AE731

Theory of Elasticity

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upcoming schedule

- Oct 2 - Material Characterization, HW3 Due
- Oct 7 - Thermoelasticity
- Oct 9 - Boundary Conditions
- Oct 14 - Fall Break (no class)

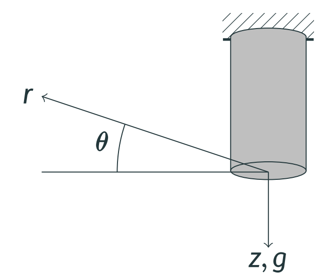
outline

- review
- material characterization
- hooke's law
- matrix relationship
- physical meaning

review

example

- Suppose a hanging cylinder has a density given by $\rho = r^2$
- Recall that self-weight is the only force acting on this problem



example

- We know that there is no traction along the outer surfaces, and we can also assume that $\sigma_{ij} = \sigma_{ij}(z)$, since the only force is acting in the z direction

- Using Cauchy's stress theorem we can find

$$
\begin{aligned}
t_j &= \sigma_{ij} n_i = 0 \\
&= \langle \sigma_{rr} n_r + \sigma_{r\theta} n_\theta + \sigma_{rz} n_z, \sigma_{\theta r} n_r + \sigma_{\theta\theta} n_\theta + \sigma_{\theta z} n_z, \sigma_{zr} n_r + \sigma_{z\theta} n_\theta + \sigma_{zz} n_z \rangle \\
&= \langle \sigma_{rr}, \sigma_{\theta r}, \sigma_{zr} \rangle = 0 \qquad \text{at } r = a
\end{aligned}
$$

example

- And on the bottom face
- $\sigma_{rz} = \sigma_{\theta_z} = \sigma_{zz} = 0$

example

- To find the stress in the z direction, we use the third equilibrium equation

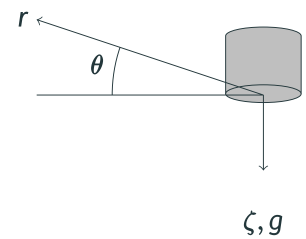
$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\tau_{rz} + F_z = 0$$

- We can substitute known values to find that

$$\frac{\partial \sigma_z}{\partial z} + r^2 g = 0$$

example

- Since we desire to find the stress at any point, we introduce a variable to indicate the coordinate of our free body diagram cut



example

- We integrate over this free body to find

$$\begin{aligned}\sigma_z &= -\int_{\zeta}^{-L} r^2 g dz \\ &= r^2 g(L - z)\end{aligned}$$

- In this case, the stress is a function of radial distance (just like the body force was)

