AE731

Theory of Elasticity

Dr. Nicholas Smith
Wichita State University, Department of Aerospace Engineering
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upcoming schedule

- Sep 4 - Displacement and Strain, Homework 1 Due
- Sep 9 - Strain Transformation
- Sep 11 - Displacement and Strain
- Sep 16 - Exam Review, Homework 2 Due
- Sep 18 - Exam 1

outline

- general deformation
- small deformation theory
- strain

chapter outline

- General description of deformations
- Assumptions for small deformations
- Definition of strain
- Strain transformation
- Principal strains
- Strain compatibility
- Strain in cylindrical and spherical coordinates

general deformation

general deformation

- When deformations are large, the deformed and un-deformed shapes can be quite different
- It can be convenient to refer to material properties in the deformed or un-deformed configuration
- Lagrangian reference: quantities are in terms of the original (un-deformed) configuration
- Eulerian reference: quantities are in terms of deformed configuration

material derivatives

- We refer to the undeformed configuration as x_i^{o} and the deformed configuration as x_i
- If some quantity, ϕ is expressed in the undeformed configuration as $\phi(x_1{}^{\text{o}}, x_2{}^{\text{o}}, x_3{}^{\text{o}}, t)$ then the material derivative is
$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t}$$

material derivatives

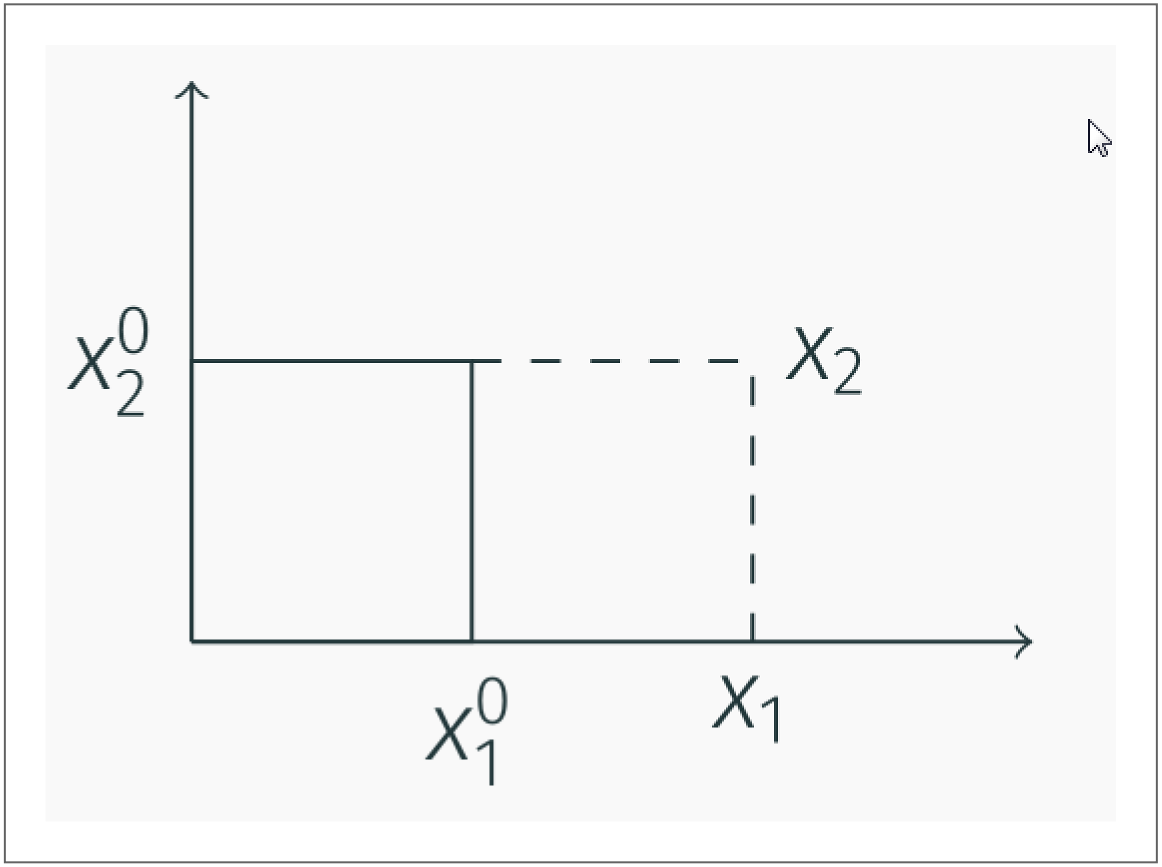
- However in Eulerian form $\bar{\phi}(x_1, x_2, x_3, t) = \phi(x_1^0, x_2^0, x_3^0, t)$ the material derivative becomes $$\frac{d\bar{\phi}}{dt} = \frac{\partial\bar{\phi}}{\partial t} + \frac{\partial\bar{\phi}}{\partial x_j}\frac{dx_j}{dt}$$

