1 A proof of focalisation

Let $\vdash_{\text{foc}} \Gamma$ mean "there is a *focalised* proof of $\vdash \Gamma$", in the sense of [Laurent04].

We define the focalised syntactic phase model as $(M, \bot, \varphi)$ where M is the free commutative monoid over formulas of MALL, $\bot = \{\Gamma \in M \mid \vdash_{\text{foc}} \Gamma\}$, and $\varphi(X) = \{X\}^\bot$ for positive atoms X. Let $[\![A]\!]$ be the interpretation of a formula A in this model.

For a formula A, let $|A|$ denote the number of main negative subformulas in A. Define Ψ_A as a $|A|$-ary monotonous operator on $\mathcal{P}(M)$ by induction:

- $\Psi_A(N) = N$ if A is negative
- $\Psi_X() = \{X^\bot\}$
- $\Psi_{B\otimes C}(B_1, \ldots, B_{|B|}, C_1, \ldots, C_{|C|}) = \Psi_B(B_1, \ldots, B_{|B|}) \cdot \Psi_C(C_1, \ldots, C_{|C|})$
- $\Psi_{B\oplus C}(B_1, \ldots, B_{|B|}, C_1, \ldots, C_{|C|}) = \Psi_B(B_1, \ldots, B_{|B|}) \cup \Psi_C(C_1, \ldots, C_{|C|})$
- $\Psi_1() = \{\emptyset\}$
- $\Psi_0() = \emptyset$

Lemma 1. *For any formula A with main negative subformulas $A_1, \ldots, A_{|A|}$, $[\![A]\!] = \Psi_A([\![A_1]\!], \ldots, [\![A_{|A|}]\!])^{\bot\bot}$.*

Proof. By induction, using positivity results from [Girard99, appendix F]: $(X^{\bot\bot} \cdot Y^{\bot\bot})^{\bot\bot} = (X \cdot Y)^{\bot\bot}$ and $(X^{\bot\bot} \cup Y^{\bot\bot})^{\bot\bot} = (X \cup Y)^{\bot\bot}$.

- If A is negative, then $[\![A]\!] = [\![A]\!]^{\bot\bot}$ because $[\![A]\!]$ is a fact.
- If $A = X$, then $[\![X]\!] = \{X\}^\bot \subseteq \{X^\bot\}^{\bot\bot}$, which is equivalent to the atomic cut rule:

$$\frac{\vdash_{\text{foc}} \Gamma, X \qquad \vdash_{\text{foc}} X^\bot, \Delta}{\vdash_{\text{foc}} \Gamma, \Delta}$$

  And clearly $\{X\} \subseteq \{X^\bot\}^\bot$, hence $[\![X]\!] = \{X^\bot\}^{\bot\bot}$.

- If $A = B \otimes C$, then

$$\begin{aligned}
[\![B \otimes C]\!] &= ([\![B]\!] \cdot [\![C]\!])^{\bot\bot} \\
&= (\Psi_B([\![B_1]\!], \ldots, [\![B_{|B|}]\!])^{\bot\bot} \cdot \Psi_C([\![C_1]\!], \ldots, [\![C_{|C|}]\!])^{\bot\bot})^{\bot\bot} \\
&= (\Psi_B([\![B_1]\!], \ldots, [\![B_{|B|}]\!]) \cdot \Psi_C([\![C_1]\!], \ldots, [\![C_{|C|}]\!]))^{\bot\bot} \\
&= \Psi_{B\otimes C}([\![B_1]\!], \ldots, [\![B_{|B|}]\!], [\![C_1]\!], \ldots, [\![C_{|C|}]\!])^{\bot\bot}
\end{aligned}$$

