1 A proof of focalisation

Let $\vdash \Gamma; \Pi$ denote a sequent of $\mathsf{LL}_{\mathrm{foc}}$ as defined in [Laurent04].

We define the focalised syntactic phase model as $(M, I, \perp, \varphi)$ where M is the free commutative monoid over formulas of LL with $?A$ and $?A, ?A$ identified, $I = \{?\Gamma \mid \Gamma \in M\}$, $\perp = \{\Gamma \in M \mid \vdash \Gamma;\}$, and $\varphi(X) = \{X\}^\perp$ for positive atoms X. Let $[\![A]\!]$ be the interpretation of a formula A in this model.

Note that $I \subseteq 1$ because of the $?w$ rule, so $[\![!A]\!] = ([\![A]\!] \cap I)^{\perp\perp}$.

To simplify the notation, let $\vdash \Gamma; N$ mean $\vdash \Gamma, N;$ when N is a negative formula. Let $\mathrm{Foc}(A) = \{\Gamma \in M \mid \vdash \Gamma; A\}$. Clearly $\mathrm{Foc}(A) \subseteq \{A\}^\perp$ by the *foc* rule, and in particular $\mathrm{Foc}(N) = \{N\}^\perp$ for N negative.

We use the decomposition of exponential connectives alluded to in [Laurent04, section 4.1]:

$$!A = \downarrow\sharp A \qquad\qquad ?A = \uparrow\flat A$$

We extend our definitions to this decomposition with $[\![\sharp A]\!] = [\![A]\!]$ and $\mathrm{Foc}(\sharp A) = \{A\}^\perp$, which is all that is needed here; this ensures the important property that $\mathrm{Foc}(N) = \mathrm{Foc}(N)^{\perp\perp}$ for N negative.

For a formula A, let $|A|$ denote the number of main negative subformulas in A (where $\sharp B$ is the main negative subformula in $!B$). Define Ψ_A as an $|A|$-ary monotonous operator on $\mathcal{P}(M)$ by induction:

- $\Psi_N(N_1) = N_1$ if N is negative
- $\Psi_X() = \{X^\perp\}$
- $\Psi_{B\otimes C}(B_1, \ldots, B_{|B|}, C_1, \ldots, C_{|C|}) = \Psi_B(B_1, \ldots, B_{|B|}) \cdot \Psi_C(C_1, \ldots, C_{|C|})$
- $\Psi_{B\oplus C}(B_1, \ldots, B_{|B|}, C_1, \ldots, C_{|C|}) = \Psi_B(B_1, \ldots, B_{|B|}) \cup \Psi_C(C_1, \ldots, C_{|C|})$
- $\Psi_1() = \{\emptyset\}$
- $\Psi_0() = \emptyset$
- $\Psi_{!B}(B_1) = B_1 \cap I$

Lemma 1. *For any formula A with main negative subformulas $A_1, \ldots, A_{|A|}$,*

$$[\![A]\!] \subseteq \Psi_A([\![A_1]\!], \ldots, [\![A_{|A|}]\!])^{\perp\perp}$$

Proof. By induction, using positivity results from [Girard99, appendix F]: $(X^{\perp\perp} \cdot Y^{\perp\perp})^{\perp\perp} \subseteq (X \cdot Y)^{\perp\perp}$ and $(X^{\perp\perp} \cup Y^{\perp\perp})^{\perp\perp} \subseteq (X \cup Y)^{\perp\perp}$.

- If A is negative, then we have $[\![A]\!] = [\![A]\!]^{\perp\perp}$ because $[\![A]\!]$ is a fact.
- If $A = X$, let $\Gamma \in \{X\}^\perp$ and $\Delta \in \{X^\perp\}^\perp$. We have

  $$\dfrac{\vdash \Gamma, X; \qquad \vdash X^\perp, \Delta;}{\vdash \Gamma, \Delta;}\ \textit{n-cut}$$

  from which $\{X\}^\perp \subseteq \{X^\perp\}^{\perp\perp}$ follows; therefore $[\![X]\!] = \{X\}^\perp \subseteq \{X^\perp\}^{\perp\perp} = \Psi_X()^{\perp\perp}$.
- If $A = B \otimes C$, then

