AE731

Theory of Elasticity

Dr. Nicholas Smith
Wichita State University, Department of Aerospace Engineering November 20, 2019

upcoming schedule

- Nov 20 - Polar Coordinates in Airy Stress, Homework 7 Due
- Nov 25 - Polar Coordinates in Airy Stress
- Nov 27 - No Class (Thanksgiving Break)
- Dec 2 - Complex Methods
- Dec 4 - Final Review, Homework 8 Due
- Dec 11 - 3:00 - 4:50 Final Exam

outline

- polar coordinates
- examples

polar coordinates

strain-displacement

- Reduced strain-displacement:

$$\epsilon_r = \frac{\partial u_r}{\partial r}, \epsilon_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right), \epsilon_z = \frac{\partial u_z}{\partial z}$$

$$\epsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$\epsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)$$

$$\epsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

strain-displacement

- Which becomes

$$\epsilon_r = \frac{\partial u_r}{\partial r}$$

$$\epsilon_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)$$

$$\epsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

integration

- When we change variables in integration, we also need to account for the proper change in dV
- $dV = dxdydz \neq drd\theta dz$
- We can find the correct dV by calculating the Jacobian

jacobian

$$dV = dxdydz = \left| \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \right| drd\theta dz$$

$$dV = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} drd\theta dz = rdrd\theta dz$$

hooke's law

- The tensor equation for Hooke's Law is valid in polar/cylindrical/spherical coordinates too
- We only need special equations when differentiating or integrating

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} - (3\lambda + 2\mu)\alpha\Delta T\delta_{ij}$$

$$\epsilon_{ij} = \frac{1+v}{E}\sigma_{ij} - \frac{v}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\delta_{ij}$$

equilibrium

- We have already found the equilibrium equations in polar coordinates, they are

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r}(\sigma_r - \sigma_\theta) + F_r = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2}{r}\tau_{r\theta} + F_\theta = 0$$

equilibrium

- The equilibrium equations can be written in terms of displacement (Navier equations)
- These are only useful when using a displacement formulation, but we are using stress functions
- Instead we need the Beltrami-Mitchell compatibility equations

compatibility

- Substituting stress-strain relations into the compatibility equations gives

$$\nabla^2(\sigma_r + \sigma_\theta) = -\frac{1}{1-v}\left(\frac{\partial F_r}{\partial r} + \frac{F_r}{r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}\right) \qquad \text{(Plane Strain)}$$

$$\nabla^2(\sigma_r + \sigma_\theta) = -(1+v)\left(\frac{\partial F_r}{\partial r} + \frac{F_r}{r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}\right) \qquad \text{(Plane Stress)}$$

airy stress functions

- When the body forces are zero, we find

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

airy stress functions

- When body forces are zero, we find the following biharmonic equation for the Beltrami-Mitchell equations

$$\nabla^4 \phi = 0$$

- Where the Laplacian is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

polar coordinates

- Recall that an Airy Stress function must satisfy the Beltrami-Mitchell compatibility equations

$$\nabla^4\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \phi = 0$$

- One method which gives several useful solutions assumes that the Airy Stress function has the form $\phi(r,\theta){=}f(r)e^{b\theta}$

polar coordinates

- Substituting this into the compatibility equations (and canceling the common $e^{b\theta}$) term gives

$$f'''' + \frac{2}{r}f''' - \frac{1-2b^2}{r^2}f'' + \frac{1-2b^2}{r^3}f' + \frac{b^2(4+b^2)}{r^4}f = 0$$

polar coordinates

- To solve this, we perform a change of variables, letting $r = e^{\xi}$, which gives

$$f'''' - 4f''' + (4 + 2b^2)f'' - 4b^2 f' + b^2(4 + b^2)f = 0$$

- We now consider f to have the form $f = e^{a\xi}$ which generates the characteristic equation

