AE731

Theory of Elasticity

Dr. Nicholas Smith
Wichita State University, Department of Aerospace Engineering August 21, 2019

upcoming schedule

- Aug 21 - Coordinate Transformation
- Aug 26 - Principal Values
- Aug 28 - Tensor Calculus
- Sep 2 - Labor Day
- Sep 4 - Displacement and Strain, Homework 1 Due

outline

- review
- examples
- index notation algebra
- group problems
- coordinate transformation
- examples

review

office hours

- TBD, Only 4 have responded, right now looks like M or W 3 - 4
- Homework will generally be due on Wednesdays
- Feel free to e-mail me for an appointment outside office hours if the time does not work for you

homework

- Homework 1 is available on Blackboard and **course website** if you want to start working on it
- Due on September 4 (I do not accept late homework)
- Covers all of Chapter 1, relatively difficult, don't wait until last minute
- Study groups help a lot (but submit your own work)

index notation

Free index vs. dummy index

- is not repeated on any term
- takes all values (1,2,3)
- e.g. $u_i = \langle u_1, u_2, u_3 \rangle$
- must match across terms in an express or equation

- is repeated on at least one term
- indicates summation over all values
- e.g. $\sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$
- can not be used more than twice in the same term ($A_{ij}B_{jk}C_{kl}$ is good, $A_{ij}B_{ij}C_{ij}$ is not)

symmetry

- Two types of symmetry: symmetry and antisymmetry
- Symmetry: $a_{ij} = a_{ji}$
- Anti-symmetry: $a_{ij} = -a_{ji}$

symmetry

- We can break any tensor up into symmetric and anti-symmetric portions
  $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

example

- Find symmetric and anti-symmetric portions of

$$\begin{bmatrix} 1 & 4 & 3 \\ 2 & 1 & 5 \\ 4 & 3 & 6 \end{bmatrix}$$

example symmetric portion

$$a_{(ij)} = \frac{1}{2}(a_{ij} + a_{ji})$$

$$a_{(ij)} = \frac{1}{2}\left(\begin{bmatrix} 1 & 4 & 3 \\ 2 & 1 & 5 \\ 4 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 4 \\ 4 & 1 & 3 \\ 3 & 5 & 6 \end{bmatrix} \right) = \begin{bmatrix} 1 & 3 & 3.5 \\ 3 & 1 & 4 \\ 3.5 & 4 & 6 \end{bmatrix}$$

example anti-symmetric portion

$$a_{(ij)} = \frac{1}{2}(a_{ij} - a_{ji})$$

$$a_{(ij)} = \frac{1}{2}\left(\begin{bmatrix} 1 & 4 & 3 \\ 2 & 1 & 5 \\ 4 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 4 \\ 4 & 1 & 3 \\ 3 & 5 & 6 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & -0.5 \\ -1 & 0 & 1 \\ 0.5 & -1 & 0 \end{bmatrix}$$

special symbols

- For convenience we define two symbols in index notation
- *Kronecker delta* is a general tensor form of the Identity Matrix

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Is also used for higher order tensors

Kronecker delta

- $\delta_{ij} = \delta_{ji}$
- δ_{ii}= 3
- $\delta_{ij}a_j$= a_i
- $\delta_{ij}a_{ij}$= a_{ii}

special symbols

- *alternating symbol* or *permutation symbol*

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of 1,2,3} \\ -1 & \text{if } ijk \text{ is an odd permutation of 1,2,3} \\ 0 & \text{otherwise} \end{cases}$$

permutation symbol

- This symbol is not used as frequently as the *Kronecker delta*
- For our uses in this course, it is enough to know that 123, 231, and 312 are even permutations
- 321, 132, 213 are odd permutations
- all other indexes are zero
- $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{mk}$

determinant

- We use the alternating symbol for writing determinants and cross-products

$$\det[a_{ij}] = |a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \epsilon_{ijk} a_{i1} a_{j2} a_{k3}$$

