AE731

Theory of Elasticity

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upcoming schedule

- Sep 21 - Equilibrium Equations
- Sep 23 - Material Characterization
- Sep 24 - Homework 3 Due
- Sep 28 - Thermoelasticity
- Sep 30 - Boundary Conditions

outline

- other stress definitions
- equilibrium equations
- spherical and cylindrical coordinates

other stress definitions

spherical and deviatoric stress

- The spherical and deviatoric stress definitions are identical to the analogous strain definitions
- Spherical stress:

$$\tilde{\sigma}_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij}$$

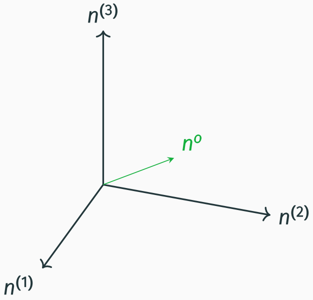
- Deviatoric stress:

$$\hat{\sigma}_{ij} = \sigma_{ij} - \tilde{\sigma}_{ij}$$

failure theories

- Many failure theories rely on some form of combined stress
- One measure is known as the *octahedral stress*
- We define a special plane whose normal forms the same angle of intersection with the three principal directions
- This plane is known as the *octahedral plane*

octahedral stress



octahedral stress

- In the principal direction we know that

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

- The normal vector for the octahedral plane in this system is

$$n^o = \frac{1}{\sqrt{3}}\langle 1, 1, 1 \rangle$$

octahedral stress

- And the octahedral normal stress can be found by

$$\begin{aligned} \sigma_{oct} &= t_i n_i \\ &= \sigma_{ij} n_j n_i \\ &= \frac{1}{3}\sigma_{kk} \end{aligned}$$

octahedral stress

- We can also find the shear stress in the octahedral plane

$$
\begin{aligned}
S^2 &= t_i t_i - N^2 \\
&= \sigma_{ij} n_j \sigma_{ik} n_k - N^2 \\
&= \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 - N^2
\end{aligned}
$$

octahedral stress

- We can simplify this to

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

- Or in terms of invariants

$$\tau_{oct} = \frac{1}{3}\sqrt{2I_1^2 - 6I_2}$$

von mises stress

- Another common stress is known as the Von Mises stress
- Von Mises stress is related to the *distortional strain energy*
- Sometimes the Von Mises stress is referred to as the effective stress

$$\sigma_e = \sigma_{VM} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

large deformation

- The stress tensor we have developed is known as the Cauchy stress tensor
- The Cauchy stress tensor is expressed in the deformed coordinate system
- This is appropriate for small deformation problems, where the un-deformed and deformed systems are nearly identical
- For large deformation problems, we may wish to define stress in terms of the un-deformed coordinate system

large deformation

- Lagrangian stress is defined as

$$\sigma^L_{pi} = \frac{\rho^0}{\rho}\sigma_{ji}\frac{\partial x^0_p}{\partial x_j}$$

large deformations

- The Cauchy stress tensor is symmetric

$$\sigma_{ij} = \sigma_{ji}$$

- Substitution of this relationship for Lagrangian stress, however, gives

$$\sigma^L_{pi}\frac{\partial x_j}{x^0_p} = \sigma^L_{pj}\frac{\partial x_i}{\partial x^0_p}$$

- Which indicates that σ^L_{ij} is not symmetric

- We can force symmetry by changing the definition to

$$\frac{\partial x_i}{\partial x_j^0}\sigma^K_{pj} = \frac{\rho^0}{\rho}\sigma_{ji}\frac{\partial x_p^0}{\partial x_j}$$

- From this we can find the Piola-Kirchoff stress, which is symmetric

$$\sigma^K_{pq} = \frac{\rho^0}{\rho}\sigma_{ji}\frac{\partial x_p^0}{\partial x_i}\frac{\partial x_q^0}{\partial x_j}$$

- This is also known as the *second Piola stress tensor* or the *Kirchoff stress tensor*
- In this course we focus on small deformations, so we will only use the Cauchy stress tensor

equilibrium equations

static equilibrium

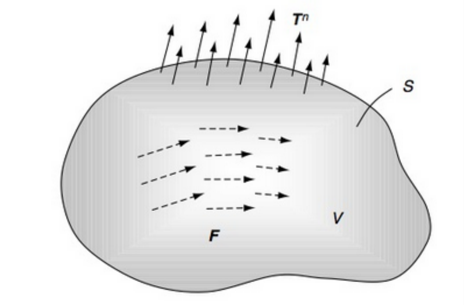


Figure 1: an arbitrary body under arbitrary remote loading with internal body forces

static equilibrium

- We primarily deal with bodies in static equilibrium
- This means that all forces and moments must sum to zero
- For a closed sub-domain of volume V and surface area S with internal body forces and applied tractions, we find

$$\iint_S t_i^n dS + \iiint_V F_i dV = 0$$

static equilibrium

- Using the Cauchy stress theorem, we can replace the traction vector with the stress tensor

$$\iint_S \sigma_{ji} n_j dS + \iiint_V F_i dV = 0$$

- We can also apply the divergence theorem to convert the surface integral to a volume integral

$$\iiint_V (\sigma_{ji,j} + F_i) dV = 0$$

static equilibrium

- Since the volume is arbitrary (we could choose any volume and the conditions for equilibrium would still hold), the integrand must vanish

$$\sigma_{ji,j} + F_i = 0$$

equilibrium equations

- Written in scalar form, the equilibrium equations are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y = 0$$
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0$$

angular momentum

- Similarly, the principle of angular momentum states that the moment forces must all sum to zero as well

$$\iint_S \epsilon_{ijk} x_j t_k^n dS + \iiint_V \epsilon_{ijk} x_j F_k dV = 0$$

- Once again we use Cauchy's stress theorem

$$\iint_S \epsilon_{ijk} x_j \sigma_{lk} n_l dS + \iiint_V \epsilon_{ijk} x_j F_k dV = 0$$

