Econometrics 1, Class Week 7

Back-up Video: Class week 7 video (click here)

Learning Outcomes

(a) Forecasting in Autoregressive Integrated Moving Average (ARIMA) models.

(b) Intertemporal Dependence.

(c) First Moments of the Stationary Distribution.

(d) Second Moments of the Stationary Distribution.

(e) Order of Integration

Prerequisites

1. Conditional and unconditional moments.

2. Law of Iterated Expectations (not strictly necessary, but may be helpful).

Univariate Time-Series Models

Models

$$
\begin{aligned}
y_t &= \alpha + \rho y_{t-1} + \epsilon_t && (1) \\
y_t &= \alpha + y_{t-1} + \epsilon_t && (2) \\
y_t &= \alpha + \epsilon_t + \mu \epsilon_{t-1} && (3) \\
\Delta y_t &= y_t - y_{t-1} \\
&= \alpha + \rho \Delta y_{t-1} + \epsilon_t + \mu \epsilon_{t-1}, && (4)
\end{aligned}
$$

where ϵ_t is white noise (WN), i.e. $\epsilon_t \overset{i.i.d.}{\sim} \mathbb{E}[\epsilon_t] = 0, \mathbb{E}[\epsilon_t^2] = \sigma^2$.

Model interpretation:

(1) : Autoregressive process of order 1, AR(1),

(2) : Random walk with drift α.

(3) : Moving Average of order 1, MA(1).

(4) : ARIMA(1,1,1).

(a) Forecasting y_{T+1} and y_{T+2}.

Estimate model parameters and residuals on the basis of a sample $\{y_t, t = 0, \cdots, T\}$; use estimates to forecast y_{T+1} and y_{T+2}.

Denote one-step forecast by $\hat{y}_{T+1|T}$, and two-step forecast by $\hat{y}_{T+2|T}$, given information up to (and including) time T.

General approach:

- Obtain expression for y_{T+s};
- replace parameters by their respective estimates and any variable dated $s > T$ by its conditional expectation, given information up to T.

AR(1)

$$
\begin{aligned}
y_{T+1} &= \alpha + \rho y_T + \epsilon_{T+1} && (5)\\
y_{T+2} &= \alpha + \rho y_{T+1} + \epsilon_{T+2} \\
&= \alpha + \rho(\alpha + \rho y_T + \epsilon_{T+1}) + \epsilon_{T+2} \\
&= \alpha(1+\rho) + \rho^2 y_T + \epsilon_{T+2} + \rho\epsilon_{T+1}. && (6)
\end{aligned}
$$

Hence, by properties of WN,

$$
\begin{aligned}
\hat{y}_{T+1|T} &= \hat{\alpha} + \hat{\rho} y_T && (7)\\
\hat{y}_{T+2|T} &= \hat{\alpha}(1+\hat{\rho}) + \hat{\rho}^2 y_T \\
&= \hat{\alpha} + \hat{\rho}\hat{y}_{T+1|T}. && (8)
\end{aligned}
$$

RW with drift: special case of AR(1), with $\rho = 1$, so

$$
\begin{aligned}
\hat{y}_{T+1|T} &= \hat{\alpha} + y_T && (9)\\
\hat{y}_{T+2|T} &= 2\hat{\alpha} + y_T \\
&= \hat{\alpha} + \hat{y}_{T+1|T}. && (10)
\end{aligned}
$$

MA(1)

$$
\begin{aligned}
y_{T+1} &= \alpha + \epsilon_{T+1} + \mu\epsilon_T && (11) \\
y_{T+2} &= \alpha + \epsilon_{T+2} + \mu\epsilon_{T+1}. && (12)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\hat{y}_{T+1|T} &= \hat{\alpha} + \hat{\mu}\hat{\epsilon}_T && (13) \\
\hat{y}_{T+2|T} &= \hat{\alpha}. && (14)
\end{aligned}
$$

ARIMA(1,1,1)

$$\begin{aligned} y_{T+1} &= y_T + \Delta y_{T+1} \\ &= y_T + \alpha + \rho \Delta y_T + \epsilon_{T+1} + \mu \epsilon_T && (15) \\ y_{T+2} &= y_T + \Delta y_{T+1} + \Delta y_{T+2} \\ &= y_{T+1} + \Delta y_{T+2} \\ &= y_{T+1} + \alpha + \rho \Delta y_{T+1} + \epsilon_{T+2} + \mu \epsilon_{T+1} \\ &= y_T + \Delta y_{T+1} + \alpha + \rho \Delta y_{T+1} + \epsilon_{T+2} + \mu \epsilon_{T+1} \\ &= y_T + \alpha + (1+\rho) \Delta y_{T+1} + \epsilon_{T+2} + \mu \epsilon_{T+1} && (16) \end{aligned}$$

Hence,

$$\begin{aligned} \hat{y}_{T+1|T} &= y_T + \hat{\alpha} + \hat{\rho} \Delta y_T + \hat{\mu} \hat{\epsilon}_T && (17) \\ \hat{y}_{T+2|T} &= \hat{y}_{T+1|T} + \hat{\alpha} + \hat{\rho}(\hat{y}_{T+1|T} - y_T) \\ &= y_T + \hat{\alpha} + (1+\hat{\rho})(\hat{y}_{T+1|T} - y_T). && (18) \end{aligned}$$