deformation

- A *deformation* is a comparison of two states. The deformation of a material point is expressed as
  $x_i = x_i(x_1^0, x_2^0, x_3^0) \quad \text{or} \quad x_i^0 = x_i^0(x_1, x_2, x_3)$
- For example, consider the 2D deformation

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 2x_1^0 \\ x_2^0 \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} x_1^0 \\ x_2^0 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}x_1 \\ x_2 \end{Bmatrix}$$

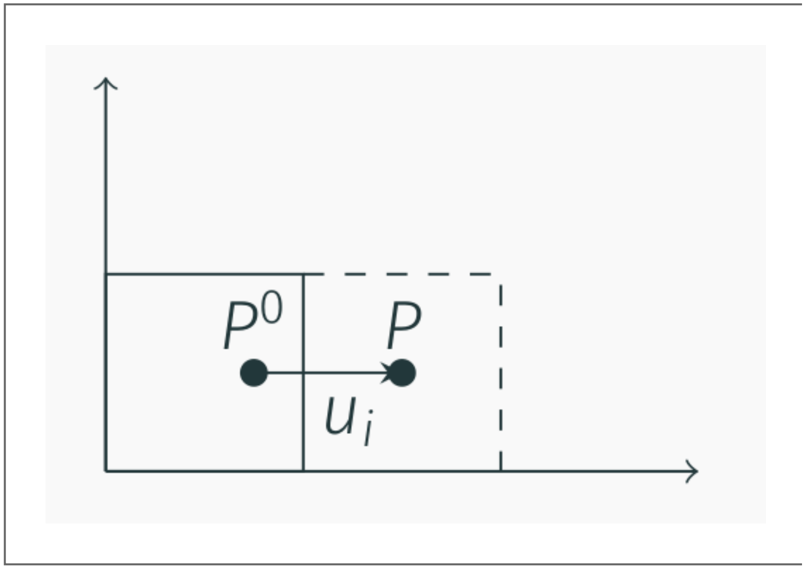
deformation



displacement

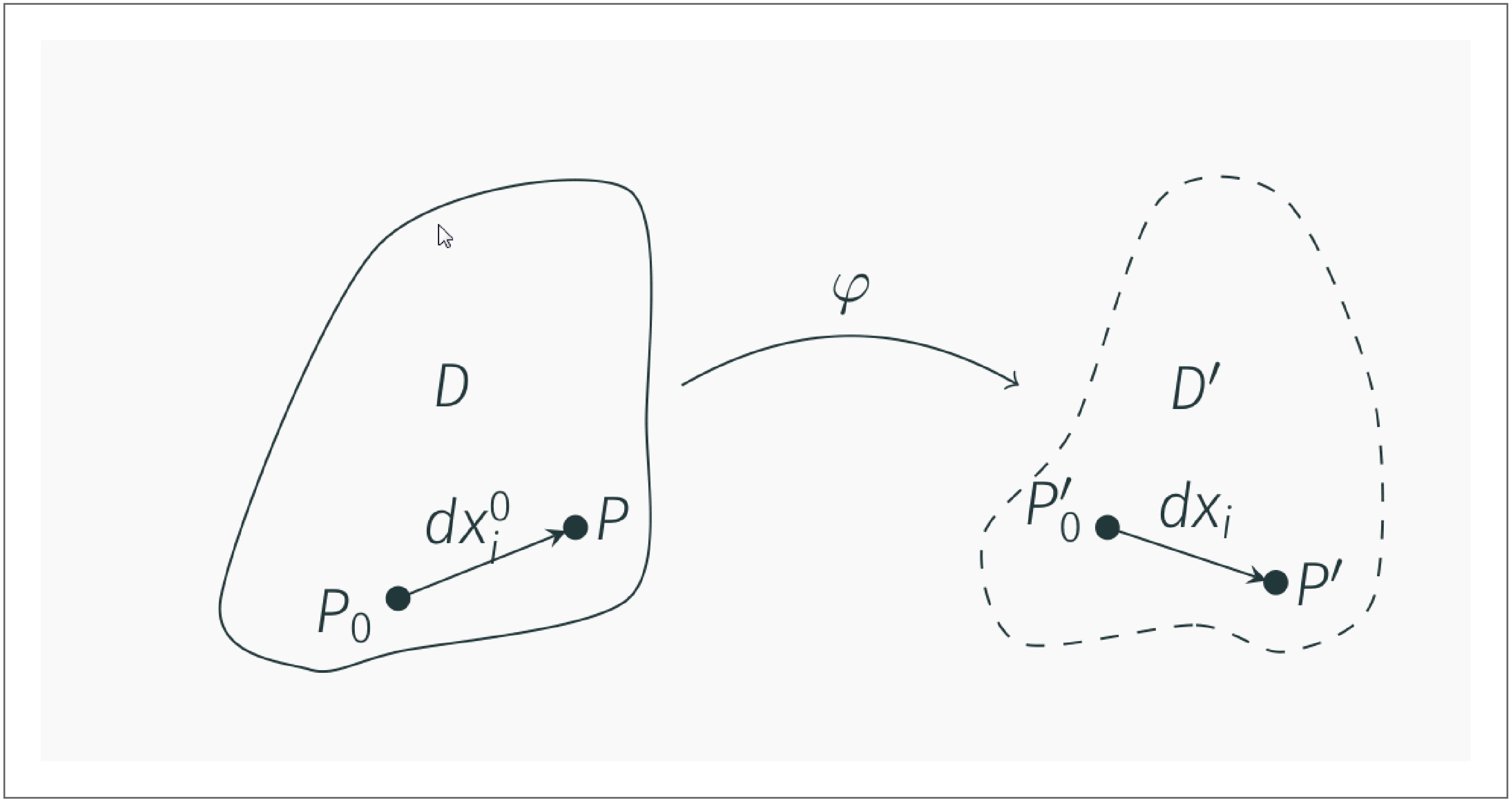
- A displacement is the shortest distance traveled when a particle moves from one location to another
- Displacement is identical in Eulerian and Lagrangian descriptions

  $u_i = (x_i - x_i^{\mathrm{o}})$



small deformation theory

deformation gradients



deformation gradients

- The position of the two points, $P_0{}'$ and P', is related by

$$P_0' = x_i(x_i^0)$$
$$P' = x_i + dx_i \qquad = x_i(x_i^0 + dx_i^0)$$

- We can approximate $x_i(x_i^0 + dx_i^0)$ with a Taylor series expansion

$$\approx x_i(x_i^0) + \frac{\partial x_i^0}{\partial x_j} dxj^0 + \frac{1}{2}\frac{\partial^2 x_i}{\partial x_j^0 \partial x_k^0} dx_j^0 dx_k^0 + \ldots$$

deformation gradients

- If the deformation is small, we can neglect higher-order terms of the expansion