- If $A = B \oplus C$, then

$$\begin{aligned}
[\![B \oplus C]\!] &= ([\![B]\!] \cup [\![C]\!])^{\bot\bot} \\
&= (\Psi_B([\![B_1]\!], \ldots, [\![B_{|B|}]\!])^{\bot\bot} \cup \Psi_C([\![C_1]\!], \ldots, [\![C_{|C|}]\!])^{\bot\bot})^{\bot\bot} \\
&= (\Psi_B([\![B_1]\!], \ldots, [\![B_{|B|}]\!]) \cup \Psi_C([\![C_1]\!], \ldots, [\![C_{|C|}]\!]))^{\bot\bot} \\
&= \Psi_{B\oplus C}([\![B_1]\!], \ldots, [\![B_{|B|}]\!], [\![C_1]\!], \ldots, [\![C_{|C|}]\!])^{\bot\bot}
\end{aligned}$$

- If $A = 1$ then $[\![1]\!] = \{\emptyset\}^{\bot\bot}$ by definition.
- If $A = 0$ then $[\![0]\!] = \emptyset^{\bot\bot}$ by definition.

$\square$

Lemma 2. *For any formula A with main negative subformulas $A_1, \dots, A_{|A|}$, $\Psi_A(\{A_1\}^\perp, \dots, \{A_{|A|}\}^\perp) \subseteq \{A\}^\perp$.*

Proof. Let $\vdash_{\mathrm{foc}(A)} A, \Gamma$ mean "there is a proof of $\vdash_{\mathrm{foc}} A, \Gamma$ in which the last rule introduces the main connective of A" if A is positive, and $\vdash_{\mathrm{foc}} A, \Gamma$ otherwise. Let $\mathrm{Foc}(A) = \{\Gamma \in M \mid \vdash_{\mathrm{foc}(A)} A, \Gamma\}$. Clearly $\mathrm{Foc}(A) \subseteq \{A\}^\perp$.

We prove by induction on A that $\Psi_A(\mathrm{Foc}(A_1), \dots, \mathrm{Foc}(A_{|A|})) = \mathrm{Foc}(A)$:

- If A is negative, the result is trivial.
- If $A = X$, then $\Psi_X() = \{X^\perp\} = \mathrm{Foc}(X)$.
- If $A = B \otimes C$, then we have

$$\begin{aligned}\Psi_A(\mathrm{Foc}(A_1), \dots, \mathrm{Foc}(A_{|A|})) &= \Psi_B(\mathrm{Foc}(B_1), \dots, \mathrm{Foc}(B_{|B|})) \cdot \Psi_C(\mathrm{Foc}(C_1), \dots, \mathrm{Foc}(C_{|C|})) \\ &= \mathrm{Foc}(B) \cdot \mathrm{Foc}(C)\end{aligned}$$

  by the induction hypothesis. Moreover,

$$\frac{\vdash_{\mathrm{foc}(B)} B, \Gamma \qquad \vdash_{\mathrm{foc}(C)} C, \Delta}{\vdash_{\mathrm{foc}(B \otimes C)} B \otimes C, \Gamma, \Delta}$$

  And this rule is invertible, hence $\mathrm{Foc}(B) \cdot \mathrm{Foc}(C) = \mathrm{Foc}(B \otimes C)$, from which the result follows.

- If $A = B \oplus C$, then we have

$$\begin{aligned}\Psi_A(\mathrm{Foc}(A_1), \dots, \mathrm{Foc}(A_{|A|})) &= \Psi_B(\mathrm{Foc}(B_1), \dots, \mathrm{Foc}(B_{|B|})) \cup \Psi_C(\mathrm{Foc}(C_1), \dots, \mathrm{Foc}(C_{|C|})) \\ &= \mathrm{Foc}(B) \cup \mathrm{Foc}(C)\end{aligned}$$

  by the induction hypothesis. Moreover,

$$\frac{\vdash_{\mathrm{foc}(B)} B, \Gamma}{\vdash_{\mathrm{foc}(B \oplus C)} B \oplus C, \Gamma, \Delta} \qquad \frac{\vdash_{\mathrm{foc}(C)} C, \Delta}{\vdash_{\mathrm{foc}(B \oplus C)} B \oplus C, \Gamma, \Delta}$$

  And these rules are complete, hence $\mathrm{Foc}(B) \cup \mathrm{Foc}(C) = \mathrm{Foc}(B \oplus C)$, from which the result follows.

