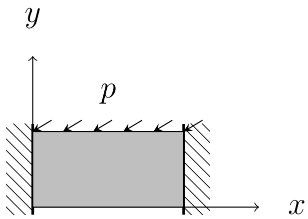
Name:

Homework 5
Due 23 Oct 2019

1. For the following figures, express the boundary conditions on each face.

   (a) **Note:** Consider for any arbitrary loading angle, θ.

   - Here we have mixed boundary conditions, with traction boundary conditions on the horizontal surfaces and displacement boundary conditions on the vertical surfaces
   - On the vertical surfaces (i.e. when $x = 0$ and $x = L$), the fixed boundary condition can be expressed as

   $$u_i = \langle 0, 0, 0 \rangle$$

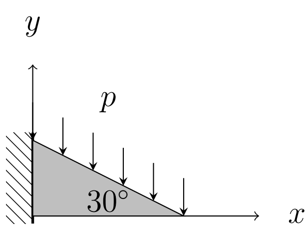
   - On the bottom horizontal surface (i.e. when $y = 0$), we have a traction-free boundary condition with a normal vector of $n_i = \langle 0, -1, 0 \rangle$. Using Cauchy's stress theorem, we find that at $y = 0$:

   $$\sigma_{12} = \sigma_{22} = \sigma_{23} = 0$$

   - on the top horizontal surface (i.e. when $y = h$), we have a traction boundary condition dependent on the applied traction p and its direction, θ. The normal vector on this surface is $n_i = \langle 0, 1, 0 \rangle$ and the traction vector is $t_i = \langle -\cos\theta, -\sin\theta, 0 \rangle$. Using Cauchy's stress theorem we find that when $y = h$

   $$\sigma_{12} = -p\cos\theta$$
   $$\sigma_{22} = -p\sin\theta$$
   $$\sigma_{23} = 0$$

   (b) **Note:** In this case the load coincides with the vertical axis

- On the vertical surface where $x = 0$ we have a displacement boundary condition where

$$u_i = \langle 0, 0, 0 \rangle$$

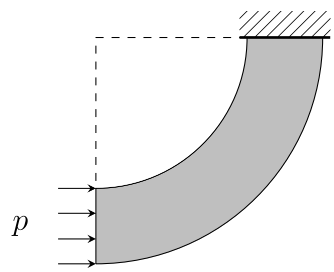
- On the bottom horizontal surface where $y = 0$ we have a traction-free boundary condition. We can use the same normal vector as in the above problem to find

$$\sigma_{12} = \sigma_{22} = \sigma_{23} = 0$$

- The third surface is along the line $y = -\sqrt{3}/3x + b$ and has a traction boundary condition of $t_i = \langle 0, -p, 0 \rangle$. If we write the normal vector as $n_i = \langle \sqrt{3}, 3, 0 \rangle$ we can use Cauchy's stress theorem to find that along the line $y = -\sqrt{3}/3x + b$

$$\begin{aligned} 0 &= \sqrt{3}\sigma_{11} + 3\sigma_{12} \\ -p &= \sqrt{3}\sigma_{12} + 3\sigma_{22} \\ 0 &= \sqrt{3}\sigma_{13} + 3\sigma_{23} \end{aligned}$$

(c) **Note:** In this case the load coincides with the horizontal axis



- Here it is convenient to use a cylindrical coordinate system and to call the inside radius a and the outside radius b.
- On the inside and outside surfaces (i.e. where $r = a$ or $r = b$) we have a traction-free boundary condition. In both cases we can use a normal vector of $n_i = \langle 1, 0, 0 \rangle$, which gives the condition that on both these surfaces

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0$$

- On the horizontal surface (i.e. where $\theta = 0$) we have the displacement boundary condition of $u_i = \langle 0, 0, 0 \rangle$
- On the vertical surface, where $\theta = -\pi/2$, we have a traction boundary of $t_i = \langle 0, -p, 0 \rangle$. The unit normal vector along this surface is $n = \langle 0, -1, 0 \rangle$, which gives stress components of

$$\begin{aligned} \sigma_{r\theta} &= 0 \\ \sigma_{\theta\theta} &= p \\ \sigma_{\theta z} &= 0 \end{aligned}$$

