \mathcal{I}} B)$$

2.1 Preservation and Creation of (Co)Limits

A functor $H : \mathcal{C} \to \mathcal{D}$ is said to *preserve* limits if, given a diagram $F : \mathcal{I} \to \mathcal{C}$

$$H \varprojlim_{I \in \mathcal{I}} FI \cong \varprojlim_{I \in \mathcal{I}} HFI$$

In other words, H preserves limits if the limit of the diagram obtained by composition with H, namely HF, corresponds with the limit of F applied to H. In particular, such a functor preserves small limits as well. A functor that preserves small (co)limits is called *(co)continuous*.

As a prominent example, the covariant homset functor $\mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$ preserve limits

$$\mathcal{C}(C, \varprojlim_{I \in \mathcal{I}} FI) \cong \varprojlim_{I \in \mathcal{I}} \mathcal{C}(C, FI) \tag{1}$$

On the other hand, the contravariant homset functor, which may be written as $\mathcal{C}(-, C) = \mathcal{C}^{\mathrm{op}}(C, -) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ carries colimits over to limits in the following sense

$$\mathcal{C}(\varinjlim_{I \in \mathcal{I}} FI, C) \cong \varprojlim_{I \in \mathcal{I}} \mathcal{C}(FI, C) \tag{2}$$

2.2 Dependent product and sum

Let us consider the set $\mathcal{I}$ (or the discrete category $\mathcal{I}$ with only identities). Then the right adjoint to the diagonal functor is called the *dependent product* $\Pi_{I\in\mathcal{I}}B(I)$ for some functor $B:\mathcal{I}\to\mathcal{C}$ and the left adjoint is called the *dependent sum* $\Sigma_{I\in\mathcal{I}}A(I)$ for some functor $A:\mathcal{I}\to\mathcal{C}$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift right=1.5, "\Delta"'] \arrow[r, phantom, "\perp"] & \mathcal{C}^{\mathcal{I}} \arrow[l, shift right=1.5, "\Sigma_{I\in\mathcal{I}}.(-)_I"'] \arrow[r, shift right=1.5, "\Pi_{I\in\mathcal{I}}.(-)_I"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right=1.5, "\Delta"'] \end{tikzcd}$$

The limit preservation (1) and colimit reverse (2) continues to hold for dependent products and sums.

$$\Pi_{I\in\mathcal{I}}\mathcal{C}(X,A(I)) \cong \mathcal{C}(X,\Pi_{I\in\mathcal{I}}A(I) \tag{3}$$

$$\mathcal{C}(\Sigma_{I\in\mathcal{I}}A(I),X) \cong \Pi_{I\in\mathcal{I}}\mathcal{C}(A(I),X) \tag{4}$$

2.2.1 Powers and CoPowers

Now we consider only constant functors and keep $\mathcal{I}$ as the discrete category. The limits and colimits of these functors are called powers and copowers which can be indicated by $\Sigma\mathcal{I}.A = \mathcal{I}\bullet A$ and $\Pi\mathcal{I}.B = B^{\mathcal{I}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift right=1.5, "\Delta"'] \arrow[r, phantom, "\perp"] & \mathcal{C}^{\mathcal{I}} \arrow[l, shift right=1.5, "\Sigma\mathcal{I}.(-)"'] \arrow[r, shift right=1.5, "\Pi\mathcal{I}.(-)"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right=1.5, "\Delta"'] \end{tikzcd}$$

Now equations (3) and (4) in turn specialise to powers and copowers

$$\mathcal{C}(X,A)^{\mathcal{I}} \cong \mathcal{C}(X,A^{\mathcal{I}}) \tag{5}$$

$$\mathcal{C}(\mathcal{I}\bullet A,X) \cong \mathcal{C}(A,X)^{\mathcal{I}} \tag{6}$$

As a consequence of (5) and (6) we get that

$$\mathcal{C}(\mathcal{I}\bullet A,B) \cong \mathcal{C}(A,B)^{\mathcal{I}} \cong \mathcal{C}(A,B^{\mathcal{I}})$$

