AE731

Theory of Elasticity

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upcoming schedule

- Aug 31 - Displacement and Strain
- Sep 2 - Strain Transformation
- Sep 3 - Homework 2 Due, Homework 1 Self-Grade Due
- Sep 7 - Exam 1 Review
- Sep 9 - Exam 1

outline

- general deformation
- small deformation theory
- strain

chapter outline

- General description of deformations
- Assumptions for small deformations
- Definition of strain
- Strain transformation
- Principal strains
- Strain compatibility
- Strain in cylindrical and spherical coordinates

general deformation

general deformation

- When deformations are large, the deformed and un-deformed shapes can be quite different
- It can be convenient to refer to material properties in the deformed or un-deformed configuration
- Lagrangian reference: quantities are in terms of the original (un-deformed) configuration
- Eulerian reference: quantities are in terms of deformed configuration

material derivatives

- We refer to the undeformed configuration as x_i^0 and the deformed configuration as x_i
- If some quantity, ϕ is expressed in the undeformed configuration as $\phi(x_1^0, x_2^0, x_3^0, t)$ then the material derivative is

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t}$$

material derivatives

- However in Eulerian form $\bar{\phi}(x_1, x_2, x_3, t) = \phi(x_1^0, x_2^0, x_3^0, t)$ the material derivative becomes

$$\frac{d\bar{\phi}}{dt} = \frac{\partial\bar{\phi}}{\partial t} + \frac{\partial\bar{\phi}}{\partial x_j}\frac{dx_j}{dt}$$

deformation

- A *deformation* is a comparison of two states. The deformation of a material point is expressed as

$$x_i = x_i(x_1^0, x_2^0, x_3^0) \quad \text{or} \quad x_i^0 = x_i^0(x_1, x_2, x_3)$$

- For example, consider the 2D deformation

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 2x_1^0 \\ x_2^0 \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} x_1^0 \\ x_2^0 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}x_1 \\ x_2 \end{Bmatrix}$$

deformation

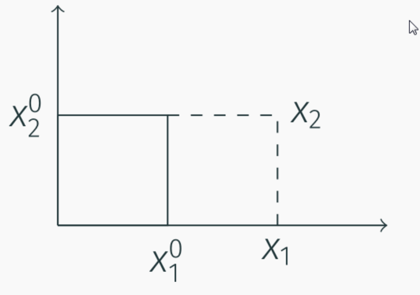


Figure 1: an illustration of deformation showing a simple square that has been stretched horizontally

displacement

A displacement is the shortest distance traveled when a particle moves from one location to another
Displacement is identical in Eulerian and Lagrangian descriptions

$$u_i = (x_i - x_i^0)$$

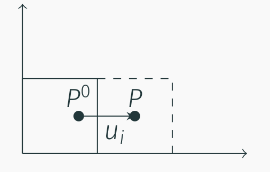


Figure 2: an illustration of displacement showing the shortest distance between a point before and after a horizontal stretch

small deformation theory

deformation gradients

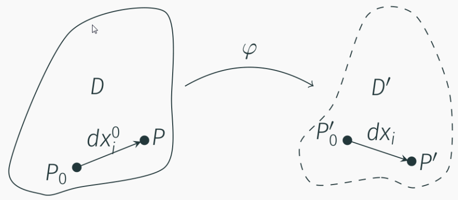


Figure 3: an arbitrary body before and after some arbitrary deformation

deformation gradients

- The position of the two points, P'_0 and P', is related by

$$\begin{aligned} P'_0 &= x_i(x_i^0) \\ P' &= x_i + dx_i \quad = x_i(x_i^0 + dx_i^0) \end{aligned}$$

- We can approximate $x_i(x_i^0, dx_i^0)$ with a Taylor series expansion

$$\approx x_i(x_i^0) + \frac{\partial x_i^0}{\partial x_j} dxj^0 + \frac{1}{2}\frac{\partial^2 x_i}{\partial x_j^0 \partial x_k^0} dx_j^0 dx_k^0 + ...$$

deformation gradients

- If the deformation is small, we can neglect higher-order terms of the expansion

$$P' = x_i + dx_i = x_i(x_i^0) + \frac{\partial x_i^0}{\partial x_j} dxj^0$$

- Which gives

$$dx_i = \frac{\partial x_i^0}{\partial x_j} dx_j^0$$

deformation gradients

- In index notation we write displacements as u_i
- The deformation gradient can be written in this notation as

$$F = u_{i,j} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

translation

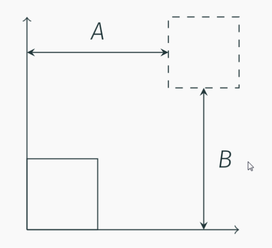


Figure 4: a square is originally on the origin but is then translated some distance A from the y axis and B from the x axis

