AE731

Theory of Elasticity

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September 30, 2021

upcoming schedule

- Sep 30 - Boundary Conditions
- Oct 1 - Homework 3 Self-grade Due, Homework 4 Due
- Oct 5 - Problem Formulation
- Oct 7 - Solution Strategies
- Oct 8 - Homework 4 Self-grade Due, Homework 5 Due
- (Oct 12) - Fall Break (No Class)

outline

- field equations
- boundary conditions
- stress formulation
- example

field equations

field equations

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \text{Strain-Displacement}$$

$$\sigma_{ij,j} + F_i = 0 \qquad \text{Equilibrium}$$

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} \qquad \text{Constitutive (Hooke's Law)}$$

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

field equations

- There are 15 unique field equations to solve for the 15 unknowns
- 3 displacements (u_i), 6 unique strain tensor terms (ϵ_{ij}), and 6 unique stress tensor terms (σ_{ij})
- These equations also depend on a knowledge of the material behavior (λ, μ) and body forces (usually gravity or zero)

compatibility equations

- If continuous, single-valued displacements are specified, differentiation will result in well-behaved strain field
- The inverse relationship, integration of a strain field to find displacement, may not always be true
- There are cases where we can integrate a strain field to find a set of discontinuous displacements

compatibility

- The compatibility equations enforce continuity of displacements to prevent this from occurring
- To enforce this condition we consider the strain-displacement relations:

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

compatibility

- and differentiate with respect to x_k and x_l

$$\epsilon_{ij,kl} = \frac{1}{2}(u_{i,jkl} + u_{j,ikl})$$

- Or

$$2\epsilon_{ij,kl} = u_{i,jkl} + u_{j,ikl}$$

compatibility

- We can eliminate the displacement terms from the equation by interchanging the indexes to generate new equations

$$2\epsilon_{ik,jl} = u_{i,jkl} + u_{k,ijl}$$
$$2\epsilon_{jl,ik} = u_{j,ikl} + u_{l,ijk}$$

- Solving for $u_{i,jkl}$ and $u_{j,ikl}$

$$u_{i,jkl} = 2\epsilon_{ik,jl} - u_{k,ijl}$$
$$u_{j,ikl} = 2\epsilon_{jl,ik} - u_{l,ijk}$$

compatibility

- Substituting these values into the equations gives

$$2\epsilon_{ij,kl} = 2\epsilon_{ik,jl} = u_{k,ijl} + 2\epsilon_{jl,ik} - u_{l,ijk}$$

- We now consider one more change of index equation

$$2\epsilon_{kl,ij} = u_{k,ijl} + u_{l,ijk}$$

compatibility

- and substituting this result gives

$$2\epsilon_{ij,kl} = 2\epsilon_{ik,jl} + 2\epsilon_{jl,ik} - 2\epsilon_{kl,ij}$$

- Or, simplified

$$\epsilon_{ij,kl} + \epsilon_{kl,ij} - \epsilon_{ik,jl} - \epsilon_{jl,ik} = 0$$

compatibility

- The so-called *Saint-Venant compatibility equations* in full are a system of 81 equations, but only six are useful (although even these six are not entirely linearly independent)
- These six are found by setting $k=l$, or in expanded form

compatibility

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}$$
$$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = 2\frac{\partial^2 \epsilon_{yz}}{\partial y \partial z}$$
$$\frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} = 2\frac{\partial^2 \epsilon_{zx}}{\partial z \partial x}$$
$$\frac{\partial^2 \epsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z}\right)$$
$$\frac{\partial^2 \epsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x}\right)$$
$$\frac{\partial^2 \epsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y}\right)$$

compatibility

- The compatibility equations are necessary to ensure that the strain field is valid and will produce a continuous displacement field
- While these equations are important and necessary in solving elasticity problems, they are not sufficient
- 15 equations with 15 "unknowns" but each of these "unknowns" could actually be a function with many more unknowns, we need to develop framework for simplifying the problem into something we can solve

boundary conditions

boundary conditions

- Boundary conditions commonly specify how a body is supported and/or how it is loaded
- Mathematically we treat this conditions as *displacements* or *tractions* at boundary points.
- Symmetry boundary conditions are also common, can reduce computational cost and simplify analytic solutions.

