AE731

Theory of Elasticity

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upcoming schedule

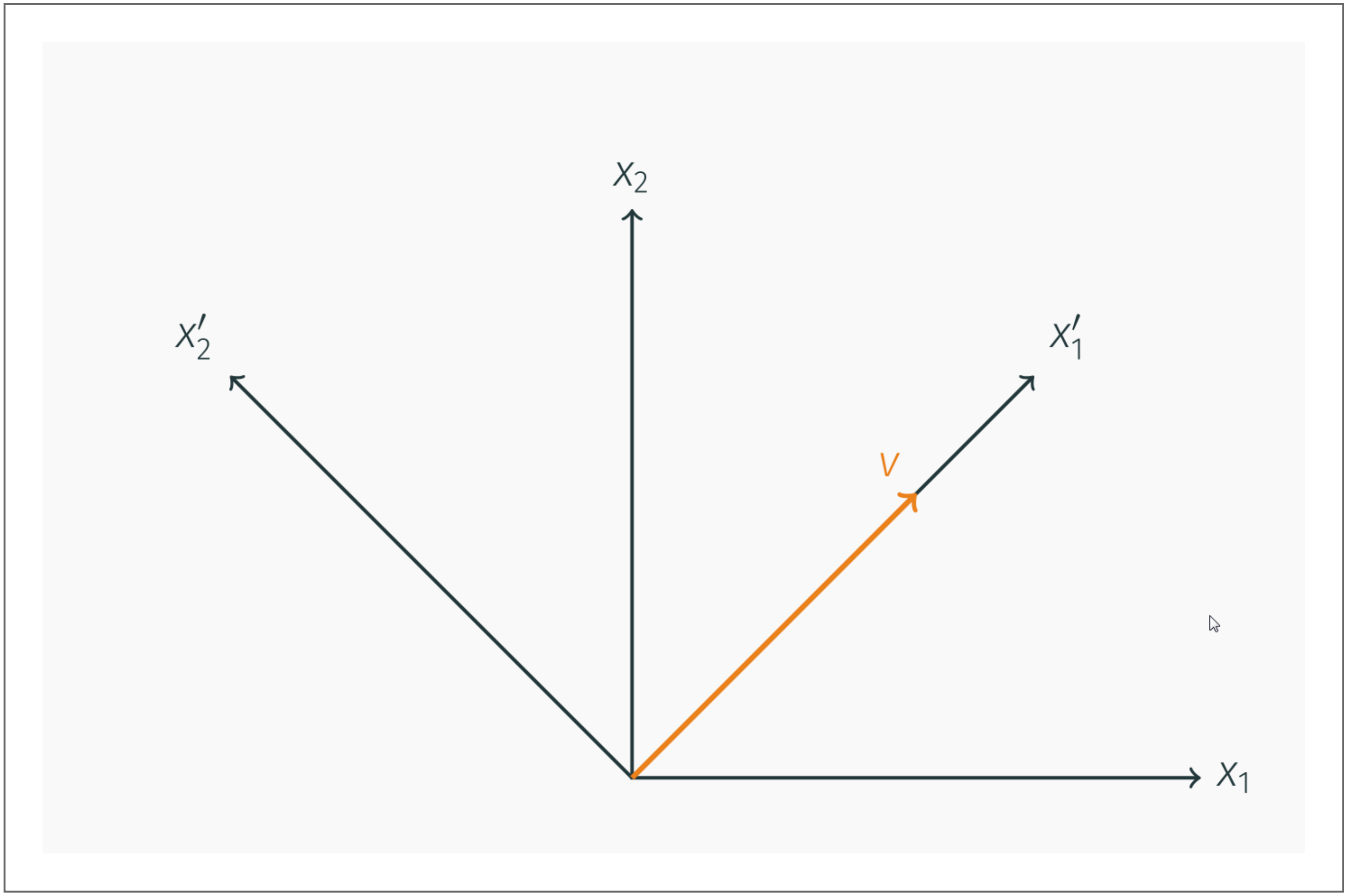
- Aug 26 - Principal Values
- Aug 28 - Tensor Calculus
- Sep 2 - Labor Day
- Sep 4 - Displacement and Strain, Homework 1 Due
- Sep 9 - Strain Transformation