$$\det[a_{ij}] = \frac{1}{6} \epsilon_{ijk} \epsilon_{pqr} a_{ip} a_{jq} a_{kr}$$

cross product

- The cross-product can be written as a determinant:

$$\hat{a} \times \hat{b} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Or in index notation

$$\hat{a} \times \hat{b} = \epsilon_{ijk} \hat{e}_i a_j b_k$$

partial derivative

- We indicate (partial) derivatives using a comma
- In three dimensions, we take the partial derivative with respect to each variable (x, y, z or x_1, x_2, x_3)
- For example a scalar property, such as density, can have a different value at any point in space
- $\rho = \rho(x_1, x_2, x_3)$

$$\rho_{,i} = \frac{\partial}{\partial x_i}\rho = \left\langle \frac{\partial \rho}{\partial x_1}, \frac{\partial \rho}{\partial x_2}, \frac{\partial \rho}{\partial x_3} \right\rangle$$

partial derivative

- Similarly, if we take the partial derivative of a vector, it produces a matrix

$$u_{i,j} = \frac{\partial}{\partial x_j} u_i = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

examples

example 1

- Write the following in conventional notation $T_{ij,j} + F_i = 0$
- The comma indicates a partial derivative
- The first index, i, is not repeated in any terms, so it is a "free index"
    - This means in a 3D coordinate system, we will have at least three equations
- The second index, j, is repeated in the first term, indicating summation.
    - We will have exactly three equations

example 1 (solution)

$T_{ij,j} + F_i = 0$

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} + F_1 = 0$$
$$\frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} + F_2 = 0$$
$$\frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} + F_3 = 0$$

example 2

- Identify whether the following expressions represent a scalar, vector, or matrix
- If index notation is used incorrectly, give a reason why
- $A_i = B_i$
- $A_i = B_i + C_i D_i$
- $\delta_{ij} A_i B_j$
- $\phi = \frac{\partial F_i}{\partial x_i}$

example 2 (solution)

- Vector equation
- Incorrect use of index notation, i used as both free and dummy index
- Scalar value (both indexes are dummy indexes)
- Scalar value (could also be written $F_{i,i}$)

index notation algebra

substitution

- When solving tensor equations, we often need to manipulate expressions
- We need to make sure the correct indexes are used when substituting, for example
- $a_i = U_{im} b_m$
- $b_i = V_{im} c_m$
- To substitute the second into the first, we need to change indexes

substitution

- We need to change the free index, i, to m in the second equation
- Since m is already used as the dummy index, we need to change that too
- $b_m = V_{mj} c_j$
- We can now make the substitution
- $a_i = U_{im} V_{mj} c_j$

multiplication

- We need to be careful with indexes when multiplying expressions
- $p = a_m b_m$ and $q = c_m d_m$
- We can express, pq, but remember the dummy index cannot be repeated more than once
- $pq \neq a_m b_m c_m d_m$
- Instead we must change the dummy index in one of the expressions first
- $pq = a_m b_m c_n d_n$

factoring

- In the following expression, we would like to factor out n, but it has different indexes
- $T_{ij}n_j - \lambda n_i = 0$
- Recall $\delta_{ij}a_j = a_i$, we can rewrite $n_i = \delta_{ij}n_j$
- $T_{ij}n_j - \lambda\delta_{ij}n_j = 0$
- $(T_{ij} - \lambda\delta_{ij})n_j = 0$

contraction

- T_{ii} is the contraction of T_{ij}
- This can often be a useful tool in solving tensor equations
- $T_{ij} = \lambda\Delta\delta_{ij} + 2\mu E_{ij}$
- $T_{ii} = \lambda\Delta\delta_{ii} + 2\mu E_{ii}$

example

- Solve the equation below for U_k in terms of P_i and a_i.

