Problems

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1.1.1 Irrational Equations

Problem 1 - Nested Radicals

Solve the equation.

Source: Fun Algebra Problem!

$$\sqrt{1+\sqrt{1+x}} = \sqrt[3]{x}$$

Solution

$$(1+\sqrt{1+x})^{\frac{1}{2}} = x^{\frac{1}{3}}$$

$$1+\sqrt{1+x} = x^{\frac{2}{3}}$$

$$1+x = \left(x^{\frac{2}{3}}-1\right)^2$$

$$1+x = x^{\frac{4}{3}} - 2x^{\frac{2}{3}} + 1$$

$$x = x^{\frac{4}{3}} - 2x^{\frac{2}{3}}$$

$$x\left(1-x^{\frac{1}{3}}\right) = -2x^{\frac{2}{3}}$$

$$1-x^{\frac{1}{3}} = -2x^{-\frac{1}{3}}$$

$$2x^{-\frac{1}{3}} + 1 - x^{\frac{1}{3}} = 0$$

$$t = x^{\frac{1}{3}} \quad (t \in \mathbb{R}), \quad 2t^{-1} + 1 - t = 0$$

$$\Rightarrow\ 2 + t - t^2 = 0 \quad \Longleftrightarrow \quad t^2 - t - 2 = 0$$

$$(t-2)(t+1) = 0 \ \Rightarrow\ t = 2 \ \text{ or } \ t = -1$$

$$\text{Since } \sqrt{1+\sqrt{1+x}} \geq 0 \Rightarrow x^{\frac{1}{3}} \geq 0 \Rightarrow t \geq 0$$

$$\therefore\ t = 2 \quad \Rightarrow \quad \boxed{x = t^3 = 8}$$

Verification:

$$\sqrt{1+\sqrt{1+8}} = \sqrt{1+3} = 2 = \sqrt[3]{8}$$

$$x = 8$$

Problem 2 - Infinite Nested Radicals

Solve the equation.

Source: Solving for Infinite x's!

$$5 = \sqrt{x + \sqrt{2x + \sqrt{x + \sqrt{2x + \sqrt{x + \sqrt{2x + \cdots}}}}}}$$

Solution

$$s = \sqrt{2x + \sqrt{x + \sqrt{2x + \cdots}}}$$

$$s = \sqrt{2x + \underbrace{\sqrt{x + s}}_{=5}} = \sqrt{2x + 5}$$

$$5 = \sqrt{x + s} \implies s = 25 - x$$

$$25 - x = \sqrt{2x + 5}$$

$$(25 - x)^2 = 2x + 5 \implies x^2 - 52x + 620 = 0$$

$$\Delta = b^2 - 4ac = (-52)^2 - 4 \cdot 1 \cdot 620 = 2704 - 2480 = 224$$

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(-52) \pm \sqrt{224}}{2} = \frac{52 \pm 4\sqrt{14}}{2} = 26 \pm 2\sqrt{14}.$$

$$s = 25 - x \geq 0 \implies x \leq 25$$

$$\boxed{x = 26 - 2\sqrt{14}}$$

Verification:

$$\sqrt{x} \approx 4.30310\ldots$$

$$\sqrt{x + \sqrt{2x}} \approx 4.96005\ldots$$

$$\sqrt{x + \sqrt{2x + \sqrt{x}}} \approx 4.99460\ldots$$

$$\sqrt{x + \sqrt{2x + \sqrt{x + \sqrt{2x}}}} \approx 4.99969\ldots$$

$$\sqrt{x + \sqrt{2x + \sqrt{x + \sqrt{2x + \sqrt{x}}}}} \approx 4.99995$$

$$\sqrt{x + \sqrt{2x + \sqrt{x + \sqrt{2x + \sqrt{x + \sqrt{2x}}}}}} \approx 4.99999\ldots$$

1.1.2 Absolute Value Equations

Problem 3 - Quadratic Absolute Value Equation

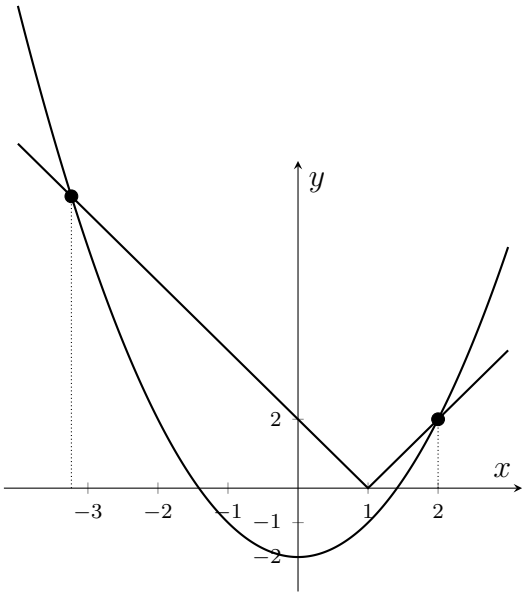
Solve the equation graphically and algebraically.

Source: Galatasaray University, Math I Midterm Exam 2022-2023, Damien Berthet

$$2|x-1| = x^2 - 2$$

Solution

Graphic method :



$$S = \{2, x_1\} \quad \text{avec} \quad x_1 < -3.$$

Calculation :

$$2|x-1| = x^2 - 2 \iff \begin{cases} 2x - 2 = x^2 - 2 \\ x > 1 \end{cases} \quad \text{or} \quad \begin{cases} -2x + 2 = x^2 - 2 \\ x < 1 \end{cases}$$

$$\iff x = 2 \text{ or } \begin{cases} x^2 + 2x - 4 = 0 \\ x < 1 \end{cases}$$

$$\iff x = 2 \text{ or } \begin{cases} x = -1 \pm \sqrt{5} \\ x < 1 \end{cases}$$

$$\boxed{S = \{2,\ -1 - \sqrt{5}\}}$$