outline

- coordinate transformation
- examples
- principal directions
- examples

coordinate transformation

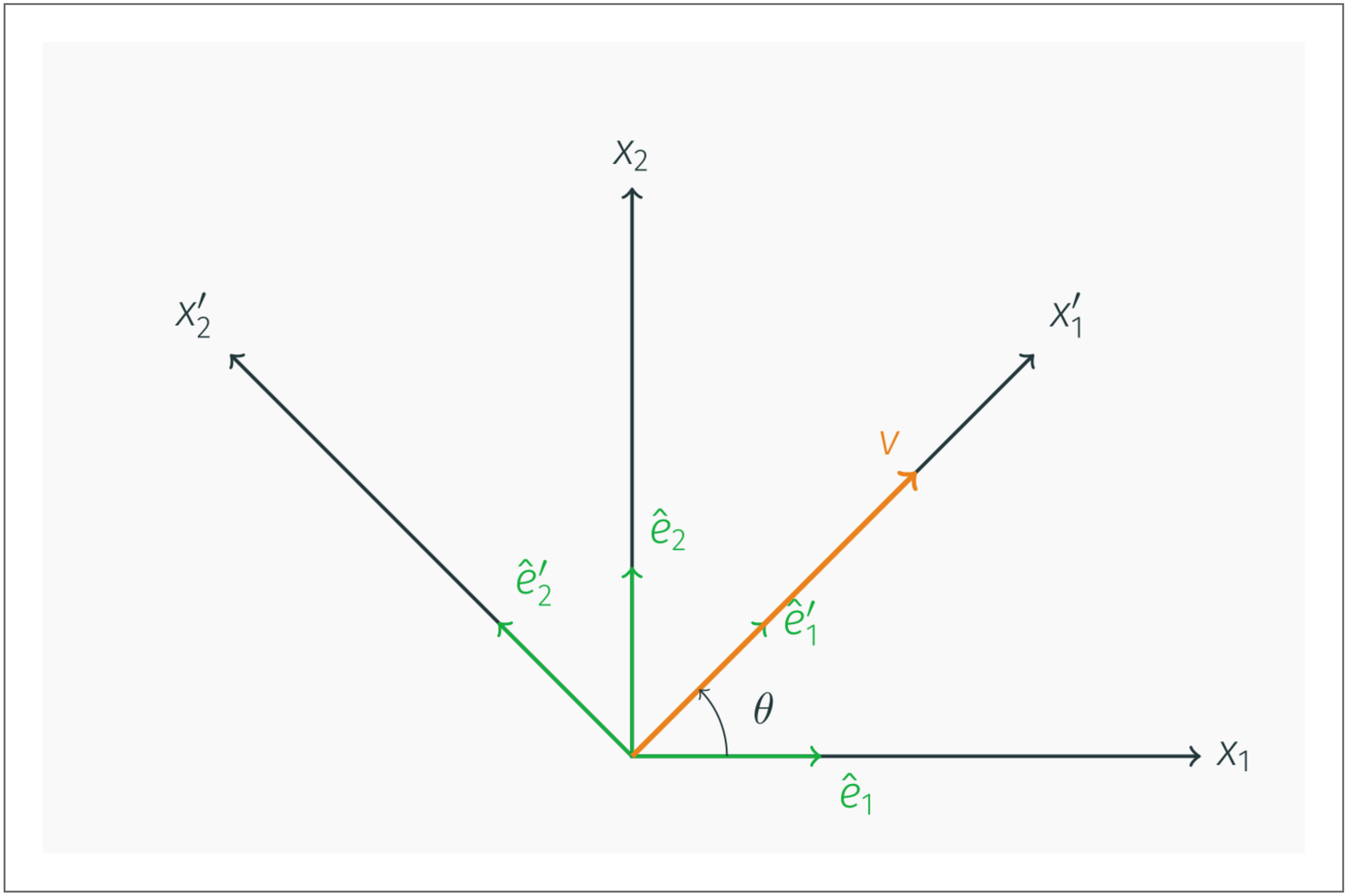
two dimensions



dimensions

- The vector, v, remains fixed, but we transform our coordinate system
- In the new coordinate system, the ${x_2}'$ portion of v is zero.
- To transform the coordinate system, we first define some unit vectors.
- $\hat{e}_1$ is a unit vector in the direction of x_1, while $\hat{e}'_1$ is a unit vector in the direction of ${x_1}'$

two dimensions



two dimensions

- For this example, let us assume $v = \langle 2, 2 \rangle$ and $\theta = 45^\circ$
- We can write the transformed unit vectors, $\hat{e}_1'$ and $\hat{e}_2'$ in terms of $\hat{e}_1$, $\hat{e}_2$ and the angle of rotation, θ.