$$\mu\left[\delta_{kj}a_ia_i+\frac{1}{1-2\nu}a_ka_j\right]U_k=P_j$$

- Multiply both sides by a_j

$$\mu\left[a_j\delta_{kj}a_ia_i+\frac{1}{1-2\nu}a_ka_ja_j\right]U_k=P_ja_j$$

- The dummy indexes can be changed

$$\mu\left[a_j\delta_{kj}a_ia_i+\frac{1}{1-2\nu}a_ka_ia_i\right]U_k=P_ja_j$$

example

- $a_j\delta_{kj} = a_k$

  $$\mu U_k \left[a_k a_i a_i + \frac{1}{1-2\nu} a_k a_i a_i\right] = P_j a_j$$

- Factoring

  $$\mu U_k a_k a_i a_i \left[1 + \frac{1}{1-2\nu}\right] = P_j a_j$$

- Simplifying

  $$\mu U_k a_k a_i a_i \left[\frac{2(1-\nu)}{1-2\nu}\right] = P_j a_j$$

example

- Solve for $U_k a_k$
  $$U_k a_k = \frac{P_j a_j (1-2\nu)}{2\mu a_i a_i (1-\nu)}$$
- This is a scalar equation, we need to find U_j, but we substitute this back into the original equation.
- First, expand the original equation
  $$\mu U_k \delta_{kj} a_i a_i + \mu U_k \frac{1}{1-2\nu} a_k a_j = P_j$$

example

- After substitution, we find

$$\mu U_j a_i a_i + \mu \frac{1}{1-2\nu} \frac{P_j a_j (1-2\nu)}{2\mu a_i a_i (1-\nu)} a_j = P_j$$

- The index j is repeated too many times, so we need to change $P_j a_j$ to a different index

$$\mu U_j a_i a_i + \frac{P_k a_k}{2 a_i a_i (1-\nu)} a_j = P_j$$

- We can now solve for U_j

$$U_j = \frac{1}{\mu a_i a_i} \left[P_j - \frac{P_k a_k}{2 a_i a_i (1-\nu)} a_j \right]$$

group problems

group 1

Identify the dummy and free indexes in each of the following expressions. Indicate the tensor order of the expression. If index notation is used incorrectly, identify why it is used incorrectly and propose a correction.

1. $a_i b_j c_k + d_{ijk}$
2. $a_{ii} b_k + c_{kk} d_j$
3. $C_{ijkl} \epsilon_{kl}$

group 2

Is it possible to factor n_i from the following equation? If so, factor it.