$$(a^2 + b^2)(a^2 - 4a + 4 + b^2) = 0$$

polar coordinates

- This has solutions

$$a = \pm ib, \ \pm 2ib$$

OR

$$b = \pm ia, \ \pm i(a-2)$$

polar coordinates

- If we consider only solutions which are periodic in θ, we find

$$
\begin{aligned}
\phi = {} & a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r \\
& + (a_4 + a_5 \log r + a_6 r^2 + a_7 r^2 \log r)\theta \\
& + \left(a_{11} r + a_{12} r \log r + \frac{a_{13}}{r} + a_{14} r^3 + a_{15} r\theta + a_{16} r\theta \log r \right) \cos\theta \\
& + \left(b_{11} r + b_{12} r \log r + \frac{b_{13}}{r} + b_{14} r^3 + b_{15} r\theta + b_{16} r\theta \log r \right) \sin\theta \\
& + \sum_{n=2}^{\infty} (a_{n1} r^n + a_{n2} r^{2+n} + a_{n3} r^{-n} + a_{n4} r^{2-n}) \cos n\theta \\
& + \sum_{n=2}^{\infty} (b_{n1} r^n + b_{n2} r^{2+n} + a_{n3} r^{-n} + b_{n4} r^{2-n}) \sin n\theta
\end{aligned}
$$

polar coordinates

- For axisymmetric problems, all field quantities are independent of θ
- This reduces the general solution to

$$\phi = a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r$$

polar coordinates

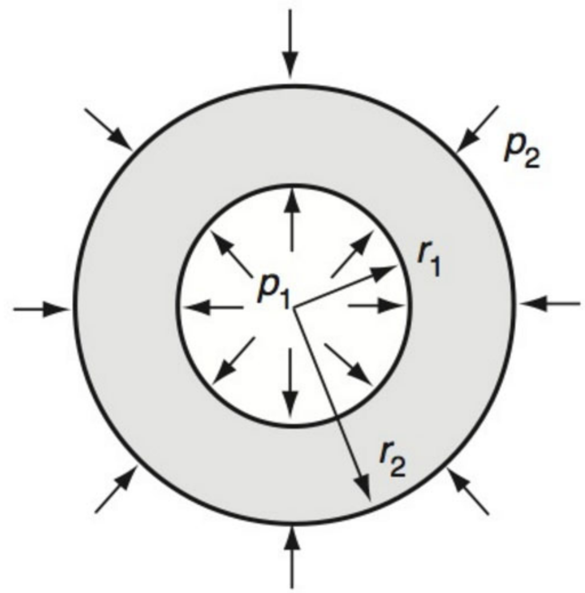
ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$\log r$	$1/r^2$	0	$-1/r^2$
θ	0	$1/r^2$	0
$r^2 \log r$	$2\log r + 1$	0	$2\log r + 3$
$r^2\theta$	2θ	-1	2θ
$r^3 \cos\theta$	$2r\cos\theta$	$2r\sin\theta$	$6r\cos\theta$
$r^3 \sin\theta$	$2r\sin\theta$	$-2r\cos\theta$	$6r\sin\theta$
$r\theta\sin\theta$	$2\cos\theta/r$	0	0
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\log r\cos\theta$	$\cos\theta/r$	$\sin\theta/r$	$\cos\theta/r$
$r\log r\sin\theta$	$\sin\theta/r$	$-\cos\theta/r$	$\sin\theta/r$
$\cos\theta/r$	$-2\cos\theta/r^3$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$2\sin\theta/r^3$

