Supplementary Material to: MRFMap: Online Probabilistic 3D Mapping using Forward Ray Sensor Models

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Abstract

This document derives the messages for the sum-product belief propagation algorithm presented in the paper [1] and should be treated as an appendix to the same.

I. Sum-Product Belief Propagation

A. Sum-Product Belief Propagation

Sum-Product belief propagation is a common message-passing algorithm for performing inference on factor graphs. By exploiting marginalisation of joint distributions using factorisation of a graph it enables computing marginal distributions very efficiently. A factor graph is a bipartite graph containing nodes corresponding to variables and factors that are connected by edges. Messages are passed between connected nodes and factors that try to influence the marginal belief of their neighbours. The passing continues until convergence (if any) is achieved.

The message sent from a variable node x to a factor f is the cumulative belief of all the incoming messages from factors to the node except the factor in question

$$\mu_{x\to f}(x) = \prod_{g\in\mathcal{F}_x\setminus f} \mu_{g\to x}(x), \tag{S1}$$

where $\mathcal{F}_x$ is the set of neighbouring factors to x. Similarly, the message sent from the factor to the node is the marginalisation of the product of the value of the factor ϕ_f with all the incoming messages from nodes other than the node in question

$$\mu_{f\to x}(x) = \sum_{\mathcal{X}_f\setminus x} \phi_f(\mathcal{X}_f) \prod_{y\in\mathcal{X}_f\setminus x} \mu_{y\to f}(y), \tag{S2}$$

where $\mathcal{X}_f$ is the set of all neighbouring nodes of f.

Upon convergence, the estimated marginal distribution of each node is proportional to the product of all messages from adjoining factors

$$p(x) \propto \prod_{g\in\mathcal{F}_x} \mu_{g\to x}(x) \tag{S3}$$

Similarly, the joint marginal distribution of the set of nodes belonging to one factor is proportional to the product of the factor and the messages from the nodes

$$p(\mathcal{X}_f) \propto \phi_f(\mathcal{X}_f) \prod_{\mathcal{X}_f} \mu_{x\to f}(x) \tag{S4}$$

B. Markov Random Field

Each ray in all the cameras generates a factor graph, the joint distribution of which is

$$p(\mathbf{o},\mathbf{d}) = \frac{1}{Z}\prod_{i\in\mathcal{X}} \phi_i(o_i) \prod_{r\in\mathcal{R}} \psi_r(\mathbf{o}_r, d_r), \tag{S5}$$

where $\mathcal{X}$ is the set of all the voxels $o_i \in \{0,1\}$ and $\mathcal{R}$ is the set of all the rays from all the cameras viewing the scene, $\mathbf{o}_r = \{o_1^r, \ldots, o_{N_r}^r\}$ is the list of all the voxels traversed by a ray r, d_r is its corresponding depth variable, and Z is the normalisation constant. The total set of all the occupancy and depth variables are summarised as $\mathbf{o} = \{o_i \mid i \in \mathcal{X}\}$ and $\mathbf{d} = \{d_r \mid r \in \mathcal{R}\}$. ϕ_i and ψ_r are the potential factors as described as follows:

1) Prior Occupancy Factor: This is simply a unary factor assigning an independent Bernoulli prior $\gamma$to the voxel occupancy label for each voxel

$$\phi_i(o_i) = \gamma^{o_i}(1-\gamma)^{1-o_i}. \tag{S6}$$

Note that these priors can be informed from predictive methods if required.

