Theory of Elasticity

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upcoming schedule

- Sep 28 - Thermoelasticity
- Sep 30 - Boundary Conditions
- Oct 1 - Homework 3 Self-grade Due, Homework 4 Due
- Oct 5 - Problem Formulation
- Oct 7 - Solution Strategies
- Oct 8 - Homework 4 Self-grade Due, Homework 5 Due
- (Oct 12) - Fall Break (No Class)

outline

- elastic constants
- thermoelasticity
- material symmetries
- poisson's ratio
- group problems

elastic constants

isotropic materials

	$\lambda =$	$\mu = G =$	$E =$	ν	$K =$
λ, μ			$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{3\lambda+2\mu}{3}$
G, E	$\frac{G(2G-E)}{E-3G}$			$\frac{E-2G}{2G}$	$\frac{GE}{3(3G-E}$
G, ν	$\frac{2G\nu}{1-2\nu}$		$2G(1+\nu)$		$\frac{2G(1+G)}{3(1-2G}$
E, ν	$\frac{\nu E}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$			$\frac{E}{3(1-2\nu}$
K, E	$\frac{3K(3K-E)}{9K-E}$	$\frac{3EK}{9K-E}$		$\frac{3K-E}{6K}$	
ν, K	$\frac{3K\nu}{1+\nu}$	$\frac{3K(1-2\nu)}{2(1+\nu)}$	$3K(1-2\nu)$		

thermoelasticity

thermal expansion

- Thermal expansion/contraction is fairly well known
- Most materials shrink at colder temperatures, but this is not always the case
- Thermal deformations will alter the strain field
- We can decompose strain into mechanical and thermal components

$$\epsilon_{ij} = \epsilon_{ij}^{M} + \epsilon_{ij}^{T}$$

thermal expansion

- Thermal strains can be written in terms of a coefficient of thermal expansion tensor

$$\epsilon_{ij}^{T} = \alpha_{ij}(T - T_0)$$

- For isotropic materials, this relationship is simplified to

$$\epsilon_{ij}^{T} = \alpha(T - T_0)\delta_{ij}$$

thermal expansion

- We can combine the previous results with Hooke's law to find

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha(T - T_0)\delta_{ij}$$

- We can also invert this relationship to find the stress
- Written in terms of Lamé constants, we find

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} - (3\lambda + 2\mu)\alpha(T - T_0)\delta_{ij}$$

example

- A modern-day alchemist is trying to make diamonds from charcoal.
- He hypothesized that it is easier to build a rigid fixture, and then force the charcoal to expand via thermal expansion, than it is to apply the necessary pressure at room temperature.
- What temperature is needed to provide a stress of 1 GPa in the charcoal, which has

$$\alpha = 5 \times 10^{-6}/^\circ\text{C}$$

, $E = 5$ GPa, $\nu = 0.3$

example

- Use stress equation

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} - (3\lambda + 2\mu)\alpha(T - T_0)\delta_{ij}$$

- Convert material properties to Lamé constants

material symmetries

monoclinic symmetry

- *Monoclinic symmetry* means the material is symmetric about one axis
- This symmetry is common in many types of crystals
- e.g. the x_i' coordinate system is given by

$$\begin{aligned}\hat{e}_1 &= \langle 1, 0, 0 \rangle \\ \hat{e}_2 &= \langle 0, 1, 0 \rangle \\ \hat{e}_3 &= \langle 0, 0, -1 \rangle\end{aligned}$$

monoclinic symmetry

- This gives

$$Q_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

monoclinic symmetry

- The transformed stress is given by

$$\sigma'_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & -\sigma_{13} \\ \sigma_{12} & \sigma_{22} & -\sigma_{23} \\ -\sigma_{13} & -\sigma_{23} & \sigma_{33} \end{bmatrix}$$