example

- We can find the average stress at some location, z

$$\bar{\sigma}_z = \frac{1}{A}\int \sigma_z dA$$

- In this case this integral becomes

$$\bar{\sigma}_z = \frac{1}{A}\int_0^R\int_0^{2\pi} r^2 g(L-z)rdrd\theta$$

$$\bar{\sigma}_z = \frac{1}{\pi R^2}\int_0^R 2\pi r^3 g(L-z)dr$$

$$\bar{\sigma}_z = \frac{2\pi}{4\pi R^2}R^4 g(L-z)$$

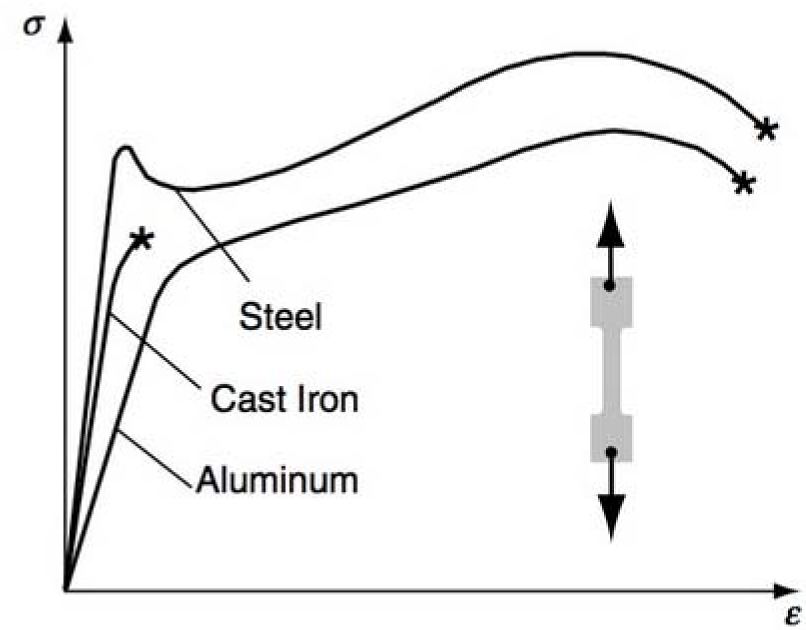
$$= \frac{R^2}{2}g(L-z)$$

material characterization

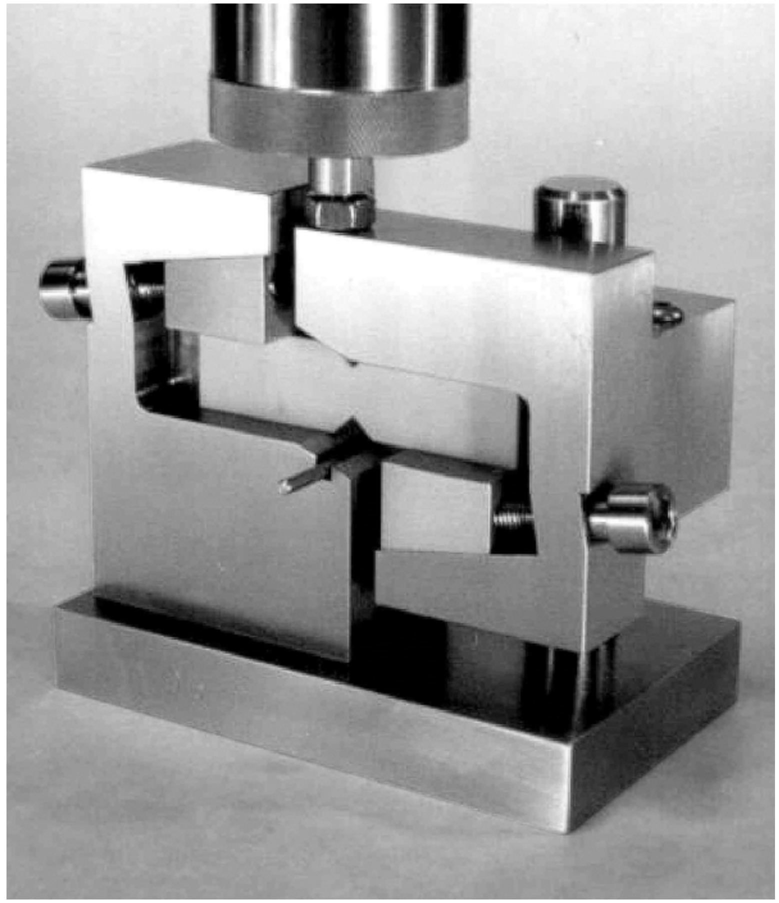
material characterization

- We have now formally defined stress and strain tensors, it is desirable to relate these two tensors to one another
- In this course we make the following assumptions about material behavior:
    - Small strains
    - Linear elastic
    - Rate-independent
    - Homogeneous

