Name:

Homework 1
Due 4 September 2019

1. Given

$$S_{ij} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 2 & 0 & 2 \end{bmatrix} \qquad \text{and} \qquad a_i = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \tag{1}$$

   Find

   (a) S_{ii}
   (b) $S_{ij}S_{ij}$
   (c) $S_{ji}S_{ji}$
   (d) $S_{jk}S_{kj}$
   (e) $a_m a_m$
   (f) $S_{mn}a_m a_n$

   Solution:

   (a) $S_{ii} = S_{11} + S_{22} + S_{33} = 1 + 1 + 2 = 4$
   (b) $S_{ij}S_{ij} = S_{11}^2 + S_{12}^2 + S_{13}^2 + ... + S_{32}^2 + S_{33}^2 = 28$
   (c) $S_{ji}S_{ji} = S_{ij}S_{ij} = 28$
   (d) $S_{jk}S_{kj} = S_{1k}S_{k1} + S_{2k}S_{k2} + S_{3k}S_{k3} = S_{11}S_{11} + S_{12}S_{21} + ... + S_{32}S_{23} + S_{33}S_{33} = 18$
   (e) $a_m a_m = a_1^2 + a_2^2 + a_3^2 = 9 + 4 + 1 = 14$
   (f) $S_{mn}a_m a_n = S_{1n}a_1 a_n + S_{2n}a_2 a_n + S_{3n}a_3 a_n = S_{11}a_1 a_1 + S_{12}a_2 a_1 + ... + S_{32}a_2 a_3 + S_{33}a_3 a_3 = 36$

2. Write the following equations in index notation

   (a) $s = A_1^2 + A_2^2 + A_3^2$
   (b) $\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = 0$

   Solution:

   (a) $s = A_1^2 + A_2^2 + A_3^2 = A_i A_i$
   (b) $\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = 0 \implies \frac{\partial^2 \phi}{\partial x_i \partial x_i} = \phi_{,ii} = 0$

3. Let f be a scalar-valued function such that $f(x_i) = \sqrt{x_i x_i}$. Find $f_{,i}$

   Solution:

- To find $f_{,k}$, we note that f is a scalar function
- The comma in the subscript indicates partial differentiation with respect to each axis in the coordinate system, which will produce a vector
- We can use the chain rule to differentiate
- $f_{,k} = \frac{1}{2}(x_i x_i)^{-\frac{1}{2}}(x_i x_i)_{,k}$
- Using the chain rule again to compute $(x_i x_i)_{,k}$ we find
- $f_{,k} = \frac{1}{2}(x_i x_i)^{-\frac{1}{2}}(x_i x_{i,k} + x_{i,k} x_i)$
- Simplifying $f_{,k} = (x_i x_i)^{-\frac{1}{2}}(x_i x_{i,k})$
- The partial derivative, $\frac{\partial x_i}{\partial x_k} = \delta_{ik}$
- Substituting $f_{,k} = (x_i x_i)^{-\frac{1}{2}}(x_i \delta_{ik})$
- We also know that $x_i \delta_{ik} = x_k$, and we can also substitute $f = \sqrt{x_i x_i}$ to find
- $f_{,k} = \frac{x_k}{f}$

4. Show (by expansion) that:

$$(AB)_{,ii} = AB_{,ii} + 2A_{,i}B_{,i} + BA_{,ii}$$

where A and B are scalars.

Solution:

- Note that although they are scalars, A and B may still be functions of x_i (and are not constants)
- To take the partial derivative, with respect to x_i, twice, we will need to utilize the chain rule
- It is convenient to re-write the equation

$$(AB)_{,ii} = [(AB)_{,i}]_{,i}$$

- Using the chain rule for the inner portion, we find

$$(AB)_{,ii} = [A_{,i}B + AB_{,i}]_{,i}$$

- Now using the chain rule for the second differentiation, we find

$$(AB)_{,ii} = A_{,ii}B + A_{,i}B_{,i} + A_{,i}B_{,i} + AB_{,ii}$$

- Simplifying

$$(AB)_{,ii} = A_{,ii}B + 2A_{,i}B_{,i} + AB_{,ii}$$

5. If S_{ij} is symmetric and A_{ij} is antisymmetric, show that $S_{ij}A_{ij} = 0$.

Solution:

- Since S_{ij} is symmetric, we know that $S_{ij} = S_{ji}$
- We also know that $A_{ij} = -A_{ji}$, since A_{ij} is antisymmetric
- Substituting S_{ji} for S_{ij} and $-A_{ji}$ for A_{ij} gives

$$S_{ij}A_{ij} = -S_{ji}A_{ji}$$

- Since both indexes are dummy indexes, which merely indicate summation, we can swap the indexes on the right hand side

$$S_{ij}A_{ij} = -S_{ij}A_{ij}$$

- Now we add $S_{ij}A_{ij}$ to both sides

$$2S_{ij}A_{ij} = 0$$

- Q.E.D.

6. For an isotropic material, which is assumed to be linear and elastic, the stress-strain relationship is given by:

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right)$$

Solve this equation for strain (ϵ_{ij}) in terms of stress (σ_{ij}). **Hint:** First find an expression for ϵ_{kk}, then use that to solve the full problem.

