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## upcoming schedule

- Oct 7 - Exam 2 Review
- Oct 8 - Homework 4 Self-grade Due, Homework 5 Due
- (Oct 12) - Fall Break (No Class)
- Oct 14 - Exam 2
- Oct 19 - Strain Energy
- Oct 21 - Virtual Work

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- exam
- group problems
- stress and equilibrium
- material behavior
- problem formulation

## exam

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- Similar format to last exam
- Three problems
- Focus on organizing your work clearly to maximize partial credit

## group problems

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## problem one - thermoelasticity

As a first-order model of the problem of freezing water in a glass bottle, we treat water as a thermoelastic solid and the glass as a fixed boundary. Find the stress and strain field in the water as a function of the elastic properties ( $E, \nu$ ) and the coefficient of thermal expansion ( $\alpha$ ).

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## problem two - inverse solution

Consider the stress field

$$\sigma = \begin{bmatrix} Ay & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Show that this is a valid solution to an elasticity problem. What problem does it solve?

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To solve the problem of torsion in prismatic bars we consider the displacement field

$$u = -\alpha yz, \quad v = -\alpha xz, \quad w = w(x, y)$$

Solve this problem using the boundary conditions for a solid square cross-section.

## stress and equilibrium

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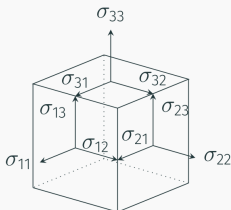
- Traction
- Stress transformation
- Principal stress
- Equilibrium

- Cauchy's stress theorem
- Max shear stress for plane stress
- Mohr's circle

## stress tensor

- To simplify the notation, we introduce the stress tensor

$$\sigma_{ij} = t_j^{(\hat{e}_i)}$$



**Figure 1:** stress tensor illustrated on a cube

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## traction

- We can find some interesting information about the traction vector by considering an arbitrary tetrahedron with some traction  $\hat{t}^{(n)}$  applied to the surface

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## traction

- If we consider the balance of forces in the  $x_1$ -direction

$$t_1 dA - \sigma_{11} dA_1 - \sigma_{21} dA_2 - \sigma_{31} dA_3 + b_1 \rho dV = 0$$

- The area components are:

$$dA_1 = n_1 dA$$

$$dA_2 = n_2 dA$$

$$dA_3 = n_3 dA$$

- And  $dV = \frac{1}{3} h dA$ .

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## traction

$$t_1 dA - \sigma_{11} n_1 dA - \sigma_{21} n_2 dA - \sigma_{31} n_3 dA + b_1 \rho \frac{1}{3} h dA = 0$$

- If we let  $h \rightarrow 0$  and divide by  $dA$

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$

- We can write this in index notation as

$$t_1 = \sigma_{i1} n_i$$

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- We find, similarly

$$t_2 = \sigma_{i2}n_i$$

$$t_3 = \sigma_{i3}n_i$$

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- We can further combine these results in index notation as

$$t_j = \sigma_{ij}n_i$$

- This means with knowledge of the nine components of  $\sigma_{ij}$ , we can find the traction vector at any point on any surface

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## maximum shear stress

- For plane stress problems, we can also use the stress transformation equations to find the maximum shear stress
- We desire to maximize this equation:

$$\tau'_{xy} = \frac{\sigma_y - \sigma_x}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

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## maximum shear stress

- Taking the derivative with respect to  $\theta$  gives

$$\frac{\partial}{\partial \theta}(\tau'_{xy}) = (\sigma_y - \sigma_x) \cos 2\theta - 2\tau_{xy} \sin 2\theta = 0$$

- Which we can use to find  $2\theta$

$$2\theta = \tan^{-1} \left( \frac{(\sigma_y - \sigma_x)}{2\tau_{xy}} \right)$$

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## maximum shear stress

- Substituting back into the original equation gives

$$\tau'_{max} = \frac{\sigma_y - \sigma_x}{2} \sin \left[ \tan^{-1} \left( \frac{(\sigma_y - \sigma_x)}{2\tau_{xy}} \right) \right] + \tau_{xy} \cos \left[ \tan^{-1} \left( \frac{(\sigma_y - \sigma_x)}{2\tau_{xy}} \right) \right]$$

- Note that

$$\begin{aligned}\sin(\tan^{-1}(x)) &= \frac{x}{\sqrt{1+x^2}} \\ \cos(\tan^{-1}(x)) &= \frac{1}{\sqrt{1+x^2}}\end{aligned}$$

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## maximum shear stress

- We note that

$$\sqrt{1 + \left( \frac{\sigma_y - \sigma_x}{2\tau_{xy}} \right)^2} = \frac{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2}}{2\tau_{xy}}$$

- And thus we find

$$\tau_{max} = \frac{(\sigma_y - \sigma_x)^2}{2\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2}} + \frac{4\tau_{xy}^2}{2\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2}}$$

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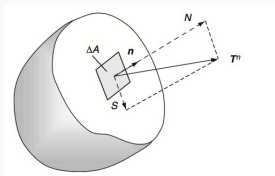
- Adding the terms and simplifying, we find

$$\tau_{max} = \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

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## tractions

- We can use what we know about principal values to find some interesting things about the tractions
- Consider the traction vector on an arbitrary internal face, and decompose into Normal and Shear components.



