Name:

Homework 6
Due 13 Nov 2019

1. Use Hooke's Law to explicitly develop the strain energy relationships in terms of stress (U_σ) and strain (U_ϵ) only from the general relationship (Equation 1).

$$U = \frac{1}{2}\sigma_{ij}\epsilon_{ij} \tag{1}$$

   - From Hooke's Law we know that

$$\begin{aligned}\sigma_{ij} &= \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}\\ \epsilon_{ij} &= \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}\end{aligned}$$

   - Substituting into Equation 1 gives

$$\begin{aligned}U_\epsilon &= \frac{1}{2}\epsilon_{ij}(\lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij})\\ U_\sigma &= \frac{1}{2}\sigma_{ij}\left(\frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}\right)\end{aligned}$$

   - Distributing the multiplication, and recognizing that $a_{ij}\delta_{ij} = a_{ii}$, we find

$$\begin{aligned}U_\epsilon &= \frac{1}{2}\lambda\epsilon_{ii}\epsilon_{jj} + \mu\epsilon_{ij}\epsilon_{ij}\\ U_\sigma &= \frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij} - \frac{\nu}{2E}\sigma_{ii}\sigma_{jj}\end{aligned}$$

2. Use the relationships in Equation 2 to prove the symmetry relationship in Hooke's Law, $C_{ijkl} = C_{klij}$.

$$\sigma_{ij} = \frac{\partial U_\epsilon}{\partial \epsilon}, \epsilon_{ij} = \frac{\partial U_\sigma}{\partial \sigma} \tag{2}$$

   - Noting that $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$, we can differentiate to find C_{ijkl}

$$\frac{\partial \sigma_{ij}}{\epsilon_{kl}} = C_{ijkl}$$

   - Similarly, we can change the indexes to find C_{klij}

$$\frac{\partial \sigma_{kl}}{\epsilon_{ij}} = C_{klij}$$

- From Equation 2, we know that

$$\sigma_{ij} = \frac{\partial U_\epsilon}{\partial \epsilon}$$
$$\frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}} = \frac{\partial^2 U_\epsilon}{\partial \epsilon_{ij} \epsilon_{kl}}$$

- And similarly

$$\sigma_{kl} = \frac{\partial U_\epsilon}{\partial \epsilon}$$
$$\frac{\partial \sigma_{kl}}{\partial \epsilon_{ij}} = \frac{\partial^2 U_\epsilon}{\partial \epsilon_{kl} \epsilon_{ij}}$$

- For a continuous function U_ϵ, we know

$$\frac{\partial^2 U_\epsilon}{\partial \epsilon_{kl} \epsilon_{ij}} = \frac{\partial^2 U_\epsilon}{\partial \epsilon_{ij} \epsilon_{kl}}$$

- Therefore

$$\frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}} = \frac{\partial \sigma_{kl}}{\partial \epsilon_{ij}}$$
$$C_{ijkl} = C_{klij}$$

3. The stress field for a beam of length $2L$ (in the x-direction) and rectangular cross-section of depth $2c$ (in the y-direction) under applied bending moments M is

$$\sigma_x = -\frac{3M}{2c^3} y$$
$$\sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

Find the strain energy density and the total strain energy for a unit thickness in the z-direction.

- In terms of stress only, we know that the strain energy density is

$$U = U_\sigma = \frac{1+\nu}{2E} \sigma_{ij} \sigma_{ij} - \frac{\nu}{2E} \sigma_{ii} \sigma_{jj}$$

- Substituting the state of stress in this problem gives

$$U = \frac{9M^2 y^2}{4c^6 E}$$

- We can integrate over the full volume to find U_T

$$\begin{aligned}
U_T &= \int_V U dV \\
&= \int_0^1 \int_{-L}^{L} \int_{-c}^{c} \frac{9M^2y^2}{4c^6E} dydxdz \\
&= \int_0^1 \int_{-L}^{L} \frac{3M^2}{4c^3E} dxdz \\
&= \int_0^1 \frac{3M^2L}{2c^3E} dz \\
&= \frac{3M^2L}{2c^3E}
\end{aligned}$$

- We may also write in terms of inertia

$$\begin{aligned}
I &= \frac{bh^3}{12} \\
&= \frac{(1)(2c)^3}{12} \\
&= \frac{8c^3}{12} \\
&= \frac{2c^3}{3}
\end{aligned}$$

- Which gives

$$U_T = \frac{M^2L}{EI}$$

4. The stress field for a rod of circular cross-section is given by

$$\begin{aligned}
\sigma_x &= \sigma_y = \sigma_z = \tau_{xy} = 0 \\
\tau_{xz} &= -\mu\alpha y \\
\tau_{yz} &= \mu\alpha x
\end{aligned}$$

Find the strain energy density and total strain energy for a rod of radius R and length L, where α is a constant and the rod axis lies along the z-axis.

