# AE731

## Theory of Elasticity

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# upcoming schedule

- Sep 11 - Displacement and Strain
- Sep 16 - Exam Review, Homework 2 Due
- Sep 18 - Exam 1

# outline

- principal strains
- special strain definitions
- strain transformation
- exam
- review

# principal strains

# principal strains

- Principal strains are found in exactly the same way as principal values discussed in Chapter 1
  $\det[e_{ij} e\delta_{ij}] = 0$
- Invariants can also be found in the same fashion as in any other tensor

$$\vartheta_1 = e_1 + e_2 + e_3$$
$$\vartheta_2 = e_1 e_2 + e_2 e_3 + e_3 e_1$$
$$\vartheta_3 = e_1 e_2 e_3$$

# principal strains

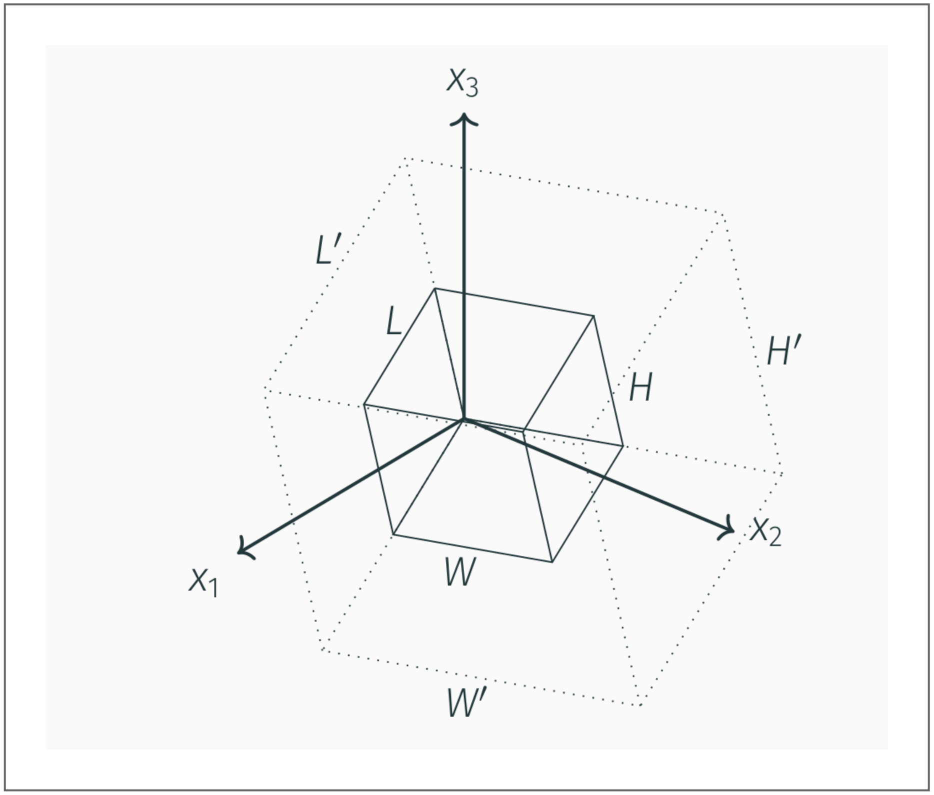
- Principal strains and invariants have some important physical meanings
- $\vartheta_1$ is called the *cubical dilation*, and is related to the change in volume of the material
- Note that in the principal direction, there are no shear strains

$$\begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix}$$

- This means that there is only extensional strain in the principal direction (i.e. a cube will remain a rectangular prism, the corners will maintain $90^\circ$ angles)

# volume change

- Consider a rectangular prism with edges oriented in the principal directions



# volume change

- The volume before deformation is $V = LWH$
- The volume after deformation is given by $V' = L'W'H'$
- We can relate the side lengths after deformation to strains

$$e_1 = \frac{L' - L}{L}$$
$$Le_1 + L = L'$$

- We can now write the volume as $V' = L(1 + e_1)W(1 + e_2)H(1 + e_3)$

# volume change

- After multiplication, the deformed volume is given as
- $V' = LWH(1 + e_1 + e_2 + e_3 + e_1e_2 + e_2e_3 + e_1e_3 + e_1e_2e_3)$
- For small strains, $e_i \ll 1$, therefore $e_1$, $e_2$, and $e_3$ will be much larger than $e_1e_2$, $e_2e_3$, $e_1e_3$ and $e_1e_2e_3$.
- $V' = LWH(1 + e_1 + e_2 + e_3)$

