AE731

Theory of Elasticity

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upcoming schedule

- Nov 4 - Strain Energy
- Nov 6 - Strain Energy
- Nov 11 - 2D Problems
- Nov 13 - Airy Stress Functions, HW 6 Due

outline

- strain energy
- uniqueness of elasticity problems
- group problems

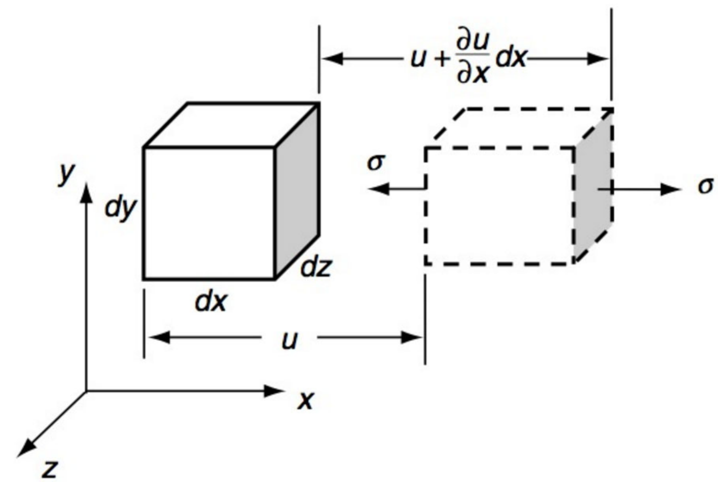
strain energy

strain energy

- Work done by surface and body forces is stored as *strain energy*
- In an elastic body, this is completely recoverable
- In one dimension, this is similar to a linear spring

strain energy

- The strain energy must be equal to the net work done
- Recall “work” is force times displacement (force in direction of displacement)



strain energy

- In uniaxial tension, the net work can be expressed as

$$dU = \int_0^{\sigma_x} \sigma d\left(u + \frac{\partial u}{\partial x}dx\right)dydz - \int_0^{\sigma_x} \sigma dudydz$$

- Or, simplifying

$$dU = \int_0^{\sigma_x} d\left(\frac{\partial u}{\partial x}dx\right)dydz$$

strain energy

- We can use strain-displacement and Hooke's Law to say

$$\frac{\partial u}{\partial x} = \epsilon_x = \frac{\sigma_x}{E}$$

- Substituting this gives

$$dU = \int_0^{\sigma_x} \frac{d\sigma}{E} dxdydz = \frac{\sigma_x^2}{2E} dxdydz$$

strain energy

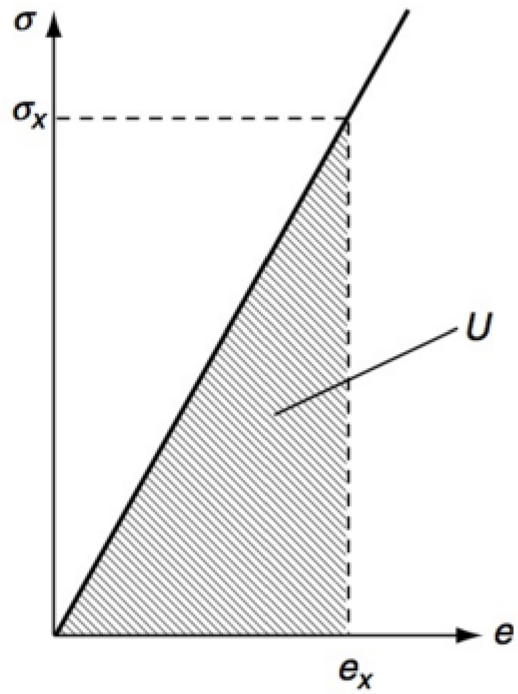
- We define the *strain energy density* as

$$U = \frac{dU}{dxdydz}$$

- In uni-axial tension, this gives

$$U = \frac{\sigma_x^2}{2E} = \frac{E\epsilon_x^2}{2} = \frac{1}{2}\sigma_x\epsilon_x$$

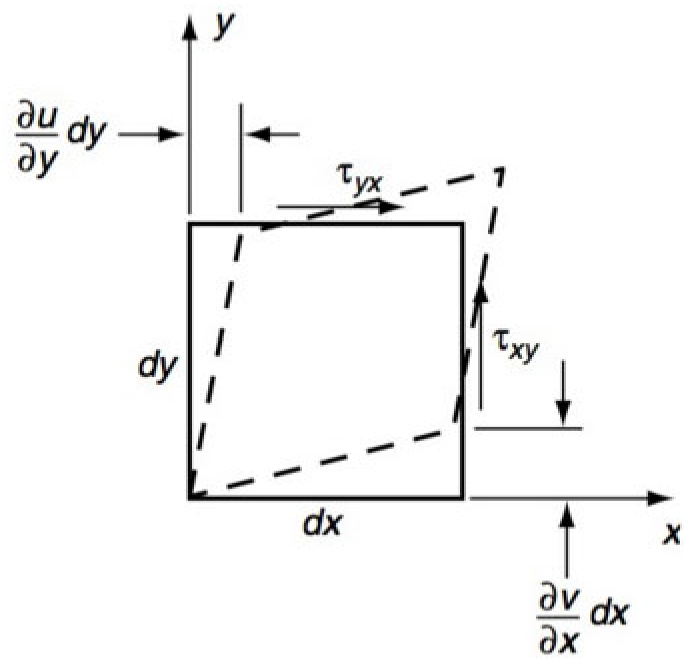
strain energy



- We can also visualize the strain energy graphically as the area under the stress-strain curve

strain energy

- We can also consider the strain energy due to a uniform shear stress



strain energy

- Following the same procedure, we find

$$dU = \frac{1}{2}\tau_{xy}dydz\left(\frac{\partial v}{\partial x}dx\right) + \frac{1}{2}\tau_{xy}dxdz\left(\frac{\partial u}{\partial y}dy\right) = \frac{1}{2}\tau_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)dxdydz$$

