Econometrics, Lecture 11A
Dynamic Linear regressions: ARDL & ECM

Ron Smith
EMS, Birkbeck, University of London

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Last time

- We looked at univariate stochastic time series models.
- The building block was an error term ε_t, which might be called: white noise, independently and identically distributed or well behaved error
- This went into autoregressive and moving average models for different orders of integration.
- We put them together to form the ARIMA model.
- Used the lag operator: $Ly_t = y_{t-1}$ and polynomials in the lag operator
- Considered stability and invertibility
- We discussed how to determine the order of integration and concluded that it was often very difficult to tell.
- But do not need to, we could leave it to the computer by estimating unrestricted models.

ARDL

- A distributed lag regression model of order q, DL(q) is:

$$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + ... + \beta_q x_{t-q} + u_t$$

  like a moving average with observed shocks, x_t.

- Combine the DL with an AR component and an iid error (though it could be MA) to give the ARDL(p,q) process:

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + ... + \alpha_p y_{t-p} + \beta_0 x_t + \beta_1 x_{t-1} + ... + \beta_q x_{t-q} + u_t$$

$$A(L^p) y_t = \alpha_0 + B(L^q) x_t + u_t$$

- If u_t is white noise, OLS gives consistent estimates of the parameters, though the estimates are not unbiased, since y_{t-1} is only uncorrelated with, not independent of, u_t.

Long-run

- Just like the AR, the ARDL(p,q)

  $$y_t = \alpha_0 + \alpha_1 y_{t-1} + ... + \alpha_p y_{t-p} + \beta_0 x_t + \beta_1 x_{t-1} + ... + \beta_q x_{t-q} + u_t$$

  is stationary, (conditional on x_t the process is stable), if all the roots (solutions), z_i, of the characteristic equation

  $$1 - \alpha_1 z - \alpha_2 z^2 - ... - \alpha_p z^p = 0 \tag{1}$$

  lie outside the unit circle. Check that $-1 < \sum \alpha_i < 1$.
- If it is stable in equilibrium, $u_t = 0$ $y_{t-i} = y$ and $x_{t-i} = x$ for all i.

  $$y = \alpha_0 + \alpha_1 y + ... + \alpha_p y + \beta_0 x + \beta_1 x + ... + \beta_q x$$

  $$y(1 - \alpha_1 - .. - \alpha_p) = \alpha_0 + (\beta_0 + \beta_1 + ... + \beta_q)x$$
- The long run relation between them will be:

  $$y = \frac{\alpha_0}{1 - \sum_{i=1}^{p} \alpha_i} + \frac{\sum_{i=0}^{q} \beta_i}{1 - \sum_{i=1}^{p} \alpha_i} x = \theta_0 + \theta_x x.$$

ARDL(1,1)

- ARDL(1,1) for illustration:

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1} + u_t. \qquad (2)$$

is stable if $-1 < \alpha_1 < 1$, and then has a long run solution:

$$y_t^* = \frac{\alpha_0}{1-\alpha_1} + \frac{\beta_0 + \beta_1}{1-\alpha_1} x_t = \theta_0 + \theta_x x_t.$$

- Estimate the long run effect of x_t on y_t from the OLS estimates of (2) as

$$\widehat{\theta}_x = \frac{\widehat{\beta}_0 + \widehat{\beta}_1}{1-\widehat{\alpha}_1}.$$

- Standard errors for the long-run coefficients can be calculated by the delta method, which is available in most programmes. Since the estimated $\widehat{\alpha}_1$ can be close to one, causing the estimate of $\widehat{\theta}_x$ to blow up (no finite sample moments), it can have poor properties.

Theoretical and empirical models

- Dynamic linear regression models arise from long run equilibrium relationships together with adjustment processes.
- The long run relationship determines an unobserved equilibrium or target value, y_t^*, but slow adjustment to equilibrium is slow. The speed of adjustment is $0 \leq \lambda \leq 1$ the proportion of the deviation from equilibrium removed in any period:

$$\begin{aligned}
y_t^* &= \theta_0 + \theta_x x_t \\
\Delta y_t &= \lambda(y_t^* - y_{t-1}) + u_t, \\
\Delta y_t &= \lambda\theta_0 + \lambda\theta_x x_t - \lambda y_{t-1} + u_t. \\
\Delta y_t &= a_0 + a_1 y_{t-1} + b x_t + u_t
\end{aligned}$$

- This Partial Adjustment Model, PAM, can be estimated by OLS and the theoretical parameters recovered: $\lambda = -a_1$; $\theta_x = b/\left(1 - a_1\right)$; $\theta_0 = a_0/\left(1 - a_1\right)$.