Solution:

- First we find ϵ_{kk}, to do this we let $j = i$

$$\sigma_{ii} = \frac{E}{1+\nu}\left(\epsilon_{ii} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ii}\right)$$

- For consistency, we can change all dummy indexes to kk. We also note that $\delta_{ii} = 3$

$$\sigma_{kk} = \frac{E}{1+\nu}\left(\epsilon_{kk} + 3\frac{\nu}{1-2\nu}\epsilon_{kk}\right)$$

- Combining like terms and simplifying gives

$$\sigma_{kk} = \frac{E}{1+\nu}\epsilon_{kk}\frac{1+\nu}{1-2\nu}$$

- Solving for ϵ_{kk}

$$\epsilon_{kk} = \frac{1-2\nu}{E}\sigma_{kk}$$

- Substituting into the original equation

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\epsilon_{ij} + \frac{\nu}{1-2\nu}\frac{1-2\nu}{E}\sigma_{kk}\delta_{ij}\right)$$

- Simplifying and multiplying the right side out

$$\sigma_{ij} = \frac{E}{1+\nu}\epsilon_{ij} + \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}$$

- And solving for ϵ_{ij}

$$\epsilon_{ij} = \frac{1+\nu}{E}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right)$$

7. Find the second-order tensor, T_{ij} with respect to a coordinate system rotated 60° counter-clockwise about the x_2 axis (as shown in Figure 1)

$$T_{ij} = \begin{bmatrix} 6 & 9 & 8 \\ 5 & 3 & 4 \\ 1 & 2 & 7 \end{bmatrix}$$

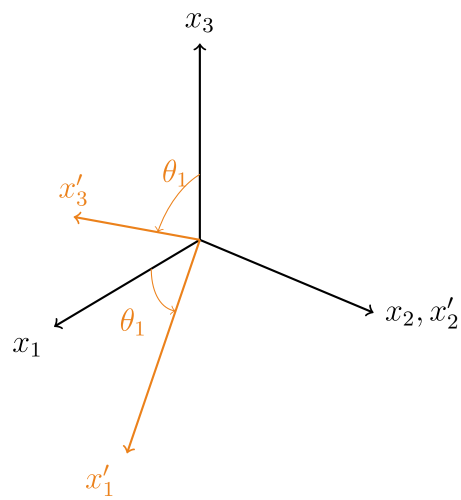

Figure 1: Axis description for Problem 7

Solution:

- Recall that the transformation tensor, Q_{ij} is given by

$$Q_{ij} = \cos(x_i', x_j)$$

- For the rotation described in this problem, this gives

$$Q_{ij} = \begin{bmatrix} \cos 60° & \cos 90° & \cos 150° \\ \cos 90° & \cos 0° & \cos 90° \\ \cos 30° & \cos 90° & \cos 60° \end{bmatrix}$$

- To find a second-order tensor in a transformed coordinate system, we use: $T'_{mn} = Q_{mi}Q_{nj}T_{ij}$ which gives

$$T'_{mn} = \begin{bmatrix} 2.85 & 2.77 & 0.82 \\ -0.96 & 3.00 & 6.33 \\ -6.18 & 8.79 & 10.15 \end{bmatrix}$$

8. For T_{ij} in Problem 7, find T''_{ij} in the coordinate system shown in Figure 2. The x'_i coordinate system is the same as shown in Problem 7, with a rotation of $\theta_1 = 60°$ about the x_2 axis. The x''_i coordinate system is obtained by rotating the x'_i coordinate system by $\theta_2 = 30°$ about the x'_3 axis.

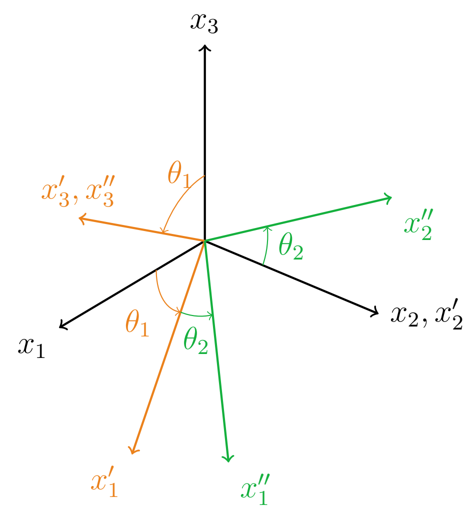

Figure 2: Axis description for Problem 8

Solution:

- We can find T''_{ij} by performing the coordinate system rotations described in Problems 7 and 8 subsequently
- We have already found Q_{ij} for problem 7, so here we identify Q^2_{ij} for the subsequent rotation, given by $Q^2_{ij} = \cos(x''_i, x'_j)$.

$$Q^2_{ij} = \begin{bmatrix} \cos 30° & \cos 60° & \cos 90° \\ \cos 120° & \cos 30° & \cos 90° \\ \cos 90° & \cos 90° & \cos 0° \end{bmatrix}$$