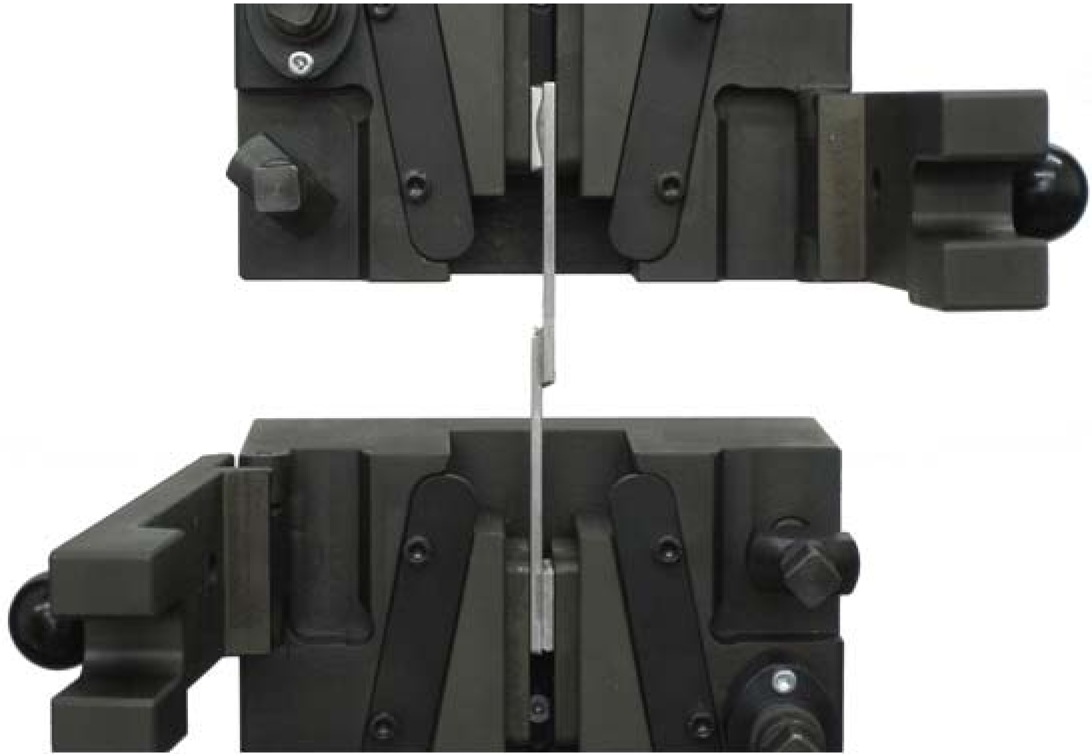
tensile test



shear tests



shear tests



hooke's law

hooke's law

- In its most general form, Hooke's Law relates the stress and strain tensors by the Cauchy stiffness tensor

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

- We could, equivalently write this equation in terms of the compliance

$$\epsilon_{ij} = S_{ijkl}\sigma_{kl}$$

hooke's law

- There are 81 terms in C_{ijkl} (3x3x3x3)
- Many of these are redundant, so a contracted notation is sometimes convenient.
- Symmetry of the stress tensor and strain tensors requires symmetries in C_{ijkl}

$$\begin{aligned}\sigma_{ij} &= C_{ijkl}\epsilon_{kl} \\ &= \sigma_{ji} \\ &= C_{jikl}\epsilon_{kl}\end{aligned}$$

- $C_{ijkl} = C_{jikl}$
- $C_{ijkl} = C_{ijlk}$

hooke's law

- Strain energy concepts further require that $C_{ijkl} = C_{klij}$
- This reduces the number of unique, unknown constants to 21
- In general, if we cannot make any assumptions about material symmetry, there are 21 parameters that we need to find to determine the linear behavior of a material.

matrix relationship

matrix form

- Due to the symmetry in σ_{ij} and ϵ_{ij}, however, many of these are redundant
- For this reason (as well as convenience in writing), many will often form a matrix equation, with σ and ϵ acting as vectors.

matrix form

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ 2\epsilon_4 \\ 2\epsilon_5 \\ 2\epsilon_6 \end{Bmatrix}$$

- In this form, the usual tensor transformation equations cannot be used. Also, be careful as σ_4, σ_5, and σ_6 do not always represent the same terms (depends on textbook/community), also sometimes the engineering form of shear ($\gamma_{12} = 2\epsilon_{12}$) is used.

matrix form

- If we consider the σ_{11} term, we find that

$$\begin{aligned}\sigma_{11} &= C_{11kl}\epsilon_{kl} \\ &= C_{1111}\epsilon_{11} + C_{1112}\epsilon_{12} + C_{1113}\epsilon_{13} + \\ &\quad C_{1121}\epsilon_{21} + C_{1122}\epsilon_{22} + C_{1123}\epsilon_{23} + \\ &\quad C_{1131}\epsilon_{31} + C_{1132}\epsilon_{32} + C_{1133}\epsilon_{33}\end{aligned}$$

- If we simplify this, we find

$$\sigma_{11} = C_{1111}\epsilon_{11} + 2C_{1112}\epsilon_{12} + 2C_{1113}\epsilon_{13} + C_{1122}\epsilon_{22} + 2C_{1123}\epsilon_{23} + C_{1133}\epsilon_{33}$$

matrix form

- In matrix form, we write the normal terms first, and we include the factor of 2 in the strain vector, giving

$$\sigma_1 = C_{11}\epsilon_1 + C_{12}\epsilon_2 + C_{13}\epsilon_3 + C_{14}\epsilon_4 + C_{15}\epsilon_5 + C_{16}\epsilon_6$$

