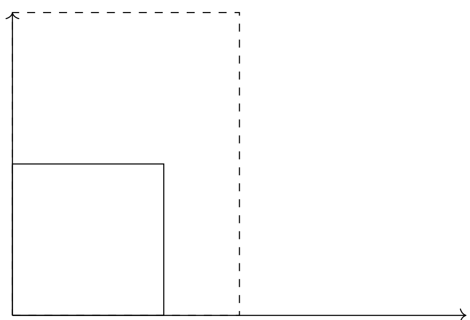
Name:

Homework 2
Due 9 Sept 2019

1. For the following prescribed displacements, sketch the deformed and un-deformed shape of a rectangle.

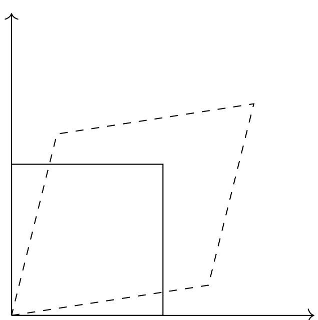
   (a)

$$u = 1.5x$$
$$v = 2y$$



   (b)

$$u = 1.3x + 0.3y$$
$$v = 0.2x + 1.2y$$



2. Determine the strain and rotation tensors from the given displacements

   (a)

$$u = 1.5x$$
$$v = 2y$$
$$w = z$$

$$e_{ij} = \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\omega_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b)

$$\begin{aligned} u &= 2x + 3yz \\ v &= xy + z^2 \\ w &= xyz \end{aligned}$$

$$e_{ij} = \frac{1}{2} \begin{bmatrix} 4 & 3z + y & 3y + yz \\ 3z + y & 2x & 2z + xz \\ 3y + yz & 2z + xz & 2xy \end{bmatrix}$$

$$\omega_{ij} = \frac{1}{2} \begin{bmatrix} 0 & 3z - y & 3y - yz \\ y - 3z & 0 & 2z - xz \\ yz - 3y & xz - 2z & 0 \end{bmatrix}$$

(c)

$$\begin{aligned} u &= xy^2 \\ v &= y^2 + z^2 \\ w &= y^3 z \end{aligned}$$

$$e_{ij} = \begin{bmatrix} y^2 & xy & 0 \\ xy & 2y & \frac{1}{2}(2z + 3y^2 z) \\ 0 & \frac{1}{2}(2z + 3y^2 z) & y^3 \end{bmatrix}$$

$$\omega_{ij} = \begin{bmatrix} 0 & xy & 0 \\ -xy & 0 & \frac{1}{2}(2z - 3y^2 z) \\ 0 & \frac{1}{2}(3y^2 z - 2z) & 0 \end{bmatrix}$$

3. Determine the displacement field from the given strain tensors (assume no rotation is present)

(a)

$$e_{ij} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

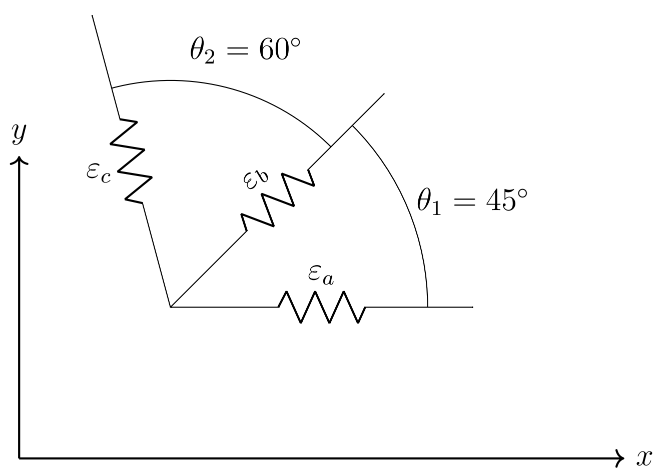
$$\begin{aligned} u &= kx \\ v &= ky \\ w &= 0 \end{aligned}$$

(b)

$$e_{ij} = \begin{bmatrix} kx^2 & z & y \\ z & ky & x \\ y & x & z \end{bmatrix}$$

$$u = \frac{1}{3}kx^3 + yz$$
$$v = \frac{1}{2}ky^2 + xz$$
$$w = \frac{1}{2}z^2 + xy$$

4. Rosette strain gages are commonly used in tensile tests to measure strain in different directions. For the rosette configuration shown below, strain is measured as $\varepsilon_a = 0.005$, $\varepsilon_b = 0.008$, and $\varepsilon_c = 0.002$. Find e_x, e_y, and e_{xy}. Note that ε_a is aligned with the x-axis.



- Recall the strain transformation relations in 2D:

$$e'_x = \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2}\cos 2\theta + e_{xy}\sin 2\theta \tag{1}$$
$$e'_y = \frac{e_x + e_y}{2} - \frac{e_x - e_y}{2}\cos 2\theta - e_{xy}\sin 2\theta \tag{2}$$
$$e'_{xy} = \frac{e_y - e_x}{2}\sin 2\theta + e_{xy}\cos 2\theta \tag{3}$$

- Since ε_a is aligned with the x-axis, we can say that $e_x = \varepsilon_a = 0.005$.
- To find e_y and e_{xy}, we need to set up a system of equations
- One set of equations we can use is (1) twice, once with $\theta_1 = 45°$ and once again with $\theta_2 = 45° + 60° = 105°$.

$$e'_x = \varepsilon_b = \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2}\cos 2\theta_1 + e_{xy}\sin 2\theta_1$$
$$e''_x = \varepsilon_c = \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2}\cos 2\theta_2 + e_{xy}\sin 2\theta_2$$

