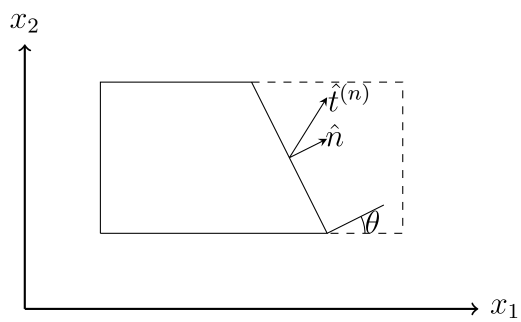
Name:

Homework 3
Due 2 Oct 2019

1. For the given states of stress, σ_{ij}, find the traction along an arbitrary interior plane as shown in the diagram.



(a)

$$\sigma_{ij} = \begin{bmatrix} k & \frac{k}{2} & 0 \\ \frac{k}{2} & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- We recall Cauchy's stress theorem:

$$\hat{t}_j^{(n)} = \sigma_{ij} n_i$$

- An interior normal shown in the above figure would be given as

$$\hat{n} = \langle \cos\theta, \sin\theta, 0 \rangle$$

- Substituting known values into Cauchy's stress theorem:

$$\begin{aligned} \hat{t}_j^{(n)} &= \begin{bmatrix} k & \frac{k}{2} & 0 \\ \frac{k}{2} & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{Bmatrix} \\ &= \frac{k}{2} \langle 2\cos\theta + \sin\theta, \cos\theta + 4\sin\theta \rangle \end{aligned}$$

(b)

$$\sigma_{ij} = \begin{bmatrix} x_1 & \frac{1}{2}x_1x_3 & \frac{1}{2}x_1^2 \\ \frac{1}{2}x_1x_3 & x_3 & 0 \\ \frac{1}{2}x_1^2 & 0 & x_2 \end{bmatrix}$$

- We follow the same procedure here, with the normal vector identical to the previous problem, we find

$$\hat{t}_j^{(n)} = \begin{bmatrix} x_1 & \frac{1}{2}x_1x_3 & \frac{1}{2}x_1^2 \\ \frac{1}{2}x_1x_3 & x_3 & 0 \\ \frac{1}{2}x_1^2 & 0 & x_2 \end{bmatrix} \begin{Bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{Bmatrix}$$
$$= \langle x_1\cos\theta + \frac{1}{2}x_1x_3\sin\theta, \frac{1}{2}x_1x_3\cos\theta + x_3\sin\theta, \frac{1}{2}x_1^2\cos\theta\rangle$$

2. Find the principal stresses and maximum shear stress for a body in *plane stress*.

   Note: In *plane stress*, $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$ which gives

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- As a plane stress problem, we can find the principal values with

$$\begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} \\ \sigma_{12} & \sigma_{22} - \lambda \end{vmatrix} = 0$$

- Multiplying out, we find

$$\lambda^2 - (\sigma_{11} + \sigma_{22})\lambda + \sigma_{11}\sigma_{22} - \sigma_{12}^2 = 0$$

- We can solve using the quadratic formula

$$\lambda = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \frac{\sqrt{(\sigma_{11} + \sigma_{22})^2 - 4(\sigma_{11}\sigma_{22} - \sigma_{12}^2)}}{2}$$

- Multiplying out inside the radical gives

$$\lambda = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \frac{\sqrt{\sigma_{11}^2 + \sigma_{22}^2 + 2\sigma_{11}\sigma_{22} - 4\sigma_{11}\sigma_{22} + 4\sigma_{12}^2}}{2}$$

- We can combine terms, simplify, and include the denominator in the radical

$$\lambda = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$$

- For a planar problem, we can find the maximum shear stress from the stress transformation equations by taking the derivative with respect to θ.

$$\sigma_{12}' = \frac{\sigma_{22} - \sigma_{11}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta$$

- And differentiating

$$\frac{\partial}{\partial\theta}(\sigma_{12}') = (\sigma_{22} - \sigma_{11})\cos 2\theta - 2\sigma_{12}\sin 2\theta = 0$$

- We can solve this for θ

$$2\theta = \tan^{-1}\left(\frac{(\sigma_{22} - \sigma_{11})}{2\sigma_{12}}\right)$$