(b) Intertemporal Dependence of AR(1): Recursive Substitution

AR(1), using L to denote the lag operator,

$$\begin{aligned} y_t &= \alpha + \rho y_{t-1} + \epsilon_t \\ &= \alpha + \rho L y_t + \epsilon_t & (19) \\ &= \alpha + \rho(\alpha + \rho y_{t-2} + \epsilon_{t-1}) + \epsilon_t \\ &= \alpha + \rho\alpha + \rho^2 y_{t-2} + \epsilon_t + \rho\epsilon_{t-1} \\ &= \cdots \\ &= \alpha(\sum_{k=0}^{s-1} \rho^k) + \rho^s y_{t-s} + \sum_{m=0}^{s-1} \rho^m \epsilon_{t-m}. & (20) \end{aligned}$$

and if $|\rho| < 1$, i.e. AR(1) is stationary, by taking limits,

$$\begin{aligned} y_t &= \frac{\alpha}{1-\rho} + \sum_{m=0}^{\infty} \rho^m \epsilon_{t-m} \\ &= \frac{\alpha}{1-\rho} + \mathrm{MA}(\infty) \\ &= \frac{1}{1-\rho L}(\alpha + \epsilon_t), & (21) \end{aligned}$$

where the last equality can be seen directly from (19).

(20) implies for RW ($\rho = 1$) with drift

$$y_t = s\alpha + y_{t-s} + \sum_{m=0}^{s-1} \epsilon_{t-m}. \quad (22)$$

(c) Expectations of the Stationary Distribution

Definition: A stochastic process is *covariance-stationary* if the unconditional moments do not depend on calendar time t.

So can only obtain unconditional moments for covariance-stationary processes, i.e. (1) and (3).

For AR(1), assuming $\mathbb{E}[y_t]$ exists, taking expectations on LHS and RHS of (1),

$$\begin{aligned} \mathbb{E}[y_t] &= \alpha + \rho\mathbb{E}[y_{t-1}] \\ &= \alpha + \rho\mathbb{E}[y_t] \\ &= \frac{\alpha}{1-\rho}. \end{aligned} \tag{23}$$

For MA(1),

$$\mathbb{E}[y_t] = \alpha. \tag{24}$$

(d) Variance, Covariance, Correlations of Stationary Distribution

Recall that additive constants (such as α) do not contribute to variances and covariances; and that multiplicative constants (such as ρ and μ) contribute multiplicatively.

AR(1): Using (20), and assuming second moment exists,

$$
\begin{aligned}
\text{var}(y_t) &= \text{var}(\alpha + \rho y_{t-1} + \epsilon_t) \\
&= \rho^2 \text{var}(y_{t-1}) + \sigma^2 \text{ b/c } \epsilon \text{ is WN} \\
&= \rho^2 \text{var}(y_t) + \sigma^2 \text{ b/c of stationarity} \\
&= \frac{\sigma^2}{1-\rho^2}. \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
\text{cov}(y_t, y_{t-s}) &= \text{cov}(\alpha(\sum_{k=0}^{s-1} \rho^k) + \rho^s y_{t-s} + \sum_{m=0}^{s-1} \rho^m \epsilon_{t-m}, y_{t-s}) \\
&= \rho^s \text{var}(y_t) \\
&= \frac{\rho^s \sigma^2}{1-\rho^2} \qquad (26) \\
\text{corr}(y_t, y_{t-s}) &= \rho^s. \qquad (27)
\end{aligned}
$$

So AR process exhibits exponentially declining, yet infinite dependence (memory).

MA(1)

$$
\begin{aligned}
\mathrm{var}(y_t) &= \mathrm{var}(\alpha + \epsilon_t + \mu\epsilon_{t-1}) \\
&= \sigma^2 + \mu^2\sigma^2 \ \text{ b/c } \epsilon \text{ is WN} \\
&= (1+\mu^2)\sigma^2. & (28) \\
\mathrm{cov}(y_t, y_{t-1}) &= \mathrm{cov}(\alpha + \epsilon_t + \mu\epsilon_{t-1}, \alpha + \epsilon_{t-1} + \mu\epsilon_{t-2}) \\
&= \mu\mathrm{var}(\epsilon_{t-1}) \\
&= \mu\sigma^2. & (29) \\
\mathrm{cov}(y_t, y_{t-s}) &= 0 \ \ \forall |s| > 1. & (30) \\
\mathrm{corr}(y_t, y_{t-1}) &= \frac{\mu}{1+\mu^2} & (31) \\
\mathrm{corr}(y, y_{t-s}) &= 0 \ \ \forall |s| > 1. & (32)
\end{aligned}
$$

So MA(1) exhibits no dependence (memory) beyond first lag.

And MA(s) exhibits no dependence beyond lag s.

(e) Integration of order 1, $I(1)$

RW with drift is $I(1)$:

$$\Delta y_t = (1 - L)y_t = \alpha + \epsilon_t. \tag{33}$$

ARIMA(1,1,1) is $I(1)$:

$$\begin{aligned} (1 - L)y_t &= \alpha + \rho(1 - L)y_{t-1} + \epsilon_t + \mu\epsilon_{t-1} \\ (1 - \rho L)(1 - L)y_t &= \alpha + (1 + \mu L)\epsilon_t. \end{aligned} \tag{34}$$