$$\begin{aligned}
[\![B \otimes C]\!] &= ([\![B]\!] \cdot [\![C]\!])^{\perp\perp} \\
&\subseteq (\Psi_B([\![B_1]\!], \ldots, [\![B_{|B|}]\!])^{\perp\perp} \cdot \Psi_C([\![C_1]\!], \ldots, [\![C_{|C|}]\!])^{\perp\perp})^{\perp\perp} && \text{by the induction hypothesis} \\
&\subseteq (\Psi_B([\![B_1]\!], \ldots, [\![B_{|B|}]\!]) \cdot \Psi_C([\![C_1]\!], \ldots, [\![C_{|C|}]\!]))^{\perp\perp} && \text{by positivity} \\
&= \Psi_{B\otimes C}([\![B_1]\!], \ldots, [\![B_{|B|}]\!], [\![C_1]\!], \ldots, [\![C_{|C|}]\!])^{\perp\perp}
\end{aligned}$$

- If $A = B \oplus C$, then

$$\begin{aligned}
[\![B \oplus C]\!] &= ([\![B]\!] \cup [\![C]\!])^{\perp\perp} \\
&\subseteq (\Psi_B([\![B_1]\!], \ldots, [\![B_{|B|}]\!])^{\perp\perp} \cup \Psi_C([\![C_1]\!], \ldots, [\![C_{|C|}]\!])^{\perp\perp})^{\perp\perp} && \text{by the induction hypothesis} \\
&\subseteq (\Psi_B([\![B_1]\!], \ldots, [\![B_{|B|}]\!]) \cup \Psi_C([\![C_1]\!], \ldots, [\![C_{|C|}]\!]))^{\perp\perp} && \text{by positivity} \\
&= \Psi_{B\oplus C}([\![B_1]\!], \ldots, [\![B_{|B|}]\!], [\![C_1]\!], \ldots, [\![C_{|C|}]\!])^{\perp\perp}
\end{aligned}$$

- If $A = 1$ then $[\![1]\!] = \{\emptyset\}^{\perp\perp}$ by definition.
- If $A = 0$ then $[\![0]\!] = \emptyset^{\perp\perp}$ by definition.
- If $A = !B$ then $[\![!B]\!] = ([\![B]\!] \cap I)^{\perp\perp} = \Psi_{!B}([\![\sharp B]\!])^{\perp\perp}$.

□

Lemma 2. *For any formula A with main negative subformulas $A_1, \ldots, A_{|A|}$,*

$$\Psi_A(\mathrm{Foc}(A_1), \ldots, \mathrm{Foc}(A_{|A|})) \subseteq \mathrm{Foc}(A)$$

Proof. By induction:

- If A is negative, $\Psi_A(\mathrm{Foc}(A)) = \mathrm{Foc}(A)$ by definition.
- If $A = X$, then $\Psi_X() = \{X^\perp\} \subseteq \mathrm{Foc}(X)$ by the *ax* rule.
- If $A = B \otimes C$, then we have

$$\begin{aligned}
\Psi_A(\mathrm{Foc}(A_1), \ldots, \mathrm{Foc}(A_{|A|})) &= \Psi_B(\mathrm{Foc}(B_1), \ldots, \mathrm{Foc}(B_{|B|})) \cdot \Psi_C(\mathrm{Foc}(C_1), \ldots, \mathrm{Foc}(C_{|C|})) \\
&\subseteq \mathrm{Foc}(B) \cdot \mathrm{Foc}(C)
\end{aligned}$$

  by the induction hypothesis; moreover,

$$\dfrac{\vdash \Gamma; B \qquad \vdash \Delta; C}{\vdash \Gamma, \Delta; B \otimes C} \otimes$$

  hence $\mathrm{Foc}(B) \cdot \mathrm{Foc}(C) \subseteq \mathrm{Foc}(B \otimes C)$, from which the result follows.

