Name:

Final Equation Sheet

Cauchy's Stress Theorem

$$t_i = \sigma_{ij} n_j \tag{1}$$

Strain-Displacement

Cartesian

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2}$$

Polar

$$\begin{aligned}\epsilon_r = \frac{\partial u_r}{\partial r}, \epsilon_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) \\ \epsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\end{aligned} \tag{3}$$

Equilibrium

Cartesian

$$\sigma_{ij,j} + F_i = 0 \tag{4}$$

Polar

$$\begin{aligned}\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r}(\sigma_r - \sigma_\theta) + F_r = 0 \\ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2}{r}\tau_{r\theta} + F_\theta = 0\end{aligned}$$

Navier

$$\mu u_{i,kk} + (\lambda + \mu) u_{k,ki} + F_i = 0 \tag{5}$$

Constitutive

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} - (3\lambda + 2\mu)\alpha \Delta T \delta_{ij} \tag{6}$$

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha \Delta T \delta_{ij} \tag{7}$$

	$\lambda =$	$\mu = G =$	$E =$	$\nu =$	$K =$
λ, μ			$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{3\lambda+2\mu}{3}$
G, E	$\frac{G(2G-E)}{E-3G}$			$\frac{E-2G}{2G}$	$\frac{GE}{3(3G-E)}$
G, ν	$\frac{2G\nu}{1-2\nu}$		$2G(1+\nu)$		$\frac{2G(1+\nu)}{3(1-2\nu)}$
E, ν	$\frac{\nu E}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$			$\frac{E}{3(1-2\nu)}$
K, E	$\frac{3K(3K-E)}{9K-E}$	$\frac{3EK}{9K-E}$		$\frac{3K-E}{6K}$	
ν, K	$\frac{3K\nu}{1+\nu}$	$\frac{3K(1-2\nu)}{2(1+\nu)}$	$3K(1-2\nu)$		

Material Constants

Compatibility

Strain

$$\begin{aligned}
\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} &= 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} \\
\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} &= 2\frac{\partial^2 \epsilon_{yz}}{\partial y \partial z} \\
\frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} &= 2\frac{\partial^2 \epsilon_{zx}}{\partial z \partial x} \\
\frac{\partial^2 \epsilon_x}{\partial y \partial z} &= \frac{\partial}{\partial x}\left(-\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z}\right) \\
\frac{\partial^2 \epsilon_y}{\partial z \partial x} &= \frac{\partial}{\partial y}\left(-\frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x}\right) \\
\frac{\partial^2 \epsilon_z}{\partial x \partial y} &= \frac{\partial}{\partial z}\left(-\frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y}\right)
\end{aligned} \tag{8}$$

Plane Stress

$$\nabla^2(\sigma_{11} + \sigma_{22}) = -(1+\nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) \tag{9}$$

Plane Stress in Polar Coordinates

$$\nabla^2(\sigma_{rr} + \sigma_{\theta\theta}) = -4\rho(1+\nu)\left(\frac{\partial F_r}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}\right) \tag{10}$$

Where, in polar coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{11}$$

Stress transformation

$$
\begin{aligned}
\sigma'_x &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\
\sigma'_y &= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \\
\tau'_{xy} &= \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta
\end{aligned} \tag{12}
$$

Strain transformation

$$
\begin{aligned}
\epsilon'_x &= \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \epsilon_{xy}\sin 2\theta \\
\epsilon'_y &= \frac{\epsilon_x + \epsilon_y}{2} - \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta - \epsilon_{xy}\sin 2\theta \\
\epsilon'_{xy} &= \frac{\epsilon_y - \epsilon_x}{2}\sin 2\theta + \epsilon_{xy}\cos 2\theta
\end{aligned} \tag{13}
$$

Airy Stress Functions

Cartesian Coordinates

$$
\begin{aligned}
F_x &= -\frac{\partial V}{\partial x} \\
F_y &= -\frac{\partial V}{\partial y}
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
\sigma_x &= \frac{\partial^2 \phi}{\partial y^2} + V \\
\sigma_y &= \frac{\partial^2 \phi}{\partial x^2} + V \\
\tau_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y}
\end{aligned} \tag{15}
$$

Polar Coordinates

$$
\begin{aligned}
\rho b_r &= -\frac{\partial V}{\partial r} \\
\rho b_\theta &= -\frac{1}{r}\frac{\partial V}{\partial \theta}
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
\sigma_r &= \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + V \\
\sigma_\theta &= \frac{\partial^2 \phi}{\partial r^2} + V \\
\tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\phi &= a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r \\
&\quad + (a_4 + a_5 \log r + a_6 r^2 + a_7 r^2 \log r)\theta \\
&\quad + \left(a_{11} r + a_{12} r \log r + \frac{a_{13}}{r} + a_{14} r^3 + a_{15} r\theta + a_{16} r\theta \log r\right) \cos\theta \\
&\quad + \left(b_{11} r + b_{12} r \log r + \frac{b_{13}}{r} + b_{14} r^3 + b_{15} r\theta + b_{16} r\theta \log r\right) \sin\theta \\
&\quad + \sum_{n=2}^{\infty} (a_{n1} r^n + a_{n2} r^{2+n} + a_{n3} r^{-n} + a_{n4} r^{2-n}) \cos n\theta \\
&\quad + \sum_{n=2}^{\infty} (b_{n1} r^n + b_{n2} r^{2+n} + a_{n3} r^{-n} + b_{n4} r^{2-n}) \sin n\theta
\end{aligned}
\tag{18}
$$