# volume change

- A "dilatation" is defined as the change in volume divided by the original volume
$$\frac{\Delta V}{V} = \frac{V' - V}{V}$$
- Substituting the relationships found earlier
$$\frac{V' - V}{V} = \frac{LWH(1 + e_1 + e_2 + e_3) - LWH}{LWH}$$
- Which simplifies to
$$e_1 + e_2 + e_3 = \vartheta_1$$

# special strain definitions

# spherical strain

- This dilatation can be used to find the so-called *spherical strain*

$$\tilde{e}_{ij} = \frac{1}{3} e_{kk} \delta_{ij} = \frac{1}{3} \vartheta_1 \delta_{ij}$$

- The *deviatoric strain* is found by subtracting the spherical strain from the strain tensor

  $$\hat{e}_{ij} = e_{ij} - \frac{1}{3} e_{kk} \delta_{ij}$$

# strain transformation

# strain transformation

- The usual tensor transformation applies to the strain tensor as well
- $e_{ij}' = Q_{im}Q_{jn}e_{mn}$
- In many instances, however, we are only concerned with the strain within a plane (for example, when using strain gages).

# strain transformation

- For an in-plane rotation (rotation about *z*-axis), we find

$$Q_{ij} = \begin{bmatrix} \cos\theta & \cos(90-\theta) & \cos 90 \\ \cos(90+\theta) & \cos\theta & \cos 90 \\ \cos 90 & \cos 90 & \cos 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# strain transformation

- If we multiply this out, for the in-plane strain terms ($e_x{}'$, $e_x{}'$, and $e_{xy}{}'$) we find

$$\begin{aligned} e'_x &= e_x \cos^2\theta + e_y \sin^2\theta + 2e_{xy}\sin\theta\cos\theta \\ e'_y &= e_x \sin^2\theta + e_y \cos^2\theta - 2e_{xy}\sin\theta\cos\theta \\ e'_{xy} &= -e_x \sin\theta\cos\theta + e_y \sin\theta\cos\theta + e_{xy}(\cos^2\theta - \sin^2\theta) \end{aligned}$$

# strain transformation

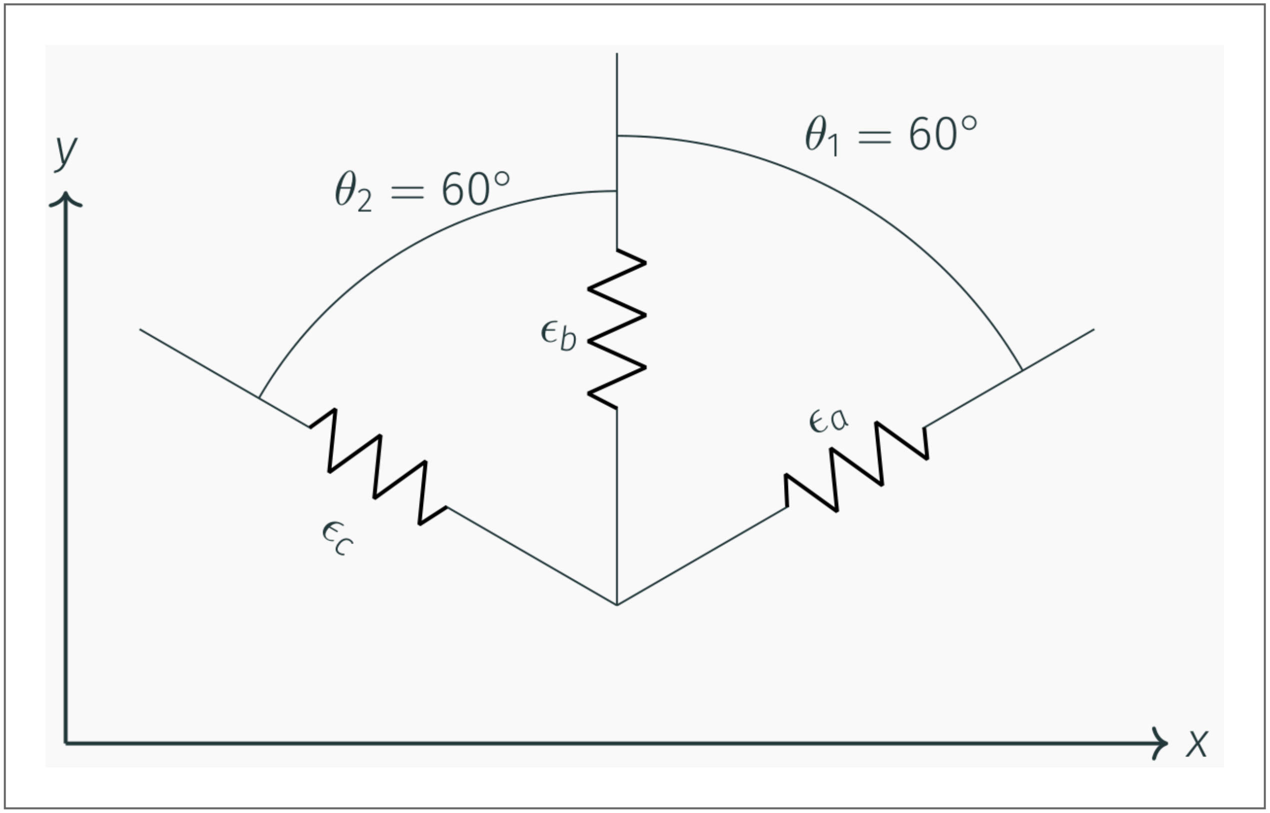
- This is often re-written using the double-angle formulas