polar coordinates

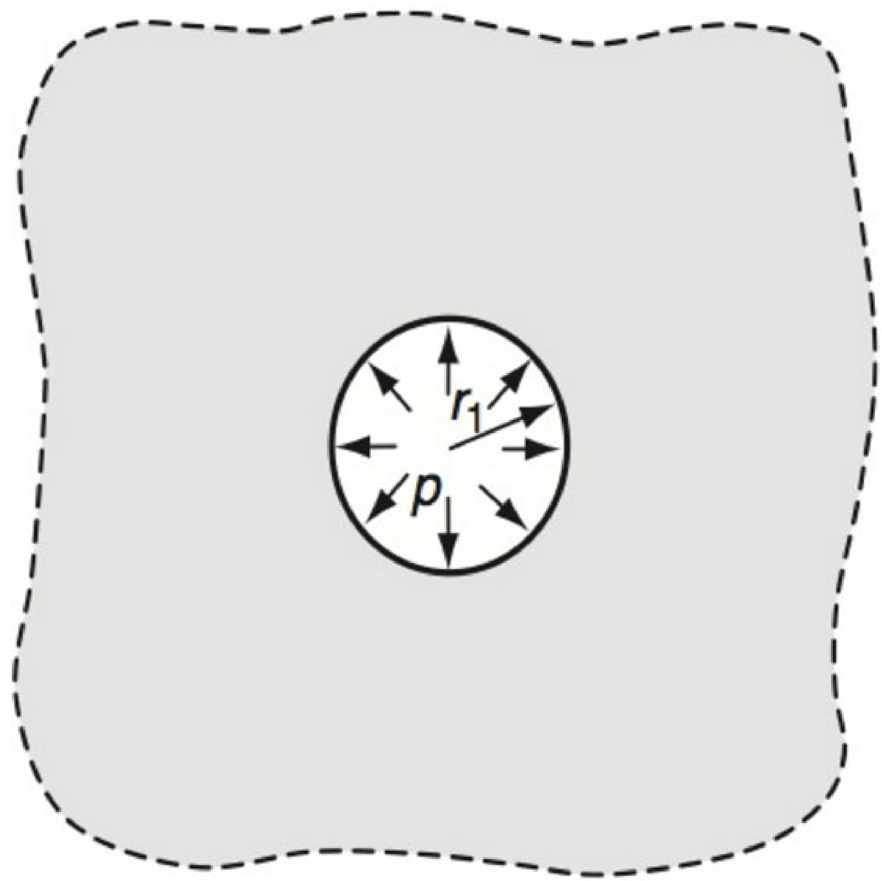
$r^4 \cos 2\theta$	0	$6r^2 \sin 2\theta$	$12r^2 \cos 2\theta$
$r^4 \sin 2\theta$	0	$-6r^2 \cos 2\theta$	$12r^2 \sin 2\theta$
$r^2 \cos 2\theta$	$-2\cos 2\theta$	$2\sin 2\theta$	$2\cos 2\theta$
$r^2 \sin 2\theta$	$-2\sin 2\theta$	$-2\cos 2\theta$	$2\sin 2\theta$
$\cos 2\theta$	$-4\cos 2\theta/r^2$	$-2\sin 2\theta/r^2$	0
$\sin 2\theta$	$-4\sin 2\theta/r^2$	$2\cos 2\theta/r^2$	0
$\cos 2\theta/r^2$	$-6\cos 2\theta/r^4$	$-6\sin 2\theta/r^4$	$6\cos 2\theta/r^4$
$\sin 2\theta/r^2$	$-6\sin 2\theta/r^4$	$6\cos 2\theta/r^4$	$6\sin 2\theta/r^4$
$r^n \cos n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$	$n(n-1)r^{n-2}\cos n\theta$
$r^n \sin n\theta$	$-n(n-1)r^{n-2}\sin n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$
$r^{n+2}\cos n\theta$	$-(n+1)(n-2)r^n \cos n\theta$	$(n+1)nr^n \sin n\theta$	$(n+2)(n+1)r^n \cos n\theta$
$r^{n+2}\sin n\theta$	$-(n+1)(n-2)r^n \sin n\theta$	$-(n+1)nr^n \cos n\theta$	$(n+2)(n+1)r^n \sin n\theta$
$\cos n\theta/r^n$	$-(n+1)n\cos n\theta/r^{n+2}$	$-(n+1)n\sin n\theta/r^{n+2}$	$(n+1)n\cos n\theta/r^{n+2}$
$\sin n\theta/r^n$	$-(n+1)n\sin n\theta/r^{n+2}$	$(n+1)n\cos n\theta/r^{n+2}$	$(n+1)n\sin n\theta/r^{n+2}$
$\cos n\theta/r^{n-2}$	$-(n+2)(n-1)\cos n\theta/r^n$	$-n(n-1)\sin n\theta/r^n$	$(n-1)(n-2)\cos n\theta/r^n$
$\sin n\theta/r^{n-2}$	$-(n+2)(n-1)\sin n\theta/r^n$	$n(n-1)\cos n\theta/r^n$	$(n-1)(n-2)\sin n\theta/r^n$

