Technology
Arts Sciences
TH Köln

Modelling and Simulation of Continuous systems

MAIT 1 WS21/22

Assignments 2-3: Car suspension and Predator-prey models

TH Köln - Campus Gummersbach

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1. Vehicle simulation

Assignment 2 required us to model, simulate and analyze a quarter car suspension system. Basically, to have a look at the behaviour of a car, its wheels and suspension when driving on a bumpy road (Figure 1-1). Particularly, a vertical movement of the car is of the interest.

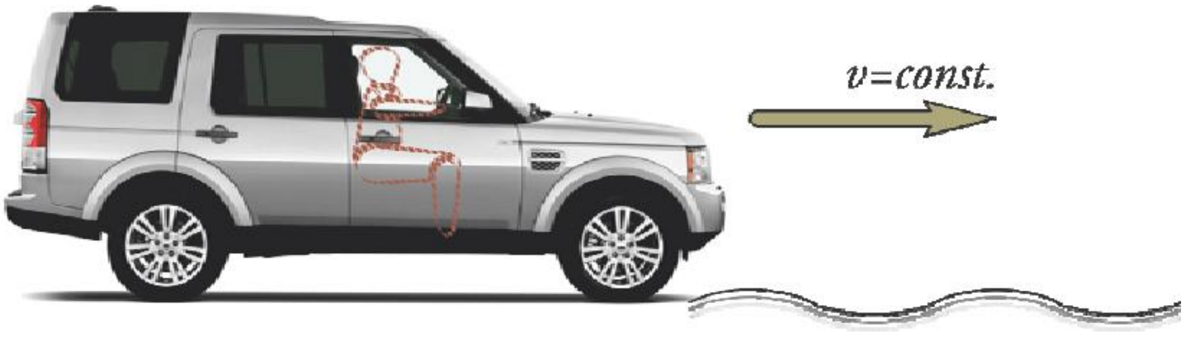

Figure 1-1: An SUV about to encounter a bumpy section of the road. Source: (Krasniqi et al., 2011)

1.1.Task

Among the tasks we had the following:

- Develop a simulation model for the vehicle – quarter car suspension system with passive suspension (without controller)
- Find the reasonable parameters for these models
- Perform simulations
- Analyze the model using methods of linear system theory

For modelling we selected Xcos module of Scilab.

1.2.Quarter car suspension model general overview

The behavior of the suspension of the vehicle can be studied using a mathematical model and running simulations on them. One of the simplest models is quarter car suspension model shown on Figure 1-2. The model consists of two masses (a mass of a wheel and axle, as well as, a quarter mass of the body of a vehicle), suspension consisting of a spring and a damper and, lastly, a spring representing the stiffness of the tire.

In this case the gravity is neglected, as well as, the mass of the car and/or its quarter is not considered since we assume the hull not to be moving. Instead, vertical movement of suspension is tracked.

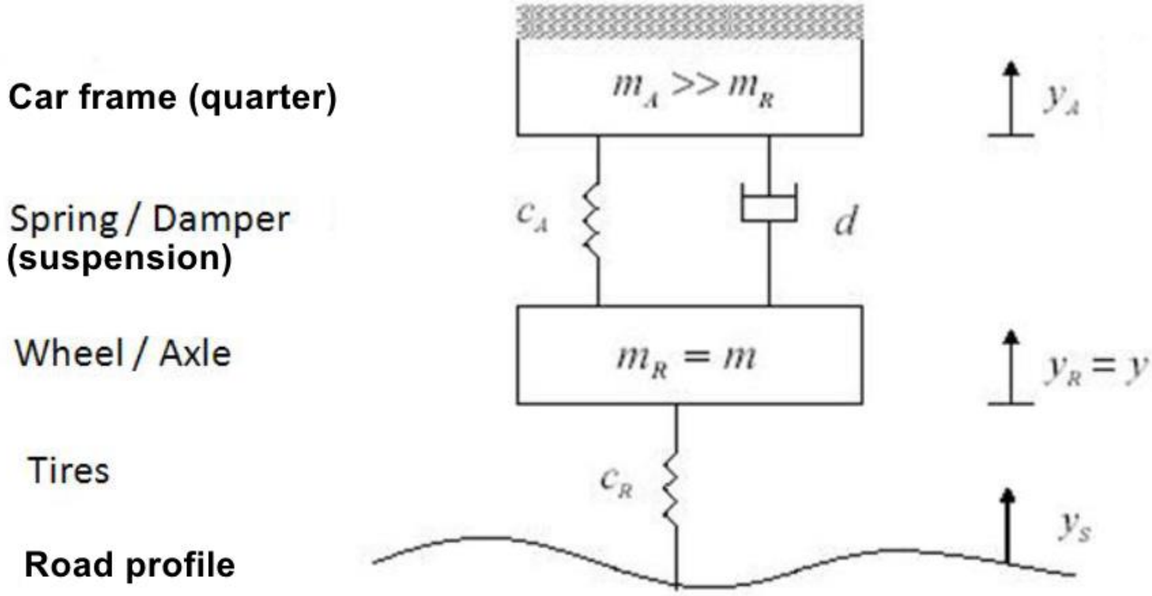

Figure 1-2: Quarter car suspension model. Adopted from the lecture slides.

On Figure 1-2 the following parameters are presented:

- $m_R\ [kg]$ – mass of the wheel and suspension
- $m_A\ [kg]$ – mass of the vehicle body (quarter)
- $c_A\ [N/m]$ – spring constant of the suspension system
- $c_R\ [N/m]$ – spring constant of the wheel and tire
- $d\ [Ns/m]$ – damping coefficient of the suspension system
- $y_A\ [m]$ – displacement of the vehicle body (output)
- $y_R\ [m]$ – displacement of the wheel (output)
- $y_S\ [m]$ – change in the road profile (input)

With the initial conditions $y(t_0) = y_0$ and $\dot{y}(t_0) = \dot{y}_0$ the following equations were considered according to Newton's second law:

$$m\ddot{y} + d\dot{y} + (c_A + c_R)y = c_R y_S$$

The detailed state analysis and equations that follow will be described in the next chapters.

1.3. Dimensionless equations

Given equations:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}\left(-dx_2 - cx_1 + k(t)\right)$$

$C_R \cdot y_S$

where:

x_1 = path

x_2 = speed

$\dot{x}_2$ = acceleration

$k(t) = C_R \cdot y_S$

k(t) is basically a force and therefore, has Newton as its unit.

Other parameters are explained on Figure 2-2

Dimensionless counterparts with units:

$\dot{x}_2 = \dot{\tilde{x}}_2 \cdot \frac{m}{s^2}$; $m = \tilde{m} \cdot kg$; $d = \tilde{d} \cdot \frac{Ns}{m}$; $x_2 = \tilde{x}_2 \cdot \frac{m}{s}$;

$c = \tilde{c} \cdot \frac{N}{m}$; $x_1 = \tilde{x}_1 \cdot m$; $k(t) = \tilde{k}(t) \cdot N$

First equation: $\dot{x}_1 = x_2$

Velocity is the first derivative of displacement. Therefore there is the same term on both sides of the equation. Both sides will cancel out easily.

Second equation: $\dot{x}_2 = \frac{1}{m}\left(-dx_2 - cx_1 + k(t)\right)$

$$\dot{\tilde{x}}_2 \cdot \frac{m}{s^2} = \frac{1}{\tilde{m}} \cdot \frac{1}{kg}\left(-\tilde{d} \cdot \frac{N\cancel{s}}{\cancel{m}} \cdot \tilde{x}_2 \cdot \frac{\cancel{m}}{\cancel{s}} - \tilde{c} \cdot \frac{N}{\cancel{m}} \cdot \tilde{x}_1 \cdot \cancel{m} + \tilde{k}(t) \cdot N\right)$$

Considering $N = \frac{kg \cdot m}{s^2}$

$$\tilde{\ddot{x}}_2 \cdot \frac{m}{s^2} = \frac{1}{\tilde{m}_1} \cdot \frac{1}{\cancel{kg}} \left(-\tilde{d}\,\tilde{\dot{x}}_2 - \tilde{c}\,\tilde{x}_1 + \tilde{k}(t) \right) \cdot \overbrace{\frac{\cancel{kg} \cdot m}{s^2}}^{N}$$

$$\tilde{\ddot{x}}_2 \cdot \frac{m}{s^2} = \frac{1}{\tilde{m}_1} \left(-\tilde{d\dot{x}_2} - \tilde{cx_1} + \tilde{k}(t) \right) \cdot \frac{m}{s^2} \quad \Big| \times \frac{s^2}{m}$$

$$\tilde{\ddot{x}}_2 = \frac{1}{\tilde{m}_1} \left(-\tilde{d\dot{x}_2} - \tilde{cx_1} + \tilde{k}(t) \right)$$

$$\boxed{\tilde{\ddot{x}}_2 = \frac{1}{\tilde{m}_1} \left(-\tilde{d\dot{x}_2} - \tilde{cx_1} + \tilde{k}(t) \right)}$$

