AE731

Theory of Elasticity

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upcoming schedule

- Sep 9 - Strain Transformation
- Sep 11 - Displacement and Strain
- Sep 16 - Exam Review, Homework 2 Due
- Sep 18 - Exam 1

outline

- example
- group problems
- principal strains
- special strain definitions
- strain transformation

example

example

- Calculate the deformation gradient, strain tensor, and rotation tensor for the given deformation

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} xy^2z \\ xz \\ z^3 \end{Bmatrix}$$

example

- Deformation gradient:

$$F = u_{i,j} = \begin{bmatrix} y^2z & 2xyz & xy^2 \\ z & 0 & x \\ 0 & 0 & 3z^2 \end{bmatrix}$$

example

- Strain tensor

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$e_{ij} = \begin{bmatrix} y^2 z & xyz + \frac{1}{2}z & \frac{1}{2}xy^2 \\ xyz + \frac{1}{2}z & 0 & \frac{1}{2}x \\ \frac{1}{2}xy^2 & \frac{1}{2}x & 3z^2 \end{bmatrix}$$

example

- Rotation tensor

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$$

$$\omega_{ij} = \begin{bmatrix} 0 & xyz - \frac{1}{2}z & \frac{1}{2}xy^2 \\ -xyz + \frac{1}{2}z & 0 & \frac{1}{2}x \\ -\frac{1}{2}xy^2 & -\frac{1}{2}x & 0 \end{bmatrix}$$

example

- As we did with the deformation gradient, we can integrate the strain tensor to find the deformation (symmetric portion)

$$e_{ij} = \begin{bmatrix} yz & xz & xy \\ xz & 2y & \frac{1}{2}x^2 \\ xy & \frac{1}{2}x^2 & 3z^2 \end{bmatrix}$$

example

- We start by integrating the diagonal terms
- $u = \int yz dx = xyz + f(y, z)$
- $v = \int 2y dy = y^2 + g(x, z)$
- $w = \int 3z^2 dz = z^3 + h(x, y)$

example

- Next we need to find the shear terms

$$
\begin{aligned}
e_{xy} &= \frac{1}{2}(u_{,y} + v_{,x}) \\
xz &= \frac{1}{2}(xz + f_{,y} + g_{,x})
\end{aligned}
$$

$$
\begin{aligned}
e_{xz} &= \frac{1}{2}(u_{,z} + w_{,x}) \\
xy &= \frac{1}{2}(xy + f_{,z} + h_{,x})
\end{aligned}
$$

$$
\begin{aligned}
e_{yz} &= \frac{1}{2}(v_{,z} + w_{,y}) \\
\frac{1}{2}x^2 &= \frac{1}{2}(g_{,z} + h_{,y})
\end{aligned}
$$

example

- Note that we cannot uniquely solve this (any arbitrary rotation ω can be added and will still produce a valid strain)
- Let $f(y, z)$=0

$$\begin{aligned} g_{,x} &= xz \\ g(x,z) &= \frac{1}{2}x^2 z \end{aligned}$$

$$\begin{aligned} h_{,x} &= xy \\ h(x,z) &= \frac{1}{2}x^2 y \end{aligned}$$

example

$$\frac{1}{2}x^2 = \frac{1}{2}(g_{,z} + h_{,y})$$
$$\frac{1}{2}x^2 = \frac{1}{2}\left(\frac{1}{2}x^2 + \frac{1}{2}x^2\right)$$

$$u = xyz$$
$$v = y^2 + \frac{1}{2}x^2 z$$
$$w = z^3 + \frac{1}{2}x^2 y$$

group problems

group 1

- Sketch the deformed and undeformed shape of a rectangle under the following deformation

$$u = 0.7x + 0.1y$$
$$v = 0.3x + 1.2y$$

group 2

- For the following deformation, find the deformation gradient, strain, and rotation

$$u = xyz$$
$$v = xy + z$$
$$z = y^2 z$$

group 3

- From the following strain field, find the displacements (you may assume no rotations)

