FTCS 3D growth rate

Assume $\Delta x = \Delta y = \Delta z$. Use p, q, r as indices. k is Fourier expansion number and $i = \sqrt{-1}$. Assume $u(t, x, y, z)$ is separable into $G(t) \cdot F_x(x) \cdot F_y(y) \cdot F_z(z)$, then

$$u^n_{pqr} = G^n \cdot e^{ikp\Delta x} \cdot e^{ikq\Delta x} \cdot e^{ikr\Delta x}$$

Then FTCS gives us:

$$\left(G^{n+1} \cdot e^{ikp\Delta x} \cdot e^{ikq\Delta x} \cdot e^{ikr\Delta x} - G^n \cdot e^{ikp\Delta x} \cdot e^{ikq\Delta x} \cdot e^{ikr\Delta x}\right) / \Delta t$$

$$= \frac{\alpha G^n}{\Delta x^2} \Big[e^{ik(p-1)\Delta x} \cdot e^{ikq\Delta x} \cdot e^{ikr\Delta x} + e^{ik(p+1)\Delta x} \cdot e^{ikq\Delta x} \cdot e^{ikr\Delta x} +$$

$$e^{ik(q-1)\Delta x} \cdot e^{ikp\Delta x} \cdot e^{ikr\Delta x} + e^{ik(q+1)\Delta x} \cdot e^{ikp\Delta x} \cdot e^{ikr\Delta x} +$$

$$e^{ik(r-1)\Delta x} \cdot e^{ikq\Delta x} \cdot e^{ikp\Delta x} + e^{ik(r+1)\Delta x} \cdot e^{ikq\Delta x} \cdot e^{ikp\Delta x} +$$

$$-6 \cdot e^{ikp\Delta x} \cdot e^{ikq\Delta x} \cdot e^{ikr\Delta x} \Big]$$

$$\left(G^{n+1} - G^n\right) \underline{e^{(p+q+r)ik\Delta x}} = \frac{\alpha G^n \Delta t}{\Delta x^2} \underline{e^{(p+q+r)ik\Delta x}} \left[\frac{3}{e^{ik\Delta x}} + 3 \cdot e^{ik\Delta x} - 6 \right]$$

$$G^{n+1} = G^n \left(1 + \frac{\alpha \Delta t}{\Delta x^2} \left(3e^{-ik\Delta x} + 3e^{ik\Delta x} - 6 \right) \right)$$

$$\frac{G^{n+1}}{G^n} = 1 + 3C\left(2\cos(k\Delta x) - 2\right) = 1 + 6C\left(\cos(k\Delta x) - 1\right)$$

Extreme value is when cosine term is -1: $1 + 6C(-2) = 1 - 12C$.

So $\boxed{0 < C \leq \frac{1}{6}}$ keeps the growth rate ≤ 1.