Now since $\mathcal{I}$ is the discrete category (it has no arrows) it can be regarded as a set! Hence, the set of natural transformation betweeen to constant functors is just the set of functions between the images of these indexed by $\mathcal{I}$

$$\mathcal{C}^{\mathcal{I}}(\Delta X,\Delta Y) \cong \mathcal{C}(X,Y)^{\mathcal{I}} \cong \mathcal{I}\to\mathcal{C}(X,Y) \tag{7}$$

Note that, since $\mathcal{I}$ and $\mathcal{C}(X,Y)$ are sets, then $\mathcal{C}(X,Y)^{\mathcal{I}}$ is the exponential object in **Set** and hence it is isomorphic to $\mathcal{I}\to\mathcal{C}(X,Y)$ which is the set of functions.

In **Set**, $\mathcal{I}\bullet A = A\times\mathcal{I}$ and $B^{\mathcal{I}}$ is the function space $\mathcal{I}\to B$.

3 Ends and Coends

Sometimes it useful to talk about limits and colimits of diagrams that have a contravariant component. These are called *ends* and *coends*. Consider a functor $S : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$. Ends and coends are respectively limits and colimits for S. It is easy to prove [4] that whenever S is "dummy" in the first variable, i.e. S factors through the second projection as in

$$\mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{\pi_2} \mathcal{C} \xrightarrow{T} \mathcal{D}$$

then the end coincides with the limit of T

$$\int_{C:\mathcal{C}} S(C,C) = \varprojlim_{C:\mathcal{C}} TC$$

Ends behave similarly to universal quantification. For a functor $S : \mathcal{C}^{\text{op}} \times \mathcal{C} \times \mathcal{D}^{\text{op}} \times \mathcal{D} \to \mathcal{E}$

$$\int_{C:\mathcal{C}} \int_{D:\mathcal{D}} S(C,C,D,D) \cong \int_{D:\mathcal{C}} \int_{C:\mathcal{D}} S(C,C,D,D) \tag{8}$$

3.1 Preservation of Ends

A functor $H : \mathcal{C} \to \mathcal{D}$ is said to *preserve the end* of a functor $S : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$

$$H \int_{C\in\mathcal{C}} S(C,C) = \int_{C\in\mathcal{C}} HS(C,C)$$

In other words, when $w : e \ddot{\to} S$ is an end of S and $Hw : He \ddot{\to} HS$ is an end for HS.

For example, as it was the case for the limits and colimits, the homset functor preserves ends in the following way

$$\mathcal{C}(X, \int_{C:\mathcal{C}} S(C,C)) = \int_{C:\mathcal{C}} (\mathcal{C}(X, S(C,C))) \tag{9}$$

and reserves ends into coends

$$\mathcal{C}(\int^{C:\mathcal{C}} S(C,C), X) = \int_{C:\mathcal{C}} (\mathcal{C}(S(C,C)), X) \tag{10}$$

3.2 Natural transformations and Ends

Natural transformations are examples of ends. Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, the end of the homset functor $\mathcal{D}^{\mathcal{C}}(F-, G-)$ is the set of natural transformations from F to G

$$\text{Nat}(F,G) = \int_{C:\mathcal{C}} \mathcal{D}(FC, GC) \tag{11}$$

4 The Yoneda Lemma

Assume a locally small category $\mathcal{C}$. For all covariant functors $F : \mathcal{C} \to \mathbf{Set}$,

$$FC \cong \mathcal{C}(C, -) \dot{\to} F \tag{12}$$

The functions witnessing the isomorphism are given by $f(x, g) = F(g)(x)$ and its inverse $f^{-1}(i) = i_C(id_C)$.

This result holds for contravariant functors as well. Say $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is our contravariant functor then

$$FC \cong \mathcal{C}^{\mathrm{op}}(C, -) \dot{\to} F \cong \mathcal{C}(-, C) \dot{\to} F$$

The homset functor $\mathcal{C}(-, -) : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is also called the Yoneda embedding, y. The fact that y is an embedding means that the y is a fully faithful functor and therefore it preserves (because it is a functor) and reflects (because it is fully faithful) isomorphisms

$$y_C \cong y_D \iff C \cong D \tag{13}$$

5 Kan Extensions

Consider a reindexing functor $J : \mathcal{C} \to \mathcal{D}$ and define the functor $- \circ J : \mathcal{E}^{\mathcal{D}} \to \mathcal{E}^{\mathcal{C}}$ also named App_J.