$$\epsilon_{ij} = \begin{bmatrix} y & x+y \\ x+y & x \end{bmatrix}$$

principal strains

principal strains

- Principal strains are found in exactly the same way as principal values discussed in Chapter 1
  $\det[e_{ij} e \delta_{ij}] = 0$
- Invariants can also be found in the same fashion as in any other tensor

$$\vartheta_1 = e_1 + e_2 + e_3$$
$$\vartheta_2 = e_1 e_2 + e_2 e_3 + e_3 e_1$$
$$\vartheta_3 = e_1 e_2 e_3$$

principal strains

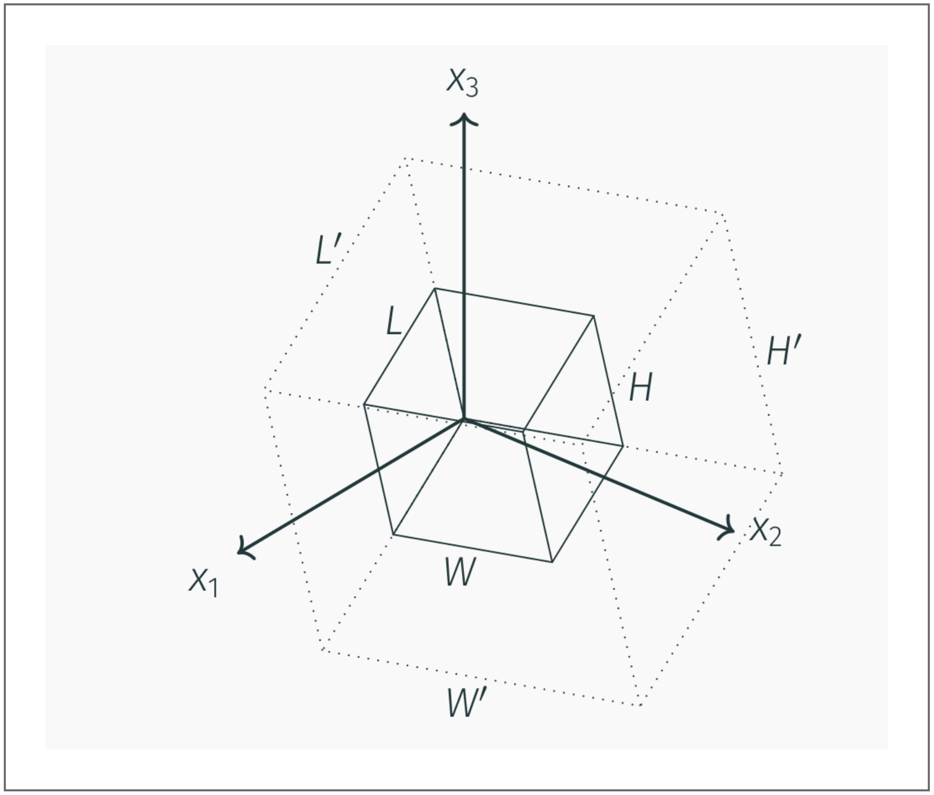
- Principal strains and invariants have some important physical meanings
- ϑ_1 is called the *cubical dilation*, and is related to the change in volume of the material
- Note that in the principal direction, there are no shear strains

$$\begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix}$$

- This means that there is only extensional strain in the principal direction (i.e. a cube will remain a rectangular prism, the corners will maintain 90° angles)

volume change

- Consider a rectangular prism with edges oriented in the principal directions



volume change

- The volume before deformation is $V = LWH$
- The volume after deformation is given by $V' = L'W'H'$
- We can relate the side lengths after deformation to strains

