[PI] Prudnikov Vol I p. 343

$D_\nu(x)$ parabolic cylinder fct.
Vol II p. 727

$$(f*g)(x) = \int_{-\infty}^{\infty} dy\, f(y)\, g(x-y)$$

$$g(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-x_0}{\sigma}\right)^2\right)$$

$$f(x) = b + c(x-E_A)^\alpha\, \Theta(x-E_A)$$

$$(f*g)(x) = \int_{-\infty}^{\infty} dy\, [b + c(y-E_A)^\alpha \underbrace{\ }_{\Theta(y-E_A)}]\, \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(x-y-x_0)^2}{2\sigma^2}}$$

$$= b + \frac{c}{\sqrt{2\pi}\,\sigma} \int_{E_A}^{\infty} dy\, (y-E_A)^\alpha\, e^{-\frac{(x-y-x_0)^2}{2\sigma^2}}$$

$$= b + \frac{c}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(x-x_0)^2}{2\sigma^2}} \int_{E_A}^{\infty} dy\, (y-E_A)^\alpha\, e^{-\frac{y^2}{2\sigma^2} + \frac{(x-x_0)y}{\sigma^2}}$$

[PI] 2.3.15.1 $\displaystyle\int_a^\infty dx\, (x-a)^{\beta-1}\, e^{-px^2-qx} = \Gamma(\beta)\,(2p)^{-\beta/2}\, e^{\frac{q^2}{8p} - \frac{a}{2}(q+ap)} \times$

$$\times D_{-\beta}\left(\frac{2ap+q}{\sqrt{2p}}\right)$$

for $\mathrm{Re}\,\beta, \mathrm{Re}\,p > 0$

here: $a = E_A$
$\beta = \alpha + 1 \Rightarrow \alpha > -1$
$p = \frac{1}{2\sigma^2}$
$q = -\frac{(x-x_0)}{\sigma^2}$

$$(f*g)(x) = b + \frac{c}{\sqrt{2\pi}\,\cancel{\sigma}}\, e^{-\frac{(x-x_0)^2}{2\sigma^2}}\, \Gamma(\alpha+1)\, \sigma^{\alpha\cancel{+1}}\, e^{\frac{(x-x_0)^2\,\cancel{2\sigma^2}}{\sigma^{\cancel{4}2}\cancel{8}\,4}}\, e^{-\frac{E_A}{2}\left[-\frac{x-x_0}{\sigma^2} + \frac{E_A}{2\sigma^2}\right]}$$

$$\times D_{-(\alpha+1)}\left(\sigma\left(\frac{E_A}{\sigma^2} - \frac{x-x_0}{\sigma^2}\right)\right)$$

$$= b + \frac{c}{\sqrt{2\pi}\,\cancel{\sigma}}\, \sigma^\alpha\, \Gamma(\alpha+1) \exp\left(-\frac{1}{4\sigma^2}\left(\underbrace{(x-x_0)^2 - 2E_A(x-x_0) + E_A^2}_{=(E_A-(x-x_0))^2}\right)\right) D_{-(\alpha+1)}\left(\frac{1}{\sigma}(E_A-(x-x_0))\right)$$

$$= b + \frac{c}{\sqrt{2\pi}}\, \sigma^\alpha\, \Gamma(\alpha+1) \exp\left(-\frac{1}{4\sigma^2}(E_A-(x-x_0))^2\right) D_{-(\alpha+1)}\left(\frac{1}{\sigma}(E_A-(x-x_0))\right)$$

$\alpha = n \in \mathbb{N}$: $\Gamma(n+1) = n!$

$$D_{-n-1}(x) = \frac{(-1)^n}{n!}\sqrt{\frac{\pi}{2}}\, e^{-x^2/4}\, \frac{d^n}{dx^n}\left[e^{x^2/2}\, \mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)\right]$$

$n = 1$:

II.8. THE PARABOLIC-CYLINDER FUNCTIONS $D_\nu(z)$

Definition:

$$D_\nu(z)=2^{\nu/2}e^{-z^2/4}\Psi\left(-\frac{\nu}{2},\frac{1}{2};\frac{z^2}{2}\right)=$$

$$=2^{(\nu-1)/2}e^{-z^2/4}z\Psi\left(\frac{1-\nu}{2},\frac{3}{2};\frac{z^2}{2}\right)=$$

$$=2^{\nu/2}e^{-z^2/4}\left[\frac{\sqrt{\pi}}{\Gamma[(1-\nu)/2]}\,{}_1F_1\left(-\frac{\nu}{2};\frac{1}{2};\frac{z^2}{2}\right)-\right.$$

$$\left.-\frac{\sqrt{2\pi}}{\Gamma(-\nu/2)}z\,{}_1F_1\left(\frac{1-\nu}{2};\frac{3}{2};\frac{z^2}{2}\right)\right].$$

$$D_\nu(z)=\frac{\Gamma(\nu+1)}{\sqrt{2\pi}}\left[e^{\nu\pi i/2}D_{-\nu-1}(iz)+e^{-\nu\pi i/2}D_{-\nu-1}(-iz)\right]=$$

$$=e^{\mp\nu\pi i}D_\nu(-z)+\frac{\sqrt{2\pi}}{\Gamma(-\nu)}e^{\mp(\nu+1)\pi i/2}D_{-\nu-1}(\pm iz).$$

$$D_{\nu+1}(z)-zD_\nu(z)+\nu D_{\nu-1}(z)=0.$$

$$\frac{d}{dz}D_\nu(z)\pm\frac{z}{2}D_\nu(z)\mp\begin{Bmatrix}\nu\\1\end{Bmatrix}D_{\nu\mp1}(z)=0.$$

$$\frac{d^n}{dz^n}\left[e^{\pm z^2/4}D_\nu(z)\right]=(-1)^n\begin{Bmatrix}(-\nu)_n\\1\end{Bmatrix}e^{\pm z^2/4}D_{\nu\mp n}(z).$$

$$D_{-1}(z)=\sqrt{\frac{\pi}{2}}\,e^{z^2/4}\operatorname{erfc}\left(\frac{z}{\sqrt{2}}\right).$$

$$D_{-2}(z)=-\sqrt{\frac{\pi}{2}}\,ze^{z^2/4}\operatorname{erfc}\left(\frac{z}{\sqrt{2}}\right)+e^{-z^2/4}.$$

$$D_{-n-1}(z)=\frac{(-1)^n}{n!}\sqrt{\frac{\pi}{2}}\,e^{-z^2/4}\frac{d^n}{dz^n}\left[e^{z^2/2}\operatorname{erfc}\left(\frac{z}{\sqrt{2}}\right)\right],$$

$$D_{-1/2}(z)=\sqrt{\frac{z}{2\pi}}\,K_{1/4}\left(\frac{z^2}{4}\right).$$

$$D_{-3/2}(z)=\frac{z^{3/2}}{\sqrt{2\pi}}\left[K_{3/4}\left(\frac{z^2}{4}\right)-K_{1/4}\left(\frac{z^2}{4}\right)\right].$$

$$D_n(z)=2^{-n/2}e^{-z^2/4}H_n\left(\frac{z}{\sqrt{2}}\right).$$

II.9. THE BESSEL FUNCTION $J_\nu(z)$, NEUMANN FUNCTION $Y_\nu(z)$ AND HANKEL FUNCTION $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$

Definitions:

$$J_\nu(z)=\frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu{}_0F_1\left(\nu+1;-\frac{z^2}{4}\right)=\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)}.$$

$$Y_\nu(z)=\frac{1}{\sin\nu\pi}\left[J_\nu(z)\cos\nu\pi-J_{-\nu}(z)\right]\qquad[\nu\neq\pm n].$$

$$Y_n(z)=\lim_{\nu\to n}Y_\nu(z)=\frac{2}{\pi}J_n(z)\ln\frac{z}{2}-\frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n}-$$

$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(n+k)!\,k!}\left[\psi(n+k+1)+\psi(k+1)\right]\left(\frac{z}{2}\right)^{n+2k},$$

25. $$\int_0^\infty \frac{xe^{-x}\,dx}{e^x+e^{-x}-1} = 0.3118211319\ldots$$

26. $$\int_{-\infty}^\infty \frac{e^{\mu x}-e^{\nu x}}{e^{qx}-1}\frac{dx}{x} = \ln\frac{\sin(\nu\pi/q)}{\sin(\mu\pi/q)}$$ $[0<\operatorname{Re}(\nu/q)<1;\ 0<\operatorname{Re}(\mu/q)<1]$.

