Theory of Elasticity

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upcoming schedule

- Nov 9 - Polar Coordinates
- Nov 11 - Micromechanics Project Presentation
- Nov 12 - Homework 7 Due
- Nov 16 - Airy Stress Review
- Nov 18 - Complex Methods

outline

- polar coordinates
- examples

polar coordinates

strain-displacement

- Reduced strain-displacement:

$$
\begin{aligned}
\epsilon_r &= \frac{\partial u_r}{\partial r}, \epsilon_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right), \epsilon_z = \frac{\partial u_z}{\partial z} \\
\epsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \\
\epsilon_{\theta z} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \\
\epsilon_{zr} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)
\end{aligned}
$$

strain-displacement

- Which becomes

$$
\begin{aligned}
\epsilon_r &= \frac{\partial u_r}{\partial r} \\
\epsilon_\theta &= \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) \\
\epsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)
\end{aligned}
$$

integration

- When we change variables in integration, we also need to account for the proper change in dV

$$dV = dxdydz \neq drd\theta dz$$

- We can find the correct dV by calculating the Jacobian

jacobian

$$dV = dxdydz = |\frac{\partial(x, y, z)}{\partial(r, \theta, z)}| drd\theta dz$$

$$dV = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} drd\theta dz = rdrd\theta dz$$

hooke's law

- The tensor equation for Hooke's Law is valid in polar/cylindrical/spherical coordinates too
- We only need special equations when differentiating or integrating

$$
\begin{aligned}
\sigma_{ij} &= \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} - (3\lambda + 2\mu)\alpha \Delta T \delta_{ij} \\
\epsilon_{ij} &= \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha \Delta T \delta_{ij}
\end{aligned}
$$

equilibrium

- We have already found the equilibrium equations in polar coordinates, they are

$$
\begin{aligned}
\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r}(\sigma_r - \sigma_\theta) + F_r &= 0 \\
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2}{r}\tau_{r\theta} + F_\theta &= 0
\end{aligned}
$$

equilibrium

- The equilibrium equations can be written in terms of displacement (Navier equations)
- These are only useful when using a displacement formulation, but we are using stress functions
- Instead we need the Beltrami-Mitchell compatibility equations

compatibility

- Substituting stress-strain relations into the compatibility equations gives

$$\nabla^2(\sigma_r + \sigma_\theta) = -\frac{1}{1-\nu}\left(\frac{\partial F_r}{\partial r} + \frac{F_r}{r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}\right) \quad \text{(Plane Strain)}$$

$$\nabla^2(\sigma_r + \sigma_\theta) = -(1+\nu)\left(\frac{\partial F_r}{\partial r} + \frac{F_r}{r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}\right) \quad \text{(Plane Stress)}$$

airy stress functions

- When the body forces are zero, we find

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$
$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2}$$
$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

airy stress functions

- When body forces are zero, we find the following biharmonic equation for the Beltrami-Mitchell equations

$$\nabla^4\phi = 0$$

- Where the Laplacian is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

polar coordinates

- Recall that an Airy Stress function must satisfy the Beltrami-Mitchell compatibility equations

$$\nabla^4\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \phi = 0$$

- One method which gives several useful solutions assumes that the Airy Stress function has the form $\phi(r,\theta) = f(r)e^{b\theta}$

polar coordinates

- Substituting this into the compatibility equations (and canceling the common $e^{b\theta}$) term gives

$$f'''' + \frac{2}{r}f''' - \frac{1-2b^2}{r^2}f'' + \frac{1-2b^2}{r^3}f' + \frac{b^2(4+b^2)}{r^4}f = 0$$

polar coordinates

- To solve this, we perform a change of variables, letting `r = e^\xi$`, which gives

$$f'''' - 4f''' + (4 + 2b^2)f'' - 4b^2 f' + b^2(4 + b^2)f = 0$$

- We now consider f to have the form $f = e^{a\xi}$ which generates the characteristic equation

$$(a^2 + b^2)(a^2 - 4a + 4 + b^2) = 0$$

polar coordinates

- This has solutions

$$\begin{aligned} a &= \pm ib, \pm 2ib \\ \text{OR} & \\ b &= \pm ia, \pm i(a - 2) \end{aligned}$$

