The last section of these notes is on "Convergence in probability". This is something you covered last term, so I chose not to go through. Feel free to come to me with any questions on it!

MSc Economics: Econometrics II

Week 4: Recursive systems of equations

[redacted]

You are given the system

$$
\begin{aligned}
y_{1,t} + \gamma_{11}x_{1,t} + \gamma_{12}x_{2,t} &= u_{1,t} \\
\beta_{21}y_{1,t} + y_{2,t} + \gamma_{23}x_{3,t} &= u_{2,t}
\end{aligned} \tag{1}
$$

Where

A1 $\mathbb{E}(X'u_i) = 0$,

A2 $\mathbb{E}u_{i,t} = 0 \quad \forall i = 1, 2; t = 1...T$

A3 $\mathbb{E}(u_{it}u_{is}) = 0$ for $t \neq s$

A4 $\mathbb{E}(u_{1,t}u_{2,t}) = 0$.

First, set the equations up in matrix form:

$$\mathbf{B}\mathbf{y}_t + \boldsymbol{\Gamma}\mathbf{x}_t = \mathbf{u}_t$$

where

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ \beta_{21} & 1 \end{bmatrix} \boldsymbol{\Gamma} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & 0 \\ 0 & 0 & \gamma_{23} \end{bmatrix} \mathbf{x}_t = \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \end{bmatrix}$$

Notation:

$$
\begin{aligned}
B_i &= \text{columns of } \mathbf{B} \text{ with a 0 in the } i\text{th row} \\
\Gamma_i &= \text{columns of } \boldsymbol{\Gamma} \text{ with a 0 in the } i\text{th row} \\
n_i &= \text{no. of endogenous variables in the } i\text{th equation} \\
m_i &= \text{no. of exogenous variables in the } i\text{th equation} \\
g_i &= n - n_i + m - m_i : \text{ no of exclusions in } i^{th} \text{ equation}
\end{aligned}
$$

Rank condition:

$$rk(\mathbf{A}_i) = rk(\begin{bmatrix} B_i & \Gamma_i \end{bmatrix}) = n-1: \; N \text{ and } S \text{ for identification}$$

Order Condition:

$$\begin{aligned} g_i &= n - n_i + m - m_i \\ g_i &\geq n-1: \; \text{is } N \text{ for identification} \\ g_i &= n-1 \Rightarrow \; \text{just-identification of eq } i \text{ if rank cond also passed} \\ g_i &> n-1 \Rightarrow \text{over-identification if rank cond also passed} \end{aligned}$$

Question a) Assess the identifiability of the two equations in terms of the order and rank conditions.

Equation 1 $y_{1,t} + \gamma_{11}x_{1,t} + \gamma_{12}x_{2,t} = u_{1,t}$

Rank cond: $rk(A_1) = rk(\begin{bmatrix} 0 & 0 \\ 1 & \gamma_{23} \end{bmatrix}) = 1$ implies eqnation 1 is identified

Order cond: $g_i = (2-1) + (3-2) = 2 > 1$ implies equation 1 is over-identified

Equation 2 $\beta_{21}y_{1,t} + y_{2,t} + \gamma_{23}x_{3,t} = u_{2,t}$

Rank cond: $rk(A_2) = rk(\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ 0 & 0 \end{bmatrix}) = 1$ implies eqnation 1 is identified

Order cond: $g_2 = (2-2) + (3-1) = 2 > 1$ implies equation 2 is over-identified as well.

Question b) Derive the R.F. of equations for $y_{1,t}$ and $y_{2,t}$

$$\begin{aligned} \mathbf{B}\,\mathbf{y}_t &= -\mathbf{\Gamma}\mathbf{x}_t + \mathbf{u}_t \\ \mathbf{y}_t &= \mathbf{B}^{-1}\mathbf{\Gamma}\mathbf{x}_t + \mathbf{B}^{-1}\mathbf{u}_t \\ &= \mathbf{\Pi}\mathbf{x}_t + \mathbf{v}_t \end{aligned} \tag{2}$$

where it is useful to note for later that

$$\begin{aligned} \mathbf{\Pi} &= \begin{bmatrix} -\gamma_{11} & -\gamma_{12} & 0 \\ \beta_{21}\gamma_{11} & \beta_{21}\gamma_{12} & -\gamma_{23} \end{bmatrix} \\ \mathbf{v}_t &= \begin{bmatrix} u_{1,t} \\ u_{2,t} - \beta_{21}u_{1,t} \end{bmatrix} \end{aligned} \tag{3}$$

