# AE731

## Theory of Elasticity

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## upcoming schedule

- Aug 19 - Intro to Elasticity
- Aug 21 - Coordinate Transformation
- Aug 26 - Principal Values
- Aug 28 - Tensor Calculus

# outline

- introduction
- syllabus and schedule
- calculus of tensors
- examples

# introduction

# about me

# education

- B.S. in Mechanical Engineering from Brigham Young University
  - Worked with ATK to develop tab-less gripping system for tensile testing composite tow specimens
  - Needed to align the specimen, as well as grip it without causing a stress concentration

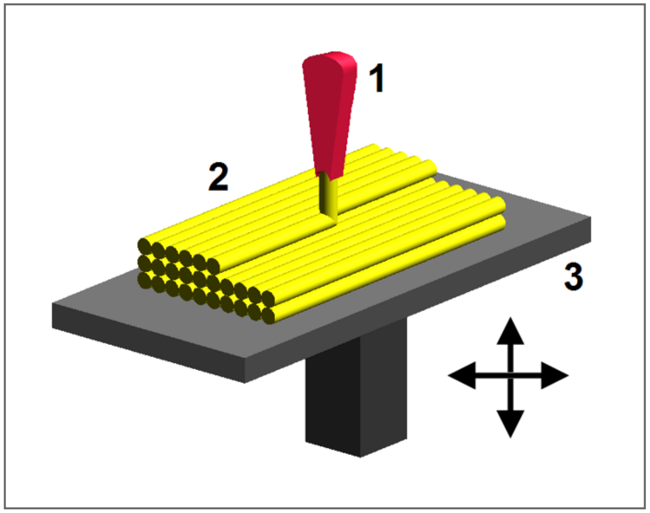
# education

- M.S. and Ph.D. from School of Aeronautics and Astronautics at Purdue University
  - Worked with Boeing to simulate mold flows
  - First ever mold simulation with anisotropic viscosity

# research

# research

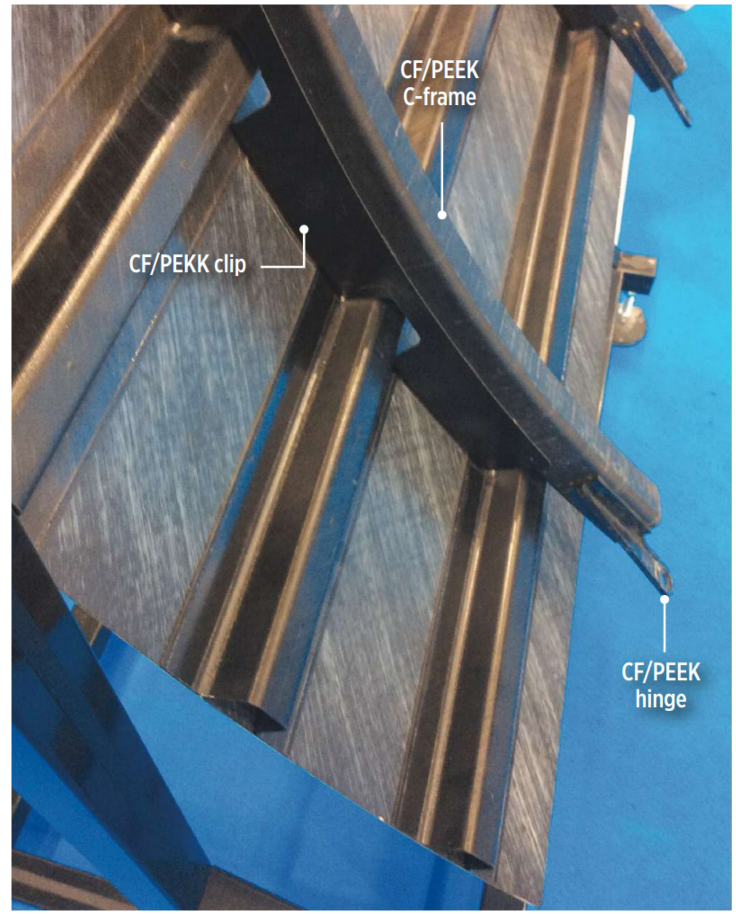
# research




- Composites are being used in 3D printing now
- Printing patterns are optimized for isotropic materials
- Sometimes composites hurt more than they help when not utilized properly