2) Ray Depth Potential Factor: A ray potential, as presented in [], creates a factor graph connecting the binary occupancy label o_i for all voxels traversed by a single ray by virtue of making a measurement. Each of these voxels has a corresponding distance from the camera origin, and thus the ray is defined to include a depth variable d_r that represents the event that the measured depth is at distance d_i^r. For this to happen all the preceding voxels ought to be empty and the corresponding voxel needs to be occupied. This leads to the definition of the joint occupancy and depth potential factor

$$\psi_r(\mathbf{o}_r, d_r) = \begin{cases} \nu_r(d_i^r) & \text{if } d_r = \sum_{i=1}^{N_r} o_i^r \prod_{j<i}(1-o_j^r) d_i^r \\ 0 & \text{otherwise} \end{cases}. \tag{S7}$$

Here $\nu_r(d_i^r)$ denotes the probability of observing depth d_i^r given the measured depth value $\mathcal{D}^r$ that we model as $\nu_r(d_i^r) = \mathcal{N}(d_i^r; \mathcal{D}_r, \sigma(d_i^r))$. Note that we model this probability to vary in mean and noise as a function of the distance from the camera, which enables utilising learnt sensor noise characteristics.

This ray potential measures how well the occupancy and the depth variables explain the depth measurement $\mathcal{D}_r$. This is more apparent when Eq. S7 is written as

$$\psi_r(\mathbf{o}_r, d_r) = \begin{cases} \nu_r(d_1^r) & \text{if } d^r = d_1^r, o_1^r = 1 \\ \nu_r(d_2^r) & \text{if } d^r = d_2^r, o_1^r = 0, o_2 = 1 \\ \vdots & \\ \nu_r(d_N^r) & \text{if } d^r = d_N^r, o_1^r = 0, \ldots, o_{N_r-1}^r = 0, o_{N_r}^r = 1 \end{cases} \tag{S8}$$

This sparse structure of the ray potential enables massive simplification of the message passing equations, as shown next.

II. Message Passing Derivation

A. Ray Depth Potential to Depth Variable Messages

Since we're only concerned about the depth variable, we marginalise out the messages from all the occupancy nodes going to the ray depth potential. Following S2 we have

$$\mu_{\psi_r \to d_r}(d_r = d_i^r) = \sum_{o_1^r} \cdots \sum_{o_{N_r}^r} \psi_r(\mathbf{o}_r, d_r) \prod_{j=1}^{N_r} \mu_{o_j^r \to \psi_r}(o_j^r). \tag{S9}$$

Naïvely evaluating this equation is not feasible. However, we can exploit the sparse diagonal nature of the ray depth potential to recursively simplify this expression. After dropping the ray index for notational convenience and abbreviating $\mu_{o_j^r \to \psi_r}(o_j^r)$ as $\mu(o_j)$ we have

$$\begin{aligned} \mu_{\psi \to d}(d = d_i) = \mu(o_1 = 1) & \overbrace{\left[\sum_{o_2} \cdots \sum_{o_N} \psi(o_1 = 1, o_2, \ldots, o_N, d = d_i) \prod_{j=2}^{N} \mu(o_j) \right]}^{\triangle} \\ + \mu(o_1 = 0) & \underbrace{\left[\sum_{o_2} \cdots \sum_{o_N} \psi(o_1 = 0, o_2, \ldots, o_N, d = d_i) \prod_{j=2}^{N} \mu(o_j) \right]}_{\square} \end{aligned} \tag{S10}$$

From Eq. S8, for the top expression $\triangle$ the ray potential term $\psi(o_1 = 1, o_2, \ldots, o_N, d = d_i)$ evaluates to $\nu(d_1)$ if $i = 1$ and 0 otherwise. Since it only depends on d_1 it can be brought out of the summation as follows:

$$\triangle = \nu(d_1) \underbrace{\sum_{o_2} \cdots \sum_{o_N} \prod_{j=2}^{N} \mu(o_j)}_{\text{evaluates to 1.}}. \tag{S11}$$

Assuming that all the incoming messages μ are normalised such that they sum to 1, the terms highlighted with the underbrace evaluate to 1. We maintain this normalisation in our implementation.