Error Correction model

- Consider the same long-run equilibrium but a different adjustment process

$$
\begin{aligned}
y_t^* &= \theta_0 + \theta_x x_t \\
\Delta y_t &= \lambda_1 \Delta y_t^* + \lambda_2 (y_{t-1}^* - y_{t-1}) + u_t, \\
\Delta y_t &= \lambda_1 \theta_x \Delta x_t + \lambda_2 (\theta_0 + \theta_x x_{t-1} - y_{t-1}) + u_t, \\
\Delta y_t &= b_0 \Delta x_t + a_0 + b_1 x_{t-1} + a_1 y_{t-1} + u_t
\end{aligned}
$$

- Again the empirical parameters can be estimated by OLS and the theoretical parameters recovered: $\lambda_2 = -a_1$, $\theta_x = -b_1 / a_1$. Or it can be estimated directly by non-linear least squares.
- It can also be written as an ARDL

$$
\begin{aligned}
y_t &= a_0 + b_0 x_t + (b_1 - b_0) x_{t-1} + (1 + a_1) y_{t-1} + u_t \\
y_t &= \alpha_0 + \beta_0 x_t + \beta_1 x_{t-1} + \alpha_1 y_{t-1} + u_t
\end{aligned}
$$

ARDL to ECM empirical models

- By subtracting y_{t-1} from both sides and adding and subtracting $\beta_0 x_{t-1}$ on the RHS we express the same statistical equation in a different form

$$\begin{aligned}
y_t &= \alpha_0 + \alpha_1 y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1} + u_t \\
y_t - y_{t-1} &= \alpha_0 + (\alpha_1 - 1) y_{t-1} + \\
&\quad \beta_0 x_t \{-\beta_0 x_{t-1} + \beta_0 x_{t-1}\} + \beta_1 x_{t-1} + u_t .
\end{aligned}$$

$$y_t - y_{t-1} = \alpha_0 + (\alpha_1 - 1) y_{t-1} + \beta_0 (x_t - x_{t-1}) + (\beta_0 + \beta_1) x_{t-1} + u_t$$

$$\Delta y_t = a_0 + b_0 \Delta x_t + a_1 y_{t-1} + b_1 x_{t-1} + u_t \qquad (3)$$

- We have reparameterized the equation where $a_0 = \alpha_0; b_0 = \beta_0; a_1 = (\alpha_1 - 1); b_1 = \beta_0 + \beta_1;$

ECM to PAM theoretical models

- In the ECM

$$\Delta y_t = \lambda_1 \Delta y_t^* + \lambda_2 (y_{t-1}^* - y_{t-1}) + u_t$$

if $\lambda_1 = \lambda_2 = \lambda$

$$\begin{aligned} \Delta y_t &= \lambda(y_t^* - y_{t-1}^*) + \lambda(y_{t-1}^* - y_{t-1}) + u_t \\ \Delta y_t &= \lambda(y_t^* - y_{t-1}) + u_t \end{aligned}$$

- An alternative parameterization, which unlike the ECM nests the partial adjustment model is:

$$\Delta y_t = \alpha_0 + (\alpha_1 - 1) y_{t-1} + (\beta_0 + \beta_1) x_t - \beta_1 \Delta x_t + u_t.$$

Reparameterizations

- When you **reparameterize** a model, as in going from ARDL to ECM, the same number of parameters (4 in this case) are estimated, just written in different ways.
- Long-run coefficients will be identical, whether you estimate it as an ARDL, ECM or by a non-linear procedure.
- The statistical properties of the model do not change, the estimated residuals, standard error of the regression and the maximised log-likelihood are identical between the different versions.
- R^2 will change, because the proportion of variation explained is measured in terms of a different dependent variable, Δy_t in the ECM rather than y_t in the ARDL.
- RESET tests that use fitted values of the dependent variable will change. Use the fitted values of Δy_t.

Restrictions and General to Specific

- When you **restrict** a model, you reduce the number of parameters estimated and such restrictions are testable.
- The number of parameters in the unrestricted model minus the number of restrictions gives the number of parameters in the restricted model: $k_U - m = k_R$
- A useful procedure in many circumstances is to start with a general model and test down to specific restricted cases.
- This general to specific procedure has the advantage that any tests on the general model are valid. Whereas if you start from the restricted model, the tests will not be valid if the model is misspecified.
- The restrictions need not be that coefficients equal zero: dropping variables.

Restricted versions of ARDL(1,1)

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1} + u_t.$$

- Static: $\alpha_1 = 0; \beta_1 = 0.$
  - $y_t = \alpha_0 + \beta_0 x_t + u_{at}.$
- First difference: $\alpha_1 = 1; \beta_1 = -\beta_0$:
  - $y_t - y_{t-1} = \alpha_0 + \beta_0(x_t - x_{t-1}) + u_{bt}$
- Partial Adjustment Model:$\beta_1 = 0$:
  - $y_t = \alpha_0 + \alpha_1 y_{t-1} + \beta_0 x_t + u_{ct}.$
- Common Factor: First order disturbance serial correlation: $\beta_1 = -\beta_0\alpha_1$:
  - $y_t - \alpha_1 y_{t-1} = \alpha_0 + \beta_0(x_t - \alpha_1 x_{t-1}) + u_{dt}$
- Random Walk with drift: $\alpha_1 = 1; \beta_1 = \beta_0 = 0.$
  - $y_t - y_{t-1} = \alpha_0 + u_{et}$

Common factor/AR1 error (Again)