$$e_1 = \frac{L' - L}{L}$$
$$Le_1 + L = L'$$

- We can now write the volume as $V' = L(1 + e_1)W(1 + e_2)H(1 + e_3)$

volume change

- After multiplication, the deformed volume is given as
- $V' = LWH(1 + e_1 + e_2 + e_3 + e_1e_2 + e_2e_3 + e_1e_3 + e_1e_2e_3)$
- For small strains, $e_i \ll 1$, therefore e_1, e_2, and e_3 will be much larger than e_1e_2, e_2e_3, e_1e_3 and $e_1e_2e_3$.
- $V' = LWH(1 + e_1 + e_2 + e_3)$

volume change

- A “dilatation” is defined as the change in volume divided by the original volume
  $$\frac{\Delta V}{V} = \frac{V' - V}{V}$$
- Substituting the relationships found earlier
  $$\frac{V' - V}{V} = \frac{LWH(1 + e_1 + e_2 + e_3) - LWH}{LWH}$$
- Which simplifies to
  $$e_1 + e_2 + e_3 = \vartheta_1$$

special strain definitions

spherical strain

- This dilatation can be used to find the so-called *spherical strain*

$$\tilde{e}_{ij} = \frac{1}{3}e_{kk}\delta_{ij} = \frac{1}{3}\vartheta_1\delta_{ij}$$

- The *deviatoric strain* is found by subtracting the spherical strain from the strain tensor
  $$\hat{e}_{ij} = e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}$$

strain transformation

strain transformation

- The usual tensor transformation applies to the strain tensor as well
- $e_{ij}' = Q_{im}Q_{jn}e_{mn}$
- In many instances, however, we are only concerned with the strain within a plane (for example, when using strain gages).

strain transformation

- For an in-plane rotation (rotation about z-axis), we find

$$Q_{ij} = \begin{bmatrix} \cos\theta & \cos(90-\theta) & \cos 90 \\ \cos(90+\theta) & \cos\theta & \cos 90 \\ \cos 90 & \cos 90 & \cos 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

strain transformation

- If we multiply this out, for the in-plane strain terms (${e_x}'$, ${e_x}'$, and ${e_{xy}}'$) we find

$$\begin{aligned} e'_x &= e_x \cos^2\theta + e_y \sin^2\theta + 2e_{xy}\sin\theta\cos\theta \\ e'_y &= e_x \sin^2\theta + e_y \cos^2\theta - 2e_{xy}\sin\theta\cos\theta \\ e'_{xy} &= -e_x \sin\theta\cos\theta + e_y \sin\theta\cos\theta + e_{xy}(\cos^2\theta - \sin^2\theta) \end{aligned}$$