examples

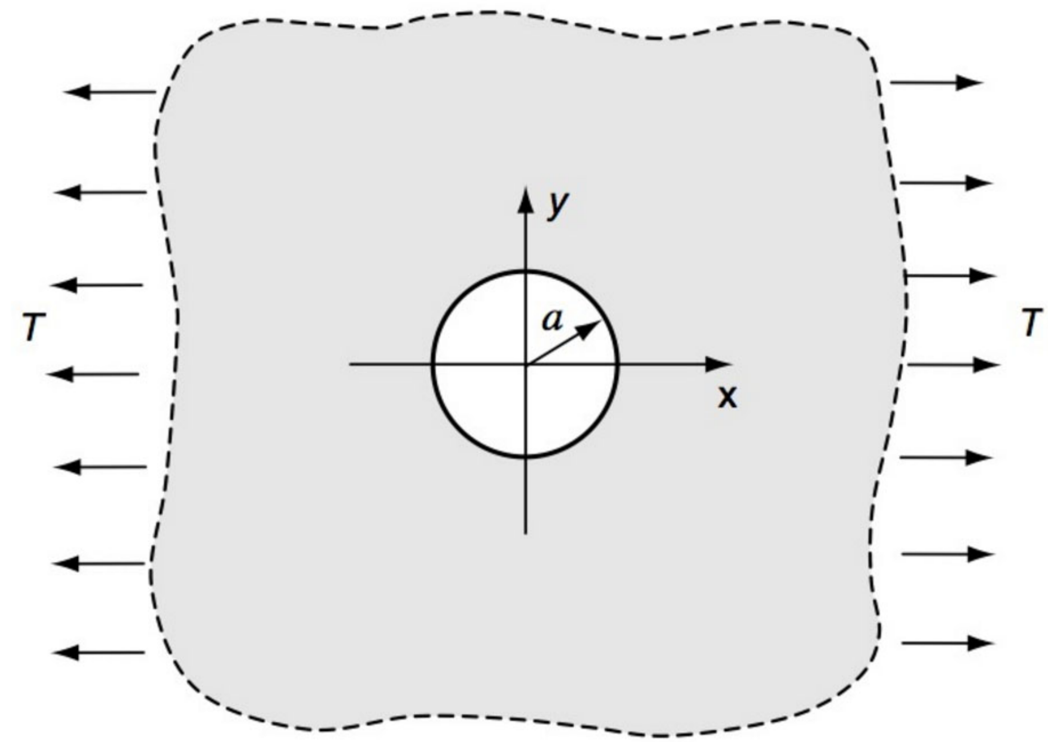
tube under uniform pressure



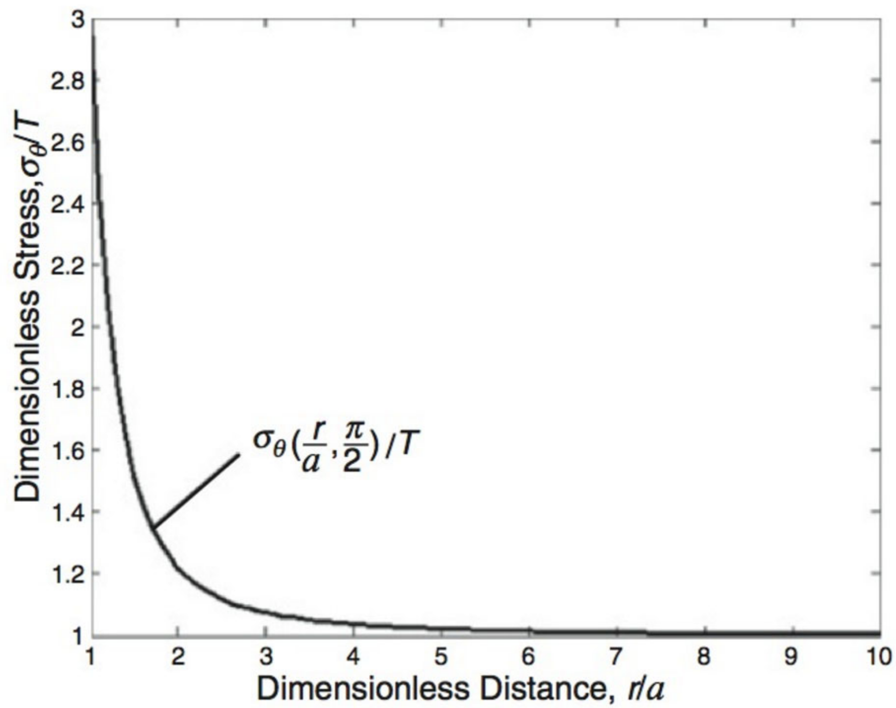