$$\hat{e}_1' = \langle \hat{e}_1 \cos\theta, \hat{e}_2 \sin\theta \rangle$$
$$\hat{e}_2' = \langle -\hat{e}_1 \sin\theta, \hat{e}_2 \cos\theta \rangle$$

two dimensions

- We can write the vector, v, in terms of the unit vectors describing our axis system
- $v = v_1\hat{e}_1 + v_2\hat{e}_2$
- (note: $\hat{e}_1 = \langle 1, 0 \rangle$ and $\hat{e}_2 = \langle 0, 1 \rangle$)
- v = $\langle 2, 2 \rangle$=2$\langle 1, 0 \rangle$+2$\langle 0, 1 \rangle$

two dimensions

- When expressed in the transformed coordinate system, we refer to $v^{'}$
- $v^{'} = \langle v_1 \cos\theta + v_2 \sin\theta, -v_1 \sin\theta + v_2 \cos\theta \rangle$
- $v' = \langle 2\sqrt{2}, 0 \rangle$
- We can recover the original vector from the transformed coordinates:
- $v = v'_1 \hat{e}'_1 + v'_2 \hat{e}'_2$
- (note: $\hat{e}'_1 = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$ and $\hat{e}'_2 = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$)
- $v = 2\sqrt{2} \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle, 0 \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle = \langle 2, 2 \rangle$

general

- Coordinate transformation can become much more complicated in three dimensions, and with higher-order tensors
- It is convenient to define a general form of the coordinate transformation in index notation
- We define Q_{ij} as the cosine of the angle between the $x_i{}'$ axis and the x_j axis.
- This is also referred to as the “direction cosine” $Q_{ij} = \cos(x_i{}', x_j)$

general

- We can use this form on our 2D transformation example

$$
\begin{aligned}
Q_{ij} &= \cos(x'_i, x_j) \\
&= \begin{bmatrix} \cos(x'_1, x_1) & \cos(x'_1, x_2) \\ \cos(x'_2, x_1) & \cos(x'_2, x_2) \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \cos(90-\theta) \\ \cos(90+\theta) & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}
\end{aligned}
$$

general

- We can transform any-order tensor using Q_{ij}
- Vectors (first-order tensors): $v_i{}' = Q_{ij}v_j$
- Matrices (second-order tensors): $\sigma_{mn}{}' = Q_{mi}Q_{nj}\sigma_{ij}$
- Fourth-order tensors: $C_{ijkl}{}' = Q_{im}Q_{jn}Q_{ko}Q_{lp}C_{mnop}$

general

- We can similarly use Q_{ij} to find tensors in the original coordinate system
- Vectors (first-order tensors): $v_i = Q_{ji}v_j{}'$
- Matrices (second-order tensors): $\sigma_{mn} = Q_{im}Q_{jn}\sigma_{ij}{}'$
- Fourth-order tensors: $C_{ijkl} = Q_{mi}Q_{nj}Q_{ok}Q_{pl}C_{mnop}{}'$

mental/emotional health warning

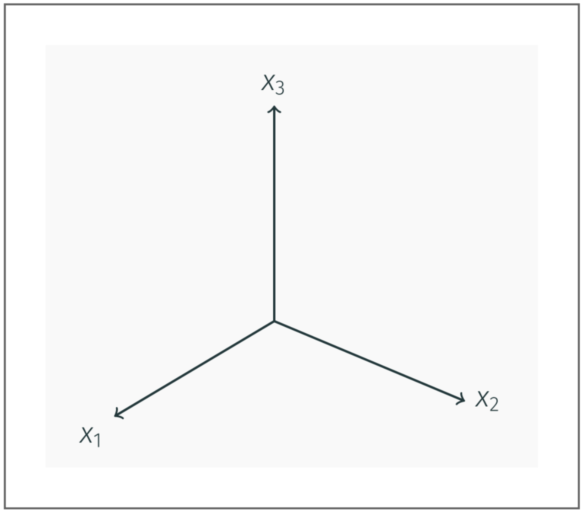
- Some texts flip the definition of Q_{ij}, and then flip their transformation law accordingly
- Any time you use tensor transformation, make sure you are following a consistent set of transformation laws

general

- We can derive some interesting properties of the transformation tensor, Q_{ij}
- We know that $v_i = Q_{ji} v_j^{'}$ and that $v_i^{'} = Q_{ij} v_j$
- If we substitute (changing the appropriate indexes) we find:
- $v_i = Q_{ji} Q_{jk} v_k$
- We can now use the Kronecker Delta to substitute $v_i = \delta_{ik} v_k$ which gives
- $\delta_{ik} v_k = Q_{ji} Q_{jk} v_k$