- Similarly we can transform the strain tensor

$$\epsilon'_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & -\epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & -\epsilon_{23} \\ -\epsilon_{13} & -\epsilon_{23} & \epsilon_{33} \end{bmatrix}$$

monoclinic symmetry

- Symmetry requires that $\sigma_{ij} = \sigma'_{ij}$, therefore

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & -C_{14} & -C_{15} \\ C_{21} & C_{22} & C_{23} & -C_{24} & -C_{25} \\ C_{31} & C_{32} & C_{33} & -C_{34} & -C_{35} \\ -C_{41} & -C_{42} & -C_{43} & C_{44} & C_{45} \\ -C_{51} & -C_{52} & -C_{53} & C_{54} & C_{55} \\ C_{61} & C_{62} & C_{63} & -C_{64} & -C_{65} \end{bmatrix}$$

monoclinic symmetry

- The only way for this equation to be satisfied is if

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{21} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{31} & C_{32} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{54} & C_{55} & 0 \\ C_{61} & C_{62} & C_{63} & 0 & 0 & C_{66} \end{bmatrix}$$

- This has only 13 independent terms

orthotropic symmetry

- *Orthotropic symmetry* is essentially monoclinic symmetry repeated about all three axes
- Composite materials are often treated as orthotropic, as are many crystals
- If we use the same method multiple times, we find that

orthotropic symmetry

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

- Which has only 9 independent terms

transversely isotropic symmetry

- *Transverse isotropy* occurs when a material is monoclinic in one axis, and perfectly symmetric (isotropic) in the other plane
- For example, many micromechanical models of composites look at only one fiber surrounded by matrix
- In the fiber direction, the material is monoclinic
- Perpendicular to the fiber, the material is the same in any direction (isotropic)

transverse isotropy

- To satisfy these conditions, the stiffness must be

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix}$$

- Here there are five independent material constants

isotropic symmetry

- An *isotropic* material has the same properties in any direction
- Therefore the stiffness matrix must be unchanged in any rotation

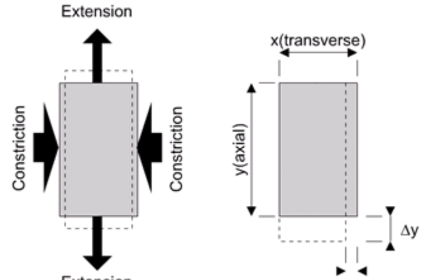
$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(C_{11} - C_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11} - C_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix}$$

poisson's ratio

poisson's ratio

- Poisson's ratio, ν, is defined as

$$\nu = -\frac{d\epsilon_{transverse}}{d\epsilon_{axial}}$$



poisson's ratio

- For isotropic materials, there is only one Poisson's ratio in the material
- For anisotropic materials (transversely isotropic, orthotropic, etc.) there are multiple
- The subscript notation for Poisson's ratios is ν_{ij} where extension is applied in direction i, with a resulting contraction in direction j

poisson's ratio

- In an orthotropic material, there are three independent Poisson's ratios, the others may be obtained from the following relationship

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}$$
$$\frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}$$
$$\frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2}$$

poisson's ratio

- In transversely isotropic materials, there are only two independent Poisson's ratios
- If the x-direction is monoclinic, then the Poisson's ratios are

$$\nu_{12} = \nu_{13}$$
$$\nu_{21} = \nu_{31}$$
$$\nu_{23} = \nu_{32}$$
$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}$$

poisson's ratio

- Physical considerations
- You will prove this in the homework, but if we require the moduli to be positive, we find that the Poisson's ratio must be

$$-1 < \nu < \frac{1}{2}$$

group problems

group one

- Consider some arbitrary, isotropic material under uni-axial tension
- What occurs when $\nu = \frac{1}{2}$?
- What about when $\nu < 0$?

group two

- Consider a $\pm 45^\circ$ laminate (which has an in-plane poisson's ratio of 0.8) bonded on top of aluminum (which has an in-plane poisson's ratio of 0.3)
- What happens when this is loaded in tension? Why might this create problems in the adhesive joining the two?

group three

- Use the table provided in these notes (or in the text) to re-write Hooke's Law in terms of Young's Modulus, E and shear modulus G.