1.4. Parameter selection

PARAMETERES	VALUES
Tire mass (m_r)	60 kg
Spring stiffness (c_a)	2000 N/m
Damping coefficient (d)	2000 Ns/m
Tire stiffness (c_r)	20000 N/m

Table 1-1: Parameters selected for a vehicle model

1.5.X-cos model

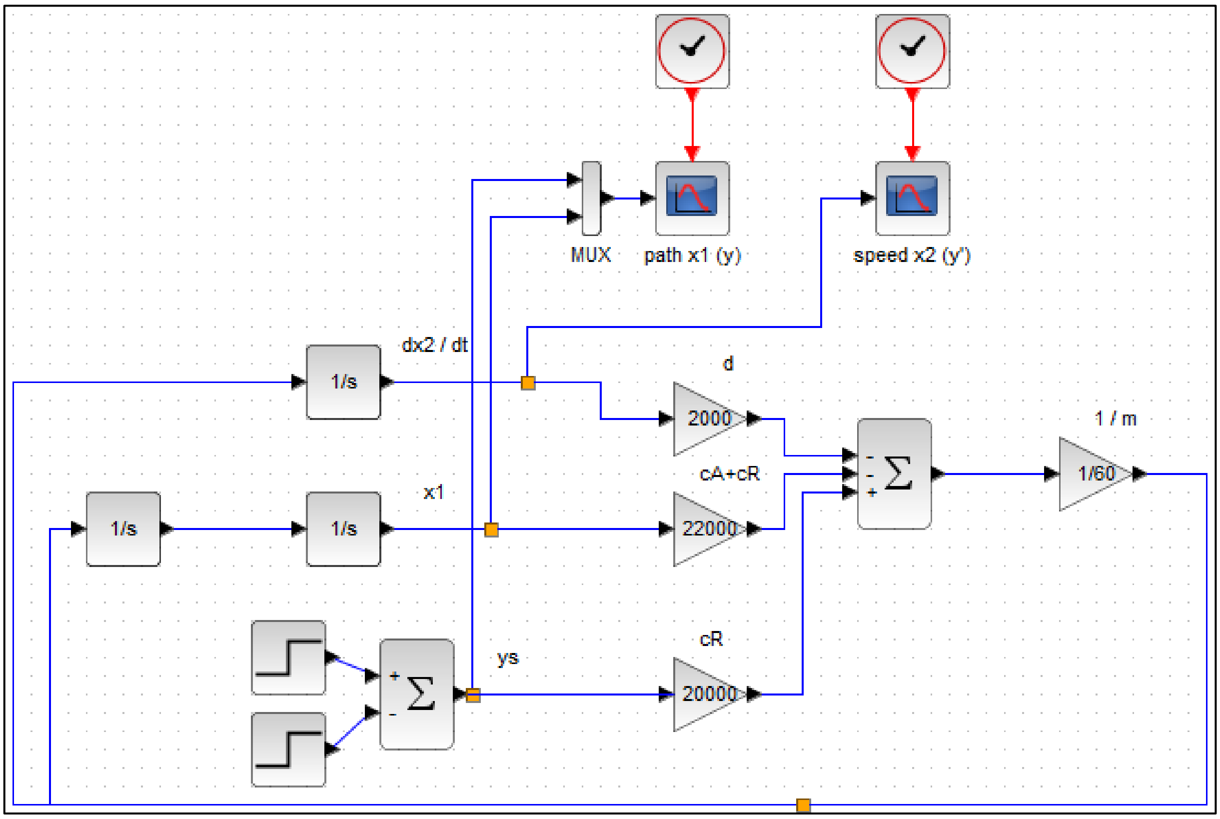

Figure 1-3: XCOS simulation - vehicle model

The simulation of the quarter car model was performed in xcos, using integrator blocks to solve the differential equations. Eight seconds were chosen as the integration time. The change in road level was modeled as a step input that jumps to 0.1 (m) after one second and returns to 0 (m) in the fourth sec-ond. The initial parameters for the simulation were chosen based on research. The mass was de-fined as 60 kg, as it includes the tire mass and the mass of the suspension system. The damping coef-ficient of the damper between chassis and tire was set to 2000 Ns/m. The stiffness c, that is, the com-bined stiff-ness of the tire and the shock absorber, was set to 22000 N/m. Based on the research, the stiffness of the tire is usually significantly greater than the stiffness cA of the shock absorber, so the stiffness of the shock absorber was defined as 2000 N/m. [(Jugulkar et al., 2016), (Tica et al., 2011)]

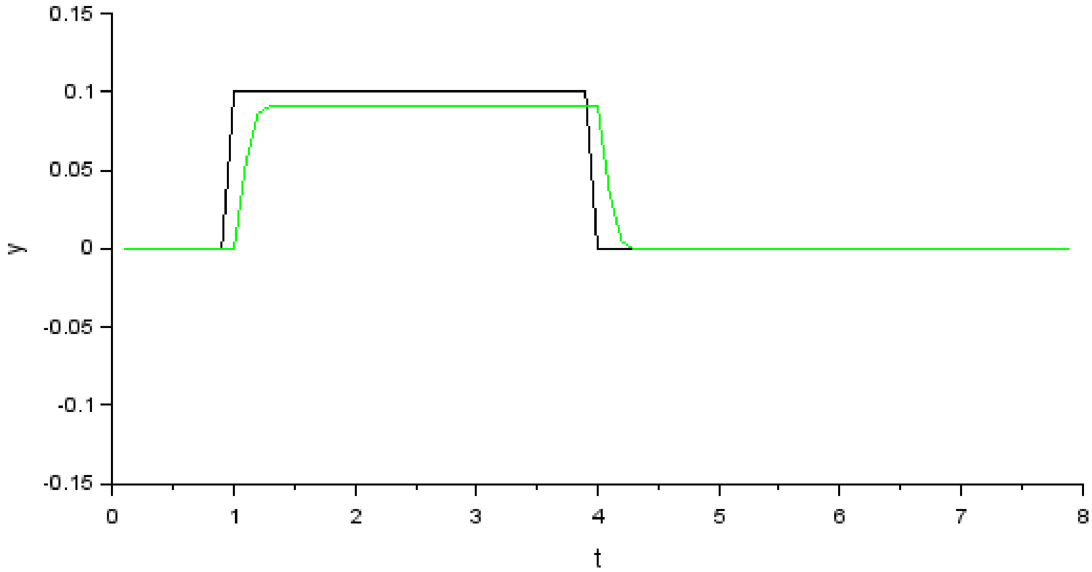

Figure 1-4: Vehicle model - street level (black) and tire path (green)

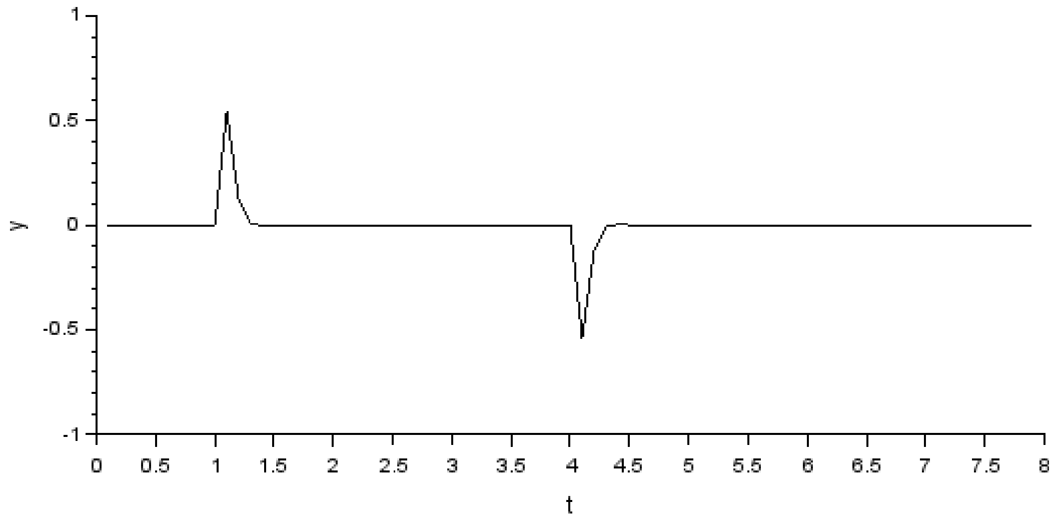

Figure 1-5: Vehicle model - vertical velocity of tire

In the simulation of the tire path, it can be seen that it follows the change in the road level with a slight delay and damping behavior, then leveling off with a small offset below the road-level. This offset results from the difference between the tire stiffness cR and the combined stiffness c. With a neglected stiffness of the shock absorber ca, the tire would completely move into the wheel housing and therefore completely following the change of the road level.