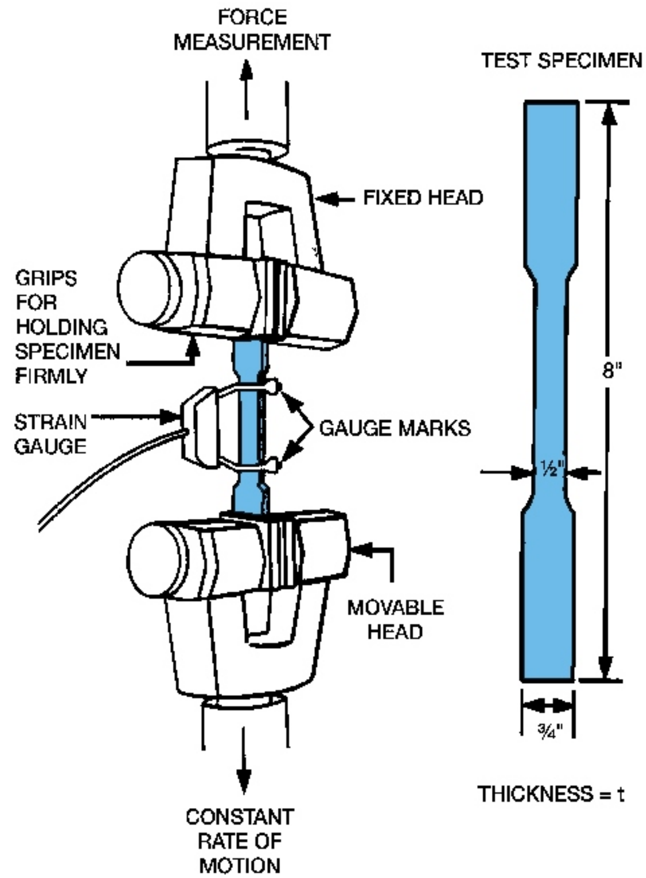
physical meaning

physical meaning

- We will discuss symmetries in greater detail in a later lecture
- An isotropic material reduces the unknown material constants to two

$$\epsilon_{ij} = \frac{1+v}{E}\sigma_{ij} - \frac{v}{E}\sigma_{kk}\delta_{ij}$$

simple tension

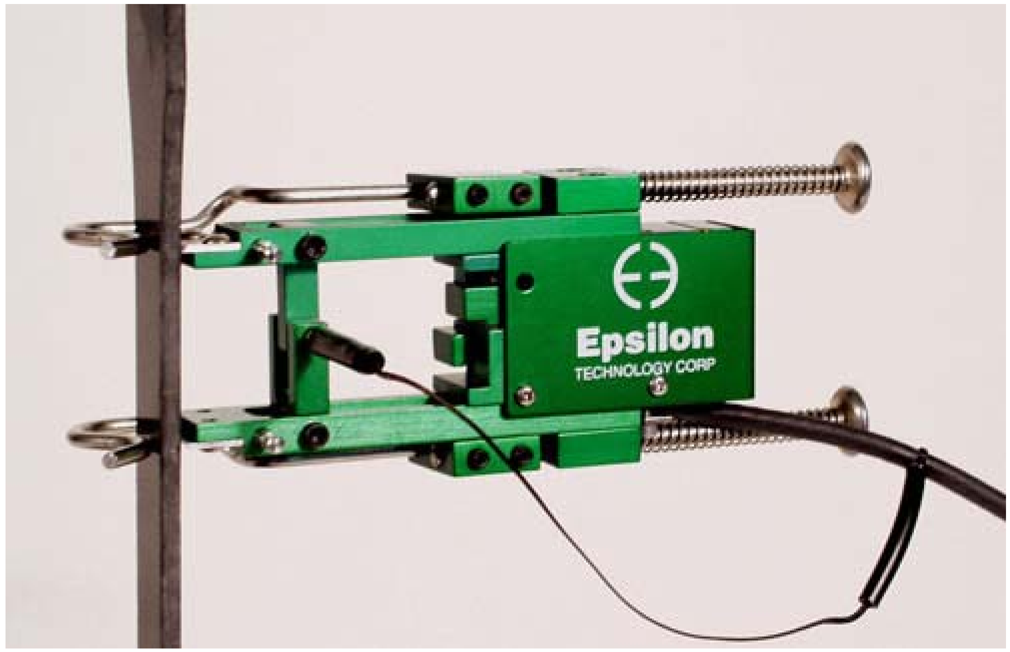


strain measurement

- How do we measure strain?
    - grip displacement
    - extensometers
    - strain gages
    - digital image correlation