$$
\begin{aligned}
e'_x &= \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2}\cos 2\theta + e_{xy}\sin 2\theta \\
e'_y &= \frac{e_x + e_y}{2} - \frac{e_x - e_y}{2}\cos 2\theta - e_{xy}\sin 2\theta \\
e'_{xy} &= \frac{e_y - e_x}{2}\sin 2\theta + e_{xy}\cos 2\theta
\end{aligned}
$$

# strain transformation

- Many times it is easy to measure the axial strain directly with strain gages, but the shear strain cannot be easily measured
- We can use an extra, off-axis strain gage, together with the strain transformation equations, to calculate the shear strain
- Many companies already do this with “rosettes” which have strain gages at specified angles built-in

# example



# example

- Given that $\epsilon_a = 0.005$, $\epsilon_b = -0.002$ and $\epsilon_c = 0.003$, find $e_x$, $e_y$, and $e_{xy}$.
- Note that $e_y = \epsilon_b = -0.002$
- Set coordinate system so that $\epsilon_b = e_x{}'$.
- Use equation for $e_x{}'$ with $\theta = 30$.

$$e_x' = \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2}\cos 60 + e_{xy}\sin 60$$

## example

- We have two unknowns in this equation, so we need another
- We can use the equation for ${e_y}'$ with $\theta = 60$ so that $\epsilon_b = {e_x}'$

$$e_y' = \frac{e_x + e_y}{2} - \frac{e_x - e_y}{2} \cos 120 - e_{xy} \sin 120$$

# example

- Substituting known values and simplifying:

$$
\begin{aligned}
0.01 + 0.002 - 0.002\cos 60 &= e_x(1+\cos 60) + e_{xy}\sin 60 \\
0.006 + 0.002 + 0.002\cos 120 &= e_x(1-\cos 120) - e_{xy}\sin 120
\end{aligned}
$$

- And solving we find $e_x = 0.006$, $e_y = -0.002$, and $e_{xy} = 0.00231$.

# exam

## exam preparations

- Exams from 2015 and 2017 are posted on Blackboard and the class website
- This year’s exam will only be 5 problems
- No equation sheet for the first exam, should remember things like coordinate transformation, Kronecker Delta
- I do not expect you to remember things like invariants, the alternating symbol, or the strain transformation equations

# exam curve example

- I grade exams problem by problem to try to avoid any bias and keep grading consistent
- Partial credit I give is meant to translate to what percentage of the problem did you do correctly
- Curve is primarily intended to correct for me writing/grading too hard (a perfect exam would have no curve)

# exam tips

- Don’t cheat
- Don’t get hung up on one problem, be mindful of the time
- If you do not know exactly how to calculate a problem, try to show that you understand the big picture
- Even when you do understand how to calculate a problem, clearly illustrate the “big picture” in a way that is clear and easy to find

# review

# topics

- Chapter 1
  - Index notation
  - Solving tensor equations
  - Coordinate transformation
  - Principal values
  - Principal directions
  - Tensor calculus