- And the divergence theorem

$$\iiint_V [(\epsilon_{ijk} x_j \sigma_{lk})_{,l} + \epsilon_{ijk} x_j F_k] dV = 0$$

angular momentum

- Expanding the derivative using the chain rule gives

$$\iiint_V [\epsilon_{ijk} x_{j,l} \sigma_{lk} \epsilon_{ijk} x_j \sigma_{lk,l} + \epsilon_{ijk} x_j F_k] dV = 0$$

- Which can be simplified (recall that $\sigma_{ji,j} + F_i = 0$)

$$\iiint_V [\epsilon_{ijk} \delta_{jl} \sigma_{lk} + \epsilon_{ijk} x_j \sigma_{lk,l} + \epsilon_{ijk} x_j F_k] dV = 0$$
$$\iiint_V [\epsilon_{ijk} \sigma_{jk} - \epsilon_{ijk} x_j F_k + \epsilon_{ijk} x_j F_k] dV = 0$$
$$\iiint_V \epsilon_{ijk} \sigma_{jk} dV = 0$$

angular momentum

- Using the same argument as before (arbitrary volume) the integrand must vanish

$$\epsilon_{ijk} \sigma_{jk} = 0$$

- Since the alternating symbol is antisymmetric in jk, σ_{jk} must be symmetric in jk for this to vanish
- And thus we have proved that the stress tensor is symmetric, thus equilibrium and angular momentum equations are satisfied when

$$\sigma_{ji,j} + F_i = 0$$

example

- Under what circumstances is the following stress field in static equilibrium?
- $\sigma_{11} = 3x_1 + k_1 x_2^2$, $\sigma_{22} = 2x_1 + 4x_2$
  $\sigma_{12} = \sigma_{21} = a + bx_1 + cx_1^2 + dx_2 + ex_2^2 + fx_1 x_2$
- We are only examining the stress field, so we neglect any internal body forces
- The first equilibrium equation gives

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$

$$3 + d + 2ex_2 + fx_1 = 0$$

example

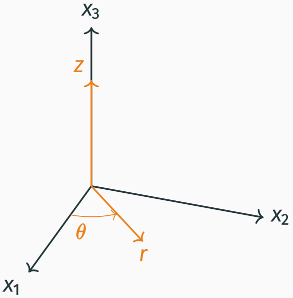
- The second equilibrium equation gives

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$$

$$b + 2cx_1 + fx_2 + 4 = 0$$

spherical and cylindrical coordinates

cylindrical coordinates



stress in cylindrical coordinates

- We can also define stress in a cylindrical coordinate system

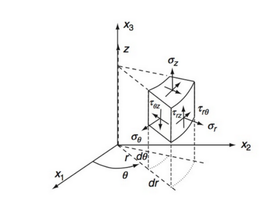


Figure 2: an illustration of the stress tensor terms in cylindrical coordinates

stress in cylindrical coordinates

- The stress tensor in cylindrical coordinates is

$$\sigma_{ij} = \begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_\theta & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_z \end{bmatrix}$$

equilibrium in cylindrical coordinates

- Using the derivative relationships developed in Chapter 1, we can express the equilibrium equations as

$$
\begin{aligned}
\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r}(\sigma_r - \sigma_\theta) + F_r = 0 \\
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} + F_\theta = 0 \\
\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\tau_{rz} + F_z = 0
\end{aligned}
$$

spherical coordinates

- We can do the same thing in spherical coordinates

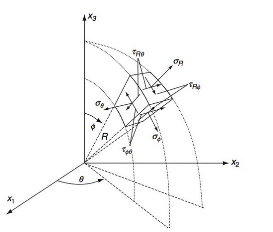


Figure 3: an illustration of the stress tensor terms in spherical coordinates

spherical coordinates

- The stress tensor in spherical coordinates is

$$\sigma_{ij} = \begin{bmatrix} \sigma_r & \tau_{r\phi} & \tau_{r\theta} \\ \tau_{r\phi} & \sigma_\phi & \tau_{\phi\theta} \\ \tau_{r\theta} & \tau_{\phi\theta} & \sigma_\theta \end{bmatrix}$$

equilibrium in spherical coordinates

- Using the derivative relationships developed in Chapter 1, we can express the equilibrium equations as

$$\begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\phi}}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r}(2\sigma_r - \sigma_\phi - \sigma_\theta + \tau_{r\phi}\cot\phi) + F_r = 0 \\ \frac{\partial \tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\phi}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial \tau_{\phi\theta}}{\partial \theta} + \frac{1}{r}[(\sigma_\phi - \sigma_\theta)\cot\phi + 3\tau_{r\phi}] + F_\phi = 0 \\ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\phi\theta}}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{1}{r}(2\tau_{\phi\theta}\cot\phi + 3\tau_{r\theta}) + F_\theta = 0 \end{aligned}$$