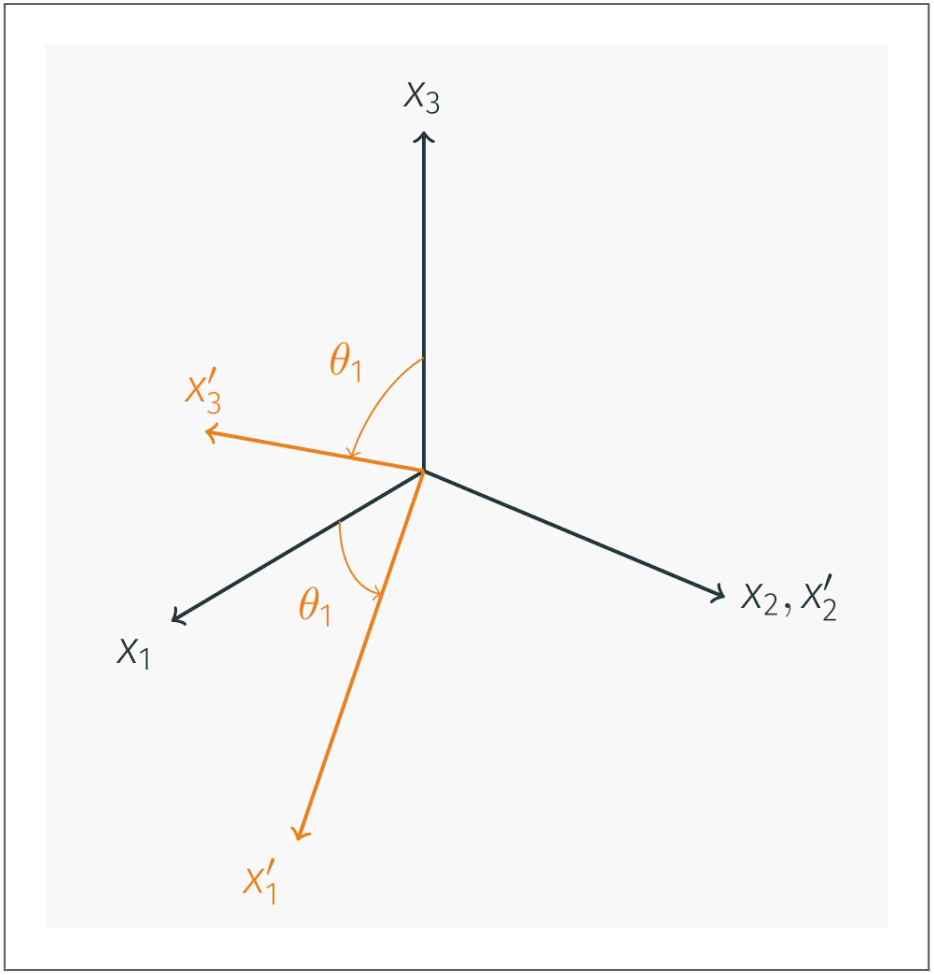
examples

example

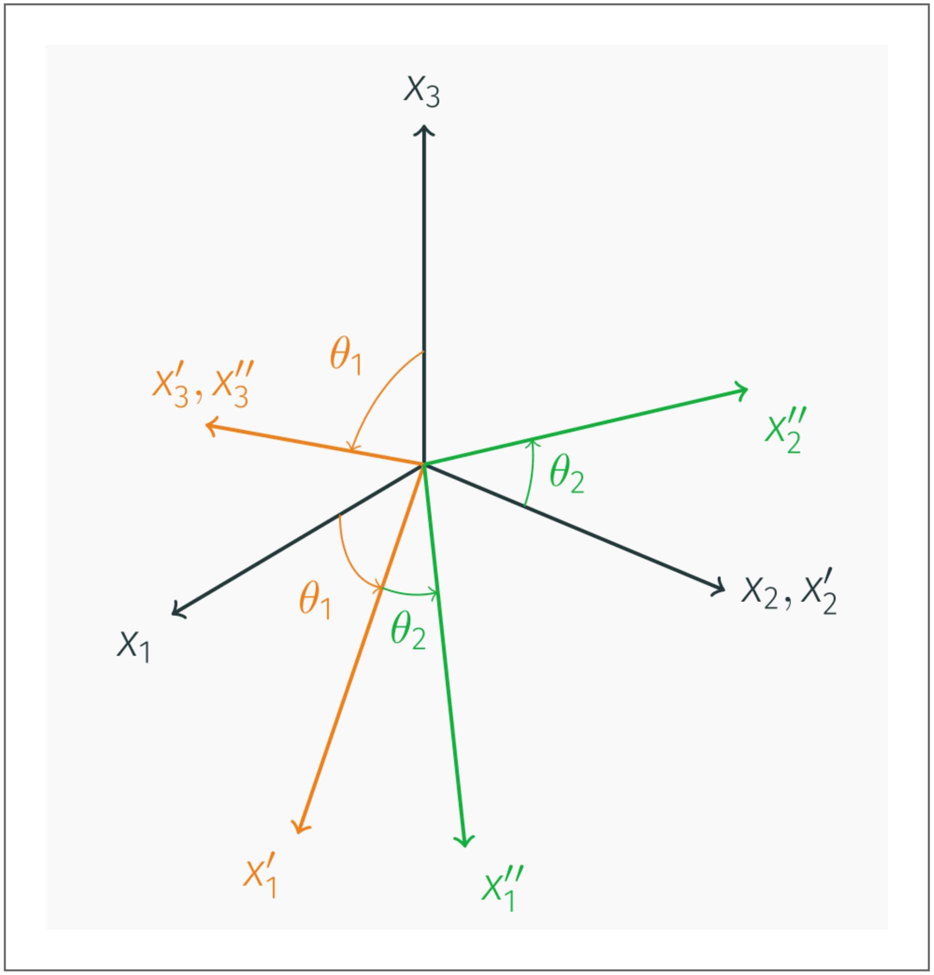


- Find Q_{ij}^1 for rotation of 60° about x_2
- Find Q_{ij}^2 for rotation of 30° about $x_3{}'$
- Find $e_i{}''$ after both rotations

example



example



example

- $Q_{ij}{}^{1} = \cos(x_i{}', x_j)$
- $Q_{ij}{}^{2} = \cos(x_i{}'', x_j{}')$

$$Q^1_{ij} = \begin{bmatrix} \cos 60 & \cos 90 & \cos 150 \\ \cos 90 & \cos 0 & \cos 90 \\ \cos 30 & \cos 90 & \cos 60 \end{bmatrix}$$

$$Q^2_{ij} = \begin{bmatrix} \cos 30 & \cos 60 & \cos 90 \\ \cos 120 & \cos 30 & \cos 90 \\ \cos 90 & \cos 90 & \cos 0 \end{bmatrix}$$

example

- We now use Q_{ij} to find $\hat{e}_i'$ and $\hat{e}_i''$
- First, we need to write $\hat{e}_i$ in a manner more consistent with index notation
- We will indicate axis direction with a superscript, e.g. $\hat{e}_1 = e_i^1$
- $e_i{}' = Q_{ij}{}^1 e_j$
- $e_i{}'' = Q_{ij}{}^2 e_j{}'$
- How do we find $e_i{}''$ in terms of e_i?
- $e_i{}'' = Q_{ij}{}^2 Q_{jk}{}^1 e_k$

principal directions

principal directions

- We defined principal directions earlier $(a_{ij} - \lambda\delta_{ij})n_j = 0$
- λ are the principal values and n_j are the principal directions
- For each eigenvalue there will be a principal direction
- We find the principal direction by substituting the solution for λ back into this equation

example

- Find the principal directions for the earlier principal values example
- Recall $\lambda = 0, 5$, let us say $\lambda_1 = 5$, we find $n_j^{(1)}$ by

$$\begin{bmatrix} 1-\lambda_1 & 2 \\ 2 & 4-\lambda_1 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = 0$$