# research

- Thermoplastic composites offer many advantages over thermoset
- Production speed, recyclability
- Also have challenges, such as bonding/welding



# classes

- AE 731 Elasticity Theory (odd years in fall)
- AE 737 Mechanics of Damage Tolerance (every year in spring)
- AE 837 Advanced Mechanics of Damage Tolerance (odd years in fall)
- AE 760AA Micromechanics and Multiscale Modeling (odd years in spring)
- AE 831 Continuum Mechanics (even years in fall)

# introductions

- Name
- Student status (Undergrad, Masters, Ph.D)
- Full time or part time student?
- One interesting thing to remember you by

# syllabus and schedule

# course website

- All class materials will be posted on the **class website**
- I will also make an effort to post to Blackboard, but it takes a little more effort, so you may need to remind me
- The website uses the same format as my presentations, so treat it like slides (navigate with arrow keys)

# course textbook

- Martin H. Saad, *Elasticity: Theory, Application, and Numerics*
- Any version is sufficient
- Homework will be given in handouts
- Textbook will be closely followed in class and is a very highly regarded elasticity text
- Includes useful MATLAB tutorials, as well as some more specialized topics we will not have time to cover in this course, but may be useful to your research

# office hours

- I will e-mail everyone in the course a Doodle link we can use to find the optimal office hours
- Let me know if you do not receive the e-mail, you may need to update your information in Blackboard
- Take advantage of office hours, this is time that I have already set aside for you
- If the regular office hours do not work for your schedule, send me an e-mail and we can work out a time to meet

# tentative outline

- Chapter 1 - Calculus of tensors
  - 19 Aug - 28 Aug (4 lectures)
- Chapter 2 - Kinematics
  - 4 Sep - 11 Sep (3 lectures)
- Exam 1 (covers Chapter 1-2)
  - 18 Sep

# tentative outline

- Chapter 3 - Stress
  - 23 Sep - 30 Sep (3 lectures)
- Chapter 4 - Constitutive equations
  - 2 Oct - 9 Oct (3 lectures)
- Chapter 5 - Solution strategies
  - 16 Oct - 23 Oct (3 lectures)
- Exam 2 (Chapters 3-5)
  - 30 Oct

# tentative outline

- Ch 6 - Energy principles
  - 4 Nov - 11 Nov (3 lectures)
- Ch 7-8 - 2D problems
  - 13 Nov - 20 Nov (3 lectures)
- Ch 10 - Complex variables
  - 2 Dec - 4 Dec
- Special topics
  - No homework
  - Not on final exam
  - Anisotropic elasticity
  - Heterogeneous materials
  - Numerical applications (finite elements)

# final exam

- Wednesday 11 December
- 3:00 - 4:50 pm
- Cannot take final exam at any other time, make travel plans accordingly
- Comprehensive

## grades

- Grade breakdown
  - Homework 15%
  - Exam 1 25%
  - Exam 2 25%
  - Final Exam 35%
- Follow a traditional grading scale

## class expectations

- Consider the cost (to you or others) of your being in class
- I ask that you refrain from distracting behaviors during class
- When you have something more important than class to take care of it, please take care of it outside of class

# calculus of tensors

# scalars

- Scalar
  - single value (at a point)
  - e.g. temperature, density

# vectors

- Vector
  - expressed in terms of coordinate system
  - one-dimensional array
  - e.g. displacement