- Once again we find the strain energy density in terms of stress

$$U = U_\sigma = \frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij} - \frac{\nu}{2E}\sigma_{ii}\sigma_{jj}$$

- Which gives

$$\begin{aligned}
U &= \frac{1+\nu}{2E}(\mu^2\alpha^2y^2 + \mu^2\alpha^2x^2) \\
U &= \frac{1+\nu}{2E}\mu^2\alpha^2(x^2+y^2) \\
U &= \frac{1+\nu}{2E}\mu^2\alpha^2r^2
\end{aligned}$$

- To maintain consistency, we recall that $\mu = \frac{E}{2(1+\nu)}$

$$U = \frac{E}{8(1+\nu)}\alpha^2 r^2$$

- We integrate over the volume to find U_T, noting that in cylindrical coordinates, $dV = rdrd\theta dz$

$$\begin{aligned}
U_T &= \int_V U dV \\
&= \int_0^L \int_0^{2\pi} \int_0^R \frac{E}{8(1+\nu)}\alpha^2 r^2 rdrd\theta dz \\
&= \int_0^L \int_0^{2\pi} \int_0^R \frac{E}{8(1+\nu)}\alpha^2 r^3 drd\theta dz \\
&= \int_0^L \int_0^{2\pi} \frac{E}{32(1+\nu)}\alpha^2 R^4 d\theta dz \\
&= \int_0^L \frac{E\pi}{16(1+\nu)}\alpha^2 R^4 dz \\
&= \frac{\alpha^2 R^4 L E\pi}{16(1+\nu)}
\end{aligned}$$

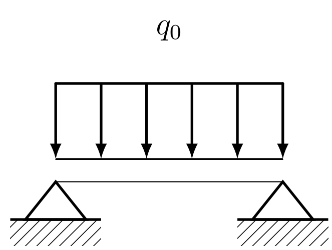

Figure 1: Beam for Problem 5

5. Use the Ritz method to approximate the solution for a simply supported Euler-Bernoulli beam of length L under a uniform load q_0. The approximation will take the form

$$w = w_0 + \sum_{j=1}^{N} c_j w_j \tag{3}$$

Compare two trial solutions, one as a polynomial

$$w_j = x^j (L - x) \tag{4}$$

And the other as a trigonometric function

$$w_j = \sin \frac{j\pi x}{L} \tag{5}$$

For each function, plot the beam deflection (w) for $N = 1$, $N = 2$ and $N = 3$. Compare to the exact solution

$$w = \frac{q_0 x}{24EI}(L^3 + x^3 - 2Lx^2) \tag{6}$$

- Note that the essential boundary conditions are

$$w(0) = 0$$
$$w(L) = 0$$

- Both given trial functions satisfy the essential boundary conditions when $w_0 = 0$, so we are left to find c_j using the total potential function
- The potential is given by

$$\Pi = U_T - W$$

- The total strain energy for an Euler-Bernoulli beam can be readily found
- Under Euler-Bernoulli assumptions, the only stress present is σ_{11} and

$$\sigma_{11} = \frac{My}{I} = Ey\frac{d^2w}{dx^2}$$

- We use the stress form of the strain energy density to find

$$U = \frac{\sigma_{11}^2}{2E} = \frac{E}{2}\left(\frac{d^2w}{dx^2}\right)^2 y^2$$

- We can find U_T as

$$\begin{aligned} U_T &= \int_0^L \left[\iint_A \frac{E}{2}\left(\frac{d^2w}{dx^2}\right) y^2 dA\right] dx \\ &= \int_0^L \frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 dx \end{aligned}$$

- And W as

$$W = \int_0^L q_0 w dx$$

- Thus the potential is

$$\Pi = \int_0^L \left(\frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 - q_0 w\right) dx$$