pressurized hole



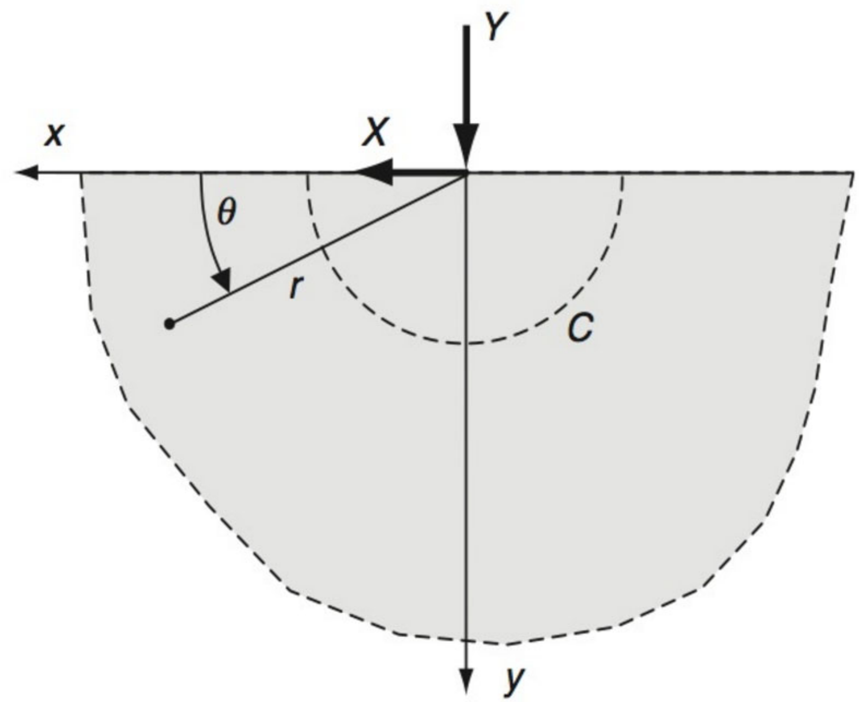
stress-free hole in tension



