AE731

Theory of Elasticity

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upcoming schedule

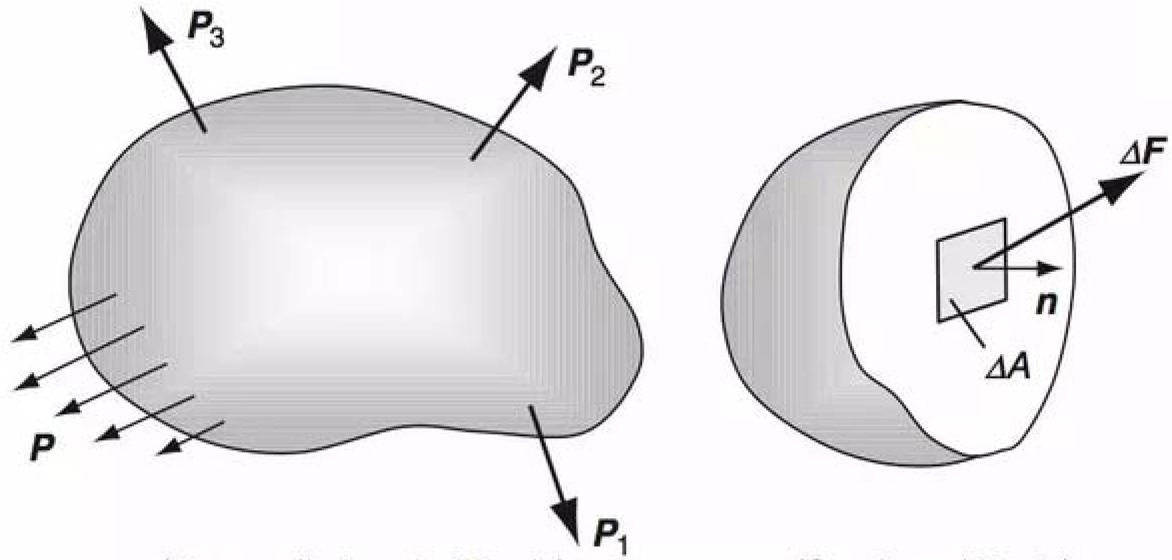
- Sep 25 - Stress Transformation
- Sep 30 - Equilibrium Equations
- Oct 2 - Material Characterization, HW3 Due
- Oct 7 - Thermoelasticity

outline

- traction vector and stress tensor
- stress transformation
- principal stresses
- maximum shear stress
- group problems

traction vector and stress tensor

traction



(Externally Loaded Body) (Sectioned Body)