- Substituting back into the original equation gives

$$\sigma'_{maxshear} = \frac{\sigma_{22} - \sigma_{11}}{2}\sin\left[\tan^{-1}\left(\frac{(\sigma_{22} - \sigma_{11})}{2\sigma_{12}}\right)\right] + \sigma_{12}\cos\left[\tan^{-1}\left(\frac{(\sigma_{22} - \sigma_{11})}{2\sigma_{12}}\right)\right]$$

- Recall

$$\sin(\tan^{-1}(x)) = \frac{x}{\sqrt{1+x^2}}$$
$$\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$$

- Also note that

$$\sqrt{1 + \left(\frac{\sigma_{22} - \sigma_{11}}{2\sigma_{12}}\right)^2} = \frac{\sqrt{(\sigma_{22} - \sigma_{11})^2 + 4\sigma_{12}^2}}{2\sigma_{12}}$$

- Which gives

$$\sigma_{maxshear} = \frac{(\sigma_{22} - \sigma_{11})^2}{2\sqrt{(\sigma_{22} - \sigma_{11})^2 + 4\sigma_{12}^2}} + \frac{4\sigma_{12}^2}{2\sqrt{(\sigma_{22} - \sigma_{11})^2 + 4\sigma_{12}^2}}$$

- Adding the terms and dividing gives

$$\sigma_{maxshear} = \sqrt{\left(\frac{\sigma_{22} - \sigma_{11}}{2}\right)^2 + \sigma_{12}^2}$$

3. The plane stress solution for a semi-infinite elastic body subjected to a point load produces the following state of stress

$$\sigma_{ij} = \frac{-2P}{\pi(x^2 + y^2)^2}\begin{bmatrix} x^2y & y^3 & 0 \\ y^3 & xy^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Calculate the maximum shear stress at any point in the body and plot contours of the result.

- Since this is a plane stress problem, and our solution above for the maximum shear stress in a body in plane strain does not take partial derivatives in x or y, the same solution is valid here

$$\sigma_{maxshear} = \sqrt{\left(\frac{\sigma_{22} - \sigma_{11}}{2}\right)^2 + \sigma_{12}^2}$$

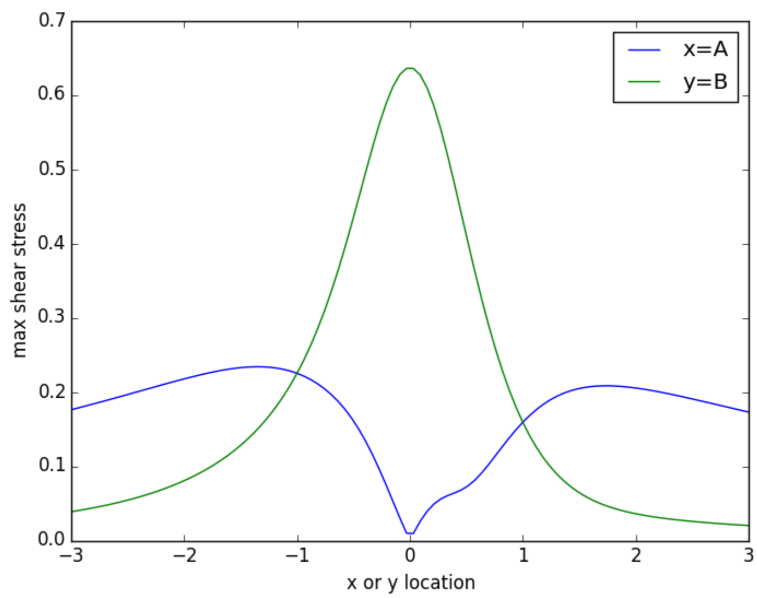
- Factoring out the terms on the outside of the matrix and substituting values gives

$$\sigma_{maxshear} = \frac{2P}{\pi(x^2+y^2)^2}\sqrt{\left(\frac{xy^2-x^2y}{2}\right)^2+(y^3)^2}$$

- We can simplify the expression slightly by factoring an xy out of the first term under the radical, which allows us to factor a y^2 out of the radical entirely

$$\sigma_{maxshear} = \frac{2yP}{\pi(x^2+y^2)^2}\sqrt{\frac{x^2}{4}\left(y-x\right)^2+y^4}$$

- We can now plot contours by letting $x = A$ and $y = B$, respectively.