boundary conditions

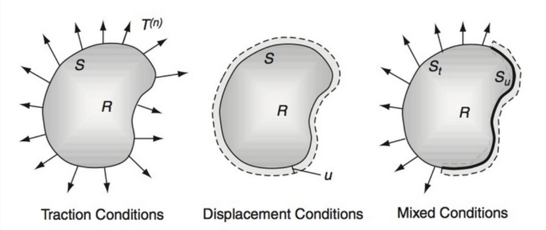


Figure 1: illustration of boundary conditions

symmetric boundaries

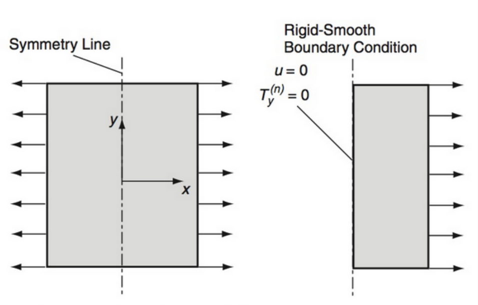


Figure 2: illustration of symmetric boundaries

coordinate systems

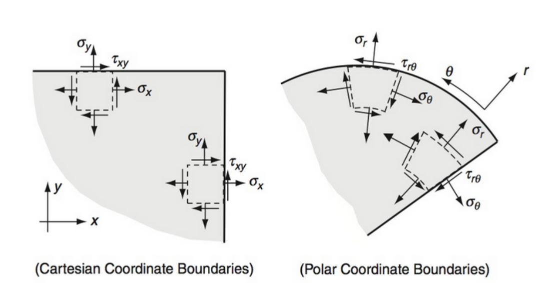
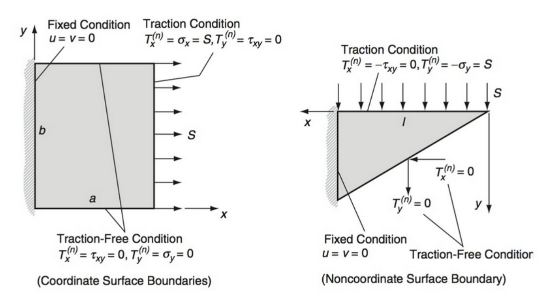


Figure 3: coordinate systems

boundaries

- In many systems, the boundaries are parallel to the coordinate system, but this is not always the case



boundaries

- We often translate traction boundary conditions into stress boundary conditions using Cauchy's Stress Theorem
- When the condition is on a face parallel to the coordinate system, this gives a zero-stress condition

$$t_j = \sigma_{ij} n_i$$

- This results in $\sigma_{xy} = \sigma_{yy} = 0$

boundaries

- When the boundary is not parallel to the coordinate system, we do not necessarily have any zero-stress conditions

$$t_x = \sigma_x n_x + \tau_{xy} n_y = 0$$
$$t_y = \tau_{xy} n_x + \sigma_y n_y = 0$$

interfaces

- When we deal with multiple materials, we must prescribe conditions at the interface of these materials
- Some common *interface conditions* are
  - *Perfectly bonded interface* where displacements and tractions are continuous at the interface
  - *Slip interface* where only normal displacements and tractions are continuous at the interface, with no tangential traction and potentially discontinuous tangential displacement

stress formulation

stress formulation

- For traction problems (i.e. traction is defined on all surfaces) it is convenient to re-formulate field equations in terms of stress only
- Since displacements are eliminated, we will need to use the compatibility equations to ensure a continuous displacement field
- It is desirable for this formulation to write the compatibility equations in terms of stress

stress formulation

- We start by using Hooke's law for each of the strain terms

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

- After some tedious algebra, we find

$$\sigma_{ij,kk}+\sigma_{kk,ij}-\sigma_{ik,jk}-\sigma_{jk,ik} = \frac{\nu}{1+\nu}(\sigma_{mm,kk}\delta_{ij}+\sigma_{mm,ij}\delta_{kk}-\sigma_{mm,jk}\delta_{ik}-\sigma_{mm,ik}$$

stress formulation

- If we also include the equilibrium equations ($\sigma_{ij,j} - F_i$) in the formulation, we find