- Substituting known values $(e_x, \theta_1, \theta_2, \varepsilon_b, \varepsilon_c)$

$$0.008 = \frac{0.005 + e_y}{2} + \frac{0.005 - e_y}{2} \cos 2(45) + e_{xy} \sin 2(45)$$
$$0.002 = \frac{0.005 + e_y}{2} + \frac{0.005 - e_y}{2} \cos 2(105) + e_{xy} \sin 2(105)$$

- Multiplying both equations by 2 and expanding

$$0.016 = 0.005 + e_y + 2e_{xy}$$
$$0.004 = 0.005 + e_y + (0.005 - e_y) \cos 210 + 2e_{xy} \sin 210$$

- Simplifying yields

$$0.011 = e_y + 2e_{xy}$$
$$-0.001 + 0.005\frac{\sqrt{3}}{2} = \left(1 + \frac{\sqrt{3}}{2}\right) e_y - e_{xy}$$

- Which gives the final solution

$$e_x = 0.00500$$
$$e_y = 0.00373$$
$$e_{xy} = 0.00363$$

- Alternatively, we can find e_y''' for $\theta_3 = 15$.

$$e_y''' = \frac{e_x + e_y}{2} - \frac{e_x - e_y}{2} \cos 2\theta_3 - e_{xy} \sin 2\theta_3$$
$$0.002 = \frac{0.005 + e_y}{2} - \frac{0.005 - e_y}{2} \cos 30 - e_{xy} \sin 30$$
$$0.004 = 0.005 + e_y - (0.005 - e_y)\frac{\sqrt{3}}{2} - e_{xy}$$
$$-0.001 + 0.005\frac{\sqrt{3}}{2} = (1 + \frac{\sqrt{3}}{2})e_y - e_{xy}$$

- Which we can see is identical to the e_x'' equation above.

(a) Rosette strain gages are generally precision manufactured, so the angles between individual gages on a rosette are very accurately controlled. However, the strain gages are usually attached by hand, and may not be perfectly aligned with the loading axis. Find e_x, e_y, and e_{xy} if ε_a is NOT aligned with the x-axis.

Hint: You may choose the angle of mis-alignment. Try to choose an angle that makes the problem less difficult.

- To solve this problem for some general mis-alignment, we would need to set up a system with three equations and three unknowns, using (1) three times. Once for the misalignment, once for the angle between the x-axis and ε_b, and a third time for the angle between the x-axis and ε_c.
- Since we are allowed to choose the angle of mis-alignment, we can set an angle of 15° below the x-axis, so that ε_c aligns with the y-axis.
- With e_y as a known value, we use (1) twice to find e_x and e_{xy}

$$
\begin{aligned}
\varepsilon_a &= \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2} \cos 2(-15) + e_{xy} \sin 2(-15) \\
\varepsilon_b &= \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2} \cos 2(30) + e_{xy} \sin 2(30)
\end{aligned}
$$

- After simplification and substitution

$$
\begin{aligned}
0.008 + \frac{\sqrt{3}}{2}(0.002) &= \left(1 + \frac{\sqrt{3}}{2}\right) e_x - e_{xy} \\
0.015 &= \frac{3}{2} e_x + \sqrt{3} e_{xy}
\end{aligned}
$$

- Which gives the solution

$$
\begin{aligned}
e_x &= 0.00673 \\
e_y &= 0.00200 \\
e_{xy} &= 0.00283
\end{aligned}
$$

- If we compare to the original solution, a mis-alignment of 15° when attaching a rosette strain gage will result in strain measurement errors of 22% for shear strain, 35% for e_x and 46% for e_y.

(b) The manufacturer of these strain gages is under pressure from investors to sell more rosettes. As a clever new engineer for Rosettes, Inc., how would you improve the rosette design to sell more rosettes?

- One improve would be to make $\theta_2 = 45°$. This would simplify calculations by giving known values to both e'_x and e'_y, so that only one angle is needed in the calculations.

5. Find the principal strains and their directions for the following states of strain

(a)

$$
e_{ij} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & -3 & 1 \\ 0 & 1 & 6 \end{bmatrix}
$$

- Principal strains can be found in the same way as principal values were found in Chapter 1, by solving $\det[e_{ij} - \lambda \delta_{ij}] = 0$

$$
\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & -3-\lambda & 1 \\ 0 & 1 & 6-\lambda \end{vmatrix} = 0
$$

- This gives

$$(2-\lambda)[(-3-\lambda)(6-\lambda)-(1)(1)]-(-1)[(-1)(6-\lambda)-(1)(0)]=0$$

- Which simplifies to

$$\lambda^3+5*\lambda^2+14\lambda-44=0$$

- And has the solutions $\lambda_1 = 6.113$, $\lambda_2 = 2.184$, and $\lambda_3 = -3.296$
- As before, we can find the principal directions by substituting the principal values back in. In this case, some rounding is necessary to find the principal directions.

$$\begin{aligned} n^{(1)} &= \langle -0.027, 0.112, 0.993\rangle \\ n^{(2)} &= \langle 0.982, -0.180, 0.047\rangle \\ n^{(3)} &= \langle -0.185, -0.977, 0.105\rangle \end{aligned}$$

(b)

$$e_{ij} = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 3 & 2 \\ -1 & 2 & 5 \end{bmatrix}$$

- Solving the characteristic equation, we find $\lambda_1 = 6.271$, $\lambda_2 = 5.341$ and $\lambda_3 = 0.388$.
- Using the same method as before, we find the principal directions to be

$$\begin{aligned} n^{(1)} &= \langle 0.169, 0.587, 0.792\rangle \\ n^{(2)} &= \langle 0.834, 0.343, -0.432\rangle \\ n^{(3)} &= \langle 0.525, -0.733, 0.432\rangle \end{aligned}$$