Live 2

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1 Unbiased or consistent

Since

$$\widehat{\beta} = \frac{\sum x_t y_t}{\sum x_t^2} = \frac{\sum x_t(\beta x_t + u_t)}{\sum x_t^2} = \beta + \frac{\sum x_t u_t}{\sum x_t^2}$$

then $\widehat{\beta}$ is unbiased if x_t and u_t are independent, allowing us to write $E(AB) = E(A)E(B)$:

$$E\left\{\frac{\sum x_t u_t}{\sum x_t^2}\right\} = E\left\{\frac{\sum x_t}{\sum x_t^2}\right\} E(u_t)$$

and $E(u_t) = 0$.

Notice that since

$$E\left\{\frac{\sum x_t u_t}{\sum x_t^2}\right\} \neq \frac{E(\sum x_t u_t)}{E(\sum x_t^2)}$$

having x_t and u_t uncorrelated $E(\sum x_t u_t) = 0$ is not enough. Independence means that u_t is also uncorrelated with $\sum x_t^2$, so it has to be uncorrelated with all the x_{t-i}, not just x_t. However

$$P\lim\left\{\frac{\sum x_t u_t/T}{\sum x_t^2/T}\right\} = \frac{P\lim\left(\sum x_t u_t/T\right)}{P\lim(\sum x_t^2/T)}$$

since as $T \to \infty$, they behave like constants. So uncorrelated is enough for consistency.

In

$$y_t = \rho y_{t-1} + u_t$$

If u_t is not serially correlated it is uncorrelated with y_{t-1}, since that was determined before u_t was realised, but it is clearly correlated with y_t. which appears in the denominator.

2 Matrices

$$Y_t = \beta_1 + \beta_2 X_{2t} + ... + \beta_k X_{kt} + u_t$$

$$\underset{T\times 1}{y} = \underset{T\times k}{X}\underset{k\times 1}{\beta} + \underset{T\times 1}{u}$$

where y and u are $T \times 1$ vectors and X is a $T \times k$ matrix.

$$\begin{bmatrix} Y_1 \\ Y_2 \\ .. \\ Y_T \end{bmatrix} = \begin{bmatrix} 1 & X_{21} & .. & X_{k1} \\ 1 & X_{22} & .. & X_{k2} \\ .. & .. & .. & .. \\ 1 & X_{2T} & .. & X_{kT} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ .. \\ \beta_k \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ .. \\ u_T \end{bmatrix}.$$

3 Method of Moments

Originally applied to the usual moments: mean, variance, skewness but extended to where you choose estimators that make some population condition hold in the sample, e.g.

$$E(X'u) = 0$$

so choose $\widehat{\beta}$ that makes

$$X'\widehat{u} = X'(y - X\widehat{\beta}) = 0$$

Generalised method of moments estimator for any function

$$E(g(y_t, \theta)) = 0$$

choose $\widehat{\theta}$ to minimise

$$T^{-1} \sum g(y_t, \theta))$$

$E(X'u) = 0$ and $E(g(y_t, \theta)) = 0$ are called orthogonality conditions.

4 Sum of squares

$$\underset{(1\times T)\,(T\times 1)}{\widehat{u}'\widehat{u}} = \underset{(1\times T)\,(T\times 1)}{\left(y - X\widehat{\beta}\right)'\left(y - X\widehat{\beta}\right)}$$

$$y'y \qquad \underset{(1\times k)(k\times T)\,(T\times k)\,(k\times 1)}{+\widehat{\beta}'X'X\widehat{\beta}} \qquad -y'X\widehat{\beta} \quad -\widehat{\beta}'X'y$$

$$\underset{(1\times T)\,(T\times k)\,(k\times 1)}{-y'X\widehat{\beta}} \quad \underset{((1\times k)(k\times T)\,(T\times 1)}{-\widehat{\beta}'X'y}$$

$$\widehat{u}'\widehat{u} = y'y + \widehat{\beta}'X'X\widehat{\beta} - 2\widehat{\beta}'X'y$$

In scalar terms (page 9 of notes) for

$$Y_t = \beta_1 + \beta_2 X_t + u_t$$

$$u'u = \sum Y_t^2 + [\beta_1^2 T + \beta_2^2 \sum X_t^2 + 2\beta_1\beta_2 \sum X_t] - 2(\beta_1 \sum Y_t + \beta_2 \sum X_t Y_t)$$

3 terms and you can see that the term $\beta' X' X \beta$ is a quadratic form

5 Derivatives

The derivative of the sum of squares is a $k \times 1$ vector

$$\begin{array}{cccc} \frac{\partial \widehat{u}'\widehat{u}}{\partial \widehat{\beta}} & = 2X'X\widehat{\beta} & -2X'y & = 0 \\ (k \times 1) & (k \times T)\,(T \times k)\,(k \times 1) & (k \times T)\,(T \times 1) & (k \times 1) \end{array}$$

From notes page 9, equations 1.8, 1.9 are

$$\frac{\partial \widehat{u}'\widehat{u}}{\partial \widehat{\beta}_1} = 2\widehat{\beta}_1 T + 2\widehat{\beta}_2 \sum X_t - 2\sum Y_t = 0$$

$$\frac{\partial \widehat{u}'\widehat{u}}{\partial \widehat{\beta}_2} = 2\widehat{\beta}_2 \sum X_t^2 + 2\widehat{\beta}_1 \sum X_t - 2\sum X_t Y_t = 0$$

$$\begin{bmatrix} \frac{\partial \widehat{u}'\widehat{u}}{\partial \widehat{\beta}_1} \\ \frac{\partial \widehat{u}'\widehat{u}}{\partial \widehat{\beta}_2} \end{bmatrix} = 2\begin{bmatrix} T & \sum X_t \\ \sum X_t & \sum X_t^2 \end{bmatrix}\begin{bmatrix} \widehat{\beta}_1 \\ \widehat{\beta}_2 \end{bmatrix} - 2\begin{bmatrix} \sum Y_t \\ \sum X_t Y_t \end{bmatrix}$$

The second derivatives are given by a $k \times k$ matrix which takes the derivative of $\frac{\partial \widehat{u}'\widehat{u}}{\partial \widehat{\beta}}$ with repect to $\widehat{\beta}'$, though it looks as if it is with respect to $\widehat{\beta}$

$$\begin{aligned} \frac{\partial \widehat{u}'\widehat{u}}{\partial \widehat{\beta}} &= 2X'X\widehat{\beta} - 2X'y \\ \frac{\partial \widehat{u}'\widehat{u}}{\partial \widehat{\beta}\partial \widehat{\beta}'} &= 2X'X \end{aligned}$$

6 Variance-Covariance matrices