2. Determine if the following strain field is compatible

$$\begin{aligned}
\epsilon_{11} &= 2x^2 + 3y^2 + z + 1 \\
\epsilon_{22} &= x^2 + 2y^2 + 3z + 2 \\
\epsilon_{33} &= 3x + 2y + z^2 + 1 \\
\epsilon_{12} &= 4xy \\
\epsilon_{13} &= \epsilon_{23} = 0
\end{aligned}$$

   - It is helpful to determine which derivatives used in the compatibility equations will be non-zero The only non-zero derivatives are

$$\begin{aligned}
\frac{\partial^2 \epsilon_x}{\partial y^2} &= 6 \\
\frac{\partial^2 \epsilon_y}{\partial x^2} &= 2 \\
\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} &= 4
\end{aligned}$$

   - Of the six compatibility equations, this leaves just one non-trivial equation

$$\begin{aligned}
\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} &= 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} \\
6 + 2 &= 2(4) \\
8 &= 8
\end{aligned}$$

   - Thus this strain field is compatible

3. The stresses in a 3-D body are

$$\begin{aligned}
\sigma_{11} &= -A(L - x)y \\
\sigma_{12} &= \frac{1}{8}A(h^2 - 4y^2) = \sigma_{21}
\end{aligned}$$

   With all other $\sigma_{ij} = 0$. Is this body in equilibrium? Are the strains compatible? What are the displacements?

   Find and plot the vertical deflection along the line $y = 0$

   Note: This problem is plane stress ($\sigma_{i3} = 0$), which is by nature, approximate. For plane stress problems, the relevant compatibility equations reduce to

$$\nabla^2(\sigma_{11} + \sigma_{22}) = -(1 + \nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$

   - Only one of the three equilibrium equations is non-trivial

$$\begin{aligned}
\sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} &= 0 \\
Ay - Ay + 0 &= 0
\end{aligned}$$

     Thus we see that equilibrium is satisfied

- Since plane stress only provides an approximation of the elasticity equations, we must use the plane stress compatibility equation, and not the full 3D Beltrami-Michell equations to check compatibility.

$$\nabla^2(\sigma_{11}+\sigma_{22}) = -(1+\nu)\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right)$$
$$\nabla^2\sigma_{11}+\nabla^2\sigma_{22} = -(1+\nu)(0+0)$$
$$0+0=0$$

  And compatibility is also satisfied

  Note: If we instead used the full Beltrami-Michell equations to check compatibility we would find

$$(1+\nu)\nabla^2\tau_{xy}+\frac{\partial^2}{\partial x\partial y}(\sigma_x+\sigma_y+\sigma_z) = 0$$
$$-A(1+\nu)+A = 0$$
$$-A\nu = 0$$

  Which is not true, so compatibility is not satisfied, but we are aware of the plane stress approximations, and are satisfied with plane stress compatibility.

- We can now use Hooke's Law to find the strain tensor from the stress tensor

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij}-\frac{\nu}{E}\sigma_{kk}$$
$$\epsilon_{11} = \frac{-A}{E}(L-x)y$$
$$\epsilon_{12} = \frac{A(1+\nu)}{8E}(h^2-4y^2)$$
$$\epsilon_{22} = \frac{A\nu}{E}(L-x)y\epsilon_{33} \qquad = \frac{A\nu}{E}(L-x)y$$

- If we neglect w and consider u and v to only be functions of x, y, we can integrate ϵ_{11} and ϵ_{22} to find

$$u = -\frac{AL}{E}xy+\frac{A}{2E}x^2y+f(y)$$
$$v = \frac{A\nu}{2E}(L-x)y^2+g(x)$$

- We usually are able to find f and g by assuming the rotation is zero, however in this case that assumption would lead to f being a function of both x and y, which is not possible (as it would give an incorrect strain for ϵ_{11}), thus the assumption of zero-rotation is invalid