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = \frac{\nu}{1+\nu}\sigma_{mm,kk}\delta_{ij} - F_{i,j} - F_{j,i}$$

- We can further simplify the equation by considering the case when i=j and noting that

$$\sigma_{ii,kk} = -\frac{1+\nu}{1-\nu}F_{i,i}$$

stress formulation

- Which we can substitute into the equation to find

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = -\frac{\nu}{1+\nu}\delta_{ij}F_{k,k} - F_{i,j} - F_{j,i}$$

- The compatibility equations in terms of stress are commonly known as the *Beltrami-Michell compatibility equations*
- When there are no body forces, we can write the six expanded form equations as

beltrami-michell

$$(1+\nu)\nabla^2\sigma_x + \frac{\partial^2}{\partial x^2}(\sigma_x + \sigma_y + \sigma_z) = 0 \ (1+\nu)\nabla^2\sigma_y + \frac{\partial^2}{\partial y^2}(\sigma_x + \sigma_y +$$

$$(1+\nu)\nabla^2\sigma_z + \frac{\partial^2}{\partial z^2}(\sigma_x + \sigma_y + \sigma_z) = 0$$

$$(1+\nu)\nabla^2\tau_{xy} + \frac{\partial^2}{\partial x \partial y}(\sigma_x + \sigma_y + \sigma_z) = 0$$

$$(1+\nu)\nabla^2\tau_{yz} + \frac{\partial^2}{\partial y \partial z}(\sigma_x + \sigma_y + \sigma_z) = 0$$

$$(1+\nu)\nabla^2\tau_{zx} + \frac{\partial^2}{\partial z \partial x}(\sigma_x + \sigma_y + \sigma_z) = 0$$

stress formulation

- When working with traction boundary problems, these compatibility equations, together with the equilibrium equations, are sufficient to solve the problem
- These partial differential equations are not easy to solve, and analytic problems approached this way are often solved only in 2D
- Solutions are also commonly based on *stress functions*, which gives a base equation form that automatically satisfies equilibrium

solution methods

- Direct method
    - Solved via direction integration
    - Limited to very simple geometries
- Inverse method
    - Choose a basic form for the solution based on our knowledge of the problem
    - Solve for coefficients
    - Usually we know the answer before we know the problem, it can be difficult to find useful problems for our solution

solution methods

- Semi-inverse method
    - Only part of the solution is assumed
    - Use direct integration to find the rest

example

Levy's problem

- Find the stresses in a semi-infinite wedge due to fluid pressure and its own self-weight

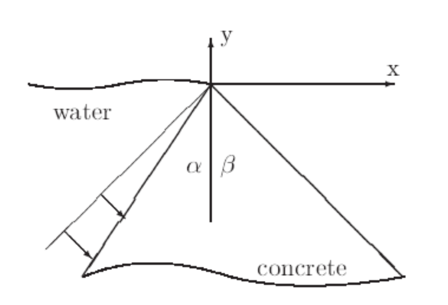


Figure 4: levy's problem

Levy's problem

- Since pressure varies linearly with depth, we will assume a linear state of stress

$$\sigma_x = a_1 x + b_1 y + c_1$$
$$\sigma_y = a_2 x + b_2 y + c_2$$
$$\tau_{xy} = a_{12} x + b_{12} y + c_{12}$$

- This leaves 9 coefficients to be determined

Levy's problem

- First let us consider the boundary conditions at the apex of the dam
- If we let the origin be at the apex of the dam, which must be traction free, we find

$$c_1 = c_2 = c_{12} = 0$$

Levy's problem

- Next let us consider the equilibrium equations

$$\sigma_{x,x} + \tau_{xy,y} + \rho b_x = 0$$
$$\tau_{xy,x} + \sigma_{y,y} + \rho b_y = 0$$

- Which in this case become

$$a_1 + b_{12} + 0 = 0$$
$$a_{12} + b_2 - \rho g = 0$$

Levy's problem

- The stresses can now be written as

$$\sigma_x = a_1 x + b_1 y$$
$$\sigma_y = a_2 x + b_2 y$$
$$\tau_{xy} = -b_2 x + \rho g x - a_1 y$$

Levy's problem

- The compatibility equations are all satisfied, as these linear functions will all go to zero when taking second derivatives
- We now consider the boundary conditions along both faces