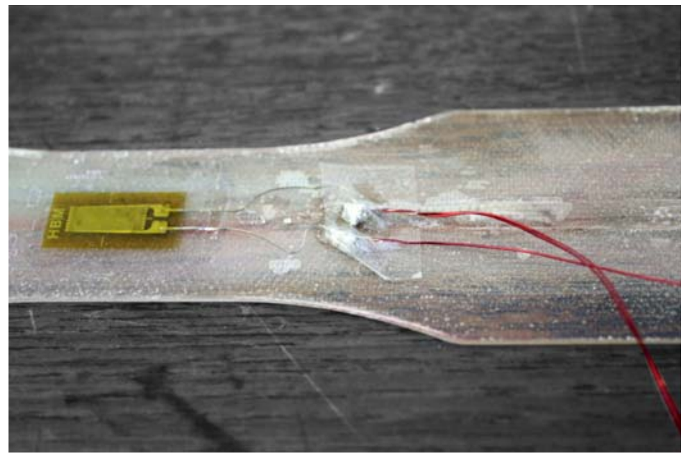
extensometers

- Reusable, quick to use, apply, and interpret
- May slip, only gives one direction, can initiate failure site



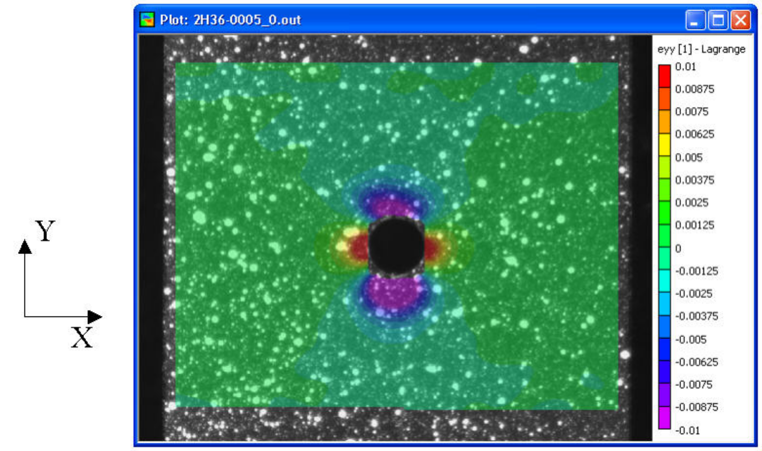
strain gages

- Can be applied in any direction, very accurate
- Must be perfectly adhered, subject to user-error in attaching, can require complicated electronics to read results



digital image correlation

- Gives full-field strain tensor
- Requires expensive equipment, software
- Cannot compute values near the edges



2H36 (2024-T3) y-direction strain due to 4% cold expansion

simple tension

- If we consider a simple tension test, if done correctly the applied stress will be

$$\sigma_{ij} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- What will the strain be?

simple tension

- Recall Hooke's law for isotropic material

$$\epsilon_{ij} = \frac{1+v}{E}\sigma_{ij} - \frac{v}{E}\sigma_{kk}\delta_{ij}$$

$$\epsilon_{ij} = \begin{bmatrix} \frac{1+v}{E}\sigma - \frac{v}{E}\sigma & 0 & 0 \\ 0 & -\frac{v}{E}\sigma & 0 \\ 0 & 0 & -\frac{v}{E}\sigma \end{bmatrix}$$

$$\epsilon_{ij} = \begin{bmatrix} \frac{1}{E}\sigma & 0 & 0 \\ 0 & -\frac{v}{E}\sigma & 0 \end{bmatrix}$$

pure shear

- In pure shear, the applied stress will be

$$\sigma_{ij} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- What will the strain be?

pure shear

- Recall Hooke's law for isotropic material

$$\epsilon_{ij} = \frac{1+v}{E}\sigma_{ij} - \frac{v}{E}\sigma_{kk}\delta_{ij}$$

$$\epsilon_{ij} = \begin{bmatrix} 0 & \frac{1+v}{E}\tau & 0 \\ \frac{1+v}{E}\tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

hydrostatic pressure

- In hydrostatic pressure (can be compression or tension), equal normal stress is applied in all three directions

$$\sigma_{ij} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

- What will the strain be?

hydrostatic pressure

- Recall Hooke's law for isotropic material

$$\epsilon_{ij} = \frac{1+v}{E}\sigma_{ij} - \frac{v}{E}\sigma_{kk}\delta_{ij}$$

$$\epsilon_{ij} = \begin{bmatrix} \frac{1+v}{E}p - \frac{3v}{E}p & 0 & 0 \\ 0 & \frac{1+v}{E}p - \frac{3v}{E}p & 0 \\ 0 & 0 & \frac{1+v}{E}p - \frac{3v}{E}p \end{bmatrix}$$

$$\epsilon_{ij} = \begin{bmatrix} \frac{1-2v}{E}p & 0 & 0 \\ 0 & \frac{1-2v}{E}p & 0 \\ & & \end{bmatrix}$$

next class

- Material symmetry (monoclinic, isotropic, orthotropic, transversely isotropic)
- Poisson's ratio in anisotropic materials
- Thermoelastic considerations