- We can now find T''_{ij} either in terms of T_{ij} or T'_{ij} (using Q^2_{ij} to indicate rotation between problem 7 and 8, and Q^1_{ij} to indicate the rotation performed in problem 7)

$$T''_{ij} = Q^2_{im} Q^2_{jn} T'_{mn}$$
$$T''_{ij} = Q^2_{im} Q^2_{jn} Q^1_{mk} Q^1_{nl} T_{kl}$$

- Which gives the solution

$$\begin{bmatrix} 3.67 & 2.38 & 3.87 \\ -1.35 & 2.18 & 5.07 \\ -0.96 & 10.71 & 10.15 \end{bmatrix}$$

9. For T_{ij} and T'_{ij} from Problem 7, and T''_{ij} from Problem 8, calculate the invariants, I_1, I_2, I_3. Comment on any findings.

   Solution: Note: The formulas used in the book are for symmetric tensors, which this example was not. The formula for II_T gives different values depending which version you use (determinants or index notation) but both methods are "invariant".

   - The invariants for a second-order tensor can be found by

   $$\begin{aligned} I_T &= T_{ii} \\ II_T &= \frac{1}{2}\left(T_{ii}T_{jj} - T_{ij}T_{ij}\right) \\ III_T &= |T_{ij}| \end{aligned}$$

   - For T_{ij} we find

   $$\begin{aligned} I_T &= 16 \\ II_T &= -14.5 \\ III_T &= -145 \end{aligned}$$

   - For T'_{ij} we find

   $$\begin{aligned} I_T &= 16 \\ II_T &= -14.5 \\ III_T &= -145 \end{aligned}$$

   - For T''_{ij} we find

   $$\begin{aligned} I_T &= 16 \\ II_T &= -14.5 \\ III_T &= -145 \end{aligned}$$

   - The invariants remain the same, even when the tensor is expressed in a new coordinate system. This is as expected, and where the name "invariants" comes from

10. For the stress tensor, σ_{ij}, find all principal values and their directions.

   $$\sigma_{ij} = \begin{bmatrix} 4.750 & 2.165 & 1.500 \\ 2.165 & 2.250 & 0.866 \\ 1.500 & 0.866 & 4.000 \end{bmatrix}$$

   Hint: Round intermediate values to two decimal places, use a graphing calculator, MATLAB, or other computer method to solve cubic equations.

   Solution:

- To find the principal values, we solve the equation $\det[\sigma_{ij} - \lambda\delta_{ij}] = 0$

$$\begin{vmatrix} 4.750-\lambda & 2.165 & 1.500 \\ 2.165 & 2.250-\lambda & 0.866 \\ 1.500 & 0.866 & 4.000-\lambda \end{vmatrix} = 0$$

- Multiplying out to find the determinant gives

$(4.75{-}\lambda)[(2.25{-}\lambda)(4{-}\lambda){-}0.750]{-}2.165[2.165(4{-}\lambda){-}0.866(1.5)]{+}1.5[0.866(2.165){-}1.5(2.25{-}\lambda)$

- Which simplifies to

$$-\lambda^3 + 11\lambda^2 - 31\lambda + 21 = 0$$

- Using a cubic equation solver, we find $\lambda_1 = 7$, $\lambda_2 = 3$, and $\lambda_3 = 1$
- To find the principal directions, we substitute each of these values back into the characteristic equation
- For λ_1:

$$\begin{bmatrix} 4.750-\lambda_1 & 2.165 & 1.500 \\ 2.165 & 2.250-\lambda_1 & 0.866 \\ 1.500 & 0.866 & 4.000-\lambda_1 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

- After substitution of λ_1 and row reduction, we find

$$\begin{bmatrix} -2.250 & 2.165 & 1.500 \\ 0 & -2.667 & 2.309 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

- We now solve for v_i, starting by assigning $v_3 = 1$, we find $v_2 = \frac{\sqrt{3}}{2}(0.866)$ and $v_1 = 1.5$
- The principal direction is traditionally normalized to a magnitude of one, which gives, for λ_1, $v_i = \langle \frac{3}{4}, \frac{\sqrt{3}}{4}, \frac{1}{2} \rangle$
- Again using the same procedure, we substitute λ_2 and row-reduce to find

$$\begin{bmatrix} 1.750 & 2.165 & 1.500 \\ 0 & -1.584 & -0.457 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

- As with λ_1, we are free to choose any value for v_3, and for convenience we choose $v_3 = 1$.
- Substituting we find $v_2 = -0.289$ and $v_1 = -0.5$.
- After normalization, we have $v_i = \langle -\frac{\sqrt{3}}{4}, -\frac{1}{4}, \frac{\sqrt{3}}{2} \rangle$
- Following the same procedure, we find, after substitution of λ_3 and row-reduction,

$$\begin{bmatrix} 3.75 & 2.165 & 1.500 \\ 0 & 0 & 0 \\ 0 & 0 & 2.4 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

- In this case, $v_3 = 0$ and we choose $v_2 = 1$, giving $v_1 = -0.577$. Normalizing we find the principal direction for λ_2 as $v_i = \langle -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \rangle$