- We now consider six cases for the displacement function w, $N = 1, 2, 3$ for the polynomial (Equation 4) and sinusoidal (Equation 5) functions **Polynomial N=1**
- For $N = 1$ we have

$$w = c_1 x(x - L)$$

- And
$$\frac{d^2w}{dx^2} = 2c_1$$
- Which gives the potential
$$\Pi = 2EILc_1^2 - \frac{L^3c_1q_0}{6}$$
- We solve for the coefficient c_1 by letting $\frac{\partial\Pi}{\partial c_1} = 0$ to find
$$c_1 = \frac{L^2q_0}{24EI}$$

Polynomial N=2

- For $N = 2$ we have
$$w = c_1x(x - L) + c_2x^2(x - L)$$
- And
$$\frac{d^2w}{dx^2} = 2c_1 + 6c_2x - 2c_2L$$
- Which gives the potential
$$\Pi = \frac{L^4c_2q_0}{4} + L^3(6EIc_2^2 - \frac{Lc_2q_0}{3} + \frac{c_1q_0}{3}) + L^2(-6EILc_2^2 + 6EIc_1c_2 - \frac{Lc_1q_0}{2}) + L(2EIL^2c_2^2 - 4EILc_1c_2$$
- We solve for the coefficients c_1, c_2 by letting $\frac{\partial\Pi}{\partial c_i} = 0$ to find
$$c_1 = \frac{L^2q_0}{24EI}$$
$$c_2 = 0$$

Polynomial N=3

- For $N = 3$ we have
$$w = c_1x(x - L) + c_2x^2(x - L) + c_3x^3(x - L)$$
- And
$$\frac{d^2w}{dx^2} = 2c_1 + 6c_2x - 2c_2L + 12c_3x^2 - 6c_3xL$$
- Which gives the potential
$$\Pi = \frac{72EIc_3^2 + c_3q_0}{5}L^5 + (18EI(c_2c_3 - Lc_3^2) - \frac{Lc_3q_0 - c_2q_0}{4})L^4 + (6EIL^2c_3^2 - 20EILc_2c_3 + 8EIc_1c_3 + 6$$

- We solve for the coefficients c_1, c_2, c_3 by letting $\frac{\partial \Pi}{\partial c_i} = 0$ to find

$$c_1 = \frac{L^2 q_0}{24EI}$$
$$c_2 = \frac{L q_0}{24EI}$$
$$c_3 = \frac{-q_0}{24EI}$$

Sinusoid N=1

- For the sinusoidal function, when $N = 1$ we have

$$w = c_1 \sin \frac{\pi x}{L}$$

- And

$$\frac{d^2 w}{dx^2} = -c_1 \frac{\pi^2}{L^2} \sin \frac{\pi x}{L}$$

- Which gives the potential

$$\Pi = \frac{\pi^4 EI c_1^2}{L^3} - \frac{c_1 q_0 L}{\pi} - \frac{c_1 q_0 L}{\pi}$$

- We solve for the coefficient c_1 by letting $\frac{\partial \Pi}{\partial c_i} = 0$ to find

$$c_1 = \frac{4L^4 q_0}{\pi^5 EI}$$

Sinusoid N=2

- For the sinusoidal function, when $N = 2$ we have

$$w = c_1 \sin \frac{\pi x}{L} + c_2 \sin \frac{2\pi x}{L}$$

- And

$$\frac{d^2 w}{dx^2} = -c_1 \frac{\pi^2}{L^2} \sin \frac{\pi x}{L} - c_2 \frac{4\pi^2}{L^2} \sin \frac{2\pi x}{L}$$

- Which gives the potential

$$\Pi = \frac{\pi^4 EI (L c_1^2 + 16 L c_2^2)}{4L^4} - \frac{q_0}{2\pi}(2Lc_1 + Lc_2) - \frac{q_0}{2\pi}(2Lc_1 - Lc_2)$$

- We solve for the coefficients c_1, c_2 by letting $\frac{\partial \Pi}{\partial c_i} = 0$ to find

$$c_1 = \frac{4L^4 q_0}{\pi^5 EI}$$
$$c_2 = 0$$

Sinusoid N=3

- For the sinusoidal function, when $N = 3$ we have

$$w = c_1 \sin \frac{\pi x}{L} + c_2 \sin \frac{2\pi x}{L} + c_3 \sin \frac{3\pi x}{L}$$

- And

$$\frac{d^2w}{dx^2} = -c_1 \frac{\pi^2}{L^2} \sin \frac{\pi x}{L} - c_2 \frac{4\pi^2}{L^2} \sin \frac{2\pi x}{L} - c_3 \frac{9\pi^2}{L^2} \sin \frac{3\pi x}{L}$$