# topics

- Chapter 2
  - Deformations
  - Deformation gradient
  - Strain and rotation
  - Strain transformation
  - Principal strains

# index notation

1. Free indexes (subscript letters not repeated in a term)
2. Dummy index (subscript letters repeated in a term)
3. Rules
    1. Indexes cannot repeat more than twice
    2. Free indexes must match on either side of equation
    3. Dummy index cannot be used as a free index

## converting to matrix

- Sometimes our expression is more complex (involves more terms)
- e.g. transformation of a matrix $a_{ij}{}' = Q_{ip}Q_{jq}a_{pq}$
  1. Re-arrange so dummy indexes are adjacent $Q_{ip}a_{pq}Q_{jq}$
  2. Identify which (if any) tensors are transposed (dummy indexes should be on the inside of adjacent terms without a transpose)

  $Q_{ip}\, a_{\,pq}\, \mathrm{Q}_{\,jq}$
  $[Q][a][Q]^T$

# example

- Convert the expression in index notation to Matrix notation
  $A_{ik}B_{jl}C_{ml}D_{mk}$

  1. Re-arrange to so that dummy indexes are in adjacent terms

     $$A_{ik}D_{mk}C_{ml}B_{jl}$$

  2. Identify which terms are transposed

     $$A_{ik}\,D_{mk}C_{\ ml}\,B_{jl}$$
     $$[A][D]^T[C][B]^T$$

# solving

- Solve the following equation for $E_k$ in terms of $C_{ij}$, $V_{ij}$, and $a_i$.
  $E_k\delta_{ik} = C_{kj}(V_{ij}a_k - E_k\delta_{ij})$
- Solve the following equation for $U_k$ in terms of $a_i$ and $P_j$
  $$\mu\left\{\delta_{kj}a_ia_i + \frac{1}{1-2\nu}a_ka_j\right\}U_k = P_j$$
  Hint: First solve for $U_ka_k$, then substitute that relationship to solve for $U_k$
- Solve the following equation for $A_{ij}$ in terms of $B_{ij}$
  $B_{ij} = A_{ij} + A_{kk}\delta_{ij}$
  Hint: First solve for $A_{kk}$ in terms of $B_{ij}$, then substitute that to solve for $A_{ij}$

# transformation

- We can express any tensor quantity in terms of a rotated coordinate system
- The direction cosines help to find the coordinates in the transformed system

  $Q_{ij} = \cos(x_i{}', x_j)$
- Any-order tensor can be expressed in this form

| | |
|---|---|
| $a' = a$ | zero order, scalar |
| $a'_i = Q_{ip} a_p$ | first order, vector |
| $a'_{ij} = Q_{ip} Q_{jq} a_{pq}$ | second order, matrix |
| $a'_{ijk} = Q_{ip} Q_{jq} Q_{kr} a_{pqr}$ | third order |
| $a'_{ijkl} = Q_{ip} Q_{jq} Q_{kr} Q_{lo} a_{pqro}$ | fourth order |

# principal values

- In the 2D coordinate transformation example, we were able to eliminate one value from a vector using coordinate transformation
- For second-order tensors, we desire to find the “principal values” where all non-diagonal terms are zero
- The direction determined by the unit vector, $n_j$, is said to be the *principal direction* or *eigenvector* of the symmetric second-order tensor, $a_{ij}$ if there exists a parameter, $\lambda$, such that
  $a_{ij}n_j = \lambda n_i$
- Where $\lambda$ is called the *principal value* or *eigenvalue* of the tensor

# principal values

- We can re-write the equation
  $(a_{ij} - \lambda\delta_{ij})n_j = 0$
- This system of equations has a non-trivial solution if and only if
  $\det[a_{ij} - \lambda\delta_{ij}]=0$
- This equation is known as the characteristic equation, and we solve it to find the principal values of a tensor

# example

- Find principal values, principal directions, and invariants for the tensor

$$c_{ij} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

# example

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & 4-\lambda \end{vmatrix} = 0$$

# example

$$\begin{aligned} (1-\lambda)[(2-\lambda)(4-\lambda)-0]-0+2[0-2(2-\lambda)] &= 0 \\ (1-\lambda)(2-\lambda)(4-\lambda)-4(2-\lambda) &= 0 \end{aligned}$$

$$\begin{aligned} (2-\lambda)[(1-\lambda)(4-\lambda)-4] &= 0 \\ (1-\lambda)(4-\lambda)-4 &= 0 \\ \lambda^2-5\lambda &= 0 \end{aligned}$$