strain transformation

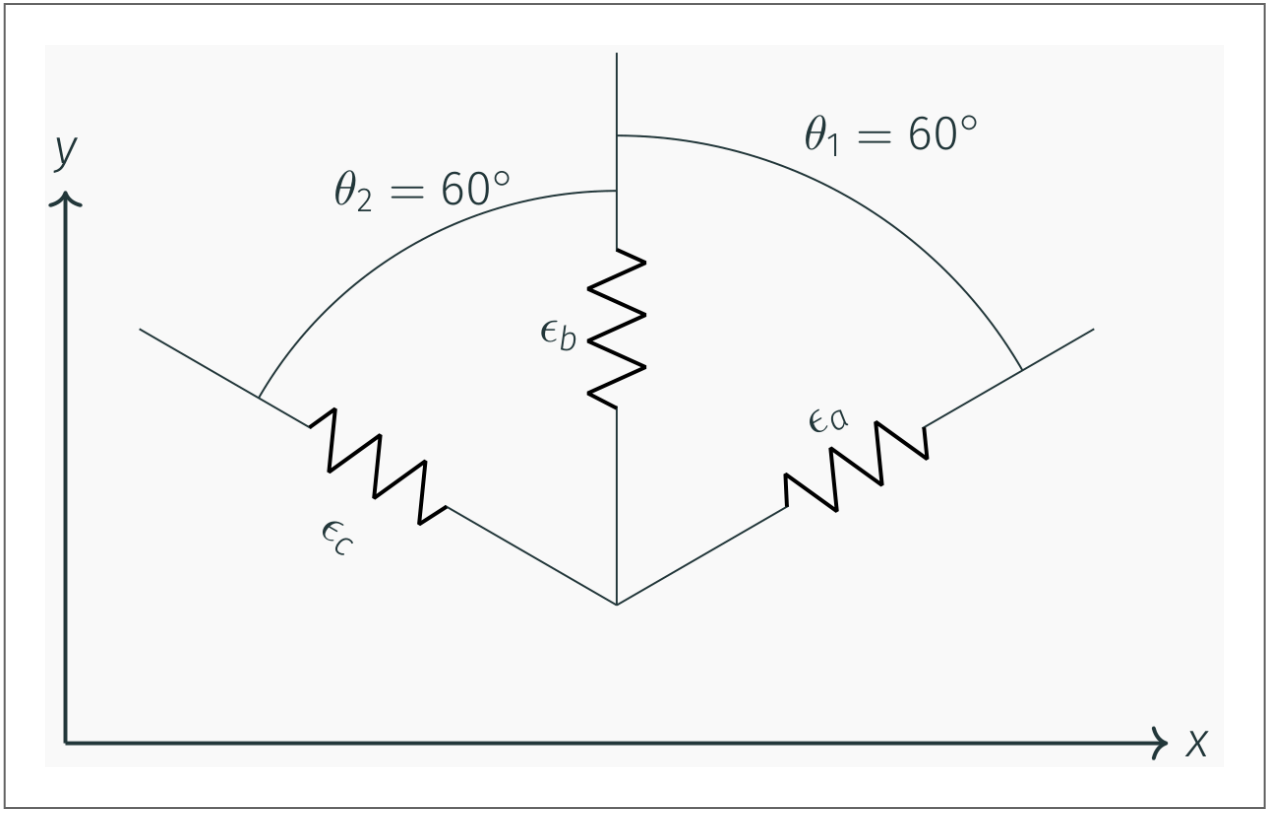
- This is often re-written using the double-angle formulas

$$
\begin{aligned}
e'_x &= \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2}\cos 2\theta + e_{xy}\sin 2\theta \\
e'_y &= \frac{e_x + e_y}{2} - \frac{e_x - e_y}{2}\cos 2\theta - e_{xy}\sin 2\theta \\
e'_{xy} &= \frac{e_y - e_x}{2}\sin 2\theta + e_{xy}\cos 2\theta
\end{aligned}
$$

strain transformation

- Many times it is easy to measure the axial strain directly with strain gages, but the shear strain cannot be easily measured
- We can use an extra, off-axis strain gage, together with the strain transformation equations, to calculate the shear strain
- Many companies already do this with “rosettes” which have strain gages at specified angles built-in

example



example

- Given that $\epsilon_a = 0.005$, $\epsilon_b = -0.002$ and $\epsilon_c = 0.003$, find e_x, e_y, and e_{xy}.
- Note that $e_y = \epsilon_b = -0.002$
- Set coordinate system so that $\epsilon_b = e_x^{'}$.
- Use equation for $e_x^{'}$ with $\theta = 30$.

$$e_x' = \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2} \cos 60 + e_{xy} \sin 60$$

example

- We have two unknowns in this equation, so we need another
- We can use the equation for $e_y{}'$ with $\theta = 60$ so that $\epsilon_b = e_x{}'$

$$e'_y = \frac{e_x + e_y}{2} - \frac{e_x - e_y}{2}\cos 120 - e_{xy}\sin 120$$

example

- Substituting known values and simplifying:

$$
\begin{aligned}
0.01 + 0.002 - 0.002\cos 60 &= e_x(1+\cos 60) + e_{xy}\sin 60 \\
0.006 + 0.002 + 0.002\cos 120 &= e_x(1-\cos 120) - e_{xy}\sin 120
\end{aligned}
$$

- And solving we find $e_x = 0.006$, $e_y = -0.002$, and $e_{xy} = 0.00231$.