- If $A = 1$, clearly $\Psi_1() = \{\emptyset\} = \mathrm{Foc}(1)$.
- If $A = 0$, clearly $\Psi_0() = \emptyset = \mathrm{Foc}(0)$.

Since $A_1, \dots, A_{|A|}$ are negative, we have

$$\begin{aligned}\Psi_A(\{A_1\}^\perp, \dots, \{A_{|A|}\}^\perp) &= \Psi_A(\mathrm{Foc}(A_1), \dots, \mathrm{Foc}(A_{|A|})) \\ &= \mathrm{Foc}(A) \\ &\subseteq \{A\}^\perp\end{aligned}$$

□

Lemma 3. *For any formula A, $[\![A]\!] \subseteq \{A\}^\perp$.*

Proof. By induction:

- If $A = X$, then $[\![X]\!] = \{X\}^\perp$ by definition.
- If $A = X^\perp$, we have $\{X^\perp\} \subseteq \{X\}^\perp$ using the axiom rule, hence $[\![X^\perp]\!] = \{X\}^{\perp\perp} \subseteq \{X^\perp\}^\perp$.

- If $A = B \& C$, we have $[\![B \& C]\!] = [\![B]\!] \cap [\![C]\!] \subseteq \{B\}^\perp \cap \{C\}^\perp$ by the induction hypothesis. Moreover,

$$\frac{\vdash_{\text{foc}} B, \Gamma \qquad \vdash_{\text{foc}} C, \Gamma}{\vdash_{\text{foc}} B \& C, \Gamma}$$

  Hence $\{B\}^\perp \cap \{C\}^\perp \subseteq \{B \& C\}^\perp$, from which the result follows.

- If $A = \top$, we have $[\![\top]\!] = M = \{\top\}^\perp$ using the rule for $\top$.

- If $A = B \parr C$, let $\Gamma \in [\![B \parr C]\!] = ([\![B]\!]^\perp \cdot [\![C]\!]^\perp)^\perp$. By the induction hypothesis, $[\![B]\!] \subseteq \{B\}^\perp$, hence $B \in \{B\}^{\perp\perp} \subseteq [\![B]\!]^\perp$, and similarly $C \in [\![C]\!]^\perp$, therefore $\vdash_{\text{foc}} B, C, \Gamma$. Moreover,

$$\frac{\vdash_{\text{foc}} B, C, \Gamma}{\vdash_{\text{foc}} B \parr C, \Gamma}$$

  Hence $\Gamma \in \{B \parr C\}^\perp$, from which the result follows.

- If $A = \perp$, we have $[\![\perp]\!] = \perp = \{\perp\}^\perp$ using the rule for $\perp$.

- Otherwise, A is a positive non-atomic formula with main negative subformulas $A_1, \ldots, A_{|A|}$. Then,

$$\begin{aligned}
[\![A]\!] &= \Psi_A([\![A_1]\!], \ldots, [\![A_{|A|}]\!])^{\perp\perp} && \text{by lemma 1} \\
&\subseteq \Psi_A(\{A_1\}^\perp, \ldots, \{A_{|A|}\}^\perp)^{\perp\perp} && \text{by the induction hypothesis and monotonicity of } \Psi_A \\
&\subseteq \{A\}^{\perp\perp\perp} = \{A\}^\perp && \text{by lemma 2}
\end{aligned}$$

□

Theorem 4 (Focalised completeness). *If a formula A of MALL is valid in all phase models, then A has a focalised proof.*

Proof. In particular $\emptyset \in [\![A]\!]$, hence $\emptyset \in \{A\}^\perp$ by lemma 3, therefore $\vdash_{\text{foc}} A$. □

Combining this with the soundness theorem, we get:

Corollary 4.1 (Focalisation). *Every provable formula A of MALL has a focalised proof.*

References

[Laurent04] Olivier Laurent. ‘A proof of the focalization property of linear logic’. In: (Apr. 2004).

[Girard99] Jean-Yves Girard. ‘On the Meaning of Logical Rules I: Syntax Versus Semantics’. In: *Computational Logic.* Springer Berlin Heidelberg, 1999, pp. 215–272. ISBN: 978-3-642-58622-4.