We want to find out whether App_J has a left and right adjoint. We call its left and right adjoint Lan_J and Ran_J respectively.

$$\mathcal{E}^{\mathcal{C}} \underset{\mathrm{Ran}_J}{\overset{\mathrm{App}_J}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathcal{E}^{\mathcal{D}} \underset{\mathrm{App}_J}{\overset{\mathrm{Lan}_J}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathcal{E}^{\mathcal{C}}$$

Now because Lan_J is the left adjoint and Ran_J to be the right one we would expect the following to be a natural isomorphisms

$$\mathcal{E}^{\mathcal{D}}(\mathrm{Lan}_J H, G) \cong \mathcal{E}^{\mathcal{C}}(H, \mathrm{App}_J G) \qquad \mathcal{E}^{\mathcal{C}}(\mathrm{App}_J H, G) \cong \mathcal{E}^{\mathcal{D}}(H, \mathrm{Ran}_J G)$$

In the MacLane [4] he is defining the right Kan extension and the proving it is the right adjoint. Here we take a different approach. By using the Yoneda lemma we derive the right adjoint which is unique up to isomorphism so it must be the right Kan extension. This proof is taken form Hinze's work on generic programming [3]. Here I have made it a bit more precise.

$$
\begin{aligned}
\mathcal{E}^{\mathcal{C}}(\mathrm{App}_J A, B) = \{ & \text{ Homsets in the exponential category are natural transformations } \} \\
& \mathrm{Nat}(\mathrm{App}_J A, B) \\
& \cong \{ \text{ Natural transformations are ends (11) } \} \\
& \int_{X:\mathcal{C}} \mathcal{C}(AJX, BX) \\
& \cong \{ \text{ Yoneda with } \mathcal{C}(A-, BX) \ \} \\
& \int_{X:\mathcal{C}} \mathrm{Nat}(\mathcal{D}(-, JX), \mathcal{C}(A-, BX)) \\
& \cong \{ \text{ by (11) } \} \\
& \int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{D}(Y, JX) \to \mathcal{C}(AY, BX) \\
& \cong \{ \text{ by (7) } \} \\
& \int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{C}(AY, BX)^{\mathcal{D}(Y,JX)} \\
& \cong \{ \text{ by (5) } \} \\
& \int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{C}(AY, BX^{\mathcal{D}(Y,JX)}) \\
& \cong \{ \text{ by (8) } \} \\
& \int_{Y:\mathcal{D}} \int_{X:\mathcal{C}} \mathcal{C}(AY, BX^{\mathcal{D}(Y,JX)}) \\
& \cong \{ \text{ Homsets preserve ends (9) } \} \\
& \int_{Y:\mathcal{D}} \mathcal{C}(AY, \int_{X:\mathcal{C}} BX^{\mathcal{D}(Y,JX)}) \\
& \cong \{ \text{ Natural transformation are ends (11) } \} \\
& \mathrm{Nat}(A-, \int_{X:\mathcal{C}} BX^{\mathcal{D}(-,JX)}) \\
& \cong \{ \text{ Homsets in the exponential category are natural transformations } \} \\
& \mathcal{E}^{\mathcal{D}}(A-, \int_{X:\mathcal{C}} BX^{\mathcal{D}(-,JX)})
\end{aligned}
$$

For all functors $J : \mathcal{C} \to \mathcal{D}$, $A : \mathcal{C} \to \mathcal{E}$ and $B : \mathcal{D} \to \mathcal{E}$

$$\mathrm{Ran}_J AY = \int_{X \in \mathcal{C}} \Pi_{\mathcal{D}(Y,JX)} AX$$

We now compute the left Kan extension. I could not find this proof is not in [3, 4].