Assuming that $i \neq 1$, the bottom expression $\square$ can be recursively expanded similar to this step. Each such expansion brings in a term of the form $\prod_{k<j} \mu(o_k = 0)\mu(o_j = 1)\nu(d_j)$, until we reach the ith term Thus

$$\mu_{\psi \to d}\left(d = d_i\right) = \prod_{k<i} \mu(o_k = 0)\Bigg[\mu(o_i = 1) \sum_{o_{i+1}} \cdots \sum_{o_N} \overbrace{\psi\left(o_1 = 0, o_2 = 0, \ldots, o_i = 1, o_{i+1}, \ldots o_N, d = d_i\right)}^{\text{evaluates to } \nu(d_i)} \prod_{j=i+1}^{N} \mu\left(o_j\right)$$

$$+ \mu(o_i = 0) \sum_{o_{i+1}} \cdots \sum_{o_N} \underbrace{\psi\left(o_1 = 0, o_2 = 0, \ldots, o_i = 0, o_{i+1}, \ldots o_N, d = d_i\right)}_{\text{evaluates to } 0} \prod_{j=i+1}^{N} \mu\left(o_j\right)\Bigg] \tag{S12}$$

where the first term evaluates to $\nu(d_i)$, and the next term evaluates to 0, giving us

$$\begin{aligned}\mu_{\psi \to d}\left(d = d_i\right) &= \prod_{k<i} \mu(o_k = 0)\mu(o_i = 1)\nu(d_i) \underbrace{\sum_{o_{i+1}} \cdots \sum_{o_N} \prod_{j=i+1}^{N} \mu\left(o_j\right)}_{\text{evaluates to } 1} \\ &= \nu(d_i)\mu(o_i = 1) \prod_{k<i} \mu(o_k = 0)\end{aligned} \tag{S13}$$

B. Depth Variable to Ray Depth Potential Messages

The message from the depth variable to the depth ray potential is irrelevant since the only factor connected to the variable is the potential itself.

C. Ray Depth Potential to Occupancy Variable Messages

Similar to the depth variable messages, we marginalise out all the variables except the node in question

$$\mu_{\psi_r \to o_i^r}\left(o_i^r = 1\right) = \sum_{d_r} \sum_{\substack{o_j^r \\ j \neq i}} \mu_{d_r \to \psi_r}(d_r)\psi_r\left(\mathbf{o}_r, d_r\right) \prod_{j=1}^{N_r} \mu_{o_j^r \to \psi_r}\left(o_j^r\right). \tag{S14}$$

After dropping the ray indices, we have

$$\mu_{\psi \to o_i}\left(o_i = 1\right) = \underbrace{\sum_{d=d_1}^{d_N} \mu(d)}_{\text{evaluates to } 1} \sum_{o_1} \cdots \sum_{o_{i-1}} \sum_{o_{i+1}} \ldots \sum_{o_N} \psi\left(o_1, \ldots, o_i = 1, \ldots, o_N, d\right) \prod_{\substack{j=1 \\ j \neq i}}^{N} \mu\left(o_j\right), \tag{S15}$$

where we use the shorthand $\mu(d) = \mu_{d_r \to \psi_r}(d_r)$, and $\mu(o) = \mu_{o_j^r \to \psi_r}(o_j^r)$. Note that as mentioned above, $\mu(d)$ sends a uniform message to the potential since it has no other factor connected to it. Thus the outermost summation evaluates to 1.

$$\mu_{\psi \to o_i}\left(o_i = 1\right) = \sum_{o_1} \cdots \sum_{o_{i-1}} \sum_{o_{i+1}} \ldots \sum_{o_N} \psi\left(o_1, \ldots, o_i = 1, \ldots, o_N, d_j\right) \prod_{\substack{j=1 \\ j \neq i}}^{N} \mu\left(o_j\right) \tag{S16}$$