$$P' = x_i + dx_i = x_i(x_i^0) + \frac{\partial x_i^0}{\partial x_j} dxj^0$$

- Which gives

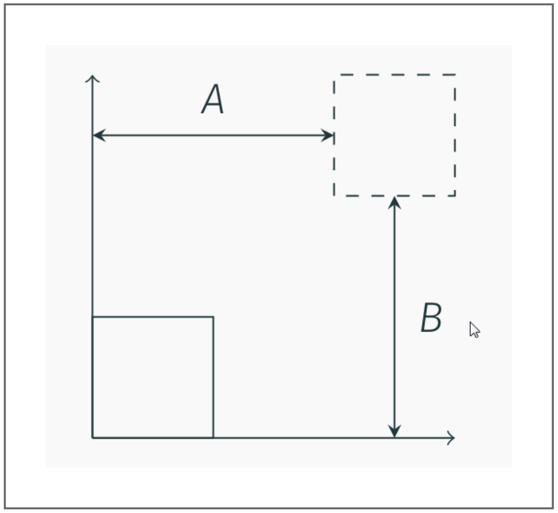
$$dx_i = \frac{\partial x_i^0}{\partial x_j} dx_j^0$$

deformation gradients

- In index notation we write displacements as u_i
- The deformation gradient can be written in this notation as

$$F = u_{i,j} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

translation



- x-displacement
  $u_1 = x_1{}^o + A$
- y-displacement
  $u_2 = x_2{}^o + B$

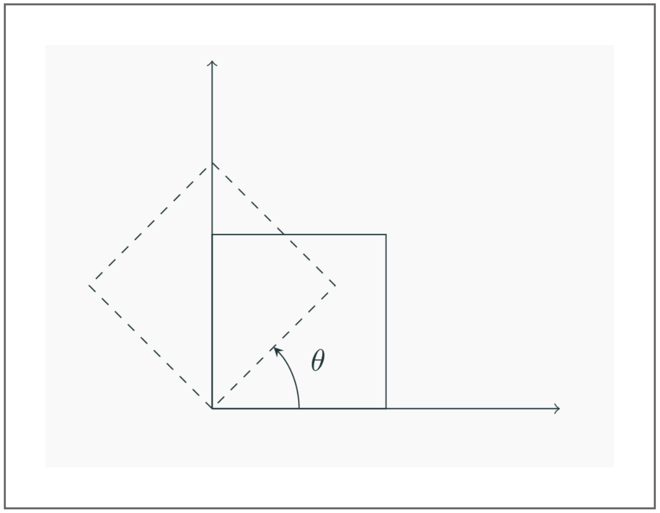
translation

- Deformation gradient

$$F = u_{i,j} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

rotation



- x-displacement
  $u_1 = x_1^{o}\cos\theta - x_2^{o}\sin\theta$
- y-displacement
  $u_2 = x_1^{o}\sin\theta + x_2^{o}\cos\theta$

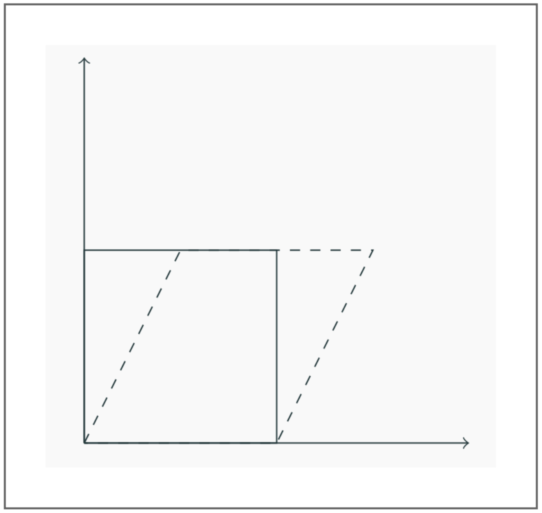
rotation

- Deformation gradient

$$F = u_{i,j} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix}$$

$$F = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

simple shear



- x-displacement
  $u_1 = x_1^0 + \frac{1}{2}x_2^0$
- y-displacement
  $u_2 = x_2{}^0$

simple shear

- Deformation gradient

$$F = u_{i,j} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

pure shear

- Sometimes it is important to eliminate rotations
- We can design an experiment with a state of pure shear by inducing this deformation