1.6.Linear system theory analysis

$$\dot{x_2} = -\frac{d}{m}x_2 - \frac{c}{m}x1 + \frac{1}{m}k(t)$$

$$\dot{x_1} = x_2$$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{m} & -\frac{d}{m} \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{c_r}{m} \end{bmatrix} k(t)$$

$$x_2 = [1 \quad 0] + 0k(t)$$

Therefore;

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{c}{m} & -\frac{d}{m} \end{bmatrix} \quad ; B = \begin{bmatrix} 0 \\ \frac{c_r}{m} \end{bmatrix}; \; C = [1 \quad 0] \, ; D = 0$$

1.6.1.Transfer Function derivation

$$(SI - A)^{-1} = \begin{bmatrix} s & 1 \\ \frac{c}{m} & s + \frac{d}{m} \end{bmatrix}^{-1}$$

$$(SI - A)^{-1} = \begin{bmatrix} s + \frac{d}{m} & 1 \\ \frac{c}{m} & s \end{bmatrix} \times \frac{1}{s^2 + \frac{d}{m}s + \frac{c}{m}}$$

$$C(SI - A)^{-1} = \begin{bmatrix} \frac{s + \frac{d}{m}}{s^2 + \frac{d}{m}s + \frac{c}{m}} & \frac{1}{s^2 + \frac{d}{m}s + \frac{c}{m}} \end{bmatrix}$$

$$C(SI - A)^{-1}B = \begin{bmatrix} \frac{s + \frac{d}{m}}{s^2 + \frac{d}{m}s + \frac{c}{m}} & \frac{1}{s^2 + \frac{d}{m}s + \frac{c}{m}} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{c_r}{m} \end{bmatrix}$$

$$C(SI - A)^{-1}B = \begin{bmatrix} \frac{1}{s^2 + \frac{d}{m}s + \frac{c}{m}} \end{bmatrix} \frac{c_r}{m}$$

$$TF = C(SI - A)^{-1}B = \frac{c_r/m}{s^2 + d/m\, s + c/m}$$

1.6.2.Eigen value calculation

$$|A - \lambda I| = 0$$

$$A - \lambda = \begin{bmatrix} 0 & 1 \\ -\frac{c}{m} & -\frac{d}{m} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -\frac{c}{m} & -\frac{d}{m} - \lambda \end{vmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{c}{m} & -\frac{d}{m} - \lambda \end{vmatrix} = 0$$

$$\lambda^2 + \frac{d}{m}\lambda + \frac{c}{m} = 0$$

Using quadratic formula, eigen values can be obtained as follows,

$$\lambda = \frac{-\frac{d}{m} \pm \sqrt{\left(\frac{d}{m}^2\right) - \left(4 \times 1 \times \frac{c}{m}\right)}}{2}$$

$$-\frac{d}{m} \pm \sqrt{\left(\frac{d}{m}^2\right) - \left(4 \times 1 \times \frac{c}{m}\right)} < 0$$

Say, $a = \frac{d}{m}; b = \frac{4c}{m}$

$$-a \pm \sqrt{a^2 - b} < 0$$

$$(b > 0), (b < a^2)$$

$$\frac{-d}{m} > 0; \ \frac{d^2}{m} - \frac{4c}{m} < 0$$

Condition for stability: $\frac{d^2}{m} - \frac{4c}{m} < 0$ i.e the eigenvalues are to be complex conjugate in left hand side of the real axis or in other words c should be greater than d for the system to be stable . And this is a always true as c which the combined stiffness of spring is normally greater then the damping constant d.

1.6.3.Substituting parameter values

$$d = 2000\ ;\ m = 60\ ;\ c = 22000$$

The transfer function would be,

$$TF = \frac{0.016s + 0.555}{s^2 + 33.33s + 366.6}$$

And the roots or eigen values would be

$$\lambda = \frac{-\frac{2000}{60} \pm \sqrt{\left(\frac{2000^2}{60}\right) - \left(4 \times 1 \times \frac{22000}{60}\right)}}{2}$$

$$\lambda = \frac{-33.3 \pm \sqrt{(1111.1) - 1466.6}}{2}$$

$$\lambda = -16.66 \pm 9.43$$

Complex roots, asymptotically stable.

1.6.4.Controllability

According to Kalman's test, if rank of controllability matrix defined as $Q = |B \ \ AB \ \ A^2B \ \ .... \ \ A^{n-1}B|$ is not equal to zero. Then, the system is controllable.

$$AB = \begin{bmatrix} 0 & 1 \\ 366.6 & -33.3 \end{bmatrix} \begin{bmatrix} 0 \\ 333.3 \end{bmatrix}$$

$$AB = 1.0e + 04 \begin{bmatrix} 0.0333 \\ -1.111 \end{bmatrix}$$

Thus, controllability matrix Q is as below,

$$\mathrm{Q} = \ |\mathrm{B\ AB}|$$

$$Q = 1.0e + 04 \begin{vmatrix} 0 & 0.0333 \\ 0.0333 & -1.111 \end{vmatrix}$$

$$Q = -\ 1.1111e + 05$$

$$Q \ \neq 0$$

As seen, the rank of the Q matrix is equal to 2 and $Q \ \neq 0$, so the system will be controllable (as rank is $\neq 0$)

1.6.5.Observability

The observability matrix can be denoted as O.

$$O = \begin{vmatrix} C \\ CA \\ CA^2 \\ . \\ CA^{n-1} \end{vmatrix}$$

$$CA = |1 \quad 0| \begin{bmatrix} 0 & 1 \\ 366.6 & -33.3 \end{bmatrix}$$

Thus, observability matrix O is as below:

$$O = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$O = 1$$

$$O \neq 0$$

As observability is not equal to 0 and rank is 2, we can say that the system is observable.

1.7.Simulation studies - Unity simulation

For a better visualization of the simulation and easy experimentation with the individual parameters, the model was additionally designed in the Unity Engine. The Unity Engine is originally a game engine, but is nowadays increasingly used in industrial contexts, for example for topics such as virtual reality. It offers many ready-made functionalities that facilitate the visualization of simulations. To achieve the greatest possible comparability with the simulation in Scilab, a manually implemented integrator (fourth order runge-kutta) was used instead of the built-in physics system, which only allows a limited insight into its functionality. The Unity Engine does not offer out-of-the-box access to its internal integrators, so an implementation by David Joiner was used (Joiner, 2018).

For the input control, the built-in animation system of the engine was used, which allows to perform movements over defined time intervals (in this case raising or lowering of the road). The graphs were realized with a Grapher addon from the Asset Store.