We intend to simplify this expression in a similar manner to the previous derivation. Breaking apart into two terms, we have

$$\mu_{\psi \to o_i}\left(o_i = 1\right) = \mu\left(o_1 = 1\right)\left[\sum_{o_2} \cdots \sum_{o_{i-1}} \sum_{o_{i+1}} \ldots \sum_{o_N} \overbrace{\psi\left(o_1 = 1, \ldots, o_i = 1, \ldots, o_N, d\right)}^{\text{evaluates to } \nu(d_1)} \prod_{\substack{j=2 \\ j \neq i}}^{N} \mu\left(o_j\right)\right]$$

$$+ \mu\left(o_1 = 0\right)\left[\sum_{o_2} \cdots \sum_{o_{i-1}} \sum_{o_{i+1}} \ldots \sum_{o_N} \psi\left(o_1 = 0, \ldots, o_i = 1, \ldots, o_N, d\right) \prod_{\substack{j=2 \\ j \neq i}}^{N} \mu\left(o_j\right)\right] \tag{S17}$$

Similar to the previous strategy, we can keep breaking it up till o_{i-1}. At that point we have

$$\mu_{\psi \to o_i}\left(o_i = 1\right) = \sum_{j}^{i-1} \mu(o_j = 1)\nu(d_j) \prod_{k<j} \mu(o_k = 0)$$

$$+ \prod_{k<i} \mu(o_k = 0)\Bigg[\sum_{o_{i+1}} \ldots \sum_{o_N} \underbrace{\psi\left(o_1 = 0, o_2 = 0, \ldots, o_{i-1} = 0, o_i = 1, \ldots, o_N, d\right)}_{\text{evaluates to } \nu(d_i)} \prod_{j=i+1}^{N} \mu\left(o_j\right)\Bigg]. \tag{S18}$$

Thus, we have

$$\mu_{\psi\to o_i}\left(o_i=1\right)=\sum_{j=1}^{i-1}\mu\left(o_j=1\right)\nu(d_j)\prod_{k<j}\mu\left(o_k=0\right)+\nu(d_i)\prod_{k<i}\mu\left(o_k=0\right) \tag{S19}$$

For the negative case we get to the same point as Eq. S18, except with $o_i=0$

$$\begin{aligned}\mu_{\psi\to o_i}\left(o_i=0\right)&=\sum_{j}^{i-1}\mu(o_j=1)\nu(d_j)\prod_{k<j}\mu(o_k=0)\\&+\prod_{k<i}\mu(o_k=0)\left[\sum_{o_{i+1}}\dots\sum_{o_N}\psi\left(o_1=0,o_2=0,\dots,o_{i-1}=0,o_i=0,\dots,o_N,d\right)\prod_{j=i+1}^{N}\mu\left(o_j\right)\right].\end{aligned} \tag{S20}$$

Observing that starting from o_{i+1} it is the same form of expansion, we then simplify and get

$$\mu_{\psi\to o_i}\left(o_i=0\right)=\sum_{j=1}^{i-1}\mu\left(o_j=1\right)\nu(d_j)\prod_{k<j}\mu\left(o_k=0\right)+\sum_{j=i+1}^{N}\mu\left(o_j=1\right)\nu(d_j)\prod_{\substack{k<j\\k\neq i}}\mu\left(o_k=0\right), \tag{S21}$$

Which for convenience can also be written as

$$\mu_{\psi\to o_i}\left(o_i=0\right)=\sum_{j=1}^{i-1}\mu\left(o_j=1\right)\nu(d_j)\prod_{k<j}\mu\left(o_k=0\right)+\frac{1}{\mu(o_i=0)}\sum_{j=i+1}^{N}\mu\left(o_j=1\right)\nu(d_j)\prod_{k<j}\mu\left(o_k=0\right). \tag{S22}$$

D. Occupancy Variable to Ray Depth Potential Messages

Since other rays can (and often do) pass through the same occupancy variable node, the outgoing message $\mu_{o_i^r\to\psi_r}$ is computed as per Eq. S1.

References

[1] K. S. Shankar and N. Michael. MRFMap: Online Probabilistic 3D Mapping usingForward Ray Sensor Models. In *Robotics: Science and Systems*, 2020.