- x-displacement

$$u_1 = x_1^0 + A$$

- y-displacement

$$u_2 = x_2^0 + B$$

translation

- Deformation gradient

$$F = u_{i,j} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

rotation

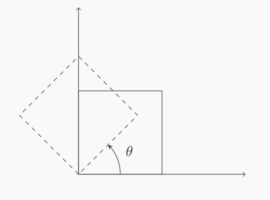


Figure 5: a square originally on the origin is rotated about the origin by some angle theta

- x-displacement

$$u_1 = x_1^0 \cos\theta - x_2^0 \sin\theta$$

- y-displacement

$$u_2 = x_1^0 \sin\theta + x_2^0 \cos\theta$$

rotation

- Deformation gradient

$$F = u_{i,j} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix}$$

$$F = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

simple shear

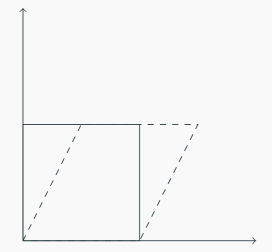


Figure 6: a square is deformed by moving the top side to the right in a simple shearing motion

- x-displacement

$$u_1 = x_1^0 + \frac{1}{2}x_2^0$$

- y-displacement

$$u_2 = x_2^0$$

simple shear

- Deformation gradient

$$F = u_{i,j} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

pure shear

- Sometimes it is important to eliminate rotations
- We can design an experiment with a state of pure shear by inducing this deformation

$$F = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

- We can integrate our usual relationship to find u_1 and u_2

pure shear

$$\frac{\partial u_1}{\partial x_1} = 1$$

$$u_1 = x_1 + f(x_2)$$

$$\frac{\partial u_1}{\partial x_2} = \frac{1}{2}$$

$$\frac{\partial f}{\partial x_2} = \frac{1}{2}$$

$$u_1 = x_1 + \frac{1}{2}x_2$$

pure shear

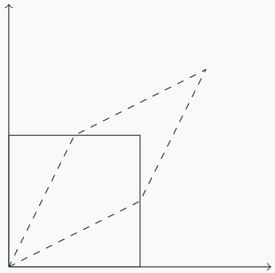
$$\frac{\partial u_2}{\partial x_2} = 1$$

$$u_2 = x_2 + g(x_1)$$

$$\frac{\partial u_2}{\partial x_1} = \frac{1}{2}$$

$$\frac{\partial g}{\partial x_1} = \frac{1}{2}$$

$$u_2 = x_2 + \frac{1}{2}x_1$$



strain

strain definition

- We can separate the deformation gradient into symmetric and antisymmetric parts

$$u_{i,j} = e_{ij} + \omega_{ij}$$

- Where

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$
$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$$

- e_{ij} is known as the strain tensor - ω_{ij} is known as the rotation tensor

geometric description

- Engineering strain

$$e^E = \frac{\Delta L}{L_0}$$

- True strain

$$e^T = \frac{\Delta L}{L_0 + \Delta L}$$

- Logarithmic strain

$$e^L = \int_{L_0}^{L} e^T = \int_{L_0}^{L} \frac{dl}{l} = \ln\left(\frac{L}{L_0}\right)$$

geometric description

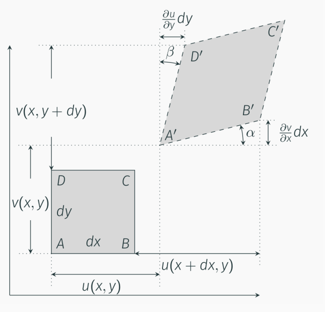
- Large strain: $\Delta L = L_0$

$$e^E = 1.00$$
$$e^T = 0.50$$
$$e^L = 0.69$$

- Small strain: $\Delta L = 0.05 L_0$

$$e^E = 0.050$$
$$e^T = 0.048$$
$$e^L = 0.049$$

geometric description



geometric description

- The extensional strain in the x-direction (engineering strain) is defined by

$$\varepsilon_x = \frac{A'B' - AB}{AB}$$

- From geometry, we can write $A'B'$ as

$$\begin{aligned} A'B' &= \sqrt{\left(dx + \frac{\partial u}{\partial x} dx\right)^2 + \left(\frac{\partial v}{\partial x} dx\right)^2} \\ &= dx\sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2} \end{aligned}$$

geometric description

- For small deformation, we assume $\frac{\partial v}{\partial x}$ is small when compared with $\frac{\partial u}{\partial x}$, which gives

$$A'B' = \left(1 + \frac{\partial u}{\partial x}\right) dx$$

geometric description

$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{\left(1 + \frac{\partial u}{\partial x}\right) dx - dx}{dx}$$
$$= \frac{\partial u}{\partial x}$$