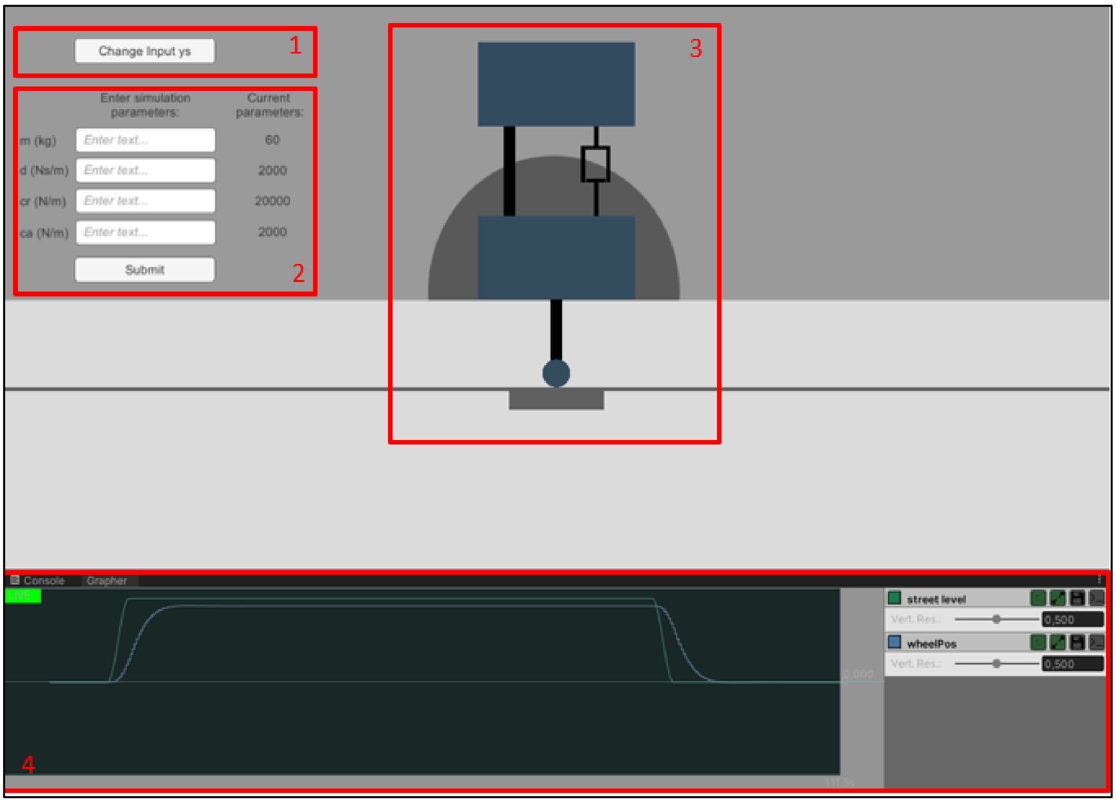

Figure 1-6: Unity Simulation Overview

The realized tool is divided into four main parts. The first one is a button that allows to switch between a rapid change of the road level (step) and a continuous change (sinusoidal). The second part allows to change the individual parameters live. The third section is the heart of the simulation and visualizes the quarter car suspension model with its movements. The last section then serves to display the behavior in the form of a graph, which plots the road level and the path of the tire.

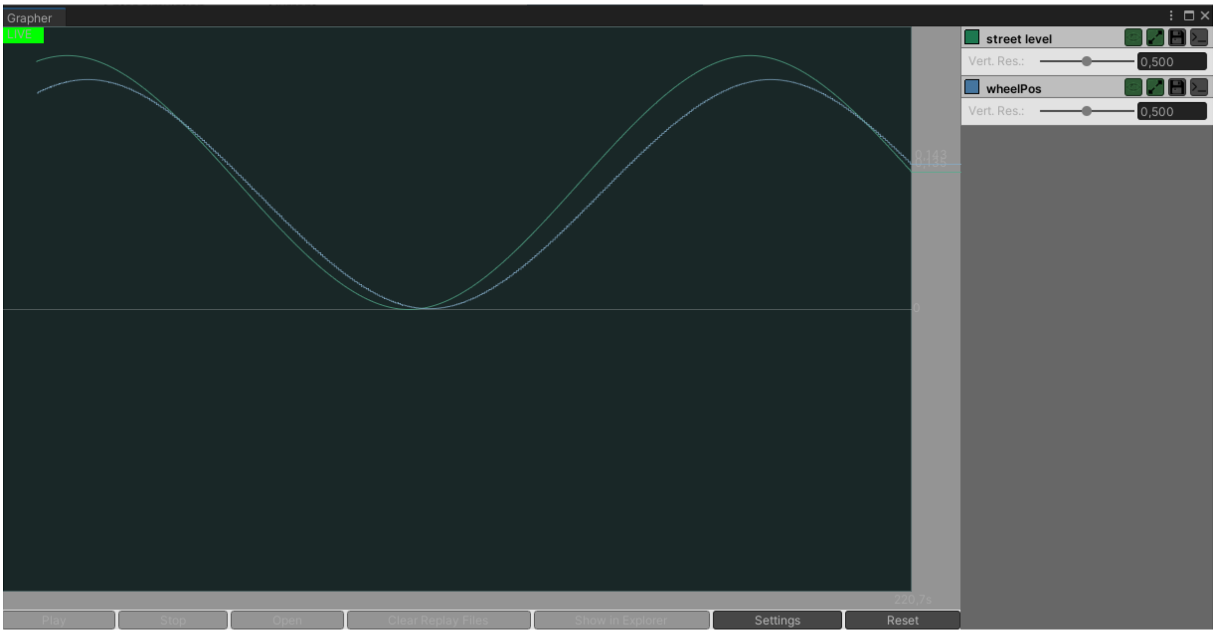

Figure 1-7: Sinusodial Input for Unity Simulation

2. Predator-prey model

2.1.Introduction

The predator-prey model simulates the dynamics of biological systems in which two species interact, one as a predator and the other as prey. Prey and predator populations depend on each other because prey serves as food for predators, and because of this interdependence, the size of one population affects the growth of the other population. This dependence can be illustrated in Figure 2-1. Thus, an increase in the population of predators leads to a decrease in the population of prey, which, respectively, leads to a decrease in the population of predators and an increase in the population of prey, and so on.

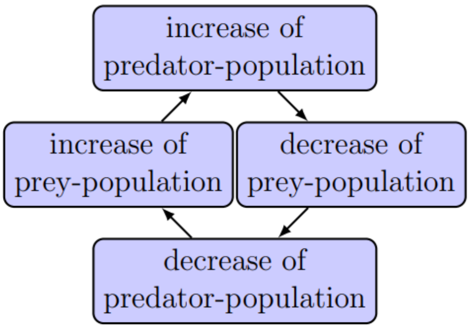

Figure 2-1: Interaction of prey and predators.

This usually ends up in constant fluctuation, as shown inFigure 2-2. Another result of the simulation could be the extinction of all predators. It is also possible that all creatures will die. [Betke, 2013].

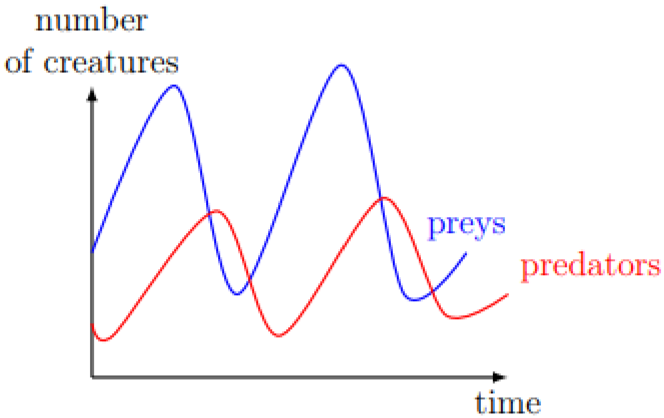

Figure 2-2: Fluctuation populations.

The interdependence between prey and predator can be represented as the following pair of first-order, non-linear, differential equations:

$$\dot{x_1} = a_1x_1 - b_1x_1x_2 - c_1x_1 - v_1u,$$
$$\dot{x_2} = b_2x_1x_2 - c_2x_2 - v_2u,$$

where:

x_1, x_2 – number of prey and predator, respectively;

$\dot{x}_1, \dot{x}_2$ – represent the instantaneous growth rates of prey and predator;

a_1– prey birth rate;

b_1– death rate of prey by predators;

b_2– birth rate of predator;

c_1, c_2 – “natural” death rate of prey and predator, respectively;

v_1, v_2 –fishing rate of prey and predator, respectively.

The following chapters will cover building a prey-predator simulation model, choosing reasonable parameters for the model, performing simulation studies, and analyzing the model using methods of linear system theory.