$$F = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

- We can integrate our usual relationship to find u_1 and u_2

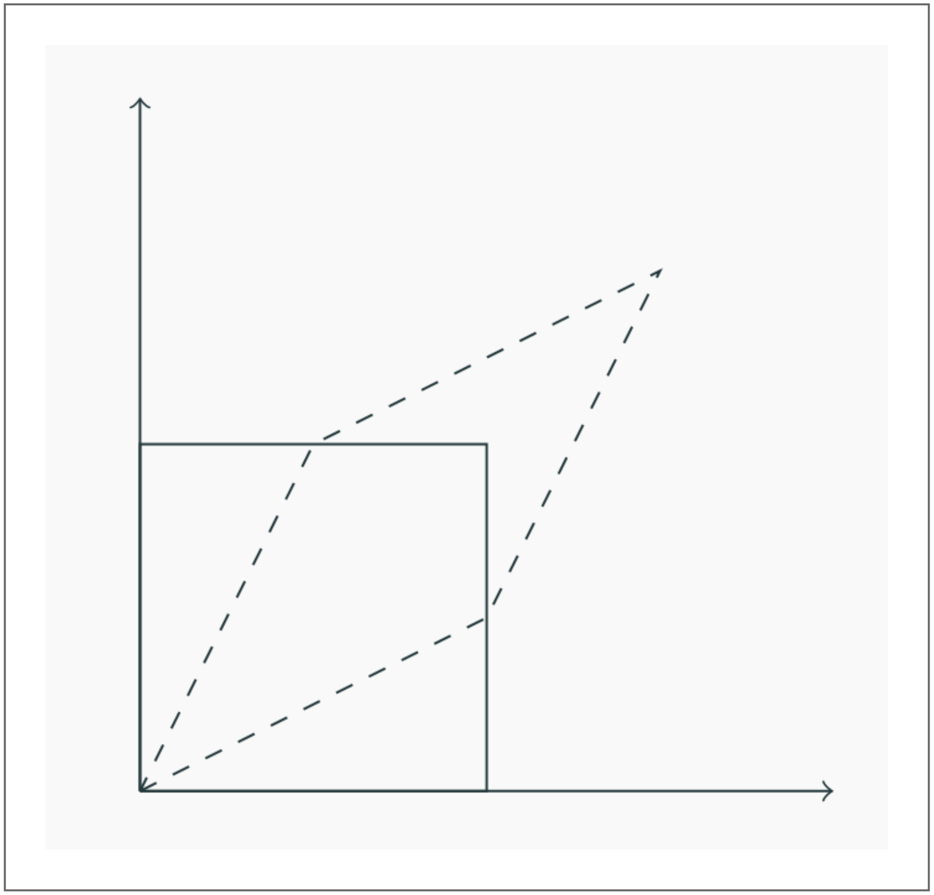
pure shear

$$\frac{\partial u_1}{\partial x_1} = 1$$
$$u_1 = x_1 + f(x_2)$$
$$\frac{\partial u_1}{\partial x_2} = \frac{1}{2}$$
$$\frac{\partial f}{\partial x_2} = \frac{1}{2}$$
$$u_1 = x_1 + \frac{1}{2}x_2$$

pure shear

$$\frac{\partial u_2}{\partial x_2} = 1$$
$$u_2 = x_2 + g(x_1)$$
$$\frac{\partial u_2}{\partial x_1} = \frac{1}{2}$$
$$\frac{\partial g}{\partial x_1} = \frac{1}{2}$$
$$u_2 = x_2 + \frac{1}{2}x_1$$

pure shear



strain

strain definition

- We can separate the deformation gradient into symmetric and antisymmetric parts
  $u_{i,j} = e_{ij} + \omega_{ij}$
- Where