polar coordinates

- If we consider only solutions which are periodic in θ, we find

$$
\begin{aligned}
\phi = {} & a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r \\
& + (a_4 + a_5 \log r + a_6 r^2 + a_7 r^2 \log r)\theta \\
& + \left(a_{11} r + a_{12} r \log r + \frac{a_{13}}{r} + a_{14} r^3 + a_{15} r\theta + a_{16} r\theta \log r \right) \cos\theta \\
& + \left(b_{11} r + b_{12} r \log r + \frac{b_{13}}{r} + b_{14} r^3 + b_{15} r\theta + b_{16} r\theta \log r \right) \sin\theta \\
& + \sum_{n=2}^{\infty} (a_{n1} r^n + a_{n2} r^{2+n} + a_{n3} r^{-n} + a_{n4} r^{2-n}) \cos n\theta \\
& + \sum_{n=2}^{\infty} (b_{n1} r^n + b_{n2} r^{2+n} + a_{n3} r^{-n} + b_{n4} r^{2-n}) \sin n\theta
\end{aligned}
$$

polar coordinates

- For axisymmetric problems, all field quantities are independent of θ
- This reduces the general solution to

$$\phi = a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r$$

polar coordinates

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$\log r$	$1/r^2$	0	$-1/r^2$
θ	0	$1/r^2$	0
$r^2 \log r$	$2\log r + 1$	0	$2\log r + 3$
$r^2\theta$	2θ	-1	2θ
$r^3 \cos\theta$	$2r\cos\theta$	$2r\sin\theta$	$6r\cos\theta$
$r^3 \sin\theta$	$2r\sin\theta$	$-2r\cos\theta$	$6r\sin\theta$
$r\theta\sin\theta$	$2\cos\theta/r$	0	0
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\log r\cos\theta$	$\cos\theta/r$	$\sin\theta/r$	$\cos\theta/r$
$r\log r\sin\theta$	$\sin\theta/r$	$-\cos\theta/r$	$\sin\theta/r$
$\cos\theta/r$	$-2\cos\theta/r^3$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$2\sin\theta/r^3$

Figure 1: table with pre-calculated airy stress terms in polar coordinates

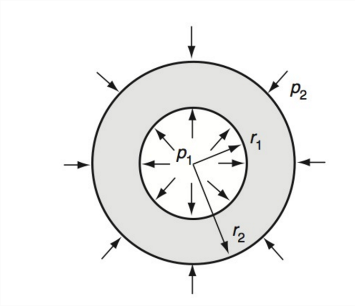
polar coordinates

$r^4\cos 2\theta$	0	$6r^2\sin 2\theta$	$12r^2\cos 2\theta$
$r^4\sin 2\theta$	0	$-6r^2\cos 2\theta$	$12r^2\sin 2\theta$
$r^2\cos 2\theta$	$-2\cos 2\theta$	$2\sin 2\theta$	$2\cos 2\theta$
$r^2\sin 2\theta$	$-2\sin 2\theta$	$-2\cos 2\theta$	$2\sin 2\theta$
$\cos 2\theta$	$-4\cos 2\theta/r^2$	$-2\sin 2\theta/r^2$	0
$\sin 2\theta$	$-4\sin 2\theta/r^2$	$2\cos 2\theta/r^2$	0
$\cos 2\theta/r^2$	$-6\cos 2\theta/r^4$	$-6\sin 2\theta/r^4$	$6\cos 2\theta/r^4$
$\sin 2\theta/r^2$	$-6\sin 2\theta/r^4$	$6\cos 2\theta/r^4$	$6\sin 2\theta/r^4$
$r^n\cos n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$	$n(n-1)r^{n-2}\cos n\theta$
$r^n\sin n\theta$	$-n(n-1)r^{n-2}\sin n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$
$r^{n+2}\cos n\theta$	$-(n+1)(n-2)r^n\cos n\theta$	$(n+1)nr^n\sin n\theta$	$(n+2)(n+1)r^n\cos n\theta$
$r^{n+2}\sin n\theta$	$-(n+1)(n-2)r^n\sin n\theta$	$-(n+1)nr^n\cos n\theta$	$(n+2)(n+1)r^n\sin n\theta$
$\cos n\theta/r^n$	$-(n+1)n\cos n\theta/r^{n+2}$	$-(n+1)n\sin n\theta/r^{n+2}$	$(n+1)n\cos n\theta/r^{n+2}$
$\sin n\theta/r^n$	$-(n+1)n\sin n\theta/r^{n+2}$	$(n+1)n\cos n\theta/r^{n+2}$	$(n+1)n\sin n\theta/r^{n+2}$
$\cos n\theta/r^{n-2}$	$-(n+2)(n-1)\cos n\theta/r^n$	$-n(n-1)\sin n\theta/r^n$	$(n-1)(n-2)\cos n\theta/r^n$
$\sin n\theta/r^{n-2}$	$-(n+2)(n-1)\sin n\theta/r^n$	$n(n-1)\cos n\theta/r^n$	$(n-1)(n-2)\sin n\theta/r^n$