2.2.Dimensionless equations

Given equations:

$$\dot{x_1} = a_1x_1 - b_1x_1x_2 - c_1x_1 - v_1u,$$
$$\dot{x_2} = b_2x_1x_2 - c_2x_2 - v_2u,$$

where:

x_1, x_2 – number of prey and predator, respectively;

$\dot{x}_1, \dot{x}_2$ – represent the instantaneous growth rates of prey and predator;

a_1– prey birth rate;

b_1– death rate of prey by predators;

b_2– birth rate of predator;

c_1, c_2 – “natural” death rate of prey and predator, respectively;

v_1, v_2 –fishing rate of prey and predator, respectively;

u – external input.

$$\dot{x_1} = \widetilde{\dot{x_1}}\frac{amount}{time};\ \dot{x_2} = \widetilde{\dot{x_2}}\frac{amount}{time};\ x_1 = \widetilde{x_1}\,amount;\ x_2 = \widetilde{x_2}\,amount;$$
$$a_1 = \widetilde{a_1}\frac{amount}{time}; b_1 = \widetilde{b_1}\frac{amount}{time}; b_2 = \widetilde{b_2}\frac{amount}{time};\ c_1 = \widetilde{c_1}\frac{amount}{time}; c_2 = \widetilde{c_2}\frac{amount}{time};$$
$$v_1 = \widetilde{v_1}\frac{amount}{time}; v_2 = \widetilde{v_2}\frac{amount}{time}; u = \tilde{u}\,amount$$

First equation after substitution:

$$\widetilde{x_1}\frac{amount}{time} = \widetilde{a_1}\frac{amount}{time}\widetilde{x_1}\,amount - \widetilde{b_1}\frac{amount}{time}\widetilde{x_1}\,amount\,\widetilde{x_2}\,amount - \widetilde{c_1}\frac{amount}{time}\widetilde{x_1}\,amount - \widetilde{v_1}\frac{amount}{time}\,\tilde{u}\,amount \mid *\frac{time}{amount}$$

$$\widetilde{x_1} = \widetilde{a_1}\widetilde{x_1}\,amount - \widetilde{b_1}\widetilde{x_1}\,\widetilde{x_2}\,amount^2 - \widetilde{c_1}\widetilde{x_1}\,amount - \widetilde{v_1}\tilde{u}\,amount$$

Second equation after substitution:

$$\widetilde{x_2}\frac{amount}{time} = \widetilde{b_2}\frac{amount}{time}\widetilde{x_1}\,amount\,\widetilde{x_2}\,amount - \widetilde{c_2}\frac{amount}{time}\widetilde{x_2}\,amount - \widetilde{v_2}\frac{amount}{time}\,\tilde{u}\,amount \mid *\frac{time}{amount}$$

$$\widetilde{x_2} = \widetilde{b_2}\widetilde{x_1}\,\widetilde{x_2}\,amount^2 - \widetilde{c_2}\widetilde{x_2}\,amount - \widetilde{v_2}\tilde{u}\,amount$$

Since the amount is a dimensionless number, we can neglect it. Then the system of dimensionless equations will look like the following:

$$\dot{\widetilde{x_1}} = \widetilde{a_1}\widetilde{x_1} - \widetilde{b_1}\widetilde{x_1}\,\widetilde{x_2} - \widetilde{c_1}\widetilde{x_1} - \widetilde{v_1}\widetilde{u}$$
$$\dot{\widetilde{x_2}} = \widetilde{b_2}\widetilde{x_1}\,\widetilde{x_2} - \widetilde{c_2}\widetilde{x_2} - \widetilde{v_2}\tilde{u}$$

2.3. Parameter selection

PARAMETERS	VALUES
a_1	1.5
b_1	0.1
c_1	1
b_2	1
c_2	3
v_1	0.01
v_2	0.02

Table 2-1: Parameters selected for predator-prey model

2.4. X-cos

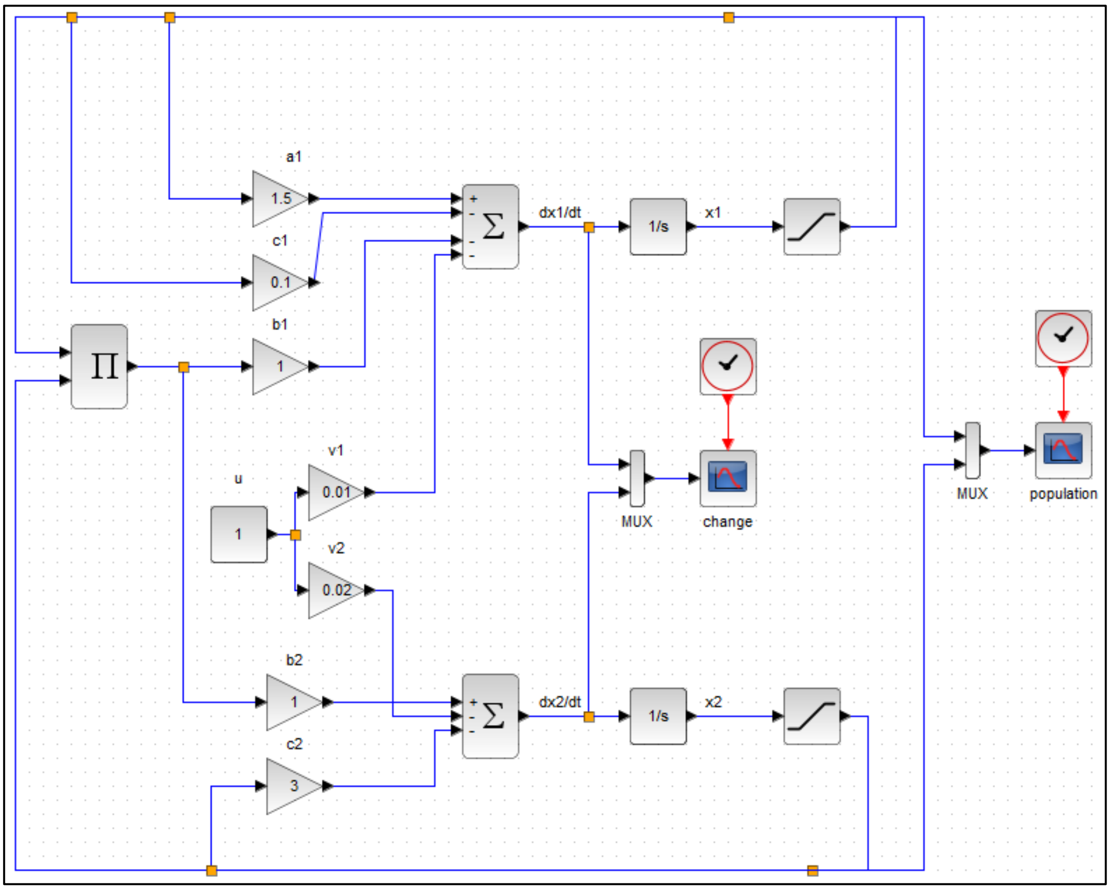

Figure 2-3: XCOS simulation - predator-prey model

The Predator-Prey model underlying the simulation is an extension of the classical Lotka-Volterra equations. The initial parameters were selected through an internet search. An integration time of 12 seconds was chosen for the simulation, which is assumed to be months in this case. The initial population of prey was set to six (assumed to be 6k) and that of predators to 2 (assumed to be 2k). The growth rate of the prey is 1.5 and that of the predators is 1. The "natural" death rate of the prey is 0.1 and that of the predators is 3. The death rate of the prey by the predators was set to 1. In addition, the fishing rate is 0.01 for the prey and 0.02 for the predators. U was set to 1. By choosing these parameters, it was possible to generate the characteristic periodic behavior of this model. The model was particularly sen-sitive to the adjustment of the external influence u during the parameter finding. [(x-engineer.org, n.d.), (Anon, 2022)]