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$$

- e_{ij} is known as the strain tensor
- ω_{ij} is known as the rotation tensor

geometric description

- Engineering strain
$$e^E = \frac{\Delta L}{L_0}$$
- True strain
$$e^T = \frac{\Delta L}{L_0 + \Delta L}$$
- Logarithmic strain
$$e^L = \int_{L_0}^{L} e^T = \int_{L_0}^{L} \frac{dl}{l} = \ln\left(\frac{L}{L_0}\right)$$

geometric description

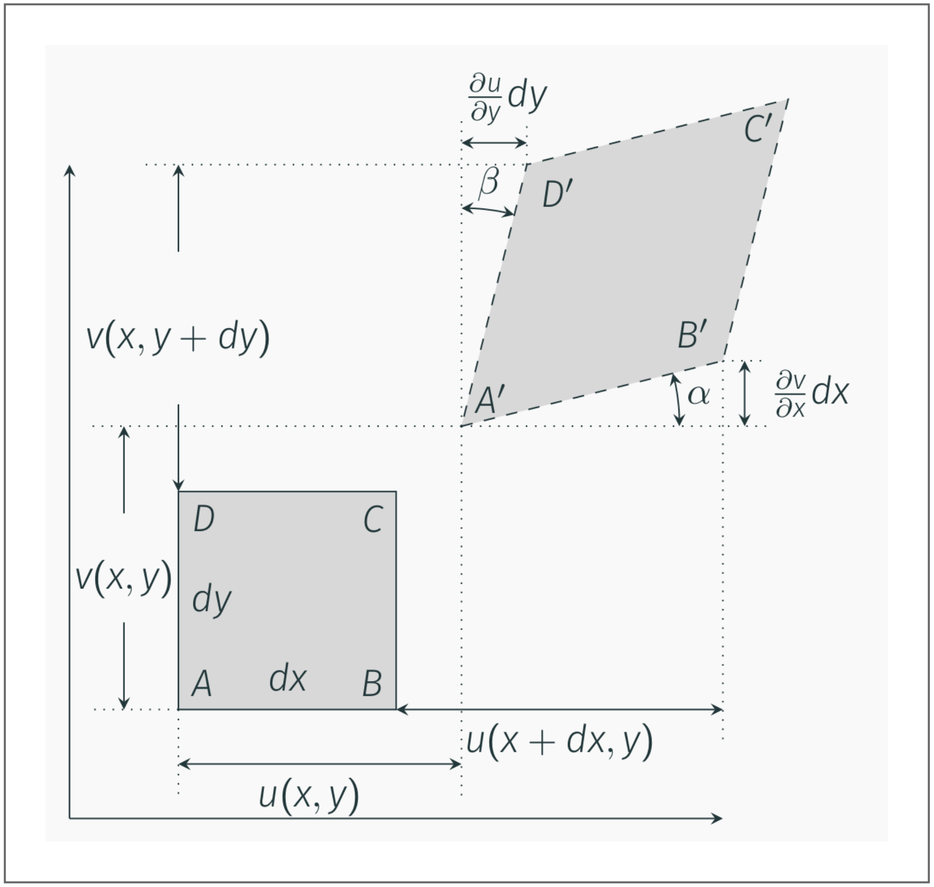
- Large strain: $\Delta L = L_0$

$$e^E = 1.00$$
$$e^T = 0.50$$
$$e^L = 0.69$$

- Small strain: $\Delta L = 0.05L_0$

$$e^E = 0.050$$
$$e^T = 0.048$$
$$e^L = 0.049$$

geometric description



geometric description

- The extensional strain in the x-direction (engineering strain) is defined by
$$\varepsilon_x = \frac{A'B' - AB}{AB}$$
- From geometry, we can write $A'B'$ as

$$A'B' = \sqrt{\left(dx + \frac{\partial u}{\partial x}dx\right)^2 + \left(\frac{\partial v}{\partial x}dx\right)^2}$$
$$= dx\sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$$

geometric description

- For small deformation, we assume $\frac{\partial v}{\partial x}$ is small when compared with $\frac{\partial u}{\partial x}$, which gives
  $$A'B' = \left(1 + \frac{\partial u}{\partial x}\right) dx$$

geometric description

$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{\left(1 + \frac{\partial u}{\partial x}\right) dx - dx}{dx}$$
$$= \frac{\partial u}{\partial x}$$