$\lambda = 5, 2, 0$

# example

- To find the principal direction for $\lambda_1 = 5$

$$\begin{bmatrix} 1-5 & 0 & 2 \\ 0 & 2-5 & 0 \\ 2 & 0 & 4-5 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

$$\begin{bmatrix} -4 & 0 & 2 \\ 0 & -3 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

# example

- To row-reduce, we can multiply row 3 by 2

$$\begin{bmatrix} -4 & 0 & 2 \\ 0 & -3 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

- Now we add this to row 1, which cancels all terms in row 1

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 2 & 0 & -1 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

# example

- We are now left with only two equations

$$-3n_2 = 0$$
$$2n_1 - n_3 = 0$$

- We know that $n_2 = 0$. If we let $n_3 = 1$, we find $n_1 = \frac{1}{2}$

$$n^1 = \langle \frac{1}{2}, 0, 1 \rangle$$

# example

- To find the principal direction for $\lambda_2 = 2$

$$\begin{bmatrix} 1-2 & 0 & 2 \\ 0 & 2-2 & 0 \\ 2 & 0 & 4-2 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

# example

- To row-reduce, we can multiply row 1 by 2

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

- Now we add this to row 3, which cancels the first term

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

# example

- We are left with two equations

$$\begin{aligned} -n_1 + 2n_3 &= 0 \\ 4n_3 &= 0 \end{aligned}$$

- We know that $n_3 = 0$, this also gives $n_1 = 0$.
- $n_2$ can be any value, we choose $n_2 = 1$
- $n^2 = \langle 0, 1, 0 \rangle$

# example

- The third principal direction can be found two ways
- We can either use the same method or use the cross-product $n^3 = n^1 \times n^2$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

- After row-reduction

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

- $n^3 = \langle -2, 0, 1 \rangle$

# partial derivatives

- We usually omit the $(x_i)$, but most variables we deal with are functions of $x_i$
- These are referred to as field variables. e.g.

$$
\begin{aligned}
2a &= a(x_1, x_2, x_3) && = a(x_i) \\
a_i &= a_i(x_1, x_2, x_3) && = a_i(x_i) \\
a_{ij} &= a_{ij}(x_1, x_2, x_3) && = a_{ij}(x_i)
\end{aligned}
$$

# partial derivatives

- We can use comma notation to simplify taking partial derivatives of field variables

$$
\begin{aligned}
a_{,i} &= \frac{\partial}{\partial x_i} a \\
a_{i,j} &= \frac{\partial}{\partial x_j} a_i \\
a_{ij,k} &= \frac{\partial}{\partial x_k} a_{ij}
\end{aligned}
$$

# tensor calculus

- Let $f$ be a scalar-valued function such that $f(x_i) = \sqrt{x_i x_i}$. Find $f_{,k}$
- We can use the chain rule to differentiate
- $f_{,k} = \frac{1}{2}(x_i x_i)^{-\frac{1}{2}}(x_i x_i)_{,k}$
- Using the chain rule again to compute $(x_i x_i)_{,k}$ we find
- $f_{,k} = \frac{1}{2}(x_i x_i)^{-\frac{1}{2}}(x_i x_{i,k} + x_{i,k} x_i)$
- Simplifying $f_{,k} = (x_i x_i)^{-\frac{1}{2}}(x_i x_{i,k})$
- The partial derivative, $\frac{\partial x_i}{\partial x_k} = \delta_{ik}$
- Substituting $f_{,k} = (x_i x_i)^{-\frac{1}{2}}(x_i \delta_{ik})$
- We also know that $x_i \delta_{ik} = x_k$, and we can also substitute $f = \sqrt{x_i x_i}$ to find
- $f_{,k} = \frac{x_k}{f}$