$$
\begin{aligned}
\mathcal{E}^{\mathcal{C}}(A, \mathrm{App}_J B) &= A \dot{\to} \mathrm{App}_J B \\
&\cong \{ \text{ Natural transformations are ends (11) } \} \\
&\int_{X:\mathcal{C}} \mathcal{C}(AX, BJX) \\
&\cong \{ \text{ by Yoneda with } \mathcal{C}(AX, B-) \ \} \\
&\int_{X:\mathcal{C}} \mathcal{D}(JX, -) \dot{\to} \mathcal{C}(AX, B-) \\
&\cong \{ \text{ Natural transformations are ends(11) } \} \\
&\int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{D}(JX, Y) \to \mathcal{C}(AX, BY) \\
&\cong \{ \text{ The set functions space is a power (7) } \} \\
&\int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{C}(AX, BY)^{\mathcal{D}(JX,Y)} \\
&\cong \{ \text{ Homsets revert powers into copowers (6) } \} \\
&\int_{X:\mathcal{C}} \int_{Y:\mathcal{D}} \mathcal{C}(\mathcal{D}(JX, Y) \bullet AX, BY) \\
&\cong \{ \text{ Switching over ends (8) } \} \\
&\int_{Y:\mathcal{D}} \int_{X:\mathcal{C}} \mathcal{C}(\mathcal{D}(JX, Y) \bullet AX, BY) \\
&\cong \{ \text{ Homsets reverse ends into coends (10) } \} \\
&\int_{Y:\mathcal{D}} \mathcal{C}(\int^{X:\mathcal{C}} \mathcal{D}(JX, Y) \bullet AX, BY) \\
&\cong \{ \text{ Natural transformations are ends (11) } \} \\
&\mathrm{Nat}(\int^{X:\mathcal{C}} \mathcal{D}(JX, -) \bullet AX, B) \\
&= \mathcal{E}^{\mathcal{D}}(\int^{X:\mathcal{C}} \mathcal{D}(JX, -) \bullet AX, B)
\end{aligned}
$$

Now by looking at what we have got we define the left Kan extension as a functor parametrised by A

$$\mathrm{Lan}_J AY = \int^{X \in \mathcal{C}} \mathcal{D}(JX, Y) \bullet AX$$

5.1 Yoneda revisited

5.1.1 Yoneda

Given a functor $F : \mathcal{C} \to \mathbf{Set}$, the Yoneda lemma says that F is isomorphic to the right Kan extension over the identity functor of F

$$F \cong \mathrm{Ran}_{\mathrm{Id}} F \tag{14}$$

We compute as follows

$$\begin{aligned} FC &\cong \mathrm{Nat}(\mathcal{C}(C,-),F) \\ &\int_{X:\mathcal{C}} \mathbf{Set}(\mathcal{C}(C,X),FX) \\ &\int_{X:\mathcal{C}} \mathcal{C}(C,X) \to FX \\ &\int_{X:\mathcal{C}} FX^{\mathcal{C}(C,X)} \\ &\mathrm{Ran}_{\mathrm{Id}} FC \end{aligned}$$

Another way of proving this is by using the fact that $\mathrm{App}_{\mathrm{Id}}$ is left adjoint to $\mathrm{Ran}_{\mathrm{Id}}$.

$$\mathcal{E}^{\mathcal{C}}(\mathrm{App}_{\mathrm{Id}} G, F) = \mathcal{E}^{\mathcal{C}}(G,F) \cong \mathcal{E}^{\mathcal{C}}(G, \mathrm{Ran}_{\mathrm{Id}} F)$$

Thus, since $\mathcal{E}^{\mathcal{C}}(-,F) \cong \mathcal{E}^{\mathcal{C}}(-,\mathrm{Ran}_{\mathrm{Id}} F)$ by Yoneda (13) we have $F \cong \mathrm{Ran}_{\mathrm{Id}} F$.

