Theory of Elasticity

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upcoming schedule

- Oct 7 - Exam 2 Review
- Oct 8 - Homework 4 Self-grade Due, Homework 5 Due
- (Oct 12) - Fall Break (No Class)
- Oct 14 - Exam 2
- Oct 19 - Strain Energy
- Oct 21 - Virtual Work

outline

- exam
- group problems
- stress and equilibrium
- material behavior
- problem formulation

exam

exam format

- Similar format to last exam
- Four problems
- Focus on organizing your work clearly to maximize partial credit

group problems

problem one - thermoelasticity

As a first-order model of the problem of freezing water in a glass bottle, we treat water as a thermoelastic solid and the glass as a fixed boundary. Find the stress and strain field in the water as a function of the elastic properties (E, ν) and the coefficient of thermal expansion (α).

problem two - inverse solution

Consider the stress field

$$\sigma = \begin{bmatrix} Ay & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Show that this is a valid solution to an elasticity problem. What problem does it solve?

problem three - semi-inverse

To solve the problem of torsion in prismatic bars we consider the displacement field

$$u = -\alpha yz, \qquad v = -\alpha xz, \qquad w = w(x, y)$$

Solve this problem using the boundary conditions for a solid square cross-section.

stress and equilibrium

topics

- Traction
- Stress transformation
- Principal stress
- Equilibrium

derivations

- Cauchy's stress theorem
- Max shear stress for plane stress
- Mohr's circle

stress tensor

- To simplify the notation, we introduce the stress tensor

$$\sigma_{ij} = t_j^{(\hat{e}_i)}$$

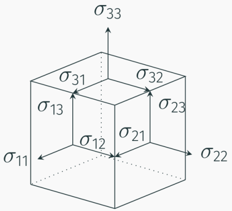


Figure 1: stress tensor illustrated on a cube

traction

- We can find some interesting information about the traction vector by considering an arbitrary tetrahedron with some traction $\hat{t}^{(n)}$ applied to the surface

traction

- If we consider the balance of forces in the x_1-direction

$$t_1 dA - \sigma_{11} dA_1 - \sigma_{21} dA_2 - \sigma_{31} dA_3 + b_1 \rho dV = 0$$

- The area components are:

$$\begin{aligned} dA_1 &= n_1 dA \\ dA_2 &= n_2 dA \\ dA_3 &= n_3 dA \end{aligned}$$

- And $dV = \frac{1}{3} h dA$.

traction

$$t_1 dA - \sigma_{11} n_1 dA - \sigma_{21} n_2 dA - \sigma_{31} n_3 dA + b_1 \rho \frac{1}{3} h dA = 0$$

- If we let $h \to 0$ and divide by dA

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$

- We can write this in index notation as

$$t_1 = \sigma_{i1} n_i$$

traction

- We find, similarly

$$t_2 = \sigma_{i2} n_i$$
$$t_3 = \sigma_{i3} n_i$$

traction

- We can further combine these results in index notation as

$$t_j = \sigma_{ij} n_i$$

- This means with knowledge of the nine components of σ_{ij}, we can find the traction vector at any point on any surface

maximum shear stress

- For plane stress problems, we can also use the stress transformation equations to find the maximum shear stress
- We desire to maximize this equation:

$$\tau'_{xy} = \frac{\sigma_y - \sigma_x}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

maximum shear stress

- Taking the derivative with respect to θ gives

$$\frac{\partial}{\partial\theta}(\tau'_{xy}) = (\sigma_y - \sigma_x) \cos 2\theta - 2\tau_{xy} \sin 2\theta = 0$$

- Which we can use to find 2θ

$$2\theta = \tan^{-1}\left(\frac{(\sigma_y - \sigma_x)}{2\tau_{xy}}\right)$$

maximum shear stress

- Substituting back into the original equation gives

$$\tau'_{max} = \frac{\sigma_y - \sigma_x}{2} \sin\left[\tan^{-1}\left(\frac{(\sigma_y - \sigma_x)}{2\tau_{xy}}\right)\right] + \tau_{xy} \cos\left[\tan^{-1}\left(\frac{(\sigma_y - \sigma_x)}{2\tau_{xy}}\right)\right]$$

- Note that

$$\sin(\tan^{-1}(x)) = \frac{x}{\sqrt{1+x^2}}$$
$$\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$$

maximum shear stress

- We note that

$$\sqrt{1 + \left(\frac{\sigma_y - \sigma_x}{2\tau_{xy}}\right)^2} = \frac{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2}}{2\tau_{xy}}$$

- And thus we find

$$\tau_{max} = \frac{(\sigma_y - \sigma_x)^2}{2\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2}} + \frac{4\tau_{xy}^2}{2\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2}}$$

maximum shear stress

- Adding the terms and simplifying, we find

$$\tau_{max} = \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

tractions

- We can use what we know about principal values to find some interesting things about the tractions
- Consider the traction vector on an arbitrary internal face, and decompose into Normal and Shear components.