- The normal strain in the y-direction is found the same way
  $$\varepsilon_y = \frac{\partial v}{\partial y}$$

geometric description

- Engineering shear strain is defined as the change in angle between two originally orthogonal directions
  $\gamma_{xy} = \frac{\pi}{2} - \angle D'A'B' = \alpha + \beta$
- For small strains, $\alpha \approx \tan\alpha$ and $\beta \approx \tan\beta$.

$$\gamma_{xy} = \frac{\frac{\partial v}{\partial x}dx}{dx + \frac{\partial u}{\partial x dx}} + \frac{\frac{\partial u}{\partial y}dy}{dy + \frac{\partial v}{\partial y dx}}$$

$$\approx \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

geometric description

- The other shear terms can be found in the same way

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$
$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

geometric description

- Engineering strain and tensor strain definitions differ only in shear terms
  $$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$
  $$e_{xy} = \frac{1}{2}\gamma_{xy}$$

example

- Calculate the deformation gradient, strain tensor, and rotation tensor for the given deformation

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} xy^2z \\ xz \\ z^3 \end{Bmatrix}$$

- Deformation gradient:

$$F = u_{i,j} = \begin{bmatrix} y^2z & 2xyz & xy^2 \\ z & 0 & x \\ 0 & 0 & 3z^2 \end{bmatrix}$$

example

- Strain tensor

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$e_{ij} = \begin{bmatrix} y^2 z & xyz + \frac{1}{2}z & \frac{1}{2}xy^2 \\ xyz + \frac{1}{2}z & 0 & \frac{1}{2}x \\ \frac{1}{2}xy^2 & \frac{1}{2}x & 3z^2 \end{bmatrix}$$

example

- Rotation tensor

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$$

$$\omega_{ij} = \begin{bmatrix} 0 & xyz - \frac{1}{2}z & \frac{1}{2}xy^2 \\ -xyz + \frac{1}{2}z & 0 & \frac{1}{2}x \\ -\frac{1}{2}xy^2 & -\frac{1}{2}x & 0 \end{bmatrix}$$

example

- As we did with the deformation gradient, we can integrate the strain tensor to find the deformation (symmetric portion)

$$e_{ij} = \begin{bmatrix} yz & xz & xy \\ xz & 2y & \frac{1}{2}x^2 \\ xy & \frac{1}{2}x^2 & 3z^2 \end{bmatrix}$$

example

- We start by integrating the diagonal terms
  $u = \int yz dx = xyz + f(y, z)$
  $v = \int 2y dy = y^2 + g(x, z)$
  $w = \int 3z^2 dz = z^3 + h(x, y)$

example

- Next we need to find the shear terms

$$e_{xy} = \frac{1}{2}(u_{,y} + v_{,x})$$

$$xz = \frac{1}{2}(xz + f_{,y} + g_{,x})$$

$$e_{xz} = \frac{1}{2}(u_{,z} + w_{,x})$$

$$xy = \frac{1}{2}(xy + f_{,z} + h_{,x})$$

$$e_{yz} = \frac{1}{2}(v_{,z} + w_{,y})$$

$$\frac{1}{2}x^2 = \frac{1}{2}(g_{,z} + h_{,y})$$

example

- Note that we cannot uniquely solve this (any arbitrary rotation ω can be added and will still produce a valid strain)
- Let $f(y, z)$=0

$$
\begin{aligned}
g_{,x} &= xz \\
g(x, z) &= \frac{1}{2}x^2 z \\
h_{,x} &= xy \\
h(x, z) &= \frac{1}{2}x^2 y
\end{aligned}
$$

example

$$\frac{1}{2}x^2 = \frac{1}{2}(g_{,z} + h_{,y})$$

$$\frac{1}{2}x^2 = \frac{1}{2}\left(\frac{1}{2}x^2 + \frac{1}{2}x^2\right)$$

$$u = xyz$$

$$v = y^2 + \frac{1}{2}x^2 z$$

$$w = z^3 + \frac{1}{2}x^2 y$$