Hint: Let $x = A$ to find contours along the y-axis, then solve again for $y = B$ to find contours along the x-axis

4. Under what circumstances (if any) will the following stress fields be in static equilibrium?

   Note: When a stress component is not given, assume it is zero.

   (a)

$$\begin{aligned}
\sigma_{11} &= 3x_1 + g_2(x_2)\\
\sigma_{22} &= 4x_2 + g_1(x_1)\\
\sigma_{12} &= a + bx_1 + cx_1^2 + dx_2 + ex_2^2 + fx_1x_2\\
\sigma_{21} &= \sigma_{12}
\end{aligned}$$

- We use the relevant static equilibrium equations

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + F_x = 0$$
$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + F_y = 0$$

- Which become

$$3 + d + 2ex_2 + fx_1 = 0$$
$$b + 2cx_1 + fx_2 + 4 = 0$$

- Since these equations must be true for all values of x_1 and x_2, e, f, and c must all be zero, leaving

$$3 + d = 0$$
$$b + 4 = 0$$

- Which gives $d = -3$, and $b = -4$
- For the stress field to be in static equilibrium, $c = e = f = 0$, $d = -3$, $b = -4$, but g_2 and g_1 have no bounds, they can be any function of x_2 or x_1 respectively and the stress field will remain in equilibrium.

(b)

$$\sigma_{13} = -Gx_2$$
$$\sigma_{23} = Gx_1$$

- Using the relevant equilibrium equations:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + F_x = 0$$
$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + F_y = 0$$
$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + F_z = 0$$

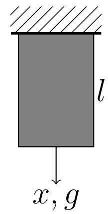
- And substituting known values

$$0 = 0$$
$$0 = 0$$
$$0 + 0 = 0$$

- Thus the stress field is in equilibrium for any value of G

5. Use the equilibrium equations to find the stress in a bar under self-weight.

   Hint: Use body forces $F_x = \rho g$, $F_y = 0$, and $F_z = 0$ to find the stress as a function of the position along the bar.



We can use Cauchy's stress theorem on the free surfaces to find most of the unknown stress tensor terms along the surface. If we also assume that stress is only a function x (or whatever direction you define gravity to act in), then we can eliminate most stress terms.

We also consider the transferse surfaces, where $n = \langle 0, 1, 0\rangle$ and $n = \langle 0, 0, 1\rangle$. Since these surfaces contain x from 0 to l, and if stress terms are only functions of x, then any stress term that is zero on these surfaces must be zero everywhere.

$$\sigma_{12} = \sigma_{22}\sigma_{23} = \sigma_{13} = \sigma_{33} = 0$$

The only non-zero stress term is σ_{11}, which agrees with what we would expect in a problem like this.

Choosing our first normal vector with $n_2 = n_3 = 0$, we find (at $x = l$):

$$\begin{aligned} t_j &= \sigma_{ij} n_i \\ &= \langle \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3, \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3, \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 \rangle \\ &= \langle \sigma_{11}, \sigma_{21}, \sigma_{31} \rangle = \langle 0 \rangle \end{aligned}$$

Thus $\sigma_{11}(l) = 0$

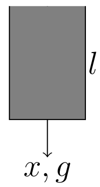
In this problem, the body force is the density times the acceleration due to gravity. Density has units of mass per volume, gravity has units of acceleration, length per time squared. Mass times acceleration gives units of force, thus ρg gives the correct units for the body force. Using the first equilibrium equation (which is the only non-trivial equation), which has the stress field term of interest for us,

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + F_x = 0$$

Substituting known values gives

$$\frac{\partial \sigma_{11}}{\partial x_1} + \rho g = 0 \tag{1}$$

Since we desire to find the stress field at any point in the body, let us now consider a free body diagram, introducing a variable x to indicate where the free body cut is made.



We can integrate (1) from l to x to find the stress field at x

$$\sigma_{11} = -\int_{L}^{x} \rho g dx \tag{2}$$

Which gives

$$\sigma_{11} = -\rho g(x - L) \tag{3}$$

Or

$$\sigma_{11} = \rho g(L - x) \tag{4}$$

Note that at the very end of the bar, when $x = L$, the stress will be 0 (this must be true because there is a free surface at that point, so we should have no traction on that surface), while the maximum magnitude of the stress will occur when $x = 0$, at the top of the bar.