5.1.2 CoYoneda

The coYoneda lemma states that

$$F \cong \mathrm{Lan}_{\mathrm{Id}} F \tag{15}$$

This can be proven by instantiating the left Kan extension with the identity functor obtaining the adjunction $\mathrm{Lan}_{\mathrm{Id}} \dashv \mathrm{App}_{\mathrm{Id}}$. From the adjunction we know that

$$\mathcal{E}^{\mathcal{C}}(\mathrm{Lan}_{\mathrm{Id}} F, G) \cong \mathcal{E}^{\mathcal{C}}(F, \mathrm{App}_{\mathrm{Id}} G) = \mathcal{E}^{\mathcal{C}}(F, G \circ \mathrm{Id}) = \mathcal{E}^{\mathcal{C}}(F,G)$$

is a natural in F and G. Because this isomorphism is natural we know that $\mathcal{E}^{\mathcal{C}}(\mathrm{Lan}_{\mathrm{Id}} F, -) \cong \mathcal{E}^{\mathcal{C}}(F,-)$ which by (13) implies $\mathrm{Lan}_{\mathrm{Id}} F \cong F$ since $\mathcal{E}^{\mathcal{C}}(-,-)$ is the Yoneda embedding.

5.2 Left and Right Shifts

As a particular case $D : 1 \to \mathcal{D}$. So $\mathrm{App}_D H = H \circ D = HD(*)$

$$\begin{tikzcd} \mathcal{E} \arrow[r, shift right, "\mathrm{Rsh}_D"'] \arrow[r, phantom, "\perp"] & \mathcal{E}^{\mathcal{D}} \arrow[l, shift right, "-D"'] \arrow[r, shift right, "-D"'] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, shift right, "\mathrm{Lsh}_D"'] \end{tikzcd}$$

$$\mathcal{E}^{\mathcal{D}}(\mathrm{Lsh}_J H, G) \cong \mathcal{E}(H, GD)$$

$$\mathcal{E}(HD, G) \cong \mathcal{E}^{\mathcal{D}}(H, \mathrm{Rsh}_J G)$$

$$\mathrm{Lsh}_D AX = \Sigma_{\mathcal{D}(D,X)} A$$

$$\mathrm{Ran}_D AX = \Pi_{\mathcal{D}(X,D)} A$$

6 Monads from Kan Extensions

6.1 The Codensity Monad

The codensity monad is just the right Kan extension of J along J

$$\mathrm{Cod}\ JX = \mathrm{Ran}_J JX = \int_{Y:\mathcal{C}} JY^{\mathcal{D}(X,JY)}$$

6.2 The Codensity Transformation

If $L \dashv R$ then both $L \circ - \vdash R \circ -$ and $- \circ R \vdash - \circ L$ are adjunctions.

$$\text{If } \mathcal{L} \underset{R}{\overset{L}{\underset{\perp}{\leftrightarrows}}} \mathcal{R} \text{ then } \mathcal{E}^{\mathcal{L}} \underset{-\circ L}{\overset{-\circ R}{\underset{\perp}{\leftrightarrows}}} \mathcal{E}^{\mathcal{R}}$$

Because of this fact there is a natural isomorphism

$$\mathcal{E}^{\mathcal{L}}(F \circ R, G) \cong \mathcal{E}^{\mathcal{R}}(F, G \circ L)$$

Now, since $F \circ R$ is $\mathrm{App}_R F$ then $\mathcal{E}^{\mathcal{L}}(F \circ R, G) \cong \mathcal{E}^{\mathcal{R}}(F, \mathrm{Ran}_R G)$. But then we know also that

$$\mathcal{E}^{\mathcal{L}}(F, G \circ L) \cong \mathcal{E}^{\mathcal{R}}(F, \mathrm{Ran}_R G)$$

Since the Yoneda embedding $\mathcal{E}^{\mathcal{R}}(-,-) : \mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is fully faithful, by (13) then $G \circ L \cong \mathrm{Ran}_R G$. (But this is also the proof that adjoints are unique up-to isomorphism). Now, if $G = R$ then we get that the monad

$$R \circ L \cong \mathrm{Ran}_R R \tag{16}$$

7 On Free Monads

Given a category $\mathcal{C}$ with coproducts, for an object $A \in \mathrm{Obj}(\mathcal{C})$ and an endofunctor $F : \mathcal{C} \to \mathcal{C}$, the initial algebra for the functor $F_A X = A + FX$ is the initial object in the category of F_A-algebras

$$\begin{tikzcd} A + F(\mu F_A) \arrow[r, dotted, "A+(\hat{f})"] \arrow[d, "\mathrm{In}"'] & A + FB \arrow[d, "{[f;\beta]}"] \\ \mu F_A \arrow[r, dotted, "\hat{f}"] & B \end{tikzcd}$$