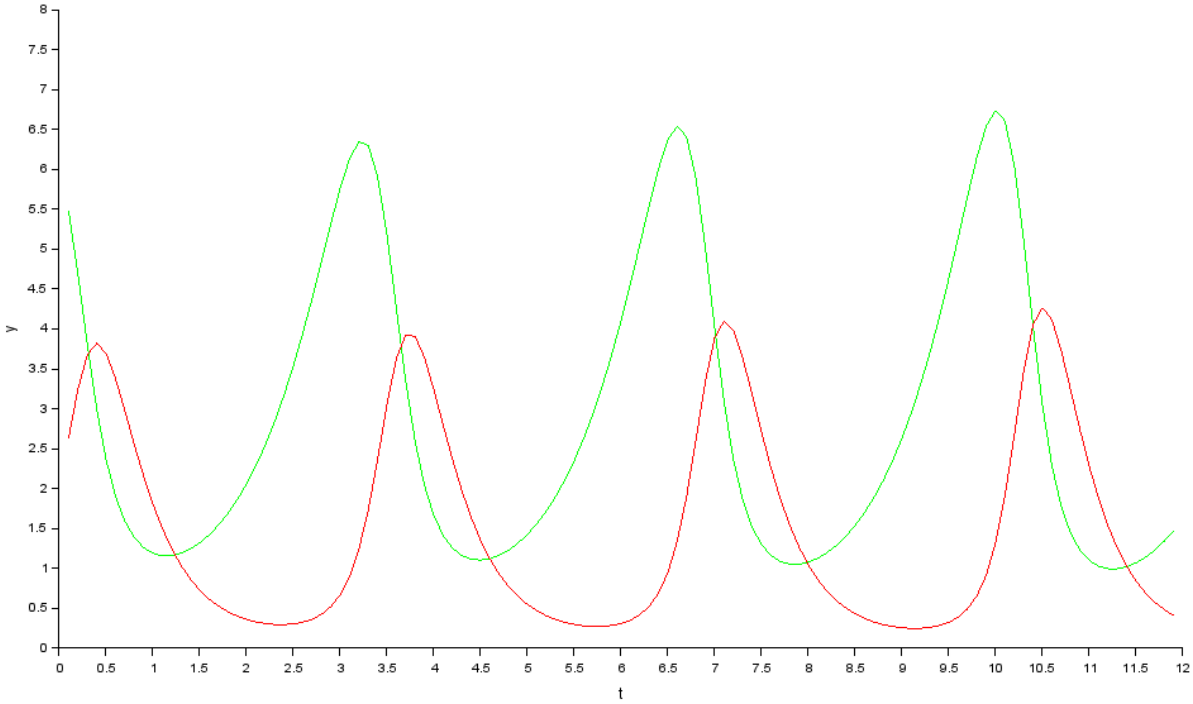

Figure 2-4: Predator-Prey model - prey population (green) and predator population (red)

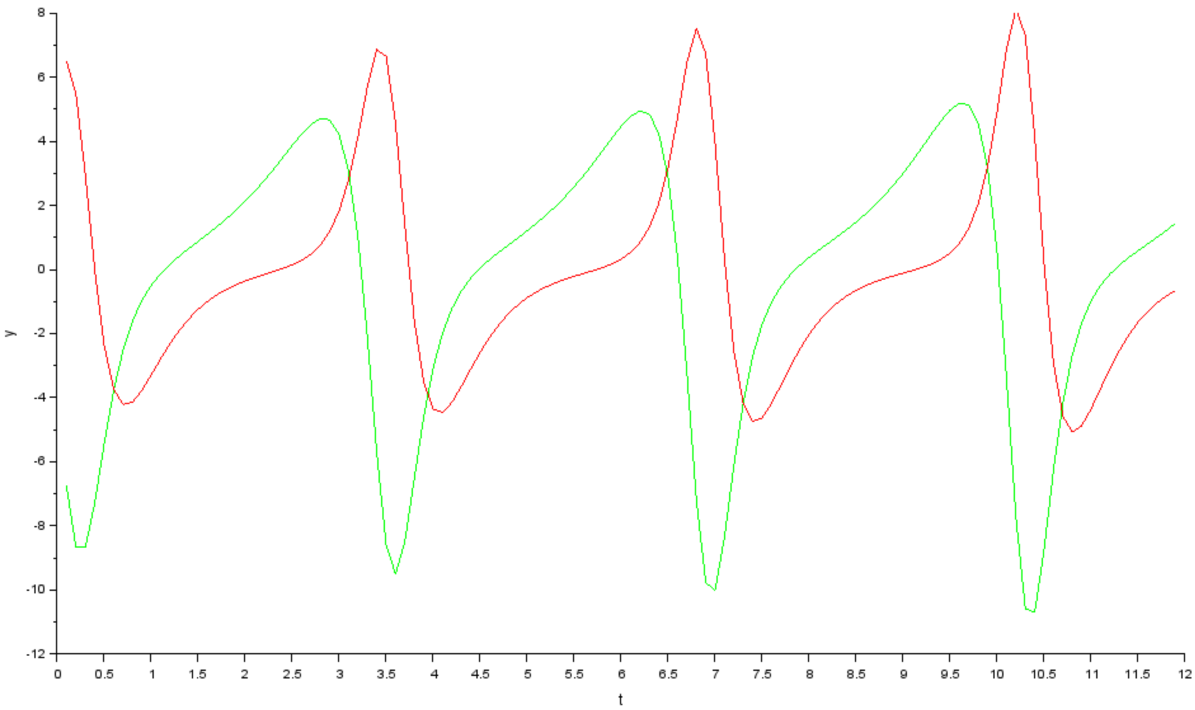

Figure 2-5: Predator-Prey model - prey (green) and predator (red) population change

The populations grow and fall asynchronously to each other and produce the typical periodic behavior through the mutual interaction. If the number of predators grows, more and more prey die. At a certain point, there is no longer enough food for the predators, and they begin to die. This in turn allows more prey to survive. In a stable ecosystem, this cycle would repeat itself permanently. However, if the system becomes unbalanced, at least one species becomes extinct. In the second simulation graph, the partial rapid changes of the population can be observed.

2.5.Linear system theory analysis

2.5.1.System state analysis

Prey:

$$\frac{dx_1}{dt} = a_1x_1 - b_1x_1x_2 - c_1x_1 - v_1u$$
$$= (a_1 - c_1)x_1 - b_1x_1x_2 - v_1u$$
$$= [(a_1 - c_1) - b_1x_2]x_1 - v_1u$$

Predator:

$$\frac{dx_2}{dt} = b_2x_1x_2 - c_2x_2 - v_2u$$
$$= [b_1x_1 - c_2]x_2 - v_2u$$

finding stationary points:

from eq():

$$\frac{dx_1}{dt} = 0 = [(a_1 - c_1) - b_1x_2]x_1$$

For $x_1 = 0$

$$b_2x_1 - c_2 = 0$$
$$x_1 = \frac{c_2}{b_2}$$

Hence the two operating state would be

1. Extinction state: $X_{s_1} = (0,0)$, Here, in this state both prey and predator would be zero or non existent.
2. Co-Existence state: $X_{s_2} = \left(\frac{c_2}{b_2}, \frac{a_1-c_1}{b_1}\right)$, Here, in this state there would be a balance between the prey and predator and both of them would be present in that particular instant of time.

2.5.2.Stability analysis

Now, we aim to analysis stability in both the extinction and co-existence state with the help of Jacobian matrix.

We have,

$$f_1 = x_1\big((a_1 - c_1) - b_1x_2\big)$$
$$f_1 = x_2(b_2x_1 - c_2)$$

using jacobian matrix principle,

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_1} \end{bmatrix} = \begin{bmatrix} (a_1 - c_1) - b_1 x_2 & b_1 x_1 \\ b_2 x_2 & b_2 x_1 - c_2 \end{bmatrix}$$

for state (0,0)

$$\begin{bmatrix} a_1 - c_1 & 0 \\ 0 & -c_2 \end{bmatrix} = \begin{bmatrix} a_1 - c_1 - \lambda_1 & 0 \\ 0 & -c_2 - \lambda_2 \end{bmatrix}$$

Then,

$\lambda_1 = a_1 - c_1$ and $\lambda_2 = -c_2$.

So, in our case $\lambda_1 = 0.5$ and $\lambda_2 = -3$. Which means that our system would be unstable in this state as λ_1 value is positive. This implies that prey and predator would not co-exisist perfectly.

So, if the system has to be stable, $a_1 < c_1$ and $-c_2$ condition has to be statisfied.

for state $\left(\frac{c_2}{b_2}, \frac{a_1 - c_1}{b_1}\right)$

$$\begin{bmatrix} (a_1 - c_1) - b_1 \frac{(a_1 - c_1)}{b_1} & b_1 \frac{c_2}{b_2} \\ b_2 \frac{(a_1 - c_1)}{b_1} & b_2 \frac{c_2}{b_2} - c_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & b_1 \frac{c_2}{b_2} \\ b_2 \frac{(a_1 - c_1)}{b_1} & 0 \end{bmatrix} = \begin{bmatrix} 0 - \lambda_1 & b_1 \frac{c_2}{b_2} \\ b_2 \frac{(a_1 - c_1)}{b_1} & 0 - \lambda_2 \end{bmatrix}$$

$$(0 - \lambda)(0 - \lambda) - \left(b_2 \frac{(a_1 - c_1)}{b_1}\right)\left(b_1 \frac{c_2}{b_2}\right)$$

$$\lambda^2 - \left(b_2 \frac{(a_1 - c_1)}{b_1}\right)\left(b_1 \frac{c_2}{b_2}\right)$$

$$\lambda^2 - (a_1 - c_1)c_2$$

$$\lambda_3 = \sqrt{(a_1 - c_1)c_2}$$

Therefore if $a_1 < c_1$ the system would be asymptotically-stable.