$$T_{ij}n_j - \lambda n_i = 0$$

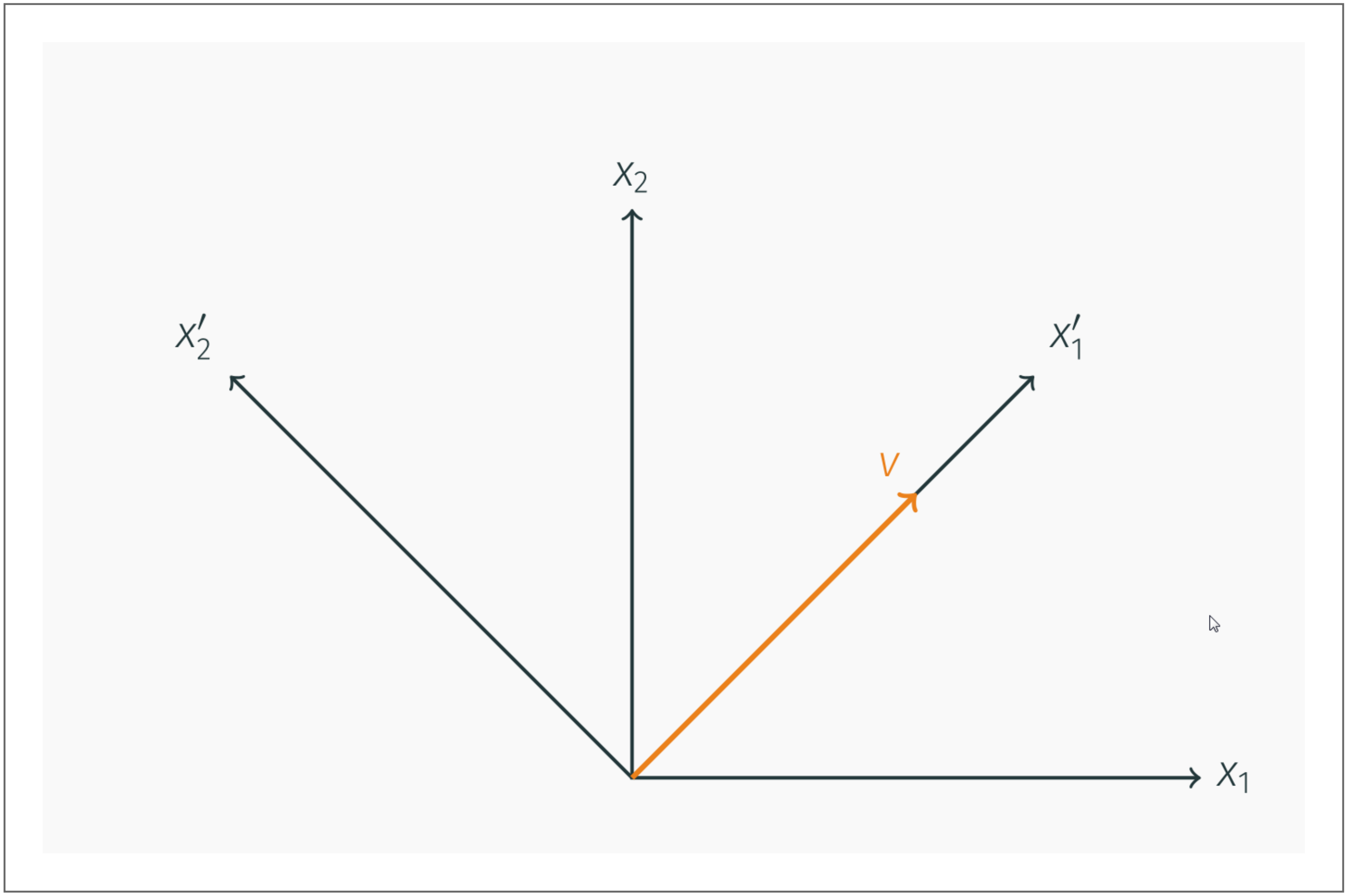
group 3

Find the symmetric, S_{ij}, and anti-symmetric, A_{ij}, portions of T_{ij}. Verify that $S_{ij} + A_{ij} = T_{ij}$