By Lambek's lemma the initial algebra In is an isomorphism which makes the carrier of the initial algebra the least fixed-point of F_A denoted by μF_A which is the free monad F^*A[1].

$$F^*A \cong A + FF^*A$$

The *unit* of the monad $\eta_A : A \to F^*A$ is given by the left injection composed with the inverse In°. Furthermore, the map representing the operations of the monad $\mathrm{op}_A : FF^*A \to F^*A$ is given by the right injection composed with In°.

To construct the *multplication* $\mu_A : F^*F^*A \to F^*A$ we will be showing that (F^*A, In) is the *free F-algebra*: the forgetful functor $U : F\text{-Alg} \to \mathcal{C}$ has a left adjoint mapping A to (F^*A, In).

The monad arising from this adjunction is called the *algebraically free* monad, which corresponds to $F*$.

$$\begin{tikzcd} F\text{-Alg} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "\mathrm{Free}"'] \arrow[loop right, "F*"] \end{tikzcd}$$

with the operations ceiling and floor witnessing the natural isomorphism on both A and B

$$\lfloor\cdot\rfloor : F\text{-Alg}(\mathrm{Free}\ A, B) \cong \mathcal{C}(A, UB) : \lceil\cdot\rceil$$

It is easy now to see that Free A is $(\mu F_A, \mathrm{In})$ (up-to isomorphism). If we assume the adjunction Free $\dashv U$ then given an F-algebra $\beta : FB \to B$ on and a map $f : A \to B$ then we can construct a unique F-algebra homomorphism $! : \mathrm{Free}\ A \to B$ such that $!\dot{\eta}_A = f$ which is what is required for Free A to be an initial F_A algebra.

Conversely, if F_A is an initial F_A algebra, then for all F_A algebras $[f;\beta] : A + FB \to B$ there is a unique homomorphism $\hat{f} = (\mu F_A, \mathrm{In}) \to (B, \beta)$ which is what is required to get an adjunction. We just have to check that the following

[1]Note that the least fixed-point of a functor does not correspond in general to the initial algebra, thus, whenever we talk about inductive data types generated by their constructors it is more correct to refer to these in terms of initial algebras which, by Lambek's Lemma, always correspond to least fixed-points.

diagram commutes which is a trivial exercise

$$
\begin{tikzcd}
F^*(A) \arrow[r, "U\hat{f}"] & UB \\
A \arrow[u, "\eta_A"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

Now we can prove that algebras for the endofunctor F are in one-to-one correspondence with *algebras for the monad* F^*, moreover there is an isomorphism of categories

$$\iota : F\text{-Alg} \cong \mathcal{C}^{F^*} : \iota^\circ$$

where $\mathcal{C}^{F^*}$ is the Eilenberg-Moore category for the monad F^*.

The isomorphisms (ι) is constructed by taking an F-algebra $\beta : FA \to A$ and constructing a F_A algebra $[id_A; \beta] : A + FA \to A$ then the F^* algebra is $U\lceil [id_A; \beta] \rceil : F^*A \to A$ recursively interpreting the structure of F^*A into A using the F-algebra.

The other direction (ι°) is as follows. Assume an algebra for the monad F^*, $g : F^*A \to A$, then $FA \xrightarrow{F(\eta_A)} FF^*A \xrightarrow{\text{In}} F^*A \xrightarrow{g} A$.

The isomorphisms on the arrows is the identity since F-algebra homomorphisms are also F^* algebra homomorphisms.

It remains to check that this is in fact an isomorphism of categories. The idea is that the equations on the algebras for the monad in the Eilenberg-Moore category force the algebras to be modular.

7.1 Free Monads and The Right Kan Extension

From the previous section we know that every monad arising from an adjunction $L \dashv R$ (hence every monad!) is isomorphic to the right Kan extension of R along R, denoted by $\text{Ran}_R R$ and also known as the *codensity monad*.