Then,

$$\lambda_3 = \sqrt{(a_1 - c_1)c_2}$$

So, in our case $\lambda_1 = 0$ Which means that our system would be unstable in this state as λ_1 value is positive. This implies that prey and predator would not co-exisist perfectly.

So, if the system has to be stable, $a_1 < c_1$ and $-c_2$ condition has to be statisfied.

2.6.Simulation studies.

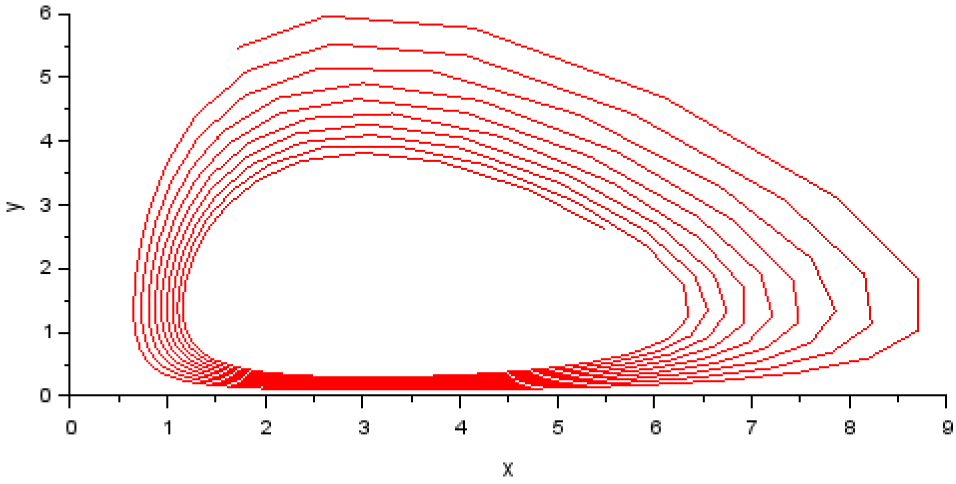

Figure 2-6: Phase plane of a dynamical system

A phase portrait is a geometric representation of the trajectories of a dynamical system in the phase plane.We can observe the movement of the system , that is the system is moving around the center stationary point. Here, we can see the system to be oscillating around the origin without converging.

2.6.1. Influence of fishing

The following section examines the impact of fisheries on the system. As a starting point, the inner equilibrium points are used. The following parameters are set for the simulation:

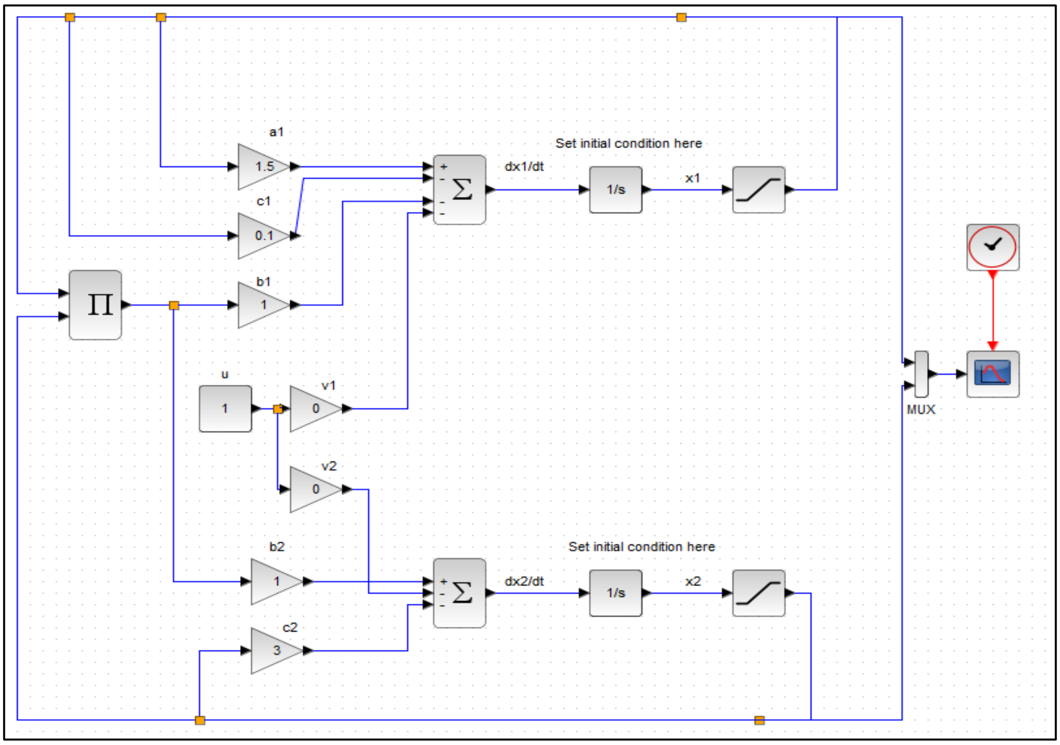

Figure 2-7: Initial Experiment Setup - Fishing

The derived equations give the following initial populations:

$x1 = \frac{c_2}{b_2} = \frac{3}{1} = 3(\text{k})\text{ preys}$

$x2 = \frac{a_1 - c_1}{b_1} = \frac{1.5 - 0.1}{1} = 1.4(\text{k})\text{ predators}$

The integration time is set to 600 month, which is equal to 50 years.