- This gives

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = 0$$

example

- We proceed to solve the equations using row-reduction, but we find

$$\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = 0$$

- This means we cannot uniquely solve for n_j
- We are only concerned with the direction, magnitude is not important
- Choose $n_2 = 1$, solve for n_1
- $n^{(1)} = \langle \frac{1}{2}, 1 \rangle$

example

- Similarly, for $\lambda_2 = 0$, we find

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = 0$$

- Which, after row-reduction, becomes

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = 0$$

- If we choose $n_2 = 1$, we find $n_1 = -2$
- This gives $n^{(2)} = \langle -2, 1 \rangle$

example

- We can assemble a transformation matrix, Q_{ij}, from the principal directions
- First we need to normalize them to unit vectors
- $||n^{(1)}|| = \sqrt{\frac{5}{4}}$
- $\hat{n}^{(1)} = \frac{2}{\sqrt{5}}\langle\frac{1}{2}, 1\rangle = \langle\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle$
- $||n^{(2)}|| = \sqrt{5}$
- $\hat{n}^{(2)} = \langle\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle$

example

- This gives

$$Q_{ij} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

- And we find $A_{mn}{}' = Q_{mi}Q_{nj}A_{ij}$

$$A'_{ij} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$$

examples

example

- Find principal values, principal directions, and invariants for the tensor

$$c_{ij} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

example

- Characteristic equation simplifies to
- $-\lambda^3 + 14\lambda^2 - 56\lambda + 64 = 0$
- Which has the solutions $\lambda = 2, 4, 8$

example

- To find the principal direction for $\lambda_1 = 8$

$$\begin{bmatrix} 8-8 & 0 & 0 \\ 0 & 3-8 & 1 \\ 0 & 1 & 3-8 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

example

- After row-reduction, we find

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -24 \\ 0 & 1 & -5 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

- This means that $n_3 = 0$ and, as a result, $n_2 = 0$.
- n_1 can be any value, we choose $n_1 = 1$ to give a unit vector.
- $n^{(1)} = \langle 1, 0, 0 \rangle$

example

- To find the principal direction for $\lambda_2 = 4$

$$\begin{bmatrix} 8-4 & 0 & 0 \\ 0 & 3-4 & 1 \\ 0 & 1 & 3-4 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

example

- After row-reduction, we find

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

- This means that $n_1 = 0$
- We also know that $n_2 = n_3$, so we choose $n_2 = n_1 = 1$
- This gives $n^{(2)} = \frac{1}{\sqrt{2}}\langle 0, 1, 1\rangle$ after normalization

example

- To find the principal direction for $\lambda_3 = 2$

$$\begin{bmatrix} 8-2 & 0 & 0 \\ 0 & 3-2 & 1 \\ 0 & 1 & 3-2 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

example

- After row-reduction, we find

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

- This means that $n_1 = 0$
- We also know that $n_2 = -n_3$, so we choose $n_2 = 1$ and $n_1 = -1$
- This gives $n^{(3)} = \frac{1}{\sqrt{2}}\langle 0, 1, -1\rangle$ after normalization

example

- In summary, for c_{ij} we have:
- $\lambda_1 = 8$ and $n^{(1)} = \langle 1, 0, 0 \rangle$
- $\lambda_2 = 4$ and $n^{(2)} = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle$
- $\lambda_3 = 2$ and $n^{(3)} = \frac{1}{\sqrt{2}} \langle 0, 1, -1 \rangle$
- We can assemble $n^{(i)}$ into a transformation tensor

$$Q_{ij} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

example

- What is ${c_{ij}}'$?
- ${c_{ij}}' = Q_{im}Q_{jn}c_{mn}$

$$c'_{ij} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

example

- We can also find the invariants for

$$c_{ij} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

- Recall:

$$\begin{aligned} I_\alpha &= a_{ii} \\ II_\alpha &= \frac{1}{2}(a_{ii}a_{jj} - a_{ij}a_{ij}) \\ III_\alpha &= \det[a_{ij}] \end{aligned}$$

example

- First invariant $I_\alpha = a_{ii} = 8 + 3 + 3 = 14$
- Second invariant

$$II_\alpha = \frac{1}{2}(a_{ii}a_{jj} - a_{ij}a_{ij})$$

$a_{ii}a_{jj} = 14 \times 14$

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + \ldots + a_{33}a_{33}$

$$II_\alpha = \frac{1}{2}(196 - 84) = 56$$

example

- Third invariant $III_\alpha = \det[a_{ij}]$ $III_\alpha = 8 \times (3 \times 3 - 1 \times 1)$=64