- If $A = B \oplus C$, then we have

$$\begin{aligned}
\Psi_A(\mathrm{Foc}(A_1), \ldots, \mathrm{Foc}(A_{|A|})) &= \Psi_B(\mathrm{Foc}(B_1), \ldots, \mathrm{Foc}(B_{|B|})) \cup \Psi_C(\mathrm{Foc}(C_1), \ldots, \mathrm{Foc}(C_{|C|})) \\
&\subseteq \mathrm{Foc}(B) \cup \mathrm{Foc}(C)
\end{aligned}$$

  by the induction hypothesis; moreover,

$$\dfrac{\vdash \Gamma; B}{\vdash \Gamma; B \oplus C} \oplus_1 \qquad \dfrac{\vdash \Delta; C}{\vdash \Delta; B \oplus C} \oplus_2$$

  hence $\mathrm{Foc}(B) \cup \mathrm{Foc}(C) \subseteq \mathrm{Foc}(B \oplus C)$, from which the result follows.

- If $A = 1$, clearly $\Psi_1() = \{\emptyset\} \subseteq \mathrm{Foc}(1)$ by the 1 rule.
- If $A = 0$, clearly $\Psi_0() = \emptyset \subseteq \mathrm{Foc}(0)$.

- If $A = !B$, then $\Psi_{!B}(\mathrm{Foc}(\sharp B)) = \{B\}^\perp \cap I$, and

$$\frac{\vdash ?\Gamma, B;}{\vdash ?\Gamma; !B}\ !$$

hence $\{B\}^\perp \cap I \subseteq \mathrm{Foc}(!B)$, from which the result follows.

□

Lemma 3. *For any formula A, $[\![A]\!] \subseteq \mathrm{Foc}(A)^{\perp\perp}$.*

Proof. By induction:

- If A is a positive formula with main negative subformulas $A_1, \ldots, A_{|A|}$, then

$$\begin{aligned}
[\![A]\!] &\subseteq \Psi_A([\![A_1]\!], \ldots, [\![A_{|A|}]\!])^{\perp\perp} && \text{by lemma 1}\\
&\subseteq \Psi_A(\mathrm{Foc}(A_1)^{\perp\perp}, \ldots, \mathrm{Foc}(A_{|A|})^{\perp\perp})^{\perp\perp} && \text{by the induction hypothesis}\\
&= \Psi_A(\mathrm{Foc}(A_1), \ldots, \mathrm{Foc}(A_{|A|}))^{\perp\perp} && \text{because } A_1, \ldots, A_{|A|} \text{ are negative}\\
&\subseteq \mathrm{Foc}(A)^{\perp\perp} && \text{by lemma 2}
\end{aligned}$$

- If $A = \sharp B$, then by the induction hypothesis $[\![\sharp B]\!] = [\![B]\!] \subseteq \mathrm{Foc}(B)^{\perp\perp} \subseteq \{B\}^\perp = \mathrm{Foc}(\sharp B)^{\perp\perp}$.
  Otherwise, it is enough to prove $[\![A]\!] \subseteq \{A\}^\perp$.
- If $A = X^\perp$, we have $\{X\} \subseteq \{X^\perp\}^\perp$, therefore $[\![X^\perp]\!] = \{X\}^{\perp\perp} \subseteq \{X^\perp\}^{\perp\perp\perp} = \{X^\perp\}^\perp$.
- If $A = B \mathbin{\&} C$, we have $[\![B \mathbin{\&} C]\!] = [\![B]\!] \cap [\![C]\!] \subseteq \{B\}^\perp \cap \{C\}^\perp$ by the induction hypothesis; moreover,

$$\frac{\vdash \Gamma, B; \qquad \vdash \Gamma, C;}{\vdash \Gamma, B \mathbin{\&} C;}\ \&$$

hence $\{B\}^\perp \cap \{C\}^\perp \subseteq \{B \mathbin{\&} C\}^\perp$, from which the result follows.