$$T_{ij} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix}$$

coordinate transformation

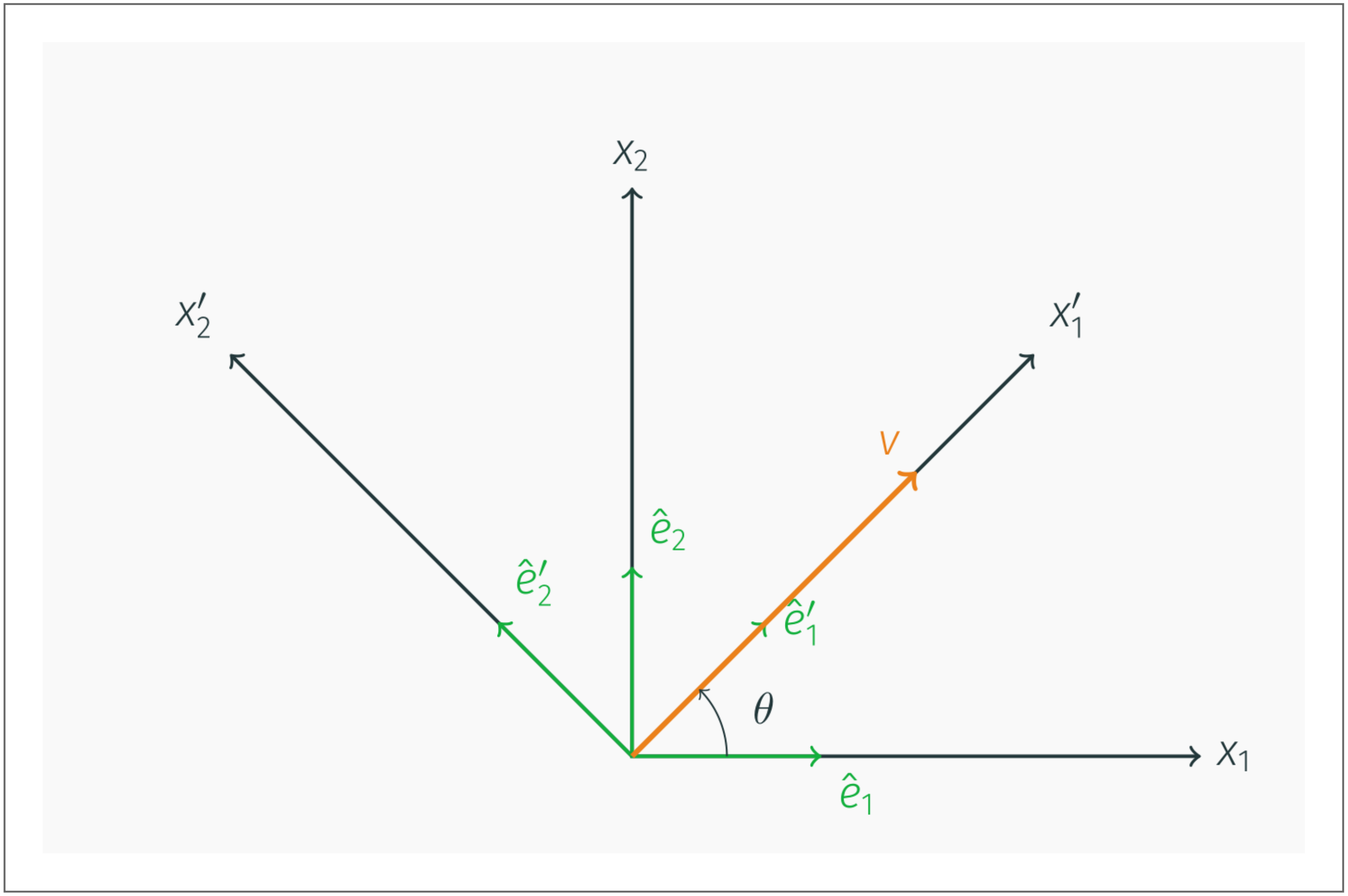
two dimensions



dimensions

- The vector, v, remains fixed, but we transform our coordinate system
- In the new coordinate system, the ${x_2}'$ portion of v is zero.
- To transform the coordinate system, we first define some unit vectors.
- $\hat{e}_1$ is a unit vector in the direction of x_1, while $\hat{e}'_1$ is a unit vector in the direction of ${x_1}'$

two dimensions



two dimensions

- For this example, let us assume $v = \langle 2, 2 \rangle$ and $\theta = 45^\circ$
- We can write the transformed unit vectors, $\hat{e}'_1$ and $\hat{e}'_2$ in terms of $\hat{e}_1$, $\hat{e}_2$ and the angle of rotation, θ.