- Which gives the potential

$$\Pi = (Lc_1^2 + 16Lc_2^2 + 81Lc_3^2) \frac{\pi^4 EI}{4L^4} - \frac{q_0}{6\pi}(6Lc_1 + 3Lc_2 + 2Lc_3) - \frac{q_0}{6\pi}(6Lc_1 - 3Lc_2 + 2Lc_3)$$

- We solve for the coefficients c_1, c_2, c_3 by letting $\frac{\partial \Pi}{\partial c_i} = 0$ to find

$$c_1 = \frac{4L^4 q_0}{\pi^5 EI}$$
$$c_2 = 0$$
$$c_3 = \frac{4L^4 q_0}{243\pi^5 EI}$$

Exact solution

- We can compare each of these sets of solutions to the exact solution

$$w = \frac{q_0 x}{24EI}(L^3 + x^3 - 2Lx^2)$$

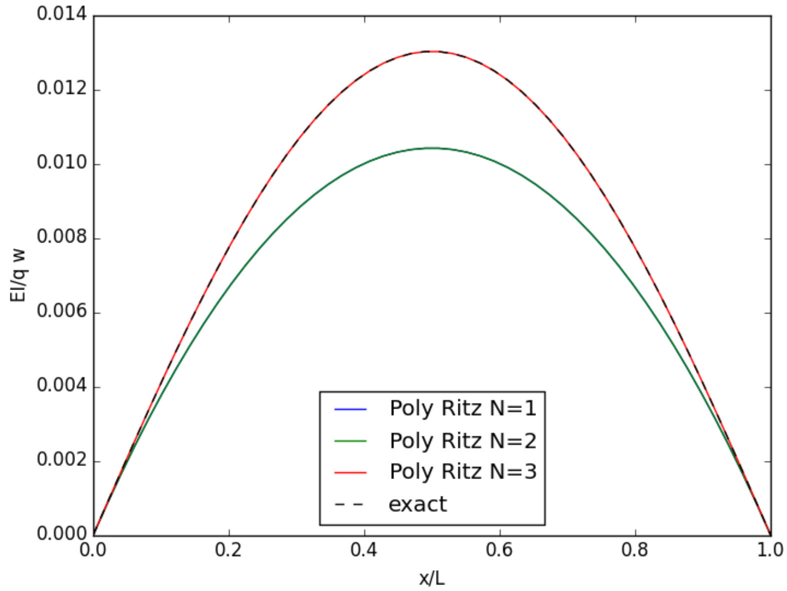

Figure 2: Note exact solution is recovered for N=3

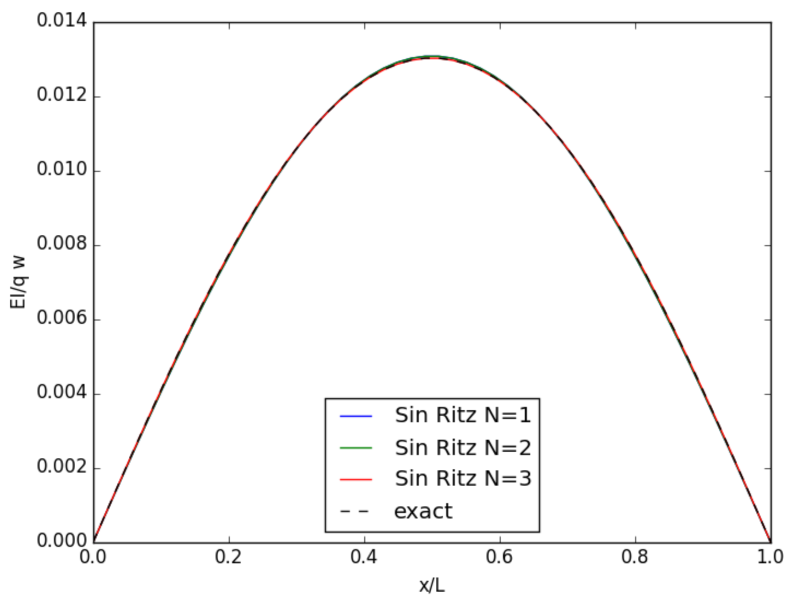

Figure 3: Exact solution is never recovered, but even at N=1 the solution is nearly exact