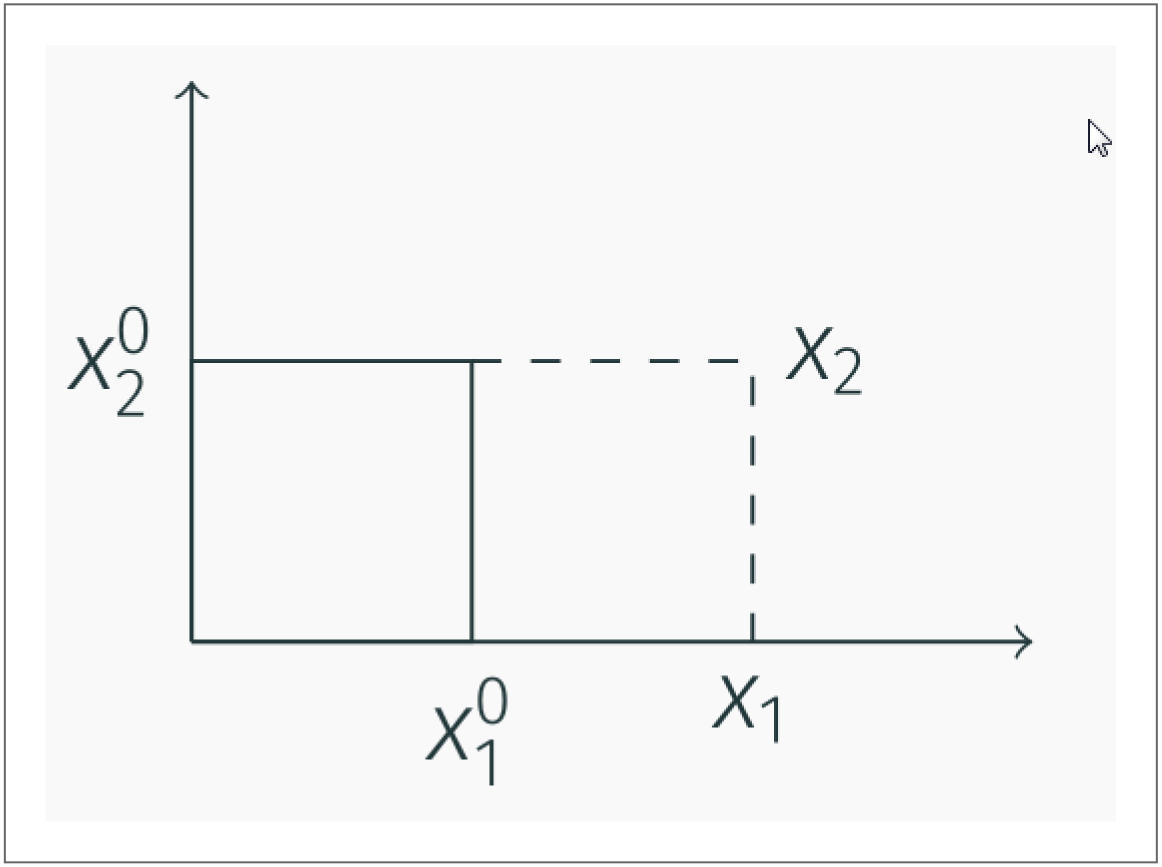
# deformation

- A *deformation* is a comparison of two states. The deformation of a material point is expressed as

$$x_i = x_i(x_1^0, x_2^0, x_3^0) \quad \text{or} \quad x_i^0 = x_i^0(x_1, x_2, x_3)$$

- For example, consider the 2D deformation

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 2x_1^0 \\ x_2^0 \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} x_1^0 \\ x_2^0 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}x_1 \\ x_2 \end{Bmatrix}$$



$X_2^0$
$X_2$
$X_1^0$
$X_1$

# displacement

- Displacement is the shortest distance traveled when a particle moves from one location to another
- It is identical in Eulerian and Lagrangian descriptions

$$u_i = (x_i - x_i^0)$$

$$\left\{\begin{array}{c} u \\ v \\ w \end{array}\right\} = \left\{\begin{array}{c} x - x^0 \\ y - y^0 \\ z - z^0 \end{array}\right\}$$

# deformation gradients

- In index notation we write displacements as $u_i$
- The deformation gradient can be written in this notation as

$$F = u_{i,j} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

# strain definition

- We can separate the deformation gradient into symmetric and antisymmetric parts
- $u_{i,j} = e_{ij} + \omega_{ij}$
- Where

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$
$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$$

- $e_{ij}$ is known as the strain tensor
- $\omega_{ij}$ is known as the rotation tensor

# example

- Find the displacements given the following strain tensor

$$e_{ij} = \begin{bmatrix} yz & xz & xy \\ xz & 2y & \frac{1}{2}x^2 \\ xy & \frac{1}{2}x^2 & 3z^2 \end{bmatrix}$$