- And the strain energy density can be expressed as

$$U = \frac{1}{2}\tau_{xy}\gamma_{xy} = \frac{\tau_{xy}^2}{2\mu} = \frac{\mu\gamma_{xy}^2}{2}$$

strain energy

- Using the conservation of energy, we can add the effects from each of these loadings to find the total strain energy

$$U = \frac{1}{2}\sigma_{ij}\epsilon_{ij}$$

- Note: Although we derived this expression with no body forces, an identical solution is found if they are included

strain energy

- To find the total strain energy in a body, we integrate the strain energy density over the volume

$$U_t = \iiint_V U dx dy dz$$

strain energy

- As we did before for the uniaxial case, we can write the strain energy density in terms of stress or strain only using Hooke's Law

$$U_\epsilon = \frac{1}{2}\lambda\epsilon_{jj}\epsilon_{kk} + \mu\epsilon_{ij}\epsilon_{ij}$$

$$U_\sigma = \frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij} - \frac{\nu}{2E}\sigma_{jj}\sigma_{kk}$$

strain energy

- If we fully expand both versions, we find that each term is squared
- This means the strain energy must be positive

strain energy

- Another interesting feature we note is that

$$\sigma_{ij} = \frac{\partial U_\epsilon}{\partial \epsilon_{ij}}$$

- and

$$\epsilon_{ij} = \frac{\partial U_\sigma}{\partial \sigma_{ij}}$$

- These relationships do not depend on stress-strain relations being linear, and are often used to derive stresses and strains in non-linear materials (*hyperelasticity*)

strain energy

- We can further use this relationship to show that

$$\frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}} = \frac{\partial \sigma_{kl}}{\partial \epsilon_{ij}}$$

$$\frac{\partial \epsilon_{ij}}{\partial \sigma_{kl}} = \frac{\partial \epsilon_{kl}}{\partial \sigma_{ij}}$$

- Going to back the general form of Hooke's Law ($\sigma_{ij} = C_{ijkl}\epsilon_{kl}$), this gives the symmetry condition

$$C_{ijkl} = C_{klij}$$

strain energy

- We can separate the strain energy into two parts, the portion caused by *volumetric* deformation and the portion caused by *distortional* deformation

$$U = U_V + U_D$$

strain energy

- The volumetric portion can be found using the spherical or hydrostatic components of stress and strain

$$U_V = \frac{1}{2}\tilde{\sigma}_{ij}\tilde{\epsilon}_{ij} = \frac{1}{6}\sigma_{jj}\epsilon_{kk}$$

strain energy

- The distortional portion can be found as

$$U_D = \frac{1}{12\mu}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right]$$

- Some failure theories make use of the distortional strain energy

uniqueness of elasticity problems

uniqueness

- In Chapter 5 we never proved if any solution was unique
- Let us assume that there exist two solutions to a given boundary value problem
- The difference of the two solutions is given as

$$\sigma_{ij} = \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}$$

$$\epsilon_{ij} = \epsilon_{ij}^{(1)} - \epsilon_{ij}^{(2)}$$

$$u_i = u_i^{(1)} - u_i^{(2)}$$

uniqueness

- Because both solutions will have the same body force, the difference solution must satisfy the equilibrium equation

$$\sigma_{ij,j} = 0$$

- We also know that the difference must give

$$T_i^n = \sigma_{ij} n_j = 0$$

On the traction boundary and

$$u_i = 0$$

On the displacement boundary

uniqueness

- Using the definition of strain energy, we can write

$$
\begin{aligned}
2\int_V U dV &= \int_V \sigma_{ij}\epsilon_{ij} dV = \int_V \sigma_{ij}(u_{i,j} - \omega_{ij}) dV \\
&= \int_V \sigma_{ij} u_{i,j} = \int_V (\sigma_{ij} u_i)_{,j} dV - \int_V \sigma_{ij,j} u_i dV \\
&= \int_S \sigma_{ij} n_j u_i dS - \int_V \sigma_{ij,j} u_i dV
\end{aligned}
$$

uniqueness

- Note that a symmetric matrix times an antisymmetric matrix =0
- We know that $\sigma_{ij} n_j = 0$ on surfaces where tractions are defined and that $u_i = 0$ on the other surfaces, so the first integral goes to zero
- We also know by equilibrium that $\sigma_{ij,j} = 0$, so the second integral will also be 0

uniqueness

- If the strain energy of the difference between two solutions is zero, then we know that
    - The stress field of the difference is zero
    - The strain field of the difference is zero
    - The displacement field of the difference is zero
- Therefore the two solutions are the same solution, and the solution is unique

group problems

uniaxial tension

- We can establish bounds on physical constants by recalling that the strain energy must always be positive and considering certain states of stress

- Uniaxial tension gives the stress state

$$\sigma_{ij} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Find the strain energy and use it to place bounds on the modulus of Elasticity, E

simple shear

- If we consider uniform simple shear

$$\sigma_{ij} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Find the strain energy and use it to place bounds on Poisson's Ratio

hydrostatic pressure

- We can also consider hydrostatic pressure

$$\sigma_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

- Find the strain energy and use it to place bounds on the hydrostatic pressure