Since the free monad is a monad, also the free monad can be transformed using the codensity transformation. Since the algebraically free monad factors as $F^* = U\text{Free}$ then by (16)

$$F^* \cong \text{Ran}_U U$$

By unfolding the definitions we get that

$$F^*A \cong \int_{Z:F\text{-Alg}} UZ^{\mathcal{C}(A,UZ)} \tag{17}$$

However, there is another way to transform the free monad. Using Yoneda (14)

$$F^*A \cong A + FF^*A \cong A + (\text{Ran}_{\text{Id}} F)F^*X \cong A + \int_{X:\mathcal{C}} FX^{\mathcal{C}(F^*A,X)} \tag{18}$$

7.2 Free Monads and the Left Kan Extension

Using CoYoneda (15)

$$F^*A \cong A + F \circ F^* \cong A + (\mathrm{Lan}_{\mathrm{Id}}F)F^*A \cong A + \int^{X:\mathbf{Set}} \mathcal{C}(X, F^*A) \bullet FX \quad (19)$$

In **Set**,

$$F^*A \cong A + \int_{X:\mathcal{C}} FX \times (X \to F^*A)$$

7.3 Algebras for the Left Kan extension

Every G-algebra is isomorphic to the algebras for the left Kan extension on G.

$$\begin{aligned}
\mathcal{D}(\mathrm{Lan}_{\mathrm{Id}}GA, A) &\cong \mathcal{D}^{\mathcal{D}}(\mathrm{Lan}_{\mathrm{Id}}G, \mathrm{Rsh}_A A) \qquad \{ \text{ by } -A \dashv \mathrm{Rsh}_{\mathrm{Id}} \ \} \\
&\cong \mathcal{D}^{\mathcal{D}}(G, \mathrm{Rsh}_A A \circ \mathrm{Id}) \qquad \{ \text{ by } \mathrm{Lan}_{\mathrm{Id}} \dashv \mathrm{App}_{\mathrm{Id}} \ \} \\
&\cong \mathcal{D}^{\mathcal{D}}(G, \mathrm{Rsh}_A A) \\
&\cong \mathcal{D}(GA, A) \qquad \{ \text{ by } -A \dashv \mathrm{Rsh}_{\mathrm{Id}} \ \}
\end{aligned}$$

This proof is worth of reminding, but a simpler way to do it is to use Yoneda (14) directly

$$\mathcal{D}(\mathrm{Lan}_{\mathrm{Id}}GA, A) \cong \mathcal{D}(GA, A)$$

7.4 The Freest Monad

Given a functor $J : \mathcal{C} \to \mathcal{D}$ and an endofunctor $F : \mathcal{C} \to \mathcal{D}$ the freest monad is defined as follows

$$F_J^{\mathrm{st}}A \cong JA + \mathrm{Lan}_J F(F^{\mathrm{st}}A)$$

Note that F is not an endofunctor and so the freest monad is in fact a *relative monad* [1]. The free monad over an endofunctor $F : \mathcal{C} \to \mathcal{C}$ is derivable by setting J to the identity functor

$$F^*A = F_{\mathrm{Id}}^{\mathrm{st}}A \cong \mathrm{Id}A + \mathrm{Lan}_{\mathrm{Id}}F(F^{\mathrm{st}}A) \cong A + F(F^{\mathrm{st}}A)$$

The last step is of course the coYoneda lemma (15). I am not sure yet if the freeest monad is derivable from the free monad.

References

[1] Thorsten Altenkirch, James Chapman, and Tarmo Uustalu. Monads need not be endofunctors. *Log. Methods Comput. Sci.*, 11(1), 2015.

[2] Steve Awodey. *Category Theory*. Oxford University Press, Inc., USA, 2nd edition, 2010.

[3] Ralf Hinze. Generic programming with adjunctions. In Jeremy Gibbons, editor, *Generic and Indexed Programming - International Spring School, SSGIP 2010, Oxford, UK, March 22-26, 2010, Revised Lectures*, volume 7470 of *Lecture Notes in Computer Science*, pages 47–129. Springer, 2010.

[4] Saunders MacLane. *Categories for the Working Mathematician.* Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.