Econometrics, Lecture 14 VAR and Cointegration examples.

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Last time

- Introduced VARs, how to estimate them and Granger Causality
- Discussed Impulse Response Functions
- Reparameterised to a VECM
- Showed the implications of cointegration for VECM.
- Looked at the Johansen procedure for determining the number of cointegrating vectors and estimating them
- Looked at the identification problem.
- This time we are going to look at examples.

Sequence

- For y_t a $m \times 1$ vector with VAR and VECM

$$
\begin{aligned}
y_t &= a + \sum_{i=1}^{p} A_i y_{t-i} + ct + \varepsilon_t \\
\Delta y_t &= a_0 - \alpha(\beta' y_{t-1} + \gamma t) + \sum_{i=1}^{p-1} \Gamma_i \Delta y_{t-i} + v_t,
\end{aligned}
$$

- We need to choose r the number of cointegrating vectors and identify α and β.
- Will use examples with $m = 2$ and $m = 3$.

Example 1: Money demand

- Consider a VAR1 in the logarithms of real money, m_t, and income, y_t, which are both I(1) with a linear trend:

$$\begin{aligned} m_t &= a_{10} + a_{11} m_{t-1} + a_{12} y_{t-1} + \gamma_1 t + \varepsilon_{1t} \\ y_t &= a_{20} + a_{21} m_{t-1} + a_{22} y_{t-1} + \gamma_2 t + \varepsilon_{2t} \end{aligned}$$

- The VECM is:

$$\begin{aligned} \Delta m_t &= a_{10} + (a_{11} - 1) m_{t-1} + a_{12} y_{t-1} + \gamma_1 t + \varepsilon_{1t} \\ \Delta y_t &= a_{20} + a_{21} m_{t-1} + (a_{22} - 1) y_{t-1} + \gamma_2 t + \varepsilon_{2t}, \end{aligned}$$

$$\begin{aligned} \Delta m_t &= a_{10} + \pi_{11} m_{t-1} + \pi_{12} y_{t-1} + \gamma_1 t + \varepsilon_{1t} \\ \Delta y_t &= a_{20} + \pi_{21} m_{t-1} + \pi_{22} y_{t-1} + \gamma_2 t + \varepsilon_{2t}. \end{aligned}$$

Cointegration

- Suppose they cointegrate so $z_t = m_t - \beta y_t$ is I(0). The cointegrating vector, CV, is $(1, -\beta)$.
- The equation is normalised by setting the coefficient of m_t to one, treating it as a demand for money function. This just identifies the CV for r=1.
- To normalise we write this

$$\begin{aligned} \Delta m_t &= a_{10} + \pi_{11}(m_{t-1} + \frac{\pi_{12}}{\pi_{11}} y_{t-1}) + \gamma_1 t + \varepsilon_{1t} \qquad (1) \\ \Delta y_t &= a_{20} + \pi_{21}(m_{t-1} + \frac{\pi_{22}}{\pi_{21}} y_{t-1}) + \gamma_2 t + \varepsilon_{2t}. \end{aligned}$$

- Restrict the long-run coefficient to be the same in each equation,

$$\frac{\pi_{12}}{\pi_{11}} = \frac{\pi_{22}}{\pi_{21}}.$$

- This says $(\pi_{11}\pi_{22} - \pi_{12}\pi_{21} = 0)$ the determinant of Π is zero, so Π is singular, not of full rank.

The VECM

- With this restriction (1) becomes:

$$\begin{aligned}\Delta m_t &= a_{10} - \alpha_1(m_{t-1} - \beta y_{t-1}) + \gamma_1 t + u_{1t} \\ \Delta y_t &= a_{20} - \alpha_2(m_{t-1} - \beta y_{t-1}) + \gamma_2 t + u_{2t}\end{aligned} \tag{2}$$

  where $-\alpha_1 = \pi_{11}$ etc.
- Thus

$$\Pi = \begin{bmatrix} -\alpha_1 & +\alpha_1\beta \\ -\alpha_2 & +\alpha_2\beta \end{bmatrix}$$

  which is clearly of rank 1, since a multiple of the first column equals the second column.
- A natural over-identifying restriction to test in this context would be that $\beta = 1$. Constant velocity of circulation.

Restricted trend

- The equation has unrestricted trend and intercept, to restrict the trend we put it in the cointegrating vector, saving one further parameter:

$$\begin{aligned}\Delta m_t &= a_{10} - \alpha_1(m_{t-1} - \beta y_{t-1} + \gamma t) + u_{1t} \\ \Delta y_t &= a_{20} - \alpha_2(m_{t-1} - \beta y_{t-1} + \gamma t) + u_{2t}\end{aligned}$$

- If y_t is weakly exogenous then $\alpha_2 = 0$, which can be tested and if accepted means income is a random walk with drift and (2) becomes

$$\begin{aligned}\Delta m_t &= a_{10} - \alpha_1(m_{t-1} - \beta y_{t-1}) + \gamma_1 t + u_{1t} \\ \Delta y_t &= a_{20} + u_{2t}\end{aligned}$$

VECM to ECM

Define $E(u_{1t}u_{2t}) = \sigma_{12}$, and noting that

$$E(u_{1t} \mid u_{2t}) = \frac{\sigma_{12}}{\sigma_{22}} u_{2t} = \frac{\sigma_{12}}{\sigma_{22}} (\Delta y_t - a_{20})$$

and defining $v_t = u_{1t} - E(u_{1t} \mid u_{2t})$, we can get the ECM treating y_t as exogenous:

$$\begin{aligned}
\Delta m_t &= (a_{10} - \frac{\sigma_{12}}{\sigma_{22}} a_{20}) + \frac{\sigma_{12}}{\sigma_{22}} \Delta y_t - \alpha_2(m_{t-1} - \beta y_{t-1}) + \gamma_1 t + v_t. \\
\Delta m_t &= a_0 + b_0 \Delta y_t + a_1 m_{t-1} + b_1 y_{t-1} + ct + v_t
\end{aligned}$$

What happens if $\alpha_2 \neq 0$?