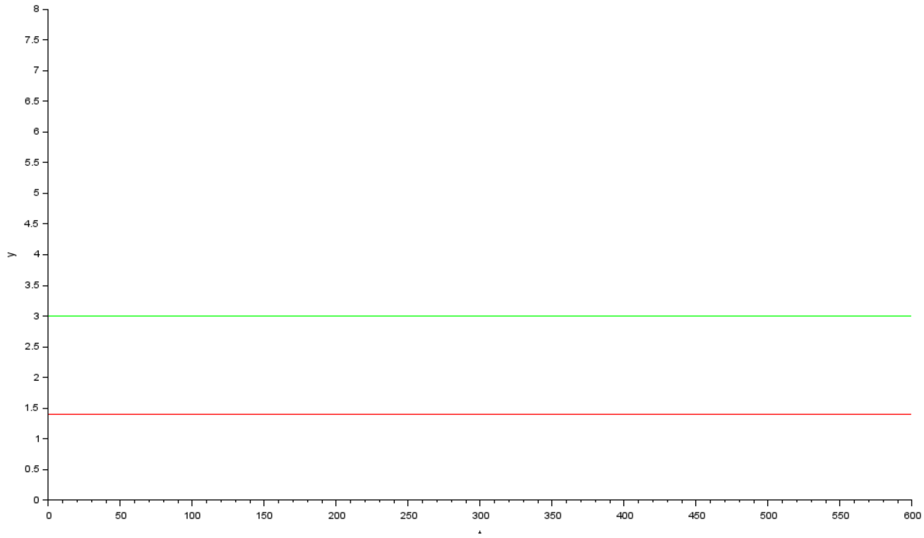

Figure 2-8: Steady State populations

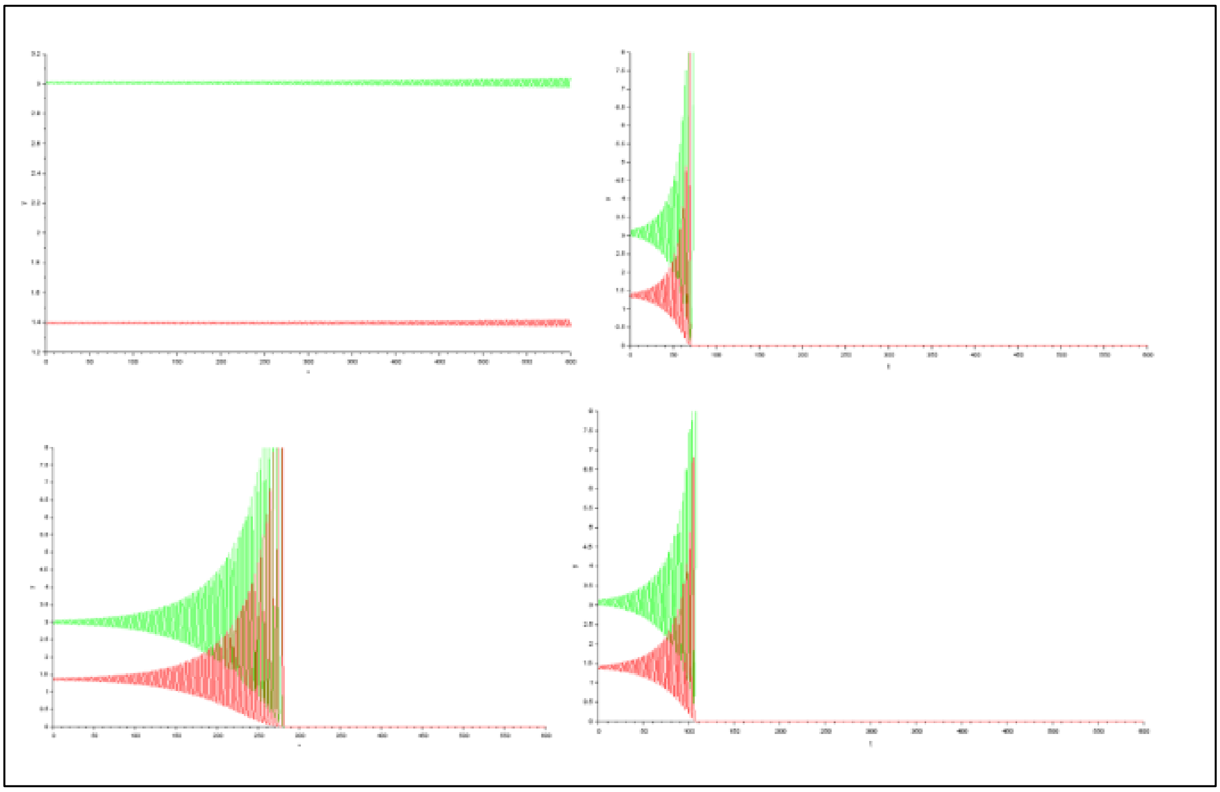

Figure 2-9: Fishing - v1 = 0.01 v2 = 0.01 (Top left) ; v1 = 0.1 v2 = 0.1 (Top right) ; v1 = 0.1 v2 = 0 (Bottom left) ; v1 = 0 v2 = 0.1 (Bottom right)

As seen in the experiments, even the smallest continuous fishing rates unbalance the system and, in the cases considered here, cause the predator to extinct and the prey to reproduce uncontrollably. In the next experiment, instead of a continuous fishing rate, we consider a punctuated fishing rate modeled by a step input. For this, a u input of 1 is applied at month 100 and stopped at month 150.

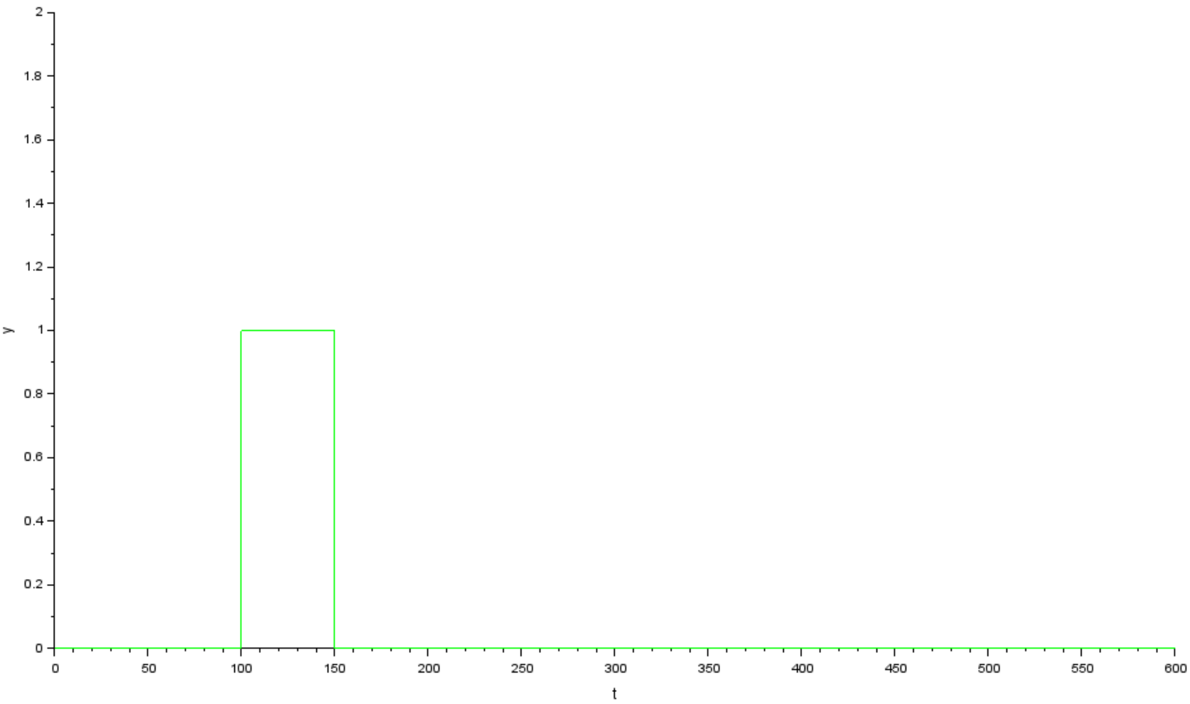

Figure 2-10: Fishing - punctual u-input

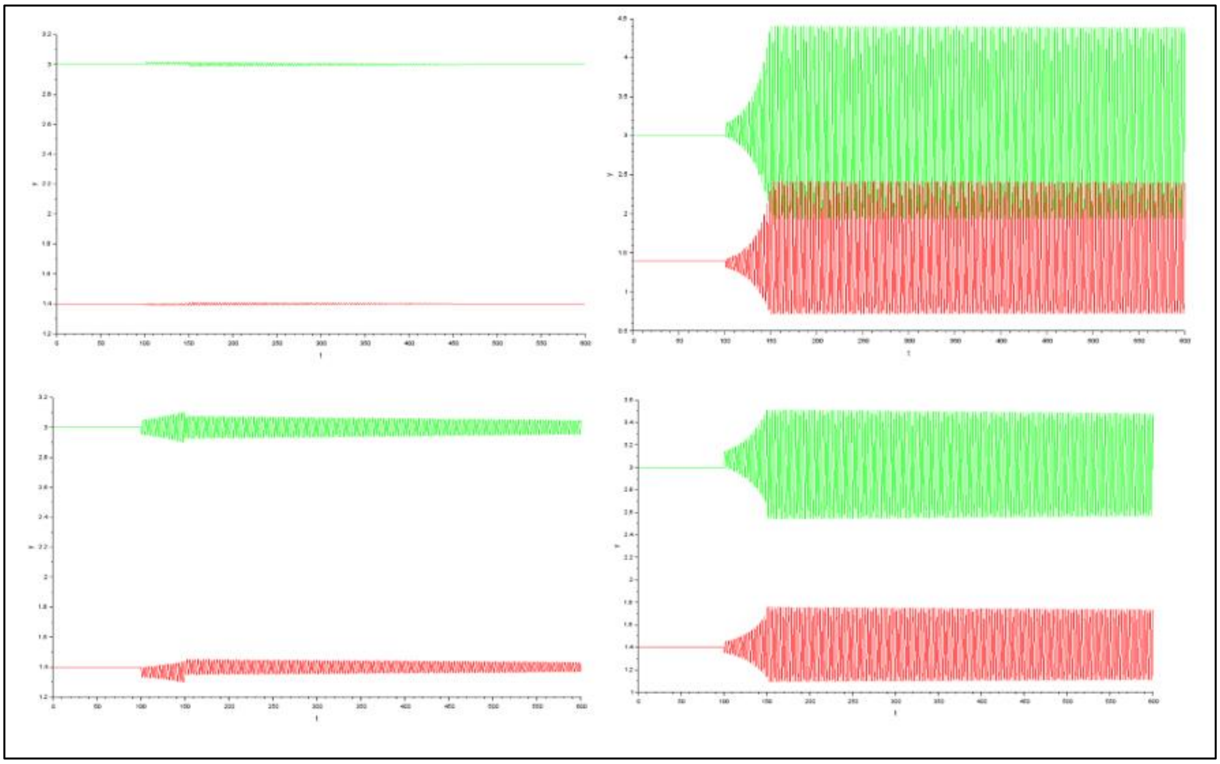

Figure 2-11: Fishing - Step - v1 = 0.01 v2 = 0.01 (Top left) ; v1 = 0.1 v2 = 0.1 (Top right) ; v1 = 0.1 v2 = 0 (Bottom left) ; v1 = 0 v2 = 0.1 (Bot-tom right)

Using the same fishing rates for a limited period of time also unbalances the system, but does not cause the extinction of any of the populations, at least during the period under consideration.

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