traction

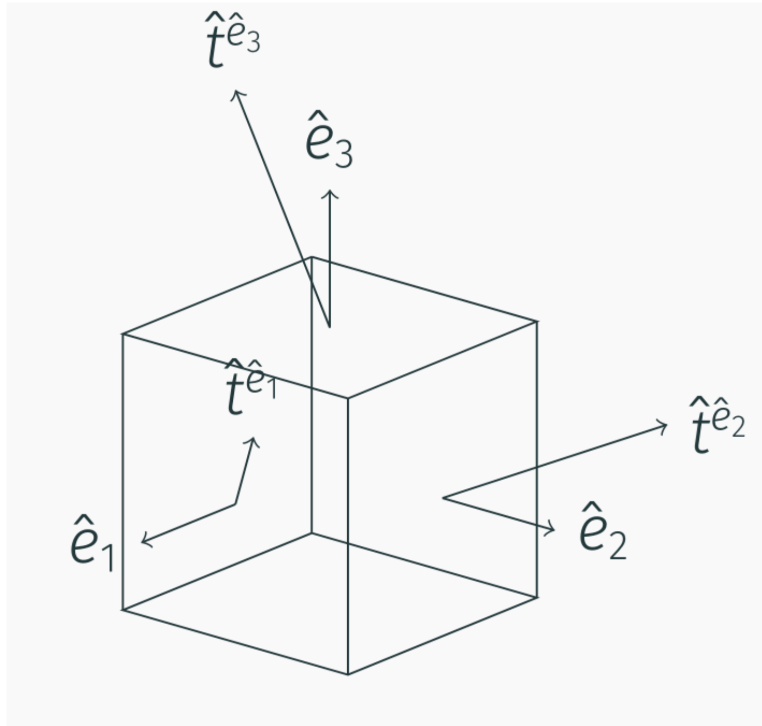
- The traction vector is defined as

$$\hat{t}^n(x, \hat{n}) = \lim_{\Delta A \to 0} \frac{\Delta \hat{f}}{\Delta A}$$

- By Newton's third law (action-reaction principle)

$$\hat{t}^n(x, \hat{n}) = -\hat{t}^n(x, -\hat{n})$$

traction



traction

- If we consider the special case where the normal vectors, $\hat{n}$, align with the coordinate system ($\hat{e}_1$,$\hat{e}_2$,$\hat{e}_3$)

- On the 1-face:

$$\hat{n} = \hat{e}_1 : \qquad \hat{t}^n = t_i^{(\hat{e}_1)}\hat{e}_i = t_1^{(\hat{e}_1)}\hat{e}_1 + t_2^{(\hat{e}_1)}\hat{e}_2 + t_3^{(\hat{e}_1)}\hat{e}_3$$

- On the 2-face:

$$\hat{n} = \hat{e}_2 : \qquad \hat{t}^n = t_i^{(\hat{e}_2)}\hat{e}_i = t_1^{(\hat{e}_2)}\hat{e}_1 + t_2^{(\hat{e}_2)}\hat{e}_2 + t_3^{(\hat{e}_2)}\hat{e}_3$$

traction

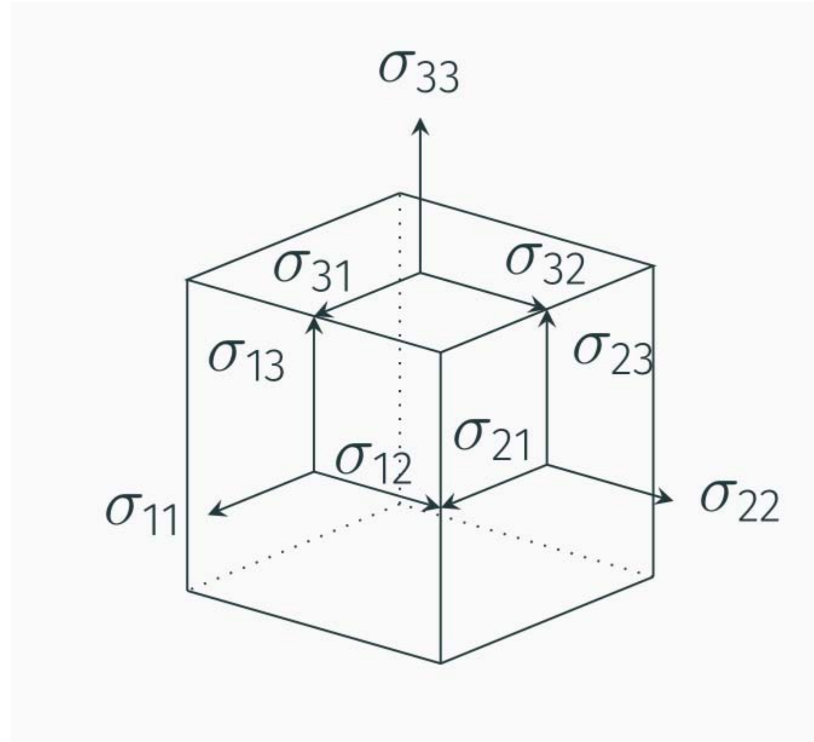
- And on the 3-face:

$$\hat{n} = \hat{e}_3 : \qquad \hat{t}^n = t_i^{(\hat{e}_3)} \hat{e}_i = t_1^{(\hat{e}_3)} \hat{e}_1 + t_2^{(\hat{e}_3)} \hat{e}_2 + t_3^{(\hat{e}_3)} \hat{e}_3$$

stress tensor

- To simplify the notation, we introduce the stress tensor

$$\sigma_{ij} = t_j^{(\hat{e}_i)}$$



traction

- We can find some interesting information about the traction vector by considering an arbitrary tetrahedron with some traction $\hat{t}^{(n)}$ applied to the surface

traction

- If we consider the balance of forces in the x_1-direction

$$t_1 dA - \sigma_{11} dA_1 - \sigma_{21} dA_2 - \sigma_{31} dA_3 + b_1 \rho dV = 0$$

- The area components are:

$$\begin{aligned} dA_1 &= n_1 dA \\ dA_2 &= n_2 dA \\ dA_3 &= n_3 dA \end{aligned}$$

- And $dV = \frac{1}{3} h dA$.

traction

$$t_1 dA - \sigma_{11} n_1 dA - \sigma_{21} n_2 dA - \sigma_{31} n_3 dA + b_1 \rho \frac{1}{3} h dA = 0$$

- If we let $h \to 0$ and divide by dA

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$

- We can write this in index notation as

$$t_1 = \sigma_{i1} n_i$$

traction

- We find, similarly

$$t_2 = \sigma_{i2} n_i$$
$$t_3 = \sigma_{i3} n_i$$

traction

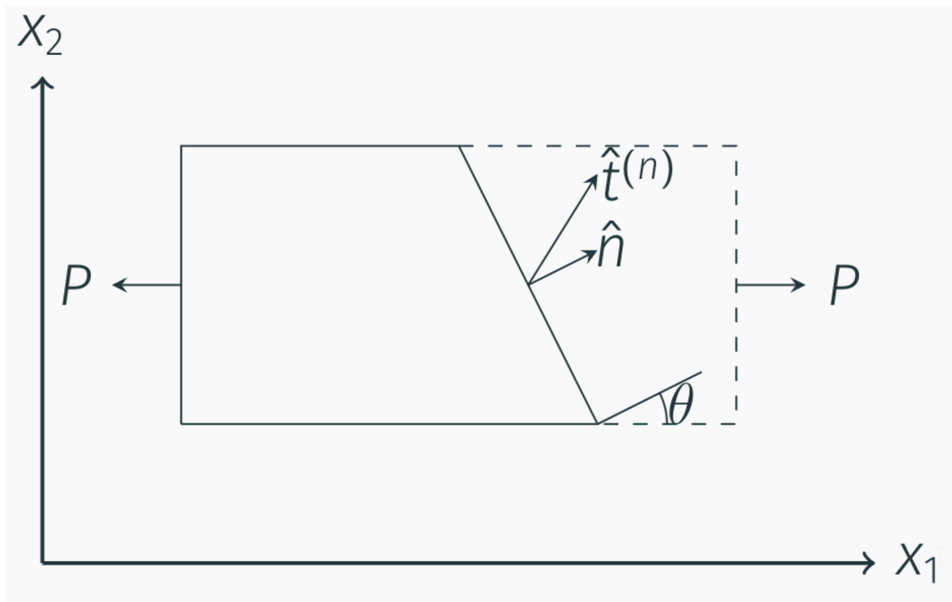
- We can further combine these results in index notation as

$$t_j = \sigma_{ij} n_i$$

- This means with knowledge of the nine components of σ_{ij}, we can find the traction vector at any point on any surface

example

- Consider a block of material with a uniformly distributed force acting on the 1-face. Find the tractions on an arbitrary interior plane



example

- First we consider a vertical cut on the interior 1-face ($n_i = \langle 1, 0, 0 \rangle$)
- Next we represent the force P as a vector, $p_i = \langle P, 0, 0 \rangle$
- Balancing forces yields

$$t_i A - p_i = 0$$

- We find $t_1 = \frac{P}{A} = \sigma_{11}$, $t_2 = 0 = \sigma_{12}$ and $t_3 = 0 = \sigma_{13}$

example

- No force is applied in the other directions, so it is trivial to find the rest of the stress tensor

$$\sigma_{ij} = \begin{bmatrix} P/A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

example

- We can now consider any arbitrary angle of interior cut.
- The normal for a cut as shown in the diagram will be $n_i = \langle \cos\theta, \sin\theta, 0 \rangle$.
- We can again use $t_j = \sigma_{ij} n_i$ to find t_j for any angle θ.