- If $A = B \parr C$, let $\Gamma \in [\![B \parr C]\!] = ([\![B]\!]^\perp \cdot [\![C]\!]^\perp)^\perp$. By the induction hypothesis, $[\![B]\!] \subseteq \{B\}^\perp$, hence $B \in \{B\}^{\perp\perp} \subseteq [\![B]\!]^\perp$, and similarly $C \in [\![C]\!]^\perp$, therefore $\vdash B, C, \Gamma;$. Moreover,

$$\frac{\vdash \Gamma, B, C;}{\vdash \Gamma, B \parr C;}\ \parr$$

hence $\Gamma \in \{B \parr C\}^\perp$, from which the result follows.

- If $A = \top$, we have $[\![\top]\!] = M = \{\top\}^\perp$ by the $\top$ rule.
- If $A = \bot$, we have $[\![\bot]\!] = \bot \subseteq \{\bot\}^\perp$ by the $\bot$ rule.
- If $A = ?B$, then $[\![?B]\!] = ([\![B]\!]^\perp \cap I)^\perp$. By the induction hypothesis, $[\![B]\!] \subseteq \mathrm{Foc}(B)^{\perp\perp}$, hence $\mathrm{Foc}(B)^\perp \subseteq [\![B]\!]^\perp$ and $([\![B]\!]^\perp \cap I)^\perp \subseteq (\mathrm{Foc}(B)^\perp \cap I)^\perp$. Moreover, $?B \in \mathrm{Foc}(B)^\perp \cap I$ because of the $?d$ rule, therefore $(\mathrm{Foc}(B)^\perp \cap I)^\perp \subseteq \{?B\}^\perp$, from which the result follows.

□

Corollary 3.1. *For any multiset of formulas $\Gamma = A_1, \ldots, A_n$, $[\![\Gamma]\!] \subseteq \{\Gamma\}^\perp$.*

Proof. By lemma 3, we have $\llbracket A_i \rrbracket \subseteq \mathrm{Foc}(A_i)^{\perp\perp} \subseteq \{A_i\}^{\perp}$ for all $1 \leq i \leq n$, hence $\{A_i\} \subseteq \{A_i\}^{\perp\perp} \subseteq \llbracket A_i \rrbracket^{\perp}$, therefore $\{\Gamma\} = \{A_1\} \cdots \{A_n\} \subseteq \llbracket A_1 \rrbracket^{\perp} \cdots \llbracket A_n \rrbracket^{\perp}$.

Thus, $\llbracket \Gamma \rrbracket = \llbracket A_1 \rrbracket \parr \cdots \parr \llbracket A_n \rrbracket = (\llbracket A_1 \rrbracket^{\perp} \cdots \llbracket A_n \rrbracket^{\perp})^{\perp} \subseteq \{\Gamma\}^{\perp}$. □

Theorem 4 (Focalised completeness)**.** *If a sequent $\vdash \Gamma$ of LL is valid in all phase models, then $\vdash \Gamma$ has a focalised proof.*

Proof. In particular $\emptyset \in \llbracket \Gamma \rrbracket$, hence $\emptyset \in \{\Gamma\}^{\perp}$ by corollary 3.1, therefore there is a proof π of $\vdash \Gamma;$ in $\mathsf{LL}_{\mathrm{foc}}$. Then, using [Laurent04, section 3.2], we get a proof π' of $\vdash \Gamma;$ in $\mathsf{LL}_{\mathrm{Foc}}$. Finally, by [Laurent04, proposition 2], π'° is a cut-free, focalised proof of $\vdash \Gamma$ in LL. □

Combining this with the soundness theorem for phase models, we get:

Corollary 4.1 (Focalisation)**.** *Every provable sequent $\vdash \Gamma$ of LL has a focalised proof.*

References

[Laurent04] Olivier Laurent. 'A proof of the focalization property of linear logic'. Apr. 2004. URL: https://web.archive.org/web/20210225023814/https://perso.ens-lyon.fr/olivier.laurent/llfoc.pdf.

[Girard99] Jean-Yves Girard. 'On the Meaning of Logical Rules I: Syntax Versus Semantics'. In: *Computational Logic*. Springer Berlin Heidelberg, 1999, pp. 215–272. ISBN: 978-3-642-58622-4.