27. $$\int_{-\infty}^\infty \frac{x^{2n}\,dx}{e^x+e^{-x}+2\cos\gamma} = \frac{(-1)^n(2n)!\,(2\pi)^{2n+1}}{(2n+1)!\sin\gamma}B_{2n+1}\left(\frac{\gamma+\pi}{2\pi}\right)$$ $[|\gamma|<\pi]$.

28. $$\int_{-\infty}^\infty \frac{x^2\,dx}{e^x+e^{-x}+2\cos\gamma} = \frac{\gamma(\pi^2-\gamma^2)}{3\sin\gamma}$$ $[|\gamma|<\pi]$.

29. $$\int_{-\infty}^\infty \frac{dx}{a^2e^x+e^{-x}+2a\cos\gamma} = \frac{\gamma}{a\sin\gamma}$$ $[|\gamma|<\pi;\ a>0]$.

30. $$\int_{-\infty}^\infty \frac{x\,dx}{a^2e^x+e^{-x}+2a\cos\gamma} = -\frac{\gamma\ln a}{a\sin\gamma}$$ $[|\gamma|<\pi;\ a>0]$.

2.3.15. Integrals of $A(x)\,e^{-px^2-qx}$.

1. $$\int_a^\infty (x-a)^{\beta-1}e^{-px^2-qx}\,dx = \Gamma(\beta)(2p)^{-\beta/2}\exp\left[\frac{q^2}{8p}-\frac{a}{2}(q+ap)\right]D_{-\beta}\left(\frac{2ap+q}{\sqrt{2p}}\right)$$
$[\operatorname{Re}\beta,\ \operatorname{Re}p>0]$ or $[\operatorname{Re}\beta,\ \operatorname{Re}q>0,\ \operatorname{Re}p=0]$ or $[\operatorname{Re}p=\operatorname{Re}q=0,\ \operatorname{Im}p\neq0,\ 0<\operatorname{Re}\beta<2]$.

2. $$\int_0^\infty e^{-i\lambda x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}\,e^{-\pi i/4}$$ $[\lambda>0]$.

3. $$\int_0^\infty x^{\alpha-1}e^{-px^2-qx}\,dx = \Gamma(\alpha)(2p)^{-\alpha/2}\exp[q^2/(8p)]\,D_{-\alpha}(q/\sqrt{2p})$$
$[\operatorname{Re}\alpha,\ \operatorname{Re}p>0]$ or $[\operatorname{Re}\alpha,\ \operatorname{Re}q>0,\ \operatorname{Re}p=0]$
или $[0<\operatorname{Re}\alpha<2,\ \operatorname{Re}p=\operatorname{Re}q=0,\ \operatorname{Im}p\neq0]$.

4. $$\int_0^\infty e^{-px^2-qx}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{p}}\exp\left(\frac{q^2}{4p}\right)\operatorname{erfc}\left(\frac{q}{2\sqrt{p}}\right)$$
$[\operatorname{Re}p>0]$ or $[\operatorname{Re}p=0,\ \operatorname{Im}p\neq0,\ \operatorname{Re}q\geqslant0]$

5. $$\int_0^\infty \frac{e^{-px^2-qx}}{\sqrt{x}}\,dx = \frac{1}{2}\sqrt{\frac{q}{p}}\exp\left(\frac{q^2}{8p}\right)K_{1/4}\left(\frac{q^2}{8p}\right)$$
$[\operatorname{Re}p>0]$ or $[\operatorname{Re}p=0,\ \operatorname{Im}p\neq0,\ \operatorname{Re}q\geqslant0]$.

6. $$\int_0^\infty \sqrt{x}\,e^{-px^2-qx}\,dx = \frac{1}{8}\left(\frac{q}{p}\right)^{3/2}\exp\left(\frac{q^2}{8p}\right)\left\{K_{3/4}\left(\frac{q^2}{8p}\right)-K_{1/4}\left(\frac{q^2}{8p}\right)\right\}$$
$[\operatorname{Re}p>0]$ or $[\operatorname{Re}p=0,\ \operatorname{Im}p\neq0,\ \operatorname{Re}q\geqslant0]$.