# matrices

- Matrix
  - two-dimensional array
  - e.g. stress, strain

# scalars, vectors, matrices

$\rho = 25$

$u = \langle x, y \rangle$

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}$$

# scalars, vectors, matrices

- In two dimensions
  - Scalars have 1 term
  - Vectors have 2 terms
  - Matrices have 4 terms
- In three dimensions
  - Scalars have 1 term
  - Vectors have 3 terms
  - Matrices have 9 terms

## tensors

- Formal definition for tensors later in the course
- Scalar = 0-order tensor
- Vector = 1st-order tensor
- Matrix = 2nd-order tensor

# tensors

- We will also use higher-order tensors in this course
- High-order tensors are difficult to write
- It can even be difficult to distinguish vectors, scalars, and matrices in some notations
- Index notation is used to address these problems

# index notation

- Use subscripts to indicate when a variable has multiple values
- $\rho$ has no subscript, and thus it must be a scalar
- $u_i$ has one subscript, $i$, indicating it has multiple values
- "Multiple" means the number of coordinate system axes, unless otherwise specified.
- $u_i = \langle u_1, u_2, u_3 \rangle$

# index notation

- $\sigma_{ij}$ has two subscripts, $i$ and $j$, meaning it spans the coordinate system in two directions.
- $\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$
- We can use this notation for any order of tensor
- In 3D, we need a fourth-order tensor to define material stiffness, we write this in index notation as
- $C_{ijkl}$

# addition

- We can do arithmetic in index notation
- $a_i + b_i = \langle a_1 + b_1, a_2 + b_2 \rangle$
- We can also use a different index in the two variables to create a matrix
- $c_{ij} = a_i + b_j = \begin{bmatrix} a_1 + b_1 & a_1 + b_2 \\ a_2 + b_1 & a_2 + b_2 \end{bmatrix}$

# multiplication

- We can multiply a scalar by a vector
- $\lambda a_i = \langle \lambda a_1, \lambda a_2 \rangle$
- Or multiply two vectors
- $c_{ij} = a_i b_j = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{bmatrix}$

# math laws

- The usual laws (commutative, associative, distributive) still apply
- $a_i + b_i = b_i + a_i$
- $a_{ij} b_k = b_k a_{ij}$
- $a_i + (b_i + c_i) = (a_i + b_i) + c_i$
- $a_i (b_{jk} c_l) = (a_i b_{jk}) c_l$
- $a_{ij} (b_k + c_k) = a_{ij} b_k + a_{ij} c_k$

# equality

- What does this mean?
  - $a_i = b_i$
  - $a_1 = b_1, a_2 = b_2$, etc.
- What about this?
  - $a_i = b_j$
  - Doesn’t make sense, incorrect use of index notation

# dummy index

- When an index is repeated in the same term, it is referred to as a “dummy index”
- The dummy index indicates summation over all axes
- e.g. $a_{ii} = a_{11} + a_{22} + a_{33}$
- Note: summation on a matrix will reduce to a scalar, summation on higher order tensors will reduce the order by 2

# dummy index

- The dummy index can be triggered by any repeated index in a .
- Summation or not?
  - $a_i + b_{ij}c_j$
  - $a_{ij} + b_{ij}$
  - $a_{ij} + b_{ij}c_j$

# matrix multiplication

- How can we write matrix multiplication in index notation?
- $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$
- $c_{11} = a_{11}b_{11} + a_{12}b_{21}$
- $c_{12} = a_{11}b_{21} + a_{12}b_{22}$
- $c_{ij} = a_{ik}b_{kj}$

# symmetry

- Symmetry can be a very powerful tool in Elasticity
- Here we define some useful forms of symmetry in index notation
- Symmetric
    - $a_{ij\ldots z} = a_{z\ldots ji}$
    - $a_{ij\ldots m\ldots n\ldots z} = a_{ij\ldots n\ldots m\ldots z}$

# anti-symmetry

- Anti-symmetric (skew symmetric)
  - $a_{ij\dots z} = -a_{z\dots ji}$
  - $a_{ij\dots m\dots n\dots z} = -a_{ij\dots n\dots m\dots z}$