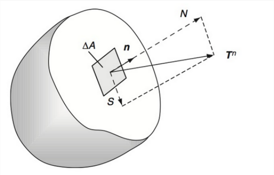


Figure 2: arbitrary body with arbitrary loading applied

tractions

- The normal component can be found using the dot product

$$N = \hat{T}^n \cdot \hat{n}$$

- The shear component can be found using the Pythagorean theorem

$$S^2 = |\hat{T}^n|^2 - N^2$$

tractions

- We now use the stress tensor in the principal direction to simplify the calculations

$$\begin{aligned} N &= \hat{T}^n \cdot \hat{n} \\ &= T_i^n n_i \\ &= \sigma_{ji} n_j n_i \\ &= \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \end{aligned}$$

tractions

- We also know that

$$
\begin{aligned}
|\hat{T}^n|^2 &= \hat{T}^n \cdot \hat{T}^n \\
&= T_i^n T_i^n \\
&= \sigma_{ji} n_j \sigma_{ki} n_k \\
&= \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2
\end{aligned}
$$

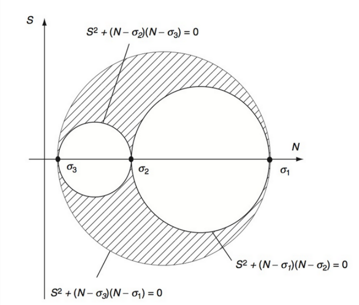
mohr's circle

- If we constrain the normal vector to be a unit vector we can formulate the following inequalities

$$
\begin{aligned}
S^2 + (N - \sigma_2)(N - \sigma_3) \geq 0 \\
S^2 + (N - \sigma_3)(N - \sigma_1) \leq 0 \\
S^2 + (N - \sigma_1)(N - \sigma_2) \geq 0
\end{aligned}
$$

- These inequalities form what is known as Mohr's circle

mohr's circle



material behavior

topics

- Hooke's Law
- Physical meaning of elastic constants
- Thermal expansion

hooke's law

- Can be written in terms of strain

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

- Or stress

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}$$

physical meaning

- Young's modulus
- Poisson's ratio
- Shear modulus
- Bulk modulus

thermoelasticity

- Separate strain into mechanical and thermal components

$$\epsilon_{ij} = \epsilon_{ij}^{M} + \epsilon_{ij}^{T}$$

- For isotropic materials:

$$\epsilon_{ij} = \alpha(T - T_0)\delta_{ij}$$

- We can combine this with Hooke's Law to find

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha(T - T_0)\delta_{ij}$$

- Or formulated in terms of stress (and Lamé constants)

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} - (3\lambda + 2\mu)\alpha(T - T_0)\delta_{ij}$$

problem formulation

topics

- Boundary conditions
- Compatibility
- Beltrami-Michell
- Navier's Equations
- Superposition

boundary conditions

- Traction
- Displacement
- Mixed

compatibility

- If continuous, single-valued displacements are specified, differentiation will result in well-behaved strain field
- The inverse relationship, integration of a strain field to find displacement, may not always be true
- There are cases where we can integrate a strain field to find a set of discontinuous displacements

compatibility

- The compatibility equations enforce continuity of displacements to prevent this from occurring
- To enforce this condition we consider the strain-displacement relations:

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

- and differentiate with respect to x_k and x_l

$$\epsilon_{ij,kl} = \frac{1}{2}(u_{i,jkl} + u_{j,ikl})$$

- Or

compatibility

- We can eliminate the displacement terms from the equation by interchanging the indexes to generate new equations

$$2\epsilon_{ik,jl} = u_{i,jkl} + u_{k,ijl}$$
$$2\epsilon_{jl,ik} = u_{j,ikl} + u_{l,ijk}$$

- Solving for $u_{i,jkl}$ and $u_{j,ikl}$

$$u_{i,jkl} = 2\epsilon_{ik,jl} - u_{k,ijl}$$
$$u_{j,ikl} = 2\epsilon_{jl,ik} - u_{l,ijk}$$

compatibility

- Substituting these values into the equations gives

$$2\epsilon_{ij,kl} = 2\epsilon_{ik,jl} - u_{k,ijl} + 2\epsilon_{jl,ik} - u_{l,ijk}$$

- We now consider one more change of index equation

$$2\epsilon_{kl,ij} = u_{k,ijl} + u_{l,ijk}$$

- and substituting this result gives

$$2\epsilon_{ij,kl} = 2\epsilon_{ik,jl} + 2\epsilon_{jl,ik} - 2\epsilon_{kl,ij}$$

- Or, simplified

beltrami-michell

- When working with stress functions, it is convenient to check compatibility of the stress function directly
- Using Hooke's Law, we can formulate compatibility in terms of stress
- These are known as the Beltrami-Michell equations

navier's equations

- Similarly, we can write the equilibrium equations in terms of displacement
- This is convenient when dealing with displacement boundary conditions
- Known as Navier's equations