- We start by integrating the diagonal terms
  $u = \int yzdx = xyz + f(y,z)$
  $v = \int 2ydy = y^2 + g(x,z)$
  $w = \int 3z^2dz = z^3 + h(x,y)$

# example

- Next we need to find the shear terms

$$
\begin{aligned}
e_{xy} &= \frac{1}{2}(u_{,y} + v_{,x}) \\
xz &= \frac{1}{2}(xz + f_{,y} + g_{,x})
\end{aligned}
$$

$$
\begin{aligned}
e_{xz} &= \frac{1}{2}(u_{,z} + w_{,x}) \\
xy &= \frac{1}{2}(xy + f_{,z} + h_{,x})
\end{aligned}
$$

$$
\begin{aligned}
e_{yz} &= \frac{1}{2}(v_{,z} + w_{,y}) \\
\frac{1}{2}x^2 &= \frac{1}{2}(g_{,z} + h_{,y})
\end{aligned}
$$

# example

- Note that we cannot uniquely solve this (any arbitrary rotation $\omega_{ij}$ can be added and will still produce a valid strain)
- Assume $\omega_{ij} = 0$

$$
\begin{aligned}
\frac{1}{2}(u_{,y} - v_{,x}) &= 0 \\
xz + f_{,y} - g_{,x} &= 0 \\
f_{,y} &= g_{,x} - xz
\end{aligned}
$$

# example

- We can now substitute this in the $e_{xy}$ expression

$$
\begin{aligned}
e_{xy} &= \frac{1}{2}(u_{,y} + v_{,x}) \\
xz &= \frac{1}{2}(xz + f_{,y} + g_{,x}) \\
2xz &= xz + g_{,x} - xz + g_{,x}) \\
2g_{,x} &= 2xz \\
g(x,z) &= \frac{1}{2}x^2 z + g_2(z)
\end{aligned}
$$

- We can substitute this into the rotation expression to find $f_{,y}$

$$
\begin{aligned}
f_{,y} &= g_{,x} - xz \\
f_{,y} &= xz - xz \\
f(y,z) &= f_2(z)
\end{aligned}
$$

# example

- Next we consider $\omega_{xz}$

$$
\begin{aligned}
\frac{1}{2}(u_{,z} - w_{,x}) &= 0 \\
xy + f_{,z} - h_{,x} &= 0 \\
h_{,x} &= xy + f_{,z}
\end{aligned}
$$

- Substituting this into $e_{xz}$ gives

$$
\begin{aligned}
e_{xz} &= \frac{1}{2}(xy + f_{,z} + xy + f_{,z}) \\
xy &= xy + f_{,z} \\
f_{,z} &= 0 \\
f_2(z) &= 0
\end{aligned}
$$

# example

- Substituting back into $\omega_{xz}$ we find

$$\begin{aligned} h_{,x} &= xy + f_{,z} \\ h_{,x} &= xy \\ h(x,y) &= \frac{1}{2}x^2y + h_2(y) \end{aligned}$$

- The last term to consider is $\omega_{yz}$

$$\begin{aligned} \frac{1}{2}(v_{,z} - w_{,y}) &= 0 \\ \frac{1}{2}x^2 + g_{2,z} - (\frac{1}{2}x^2 + h_{2,y}) &= 0 \\ \frac{1}{2}x^2 + g_{2,z} &= \frac{1}{2}x^2 + h_{2,y} \end{aligned}$$

# example

- Substituting into $e_{yz}$

$$
\begin{aligned}
e_{yz} &= \frac{1}{2}(v_{,z} + w_{,y}) \\
\frac{1}{2}x^2 &= \frac{1}{2}(\frac{1}{2}x^2 + g_{2,z} + \frac{1}{2}x^2 + h_{2,y}) \\
\frac{1}{2}x^2 &= \frac{1}{2}x^2 + g_{2,z} \\
g_{2,z} &= 0 \\
g(x, z) &= \frac{1}{2}x^2 z
\end{aligned}
$$

- And substituting back into $\omega_{yz}$

$$
\begin{aligned}
\frac{1}{2}x^2 &= \frac{1}{2}x^2 + h_{2,y} \\
h_(2, y) &= 0 \\
h_2(y) &= 0
\end{aligned}
$$

# example

$$u = xyz$$

$$v = y^2 + \frac{1}{2}x^2 z$$

$$w = z^3 + \frac{1}{2}x^2 y$$