Restricted model is:

$$y_t = \alpha + \beta x_t + v_t; \;\; v_t = \rho v_{t-1} + \varepsilon_t$$

noting that

$$v_t = y_t - \alpha - \beta x_t; \text{ and } \;\; v_{t-1} = y_{t-1} - \alpha - \beta x_{t-1}$$

$$y_t = \alpha + \beta x_t + \rho(y_{t-1} - \alpha - \beta x_{t-1}) + \varepsilon_t$$

$$\begin{aligned} y_t &= \alpha(1-\rho) + \beta x_t + \rho y_{t-1} - \beta\rho x_{t-1} + \varepsilon_t \\ (1-\rho L)y_t &= (1-\rho L)(\alpha + \beta x_t) + \varepsilon_t \end{aligned}$$

The model with AR1 errors is not linear in the parameters and is estimated by Generalised Least Squares or Maximum Likelihood.

Unit long-run coefficient

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1} + u_t.$$

$$\begin{aligned}\theta_x &= \frac{\beta_0 + \beta_1}{1 - \alpha_1} = 1, \\ \beta_1 + \beta_0 + \alpha_1 - 1 &= 0\end{aligned}$$

$$\begin{aligned} y_t - y_{t-1} &= \alpha_0 + \beta_0 (x_t - x_{t-1}) + (\alpha_1 - 1) y_{t-1} + (\beta_0 + \beta_1) x_{t-1} + u_t. \\ \Delta y_t &= \alpha_0 + \beta_0 \Delta x_t + (\alpha_1 - 1)(y_{t-1} - x_{t-1}) \\ &\quad + (\beta_0 + \beta_1 + \alpha_1 - 1) \, x_{t-1} + u_t. \end{aligned}$$

Restricted:

$$\Delta y_t = a_0 + b_0 \Delta x_t + a_1 (y_{t-1} - x_{t-1}) + u_t.$$

Restricted form is not unique

Often you impose restrictions by substituting out parameters. Starting from the ECM and the long run unit coefficient restriction we can substitute out for β_1 and rearrange. This gives a statistically identical but different form of the restricted model.

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1} + u_t.$$

$$\begin{aligned} \beta_1 + \beta_0 + \alpha_1 - 1 &= 0 \\ \beta_1 &= 1 - \beta_0 - \alpha_1 \end{aligned}$$

$$\begin{aligned} y_t &= \alpha_0 + \alpha_1 y_{t-1} + \beta_0 x_t + (1 - \beta_0 - \alpha_1) x_{t-1} + u_t. \\ y_t - x_{t-1} &= \alpha_0 + \alpha_1 (y_{t-1} - x_{t-1}) + \beta_0 (x_t - x_{t-1}) + u_t. \end{aligned}$$

Cobb-Douglas (Again)

- Restriction and reparameterization are not just time series things. In first lecture we had data for firms on output i, Q_i, employment E_i, and capital stock, K_i, $i = 1, 2, ..., N$.
- Cobb-Douglas Production Function

$$Q_i = AK_i^{\beta_1} E_i^{\beta_2} e^{u_i}$$

- Made it linear by taking logs

$$\begin{aligned} \ln Q_i &= \ln A + \beta_1 \ln K_i + \beta_2 \ln E_i + u_i, \\ q_i &= a + \beta_1 k_i + \beta_2 e_i + u_i \end{aligned}$$

- Reparameterize to test for constant returns to scale

$$q_i - e_i = a + \beta_1 (k_i - e_i) + (\beta_1 + \beta_2 - 1) e_i + u_i$$

ARDL(1,1,1)

- The empirical ECM with two exogenous variables is:

$$\Delta y_t = a_0 + a_1 y_{t-1} + b_0 \Delta x_t + b_1 x_{t-1} + c_0 \Delta z_t + c_1 z_{t-1} + \varepsilon_t .$$

- The long run relationship, which exists if $a_1 \neq 0$, is

$$\begin{aligned} y_t^* &= -\frac{a_0}{a_1} - \frac{b_1}{a_1} x_t - \frac{c_1}{a_1} z_t \\ y_t^* &= \theta_0 + \theta_x x_t + \theta_z z_t . \end{aligned}$$

- The theoretical ECM is

$$\Delta y_t = \lambda_1 \Delta y_t^* + \lambda_2 (y_{t-1}^* - y_{t-1}) + u_t$$

- Now there are 6 empirical parameters, $(a_0, a_1, b_0, b_1, c_0, c_1)$ but only 5 theoretical parameters $(\lambda_1, \lambda_2, \theta_0, \theta_x, \theta_z)$.
- What is the restriction?

Next time

- Suppose y_t and x_t are I(1) then in general any linear combination of them, $y_t - \beta x_t = z_t$, will also be I(1).
- If there is a linear combination that is I(0), they are said to cointegrate.
- If they cointegrate, they have a common stochastic trend, $\sum \varepsilon_t$, random walk type component, which is cancelled out by the linear combination; and this linear combination is called the cointegrating vector, $(1, -\beta)$
- The cointegrating relationship is often interpreted as an equilibrium relationship and z_t a measure of disequilibrium.
- We will consider estimation and testing for cointegrating relationships.