Look at the sturcuture of $\mathbf{v}_t$: $y_{2,t}$ responds to both its own shock, u_{2t}, and the shock to $y_{1,t}$, i.e. u_{1t}, in period t. However, $y_{1,t}$ is not affected by shock the period t shcok to $y_{2,t}$. Therefore recursive systems imply a *causal* ordering between the variables: we can talk about $y_{1,t}$ causing $y_{2,t}$ in this model.

In a macroeconomic example (2) could represent a VARX, with the first two exogenous variables being lags of the two endogenous variables, for example the output-gap and the risk-free rate.[1] Then $x_{3,t}$ would be an exogenous variable, such as lagged inflation. Psaradakis, Sola, Ravn (2004) use such a set of variables (though not the type of structure in (1)). Then the restirctions imposed by (1) imply that the output gap depends on its own lag, the lagged risk-free rate, and its own shock u_{1t}. In equation 2, the central bank responds to the current outputgap (which includes the shock u_{1t}), and lagged inflation. We can see this in the reduced form through the second row of $\mathbf{v}_t$.

Returning to econometric issues, notice that in an *over*identified system, there is not a unique estimator of the parameters of the S.F. system from the estimated values of the R.F. coefficients.

$$\frac{\hat{\pi}_{21}}{-\hat{\pi}_{11}} = \hat{\beta}_{21} \quad ; \quad \frac{\hat{\pi}_{22}}{-\hat{\pi}_{21}} = \hat{\beta}_{21}$$

with the two not necessarily equal.

Question c) Show that the OLS estimates of the coefficients of each equation of the S.F are consistent.

Intuition: Clearly the first equation of (1) can be consistently estimated by OLS. The restrictions on $\mathbf{B}$ exclude $y_{2,t}$ from the first equation while the $x_{i,t}$ are exogenous. Hence all the regressors appearing in the first equation are exogenous w.r.t $u_{1,t}$ so the key Gauss Markov assumption $\mathbb{E}(X'u) = 0$ is met for the first equation.

Now consider the second equation of (1): To use OLS consistently on this equation we would require that $y_{1,t}$, which appears as a regressor in the second equation is uncorrelated with $u_{2,t}$, the error in the second equation. But notice $y_{1,t}$ is a function only of predetermined, or exogenous, variables, and the stochastic process $u_{1,t}$. Also,

[1] Actually, for a VARX we would have to replace A1, exogeneity of the X's, with the weaker assumption $\mathbb{E}(x_{i,t}i_{j,t}) = 0 \ \forall i, j$.

the two stochastic processes $u_{i,t}$ are independent. Because of this independence and the exclusion of $y_{2,t}$ from the first equation, $y_{1,t}$ is predetermined, or exogenous, w.r.t. $u_{2,t}$. I.e introducting $y_{1,t}$ into the second equation only introduces more information which is exogenous w.r.t $u_{2,t}$, so we can estimate this equation by OLS also.

$$\begin{aligned}
\beta_{21}y_{1,t} + y_{2,t} + \gamma_{23}x_{3,t} &= u_{2,t} \\
y_{2t} &= -\beta_{21}(-\gamma_{11}x_{1,t} - \gamma_{12}x_{2t} + u_{1t}) - \gamma_{23}x_{3,t} + u_{2,t} \\
&= \pi_{11}x_{1t} + \pi_{12}x_{2t} - \gamma_{23}x_{3t} - \beta_{21}u_{1t} + u_{2t} \qquad (4)
\end{aligned}$$

Let $X_{1,t} = \begin{bmatrix} x_{1,t} & x_{2,t} & x_{3,t} & u_{1,t} \end{bmatrix}$ be the 1.4 set of regressors in (4). Then to establish consistency we must check the condition $\mathbb{E}(X_{1,t}u_{2,t}) = 0$:

$$\begin{aligned}
\mathbb{E}(x_{i,t}u_{2,t}) &= 0, \quad i = 1,2,3 \quad \text{by A1} \\
\mathbb{E}(u_{1,t}u_{2,t}) &= \mathbb{E}[u_{1t}u_{2t}] \\
&= 0 \qquad\qquad\qquad \text{by A4} \\
\Rightarrow \mathbb{E}(X_{1,t}u_{2,t}) &= 0
\end{aligned}$$

More details on consistency: the role of $\mathbb{E}(x_t u_t) = 0$.