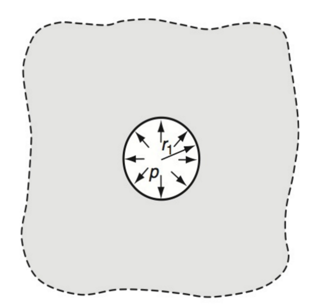
Figure 2: continued table of polar coordinate airy stress terms

examples

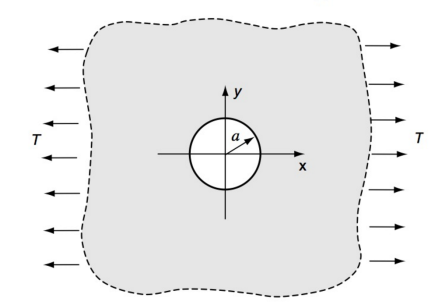
tube under uniform pressure



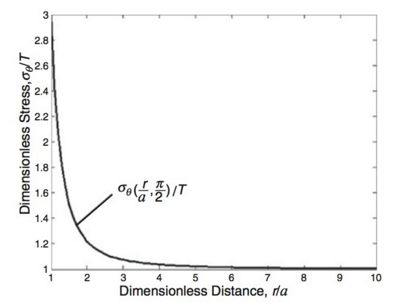
pressurized hole



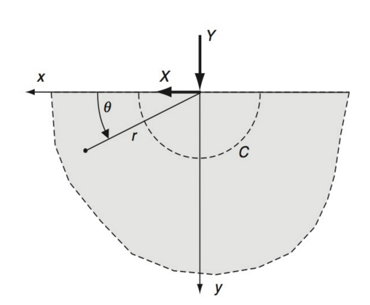
stress-free hole in tension



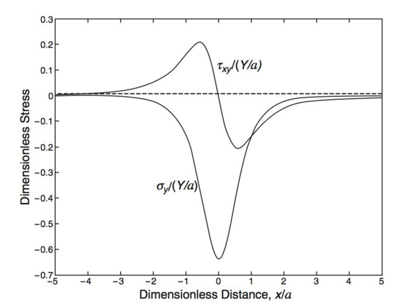
stress-free hole in tension



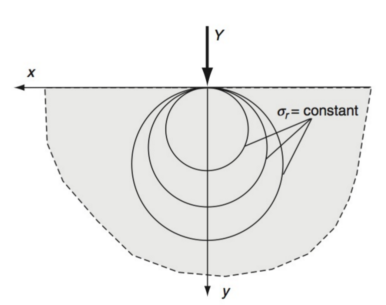
concentrated force



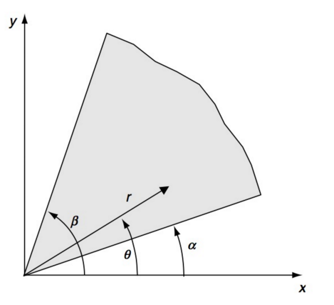
concentrated force



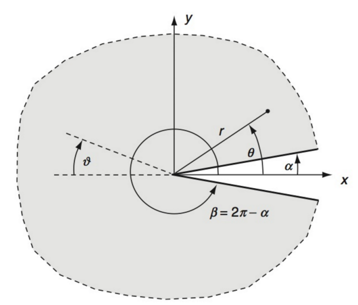
concentrated force



wedge



notch/crack



curved beam

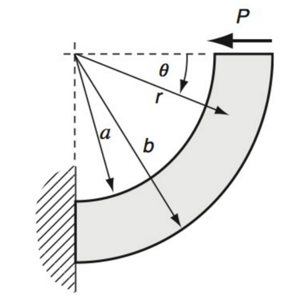


Figure 3: solution for a curved beam

rotating disk

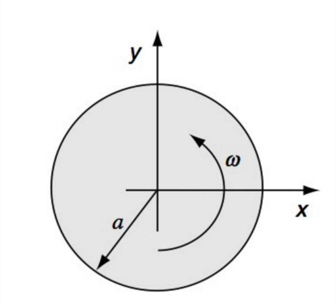


Figure 4: rotating disk problem