Example 2: Old exam questions

- A second-order cointegrating vector error-correction model (VECM), with unrestricted intercepts and restricted trends, was estimated on quarterly US data from 1947Q3 to 1988Q4.
- The variables included were
  - the logarithm of real consumption (c_t),
  - the logarithm of real investment (i_t), and
  - the logarithm of real income (q_t).
- Q: The VECM was estimated with unrestricted intercepts and restricted trends. What does this mean?
- A:

  $$\Delta y_t = a_0 - \alpha(\beta' y_{t-1} + ct) + \Gamma \Delta y_{t-1} + \varepsilon_t,$$

  The intercepts a_0 lie outside the error correction term and the trends ct are restricted to lie within it. Whereas one estimates 3 intercepts, one only estimates r trend coefficients, giving $3 - r$ restrictions.

Number of cointegrating vectors, r

- The Johansen maximal eigenvalue tests were:

H_o	H_1	$Statistic$	$10\% CV$
$r = 0$	$r = 1$	34.6	23.1
$r \leq 1$	$r = 2$	15.8	17.2
$r \leq 2$	$r = 3$	3.3	10.5

- The Johansen Trace Tests were:

H_o	H_1	$Statistic$	$10\% CV$
$r = 0$	$r \geq 1$	53.7	39.3
$r \leq 1$	$r \geq 2$	19.1	23.1
$r \leq 2$	$r = 3$	3.3	10.5

Q: How many cointegrating vectors do the tests indicate? A: One
Q: If there are r cointegrating vectors, how many restrictions on each vector do you need to identify it? A: r

Estimated Cointegrating vectors

- Assuming that $r = 2$, the following two just-identified cointegrating vectors z_{1t} and z_{2tt} (standard errors in parentheses) were estimated:

c	i	q	t
1	0	-1.13	0.0003
		(0.16)	(0.0006)
0	1	-1.14	0.0007
		(0.26)	(0.001)

- Q: Interpret the just identifying restrictions used above.
- A: Investment does not appear in the consumption function and consumption does not appear in the investment function.

$$
\begin{aligned}
c_t &= 1.13q_t - 0.0003t + z_{1t} \\
i_t &= 1.14q_t - 0.0007t + z_{2t}
\end{aligned}
$$

Just identifying restrictions

- For a $r \times m$ matrix of cointegrating vectors

$$z_t = \beta y_t$$

  we need r^2 restrictions on β.
- Many programs automatically provide the restrictions by assuming, as above,

$$\beta = \begin{bmatrix} I_r & B \end{bmatrix}$$

  making the first r variables "dependent variables" determine by the last $m - r$.
- Order matters.

Testing overidentifying restrictions

- The system MLL= 1552.9. It was re-estimated subject to the over-identifying restrictions that:
  - (i) both coefficients of q equal one, giving a MLL=1552.3; and
  - (ii) the coefficients of q equal one and the trend coefficients equal zero, giving a MLL=1548.1.
- Q: Test the two sets of overidentifying restrictions. 5% asymptotic (bootstrap) critical values, CV, are $\chi^2(2) = 5.99$ $(8.46), \chi^2(4) = 9.49$ (14.23). Comment on the difference between the results using the two sets of CVs.
- (i) 2(1552.9-1552.3)=1.2$<\chi^2(2)$, do not reject H_0 using either CV (ii) 2(1552.9-1548.1)=9.6$>\chi^2(4)$ reject H_0 with asymptotic but not with bootstrap CV
- Small sample critical values given by the bootstrap are bigger than the asymptotic values and so one is less likely to reject using the bootstrap critical values.

VECM estimates [t stats], just identified system (constants included. not reported)

	Δc_t	Δi_t	Δq_t
$z_{1,t-1}$	0.075068	0.262958	0.192686
	[2.74240]	[3.20914]	[4.63684]
$z_{2,t-1}$	-0.011232	-0.171416	0.009323
	[-0.67114]	[-3.42157]	[0.36694]
Δc_{t-1}	-0.209469	-0.171819	0.094535
	[-2.31259]	[-0.63368]	[0.68749]
Δi_{t-1}	0.022574	0.334330	0.156990
	[0.72374]	[3.58069]	[3.31537]
Δq_{t-1}	0.212411	0.697502	0.126186
	[3.17484]	[3.48267]	[1.24236]
R^2	0.146291	0.405637	0.320507
SER	0.007527	0.022533	0.011427

Granger Causality

- Q: Do you think investment is Granger Causal for Consumption?
- A: The fact that both $z_{2,t-1}$ (which is a function of lagged investment) and Δi_{t-1} are individually insignificant in the consumption equation suggests that investment may be Granger non-causal for consumption, though the two terms could be jointly significant.

Next time

- The concept of identification and observational equivalence, introduced with respect to cointegrating, long-run, relationships is fundamental to econometrics.
- Next time we extend it to simultaneous systems where variables like prices and quantities are jointly determined.
- In such systems neither price nor quantity are exogenous, so we need other ways to estimate the equations than OLS.