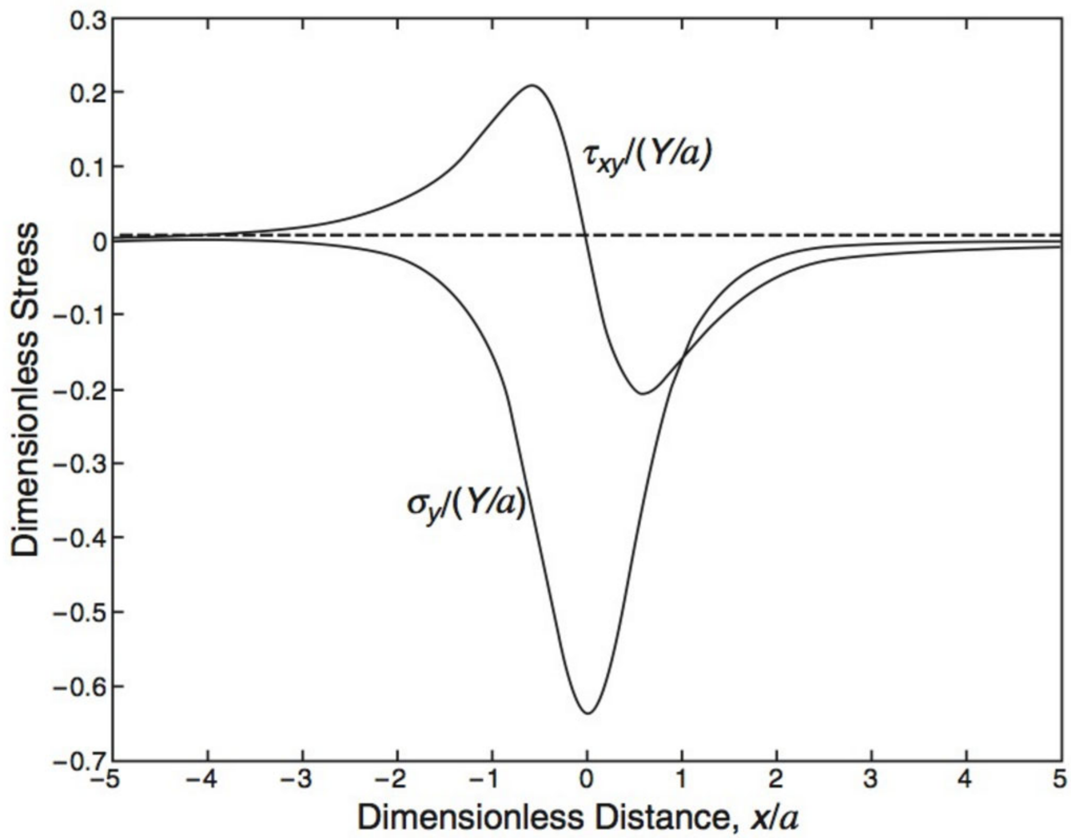
stress-free hole in tension



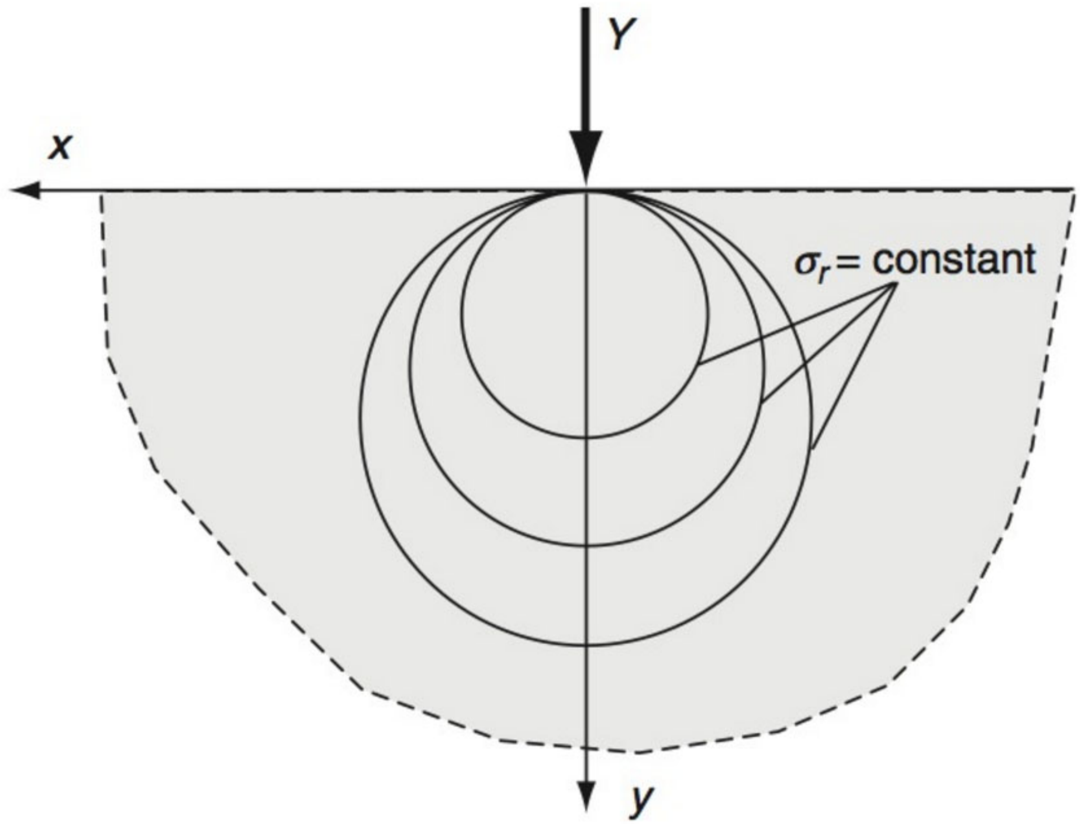
concentrated force