$$\hat{e}'_1 = \langle \hat{e}_1 \cos\theta, \hat{e}_2 \sin\theta \rangle$$
$$\hat{e}'_2 = \langle -\hat{e}_1 \sin\theta, \hat{e}_2 \cos\theta \rangle$$

two dimensions

- We can write the vector, v, in terms of the unit vectors describing our axis system
- $v = v_1\hat{e}_1 + v_2\hat{e}_2$
- (note: $\hat{e}_1 = \langle 1, 0\rangle$ and $\hat{e}_2 = \langle 0, 1\rangle$)
- v = $\langle 2, 2\rangle$=2$\langle 1, 0\rangle$+2$\langle 0, 1\rangle$

two dimensions

- When expressed in the transformed coordinate system, we refer to $v^{'}$
- $v^{'} = \langle v_1 \cos\theta + v_2 \sin\theta, -v_1 \sin\theta + v_2 \cos\theta \rangle$
- $v' = \langle 2\sqrt{2}, 0 \rangle$
- We can recover the original vector from the transformed coordinates:
- $v = v_1' \hat{e}_1' + v_2' \hat{e}_2'$
- (note: $\hat{e}_1' = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$ and $\hat{e}_2' = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$)
- $v = 2\sqrt{2} \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle, 0 \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle = \langle 2, 2 \rangle$

general

- Coordinate transformation can become much more complicated in three dimensions, and with higher-order tensors
- It is convenient to define a general form of the coordinate transformation in index notation
- We define Q_{ij} as the cosine of the angle between the $x_i{}'$ axis and the x_j axis.
- This is also referred to as the “direction cosine” $Q_{ij} = \cos(x_i{}', x_j)$

general

- We can use this form on our 2D transformation example

$$
\begin{aligned}
Q_{ij} &= \cos(x'_i, x_j) \\
&= \begin{bmatrix} \cos(x'_1, x_1) & \cos(x'_1, x_2) \\ \cos(x'_2, x_1) & \cos(x'_2, x_2) \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \cos(90-\theta) \\ \cos(90+\theta) & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}
\end{aligned}
$$

general

- We can transform any-order tensor using Q_{ij}
- Vectors (first-order tensors): $v_i{}' = Q_{ij}v_j$
- Matrices (second-order tensors): $\sigma_{mn}{}' = Q_{mi}Q_{nj}\sigma_{ij}$
- Fourth-order tensors: $C_{ijkl}{}' = Q_{im}Q_{jn}Q_{ko}Q_{lp}C_{mnop}$

general

- We can similarly use Q_{ij} to find tensors in the original coordinate system
- Vectors (first-order tensors): $v_i = Q_{ji}v_j{}'$
- Matrices (second-order tensors): $\sigma_{mn} = Q_{im}Q_{jn}\sigma_{ij}{}'$
- Fourth-order tensors: $C_{ijkl} = Q_{mi}Q_{nj}Q_{ok}Q_{pl}C_{mnop}{}'$

mental/emotional health warning

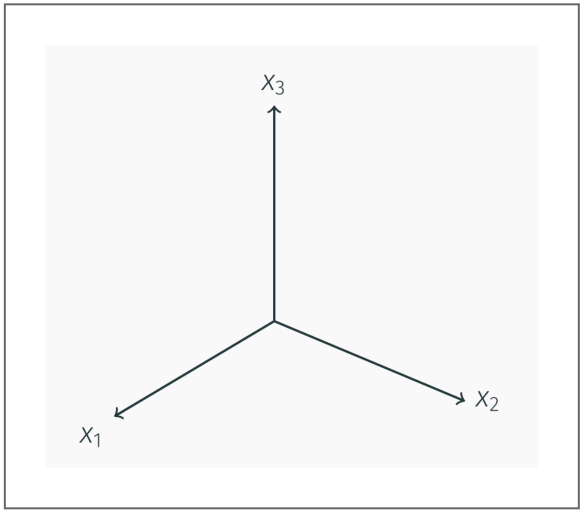
- Some texts flip the definition of Q_{ij}, and then flip their transformation law accordingly
- Any time you use tensor transformation, make sure you are following a consistent set of transformation laws

general

- We can derive some interesting properties of the transformation tensor, Q_{ij}
- We know that $v_i = Q_{ji}v_j^{'}$ and that $v_i^{'} = Q_{ij}v_j$
- If we substitute (changing the appropriate indexes) we find:
- $v_i = Q_{ji}Q_{jk}v_k$
- We can now use the Kronecker Delta to substitute $v_i = \delta_{ik}v_k$ which gives
- $\delta_{ik}v_k = Q_{ji}Q_{jk}v_k$