- Instead we can examine the equation for ϵ_{12}

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$
$$\frac{A(1+\nu)}{8E}(h^2-4y^2) = \frac{1}{2}\left(-\frac{AL}{E}x+\frac{A}{2E}x^2+f'(y)-\frac{A\nu}{2E}y^2+g'(x)\right)$$

Where we note that all terms are either constant or a function of x or y only, thus we can separate this into two equations (here we include the constant term with the y equation, but it could instead be included in the equations for x)

$$\frac{A(1+\nu)}{4E}(h^2-4y^2) = f'(y) - \frac{A\nu}{2E}y^2$$
$$0 = -\frac{AL}{E}x + \frac{A}{2E}x^2 + g'(x)$$

Which, after solving and integrating, gives

$$f(y) = \frac{A(1+\nu)}{4E}h^2y - \frac{A(1+\nu)}{3E}y^3 + \frac{A\nu}{6E}y^3$$
$$g(x) = \frac{AL}{2E}x^2 - \frac{A}{6E}x^3$$

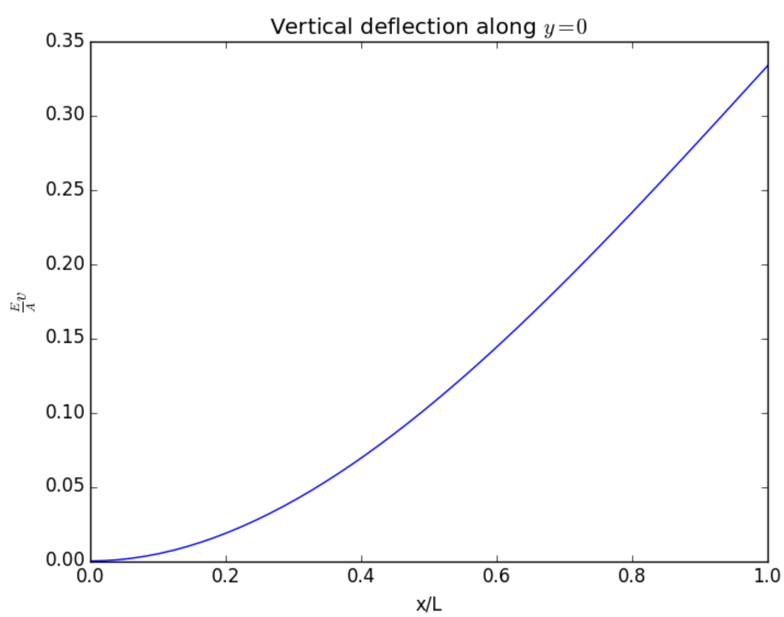
- Substituting f and g back into the expressions for u and v gives

$$u = -\frac{AL}{E}xy + \frac{A}{2E}x^2y + \frac{A(1+\nu)}{4E}h^2y - \frac{A(1+\nu)}{3E}y^3 + \frac{A\nu}{6E}y^3$$
$$v = \frac{A\nu}{2E}(L-x)y^2 + \frac{AL}{2E}x^2 - \frac{A}{6E}x^3$$

- To plot the vertical deflection along the line $y = 0$ we substitute $y =$ into the expression for v

$$v = \frac{A}{6E}(3Lx^2 - x^3)$$

We can normalize by plotting $\frac{E}{A}v$ on the vertical axis and $\frac{x}{L}$ on the horizontal axis



4. Check equilibrium and compatibility conditions for the following stress field

$$\sigma_{11} = Py\left(1 - \frac{y^2}{3b^2}\right)$$

With all other $\sigma_{ij} = 0$

- The equilibrium equations are easily shown to be satisfied, since σ_{11} is a function of y-only, all derivatives used in the equilibrium equations go to zero.
- The first compatibility equation from the Beltrami-Michell equations is

$$\begin{aligned}(1+\nu)\nabla^2\sigma_x + \frac{\partial^2}{\partial x^2}(\sigma_x + \sigma_y + \sigma_z) &= 0 \\ (1+\nu)\left(\frac{-2Py}{3b^2}\right) &= 0\end{aligned}$$

  Since b is a constant and y is a variable, compatibility can not be satisfied at all points, so this equation is not compatible