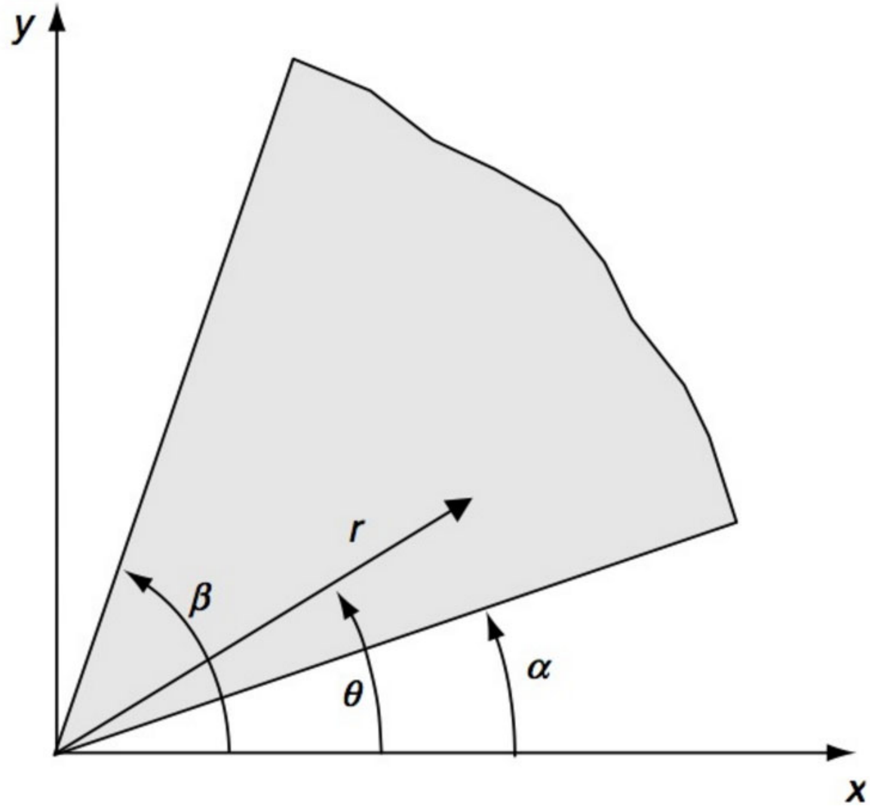
concentrated force



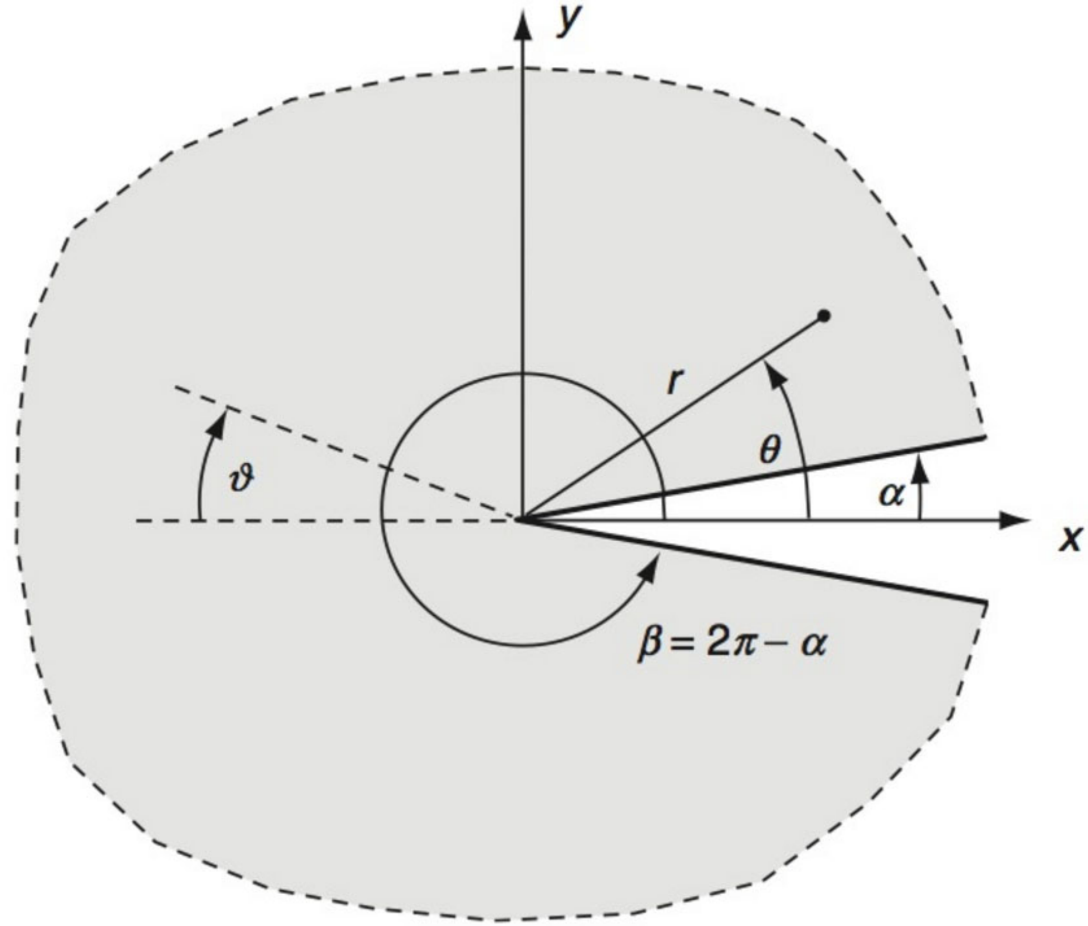