## symmetry

- Useful identity
  - If $a_{ij...m...n...k}$ is symmetric in $mn$ and $b_{pq...m...n...r}$ is antisymmetric in $mn$, then the product is zero
  - $a_{ij...m...n...k}b_{pq...m...n...r} = 0$
- We can also write any tensor as the sum of its symmetric and anti-symmetric parts
  - $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

# symmetry

- This textbook uses a special shortcut notation for the symmetric and anti-symmetric portions of a tensor
  - Symmetric: $a_{(ij)} = \frac{1}{2}(a_{ij} + a_{ji})$
  - Anti-symmetric: $a_{[ij]} = \frac{1}{2}(a_{ij} - a_{ji})$

# special symbols

- For convenience we define two symbols in index notation
- *Kronecker delta* is a general tensor form of the Identity Matrix

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Is also used for higher order tensors

# Kronecker delta

- $\delta_{ij} = \delta_{ji}$
- $\delta_{ii} = 3$
- $\delta_{ij} a_j = a_i$
- $\delta_{ij} a_{ij} = a_{ii}$

# special symbols

- *alternating symbol* or *permutation symbol*

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of 1,2,3} \\ -1 & \text{if } ijk \text{ is an odd permutation of 1,2,3} \\ 0 & \text{otherwise} \end{cases}$$

# permutation symbol

- This symbol is not used as frequently as the *Kronecker delta*
- For our uses in this course, it is enough to know that 123, 231, and 312 are even permutations
- 321, 132, 213 are odd permutations
- all other indexes are zero
- $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{mk}$

# determinant

- We use the alternating symbol for writing determinants and cross-products

$$\det[a_{ij}] = |a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \epsilon_{ijk} a_{i1} a_{j2} a_{k3}$$

$$\det[a_{ij}] = \frac{1}{6} \epsilon_{ijk} \epsilon_{pqr} a_{ip} a_{jq} a_{kr}$$

# cross product

- The cross-product can be written as a determinant:

$$\hat{a} \times \hat{b} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Or in index notation

$$\hat{a} \times \hat{b} = \epsilon_{ijk} \hat{e}_i a_j b_k$$

# partial derivative

- We indicate (partial) derivatives using a comma
- In three dimensions, we take the partial derivative with respect to each variable ($x, y, z$ or $x_1, x_2, x_3$)
- For example a scalar property, such as density, can have a different value at any point in space
- $\rho = \rho(x_1, x_2, x_3)$

$$\rho_{,i} = \frac{\partial}{\partial x_i}\rho = \left\langle \frac{\partial \rho}{\partial x_1}, \frac{\partial \rho}{\partial x_2}, \frac{\partial \rho}{\partial x_3} \right\rangle$$

# partial derivative

- Similarly, if we take the partial derivative of a vector, it produces a matrix

$$u_{i,j} = \frac{\partial}{\partial x_j} u_i = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

# examples

# example 1

- Write the following in conventional notation $T_{ij,j} + F_i = 0$
- The comma indicates a partial derivative
- The first index, $i$, is not repeated in any terms, so it is a "free index"
  - This means in a 3D coordinate system, we will have at least three equations
- The second index, $j$, is repeated in the first term, indicating summation.
  - We will have exactly three equations

# example 1 (solution)

$T_{ij,j} + F_i = 0$

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} + F_1 = 0$$
$$\frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} + F_2 = 0$$
$$\frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} + F_3 = 0$$

# example 2

- Identify whether the following expressions represent a scalar, vector, or matrix
- If index notation is used incorrectly, give a reason why
- $A_i = B_i$
- $A_i = B_i + C_i D_i$
- $\delta_{ij} A_i B_j$
- $\phi = \frac{\partial F_i}{\partial x_i}$

# example 2 (solution)

- Vector equation
- Incorrect use of index notation, $i$ used as both free and dummy index
- Scalar value (both indexes are dummy indexes)
- Scalar value (could also be written $F_{i,i}$)