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$r^2 \ln r$	$2 \ln r + 1$	0	$2 \ln r + 3$
$\ln r$	r^{-2}	0	$-r^{-2}$
θ	0	r^{-2}	0
$r^3 \cos\theta$	$2 r \cos\theta$	$2 r \sin\theta$	$6 r \cos\theta$
$r\theta \cos\theta$	$-2 r^{-1} \sin\theta$	0	0
$r \ln r \cos\theta$	$r^{-1} \cos\theta$	$r^{-1} \sin\theta$	$r^{-1} \cos\theta$
$r^{-1} \cos\theta$	$-2 r^{-3} \cos\theta$	$-2 r^{-3} \sin\theta$	$2 r^{-3} \cos\theta$
$r^3 \sin\theta$	$2 r \sin\theta$	$-2 r \cos\theta$	$6 r \sin\theta$
$r\theta \sin\theta$	$2 r^{-1} \cos\theta$	0	0
$r \ln r \sin\theta$	$r^{-1} \sin\theta$	$-r^{-1} \cos\theta$	$r^{-1} \sin\theta$
$r^{-1} \sin\theta$	$-2 r^{-3} \sin\theta$	$2 r^{-3} \cos\theta$	$2 r^{-3} \sin\theta$
$r^{n+2} \cos(n\theta)$	$-(n+1)(n+2) r^n \cos(n\theta)$	$n(n+1) r^n \sin(n\theta)$	$(n+1)(n+2) r^n \cos(n\theta)$
$r^{-n+2} \cos(n\theta)$	$-(n+2)(n-1) r^{-n} \cos(n\theta)$	$-n(n-1) r^{-n} \sin(n\theta)$	$(n-1)(n-2) r^{-n} \cos(n\theta)$
$r^n \cos(n\theta)$	$-n(n-1) r^{n-2} \cos(n\theta)$	$n(n-1) r^{n-2} \sin(n\theta)$	$n(n-1) r^{n-2} \cos(n\theta)$
$r^{-n} \cos(n\theta)$	$-n(n+1) r^{-n-2} \cos(n\theta)$	$-n(n+1) r^{-n-2} \sin(n\theta)$	$n(n+1) r^{-n-2} \cos(n\theta)$
$r^{n+2} \sin(n\theta)$	$-(n+1)(n+2) r^n \sin(n\theta)$	$-n(n+1) r^n \cos(n\theta)$	$(n+1)(n+2) r^n \sin(n\theta)$
$r^{-n+2} \sin(n\theta)$	$-(n+2)(n-1) r^{-n} \sin(n\theta)$	$n(n-1) r^{-n} \cos(n\theta)$	$(n-1)(n-2) r^{-n} \sin(n\theta)$
$r^n \sin(n\theta)$	$-n(n-1) r^{n-2} \sin(n\theta)$	$-n(n-1) r^{n-2} \cos(n\theta)$	$n(n-1) r^{n-2} \sin(n\theta)$
$r^{-n} \sin(n\theta)$	$-n(n+1) r^{-n-2} \sin(n\theta)$	$n(n+1) r^{-n-2} \cos(n\theta)$	$n(n+1) r^{-n-2} \sin(n\theta)$

Compatibility

$$\nabla^4 \phi = -2 \frac{\kappa - 1}{\kappa + 1} \nabla^2 V \tag{19}$$

$$
\begin{aligned}
\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \text{(Cartesian coordinates)} \\
\nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \qquad \text{(Polar Coordinates)}
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\kappa &= 3 - 4\nu \qquad \text{plain strain} \\
\kappa &= \frac{3-\nu}{1+\nu} \qquad \text{plane stress}
\end{aligned}
\tag{21}
$$

Conversion from Cartesian to Polar

Displacement

$$
\begin{aligned}
u_r &= u\cos\theta + v\sin\theta \\
u_\theta &= -u\sin\theta + v\cos\theta
\end{aligned}
\tag{22}
$$

Stress

$$
\begin{aligned}
\sigma_r &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \\
\sigma_\theta &= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta \\
\sigma_{r\theta} &= -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)
\end{aligned}
\tag{23}
$$

Useful Identities

$$
\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
&= 2\cos^2\theta - 1 \\
&= 1 - 2\sin^2\theta
\end{aligned}
\tag{24}
$$