concentrated force



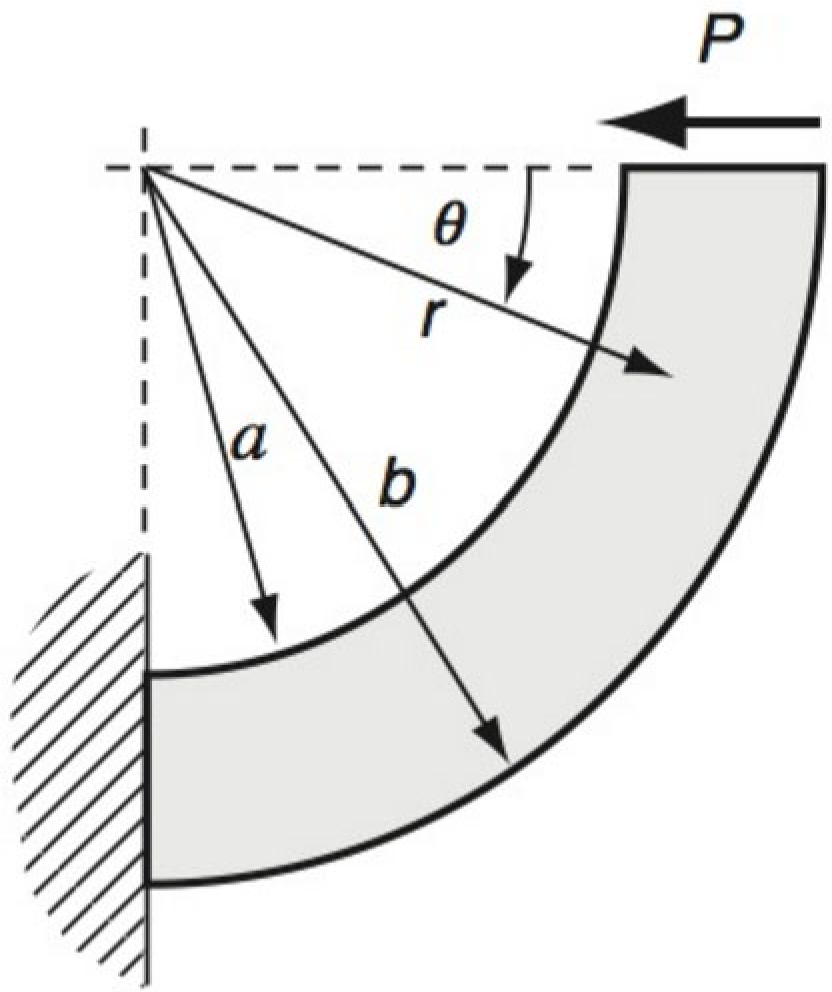
wedge



notch/crack



curved beam



rotating disk