Proving consistency from first principles is a little intricate; without being completely formal, we can gain some insight into the importance of the $\mathbb{E}(x'u) = 0$ condition through the following argument, based on Ruud pp.256-264. We need additional assumptions

A5: Population full rank. $\mathbb{E}(x_n x_n') = \mathbf{D}$, some nonsingular matrix $\mathbf{D}$.

A6: IID. Let (y_n, x_n) be i.i.d. from their joint distribution.[2]

The key idea behind consistency is Chebychev's Law of Large Numbers (LLN):

LLN for x_i, $i = 1 \ldots T$, i.i.d. R.V.s with $\mathbb{E}(x_i) = x_0$

$$\bar{x}_T \to^p x_0 \qquad (5)$$

[2]it is quite possible to relax this but it makes the argument more complicated

We are familiar with this idea that as sample size grows, the variance of the sample mean goes to zero, so that the sample mean 'converges in probability' to a constant, the population expectation.

We use the LLN to demonstrate consistency by rewriting our OLS estimators in terms of sample averages. As an example, consider the first equation of (1). Write the OLS estimation of the first line of (1) as

$$y_{1,t} = z_{1,t}'\theta_1 + u_{1,t}$$

where $z_{1,t} = [x_{1,t}, x_{2,t}]'$ then

$$\begin{aligned}
\theta_1 &= [z_1'z_1]^{-1}z_1'y_1 \\
&= [\sum_t z_{1,t}z_{1,t}']^{-1}\sum_t z_{i,1}y_{1,t} \\
&= [\frac{1}{T}\sum_t z_{1,t}z_{1,t}']^{-1}\frac{1}{T}\sum_t z_{i,1}y_{1,t}
\end{aligned}$$

The first block is the sample average of the products of the elements of $z_{1,t}$. given the LLN, we can say that

$$\begin{aligned}
[\frac{1}{T}\sum_t z_{1,t}z_{1,t}'] &\to^p \mathbf{D} \\
\text{or, equvialently, we say} \quad plim(\mathbb{E}_N[z_{1,t}z_{1,t}']) &= \mathbf{D} \qquad (6)
\end{aligned}$$

a non-singular matrix. Probability limits survive continuous transformations so

$$[\frac{1}{T}\sum_t z_{1,t}z_{1,t}']^{-1} \to^p \mathbf{D}^{-1}$$

Next we want to deal with the sample average of the products $z_{1,t}'y_{1,t}$. However, it is better to look at the object

$$\frac{1}{T}z_1'(y_1 - z_1\theta_0) = \mathbb{E}_N[z_{1,t}(y_{1,t} - z_{1,t}'\theta_0)]$$

again by the LLN

$$\begin{aligned}
\mathbb{E}_N[z_{1,t}(y_{1,t} - z_{1,t}'\theta_0)] &\to^p \mathbb{E}[z_{1,t}(y_{1,t} - z_{1,t}'\theta_0)] \\
&= 0
\end{aligned}$$

because the object inside the (...) is just $u_{1,t}$ and we know the regressors are exogenous w.r.t. this error. So here we see the key assumption, $\mathbb{E}[z_{1,t}u_{1,t}] = 0$, working in practice, i.e. underpinning our deomonstration of consistency...which we complete by observing that

$$\begin{aligned}\mathbb{E}_N[z_{1,t}y_{1,t}] &= \mathbb{E}_N[z_{1,t}(y_{1,t} - z'_{1,t}\theta_0)] + \mathbb{E}_N[z_{1,t}z'_{1,t}]\theta_0 \\ &\to^p 0 + plim(\mathbb{E}_N[z_{1,t}z'_{1,t}])\theta_0 \\ &= \mathbf{D}\theta_0\end{aligned}$$

Hence

$$\hat{\theta} = \mathbb{E}_N[z_{1,t}z'_{1,t}]^{-1}\mathbb{E}_N[z_{1,t}y_{1,t}] \quad \to^p \quad \mathbf{D}^{-1}\mathbf{D}\theta_0 = \theta_0$$