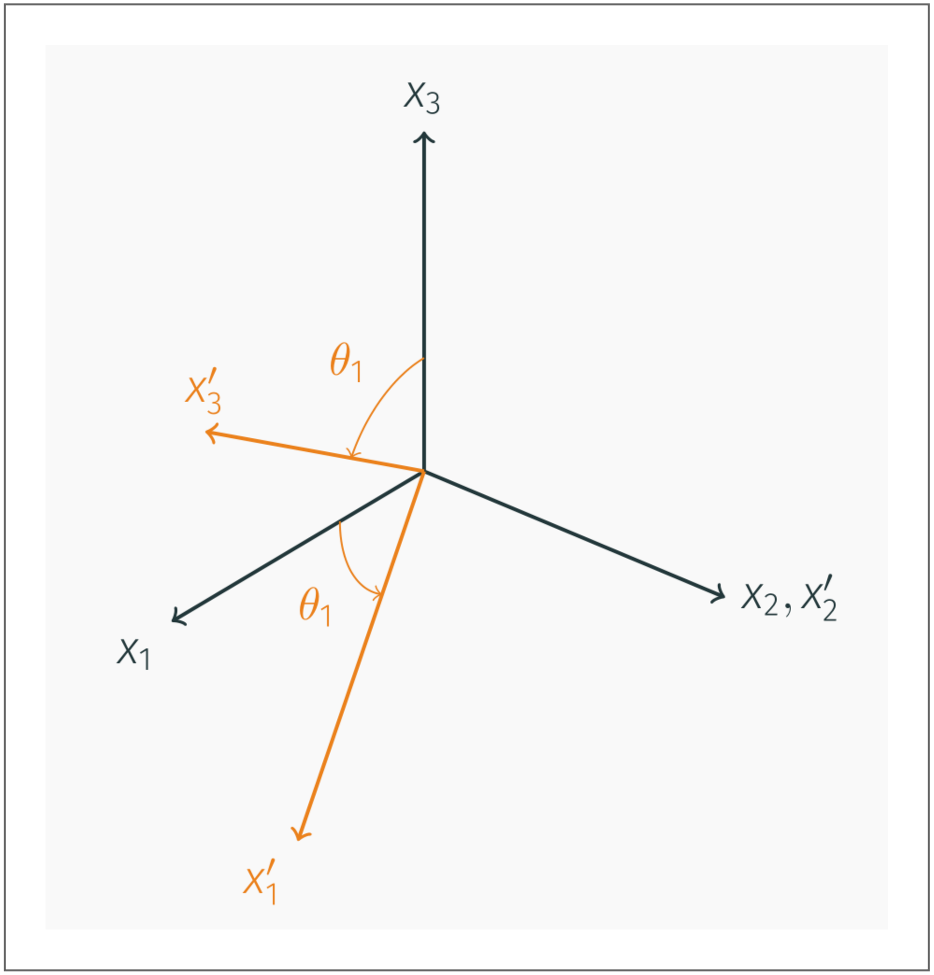
examples

example

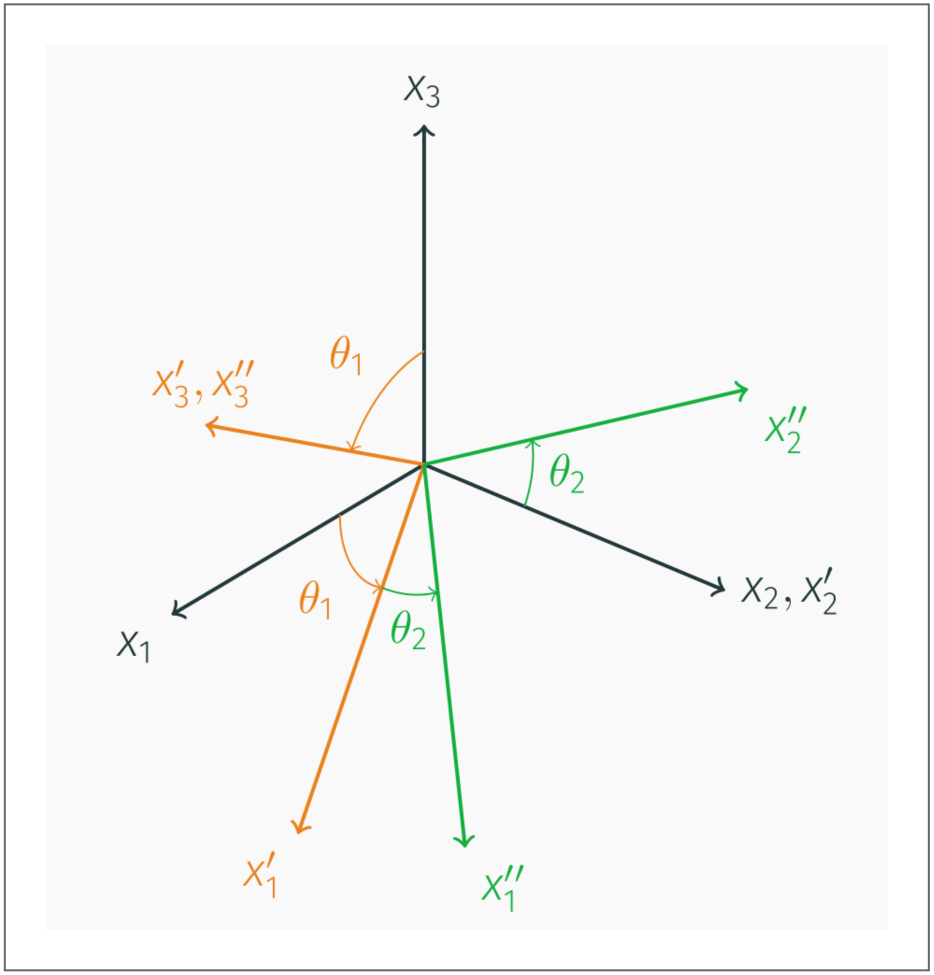


- Find Q_{ij}^1 for rotation of 60° about x_2
- Find Q_{ij}^2 for rotation of 30° about $x_3{}'$
- Find e_i'' after both rotations

example



example



example

- $Q_{ij}{}^1 = \cos(x_i{}', x_j)$
- $Q_{ij}{}^2 = \cos(x_i{}'', x_j{}')$

$$Q_{ij}^1 = \begin{bmatrix} \cos 60 & \cos 90 & \cos 150 \\ \cos 90 & \cos 0 & \cos 90 \\ \cos 30 & \cos 90 & \cos 60 \end{bmatrix}$$

$$Q_{ij}^2 = \begin{bmatrix} \cos 30 & \cos 60 & \cos 90 \\ \cos 120 & \cos 30 & \cos 90 \\ \cos 90 & \cos 90 & \cos 0 \end{bmatrix}$$

example

- We now use Q_{ij} to find $\hat{e}'_i$ and $\hat{e}''_i$
- First, we need to write $\hat{e}_i$ in a manner more consistent with index notation
- We will indicate axis direction with a superscript, e.g. $\hat{e}_1 = e_i^1$
- $e_i{}' = Q_{ij}{}^1 e_j$
- $e_i{}'' = Q_{ij}{}^2 e_j{}'$
- How do we find $e_i{}''$ in terms of e_i?
- $e_i{}'' = Q_{ij}{}^2 Q_{jk}{}^1 e_k$