- The normal strain in the y-direction is found the same way

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

geometric description

- Engineering shear strain is defined as the change in angle between two originally orthogonal directions

$$\gamma_{xy} = \frac{\pi}{2} - \angle D'A'B' = \alpha + \beta$$

- For small strains, $\alpha \approx \tan \alpha$ and $\beta \approx \tan \beta$.

$$\gamma_{xy} = \frac{\frac{\partial v}{\partial x} dx}{dx + \frac{\partial u}{\partial x} dx} + \frac{\frac{\partial u}{\partial y} dy}{dy + \frac{\partial v}{\partial y} dy}$$
$$\approx \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

geometric description

- The other shear terms can be found in the same way

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$
$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

geometric description

- Engineering strain and tensor strain definitions differ only in shear terms

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$
$$e_{xy} = \frac{1}{2}\gamma_{xy}$$

example

- Calculate the deformation gradient, strain tensor, and rotation tensor for the given deformation

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} xy^2z \\ xz \\ z^3 \end{Bmatrix}$$

- Deformation gradient:

$$F = u_{i,j} = \begin{bmatrix} y^2z & 2xyz & xy^2 \\ z & 0 & x \\ 0 & 0 & 3z^2 \end{bmatrix}$$

example

- Strain tensor

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$e_{ij} = \begin{bmatrix} y^2z & xyz + \frac{1}{2}z & \frac{1}{2}xy^2 \\ xyz + \frac{1}{2}z & 0 & \frac{1}{2}x \\ \frac{1}{2}xy^2 & \frac{1}{2}x & 3z^2 \end{bmatrix}$$

example

- Rotation tensor

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$$

$$\omega_{ij} = \begin{bmatrix} 0 & xyz - \frac{1}{2}z & \frac{1}{2}xy^2 \\ -xyz + \frac{1}{2}z & 0 & \frac{1}{2}x \\ -\frac{1}{2}xy^2 & -\frac{1}{2}x & 0 \end{bmatrix}$$

example

- As we did with the deformation gradient, we can integrate the strain tensor to find the deformation (symmetric portion)

$$e_{ij} = \begin{bmatrix} yz & xz & xy \\ xz & 2y & \frac{1}{2}x^2 \\ xy & \frac{1}{2}x^2 & 3z^2 \end{bmatrix}$$

example

- We start by integrating the diagonal terms

$$u = \int yz dx = xyz + f(y, z)$$
$$v = \int 2y dy = y^2 + g(x, z)$$
$$w = \int 3z^2 dz = z^3 + h(x, y)$$

example

- Next we need to find the shear terms

$$e_{xy} = \frac{1}{2}(u_{,y} + v_{,x})$$
$$xz = \frac{1}{2}(xz + f_{,y} + g_{,x})$$

$$e_{xz} = \frac{1}{2}(u_{,z} + w_{,x})$$
$$xy = \frac{1}{2}(xy + f_{,z} + h_{,x})$$

$$e_{yz} = \frac{1}{2}(v_{,z} + w_{,y})$$
$$\frac{1}{2}x^2 = \frac{1}{2}(g_{,z} + h_{,y})$$

example

- Note that we cannot uniquely solve this (any arbitrary rotation ω can be added and will still produce a valid strain)
- Let $f(y, z) = 0$

$$g_{,x} = xz$$
$$g(x, z) = \frac{1}{2}x^2 z$$
$$h_{,x} = xy$$
$$h(x, z) = \frac{1}{2}x^2 y$$

example

$$\frac{1}{2}x^2 = \frac{1}{2}\left(g_{,z} + h_{,y}\right)$$
$$\frac{1}{2}x^2 = \frac{1}{2}\left(\frac{1}{2}x^2 + \frac{1}{2}x^2\right)$$
$$u = xyz$$
$$v = y^2 + \frac{1}{2}x^2 z$$
$$w = z^3 + \frac{1}{2}x^2 y$$