**Figure 2:** arbitrary body with arbitrary loading applied

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- The normal component can be found using the dot product

$$N = \hat{T}^n \cdot \hat{n}$$

- The shear component can be found using the Pythagorean theorem

$$S^2 = |\hat{T}^n|^2 - N^2$$

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- We now use the stress tensor in the principal direction to simplify the calculations

$$\begin{aligned} N &= \hat{T}^n \cdot \hat{n} \\ &= T_i^n n_i \\ &= \sigma_{ji} n_j n_i \\ &= \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \end{aligned}$$

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- We also know that

$$\begin{aligned}
 |\hat{T}^n|^2 &= \hat{T}^n \cdot \hat{T}^n \\
 &= T_i^n T_i^n \\
 &= \sigma_{ji} n_j \sigma_{ki} n_k \\
 &= \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2
 \end{aligned}$$

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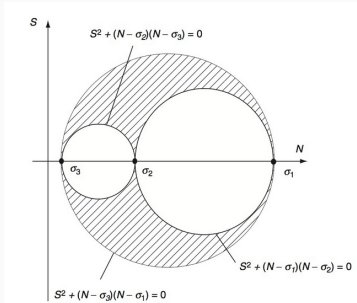
## mohr's circle

- If we constrain the normal vector to be a unit vector we can formulate the following inequalities

$$\begin{aligned}
 S^2 + (N - \sigma_2)(N - \sigma_3) &\geq 0 \\
 S^2 + (N - \sigma_3)(N - \sigma_1) &\leq 0 \\
 S^2 + (N - \sigma_1)(N - \sigma_2) &\geq 0
 \end{aligned}$$

- These inequalities form what is known as Mohr's circle

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## material behavior

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- Hooke's Law
- Physical meaning of elastic constants
- Thermal expansion

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## hooke's law

- Can be written in terms of strain

$$\epsilon_{ij} = \frac{1 + \nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij}$$

- Or stress

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

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- Young's modulus
- Poisson's ratio
- Shear modulus
- Bulk modulus

## thermoelasticity

- Separate strain into mechanical and thermal components

$$\epsilon_{ij} = \epsilon_{ij}^M + \epsilon_{ij}^T$$

- For isotropic materials:

$$\epsilon_{ij} = \alpha(T - T_0)\delta_{ij}$$

- We can combine this with Hooke's Law to find

$$\epsilon_{ij} = \frac{1 + \nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij} + \alpha (T - T_0) \delta_{ij}$$

- Or formulated in terms of stress (and Lamé constants)

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} - (3\lambda + 2\mu) \alpha (T - T_0) \delta_{ij}$$

## problem formulation

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- Boundary conditions
- Compatibility
- Beltrami-Michell
- Navier's Equations
- Superposition

## boundary conditions

- Traction
- Displacement
- Mixed

- If continuous, single-valued displacements are specified, differentiation will result in well-behaved strain field
- The inverse relationship, integration of a strain field to find displacement, may not always be true
- There are cases where we can integrate a strain field to find a set of discontinuous displacements

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## compatibility

- The compatibility equations enforce continuity of displacements to prevent this from occurring
- To enforce this condition we consider the strain-displacement relations:

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

- and differentiate with respect to  $x_k$  and  $x_l$

$$\epsilon_{ij,kl} = \frac{1}{2}(u_{i,jkl} + u_{j,ikl})$$

- Or

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- We can eliminate the displacement terms from the equation by interchanging the indexes to generate new equations

$$2\epsilon_{ik,jl} = u_{i,jkl} + u_{k,ijl}$$

$$2\epsilon_{jl,ik} = u_{j,ikl} + u_{l,ijk}$$

- Solving for  $u_{i,jkl}$  and  $u_{j,ikl}$

$$u_{i,jkl} = 2\epsilon_{ik,jl} - u_{k,ijl}$$

$$u_{j,ikl} = 2\epsilon_{jl,ik} - u_{l,ijk}$$

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- Substituting these values into the equations gives

$$2\epsilon_{ij,kl} = 2\epsilon_{ik,jl} - u_{k,ijl} + 2\epsilon_{jl,ik} - u_{l,ijk}$$

- We now consider one more change of index equation

$$2\epsilon_{kl,ij} = u_{k,ijl} + u_{l,ijk}$$

- and substituting this result gives

$$2\epsilon_{ij,kl} = 2\epsilon_{ik,jl} + 2\epsilon_{jl,ik} - 2\epsilon_{kl,ij}$$

- Or, simplified

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- When working with stress functions, it is convenient to check compatibility of the stress function directly
- Using Hooke's Law, we can formulate compatibility in terms of stress
- These are known as the Beltrami-Michell equations

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## navier's equations

- Similarly, we can write the equilibrium equations in terms of displacement
- This is convenient when dealing with displacement boundary conditions
- Known as Navier's equations

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