$$
\begin{aligned}
t_1 &= \frac{P}{A} \cos\theta \\
t_2 &= 0 \\
t_3 &= 0
\end{aligned}
$$

stress transformation

stress transformation

- Stress transformation equations are identical to the strain transformation equations
- Both stress and strain are tensor, and transform in the same fashion
- Rotation about z-axis gives

$$Q_{ij} = \begin{bmatrix} \cos\theta & \cos(90-\theta) & \cos 90 \\ \cos(90+\theta) & \cos\theta & \cos 90 \\ \cos 90 & \cos 90 & \cos 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

stress transformation

- We recall that $\sigma_{ij}{}' = Q_{im} Q_{jn} Q_{mn}$
- Which gives

$$\sigma'_x = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$
$$\sigma'_y = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta$$
$$\tau'_{xy} = -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

stress transformation

- As with the strain transformation equations, these are often re-written using the double-angle formulae.

$$
\begin{aligned}
\sigma'_x &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\
\sigma'_y &= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \\
\tau'_{xy} &= \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta
\end{aligned}
$$

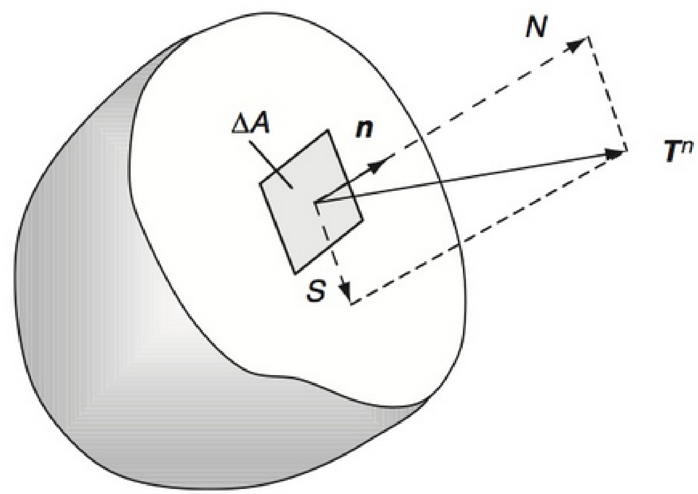
principal stresses

principal stresses

- Principal stresses can be found in the same fashion as principal values and principal strains det[$\sigma_{ij} - \sigma\delta_{ij}$]=0

tractions

- We can use what we know about principal values to find some interesting things about the tractions
- Consider the traction vector on an arbitrary internal face, and decompose into Normal and Shear components.



tractions

- The normal component can be found using the dot product

$$N = \hat{T}^{n} \cdot \hat{n}$$

- The shear component can be found using the Pythagorean theorem

$$S^2 = |\hat{T}^{n}|^2 - N^2$$

tractions

- We now use the stress tensor in the principal direction to simplify the calculations

$$
\begin{aligned}
N &= \hat{T}^{n} \cdot \hat{n} \\
&= T_i^n n_i \\
&= \sigma_{ji} n_j n_i \\
&= \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2
\end{aligned}
$$

tractions

- We also know that

$$
\begin{aligned}
|\hat{T}^n|^2 &= \hat{T}^n \cdot \hat{T}^n \\
&= T_i^n T_i^n \\
&= \sigma_{ji} n_j \sigma_{ki} n_k \\
&= \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2
\end{aligned}
$$

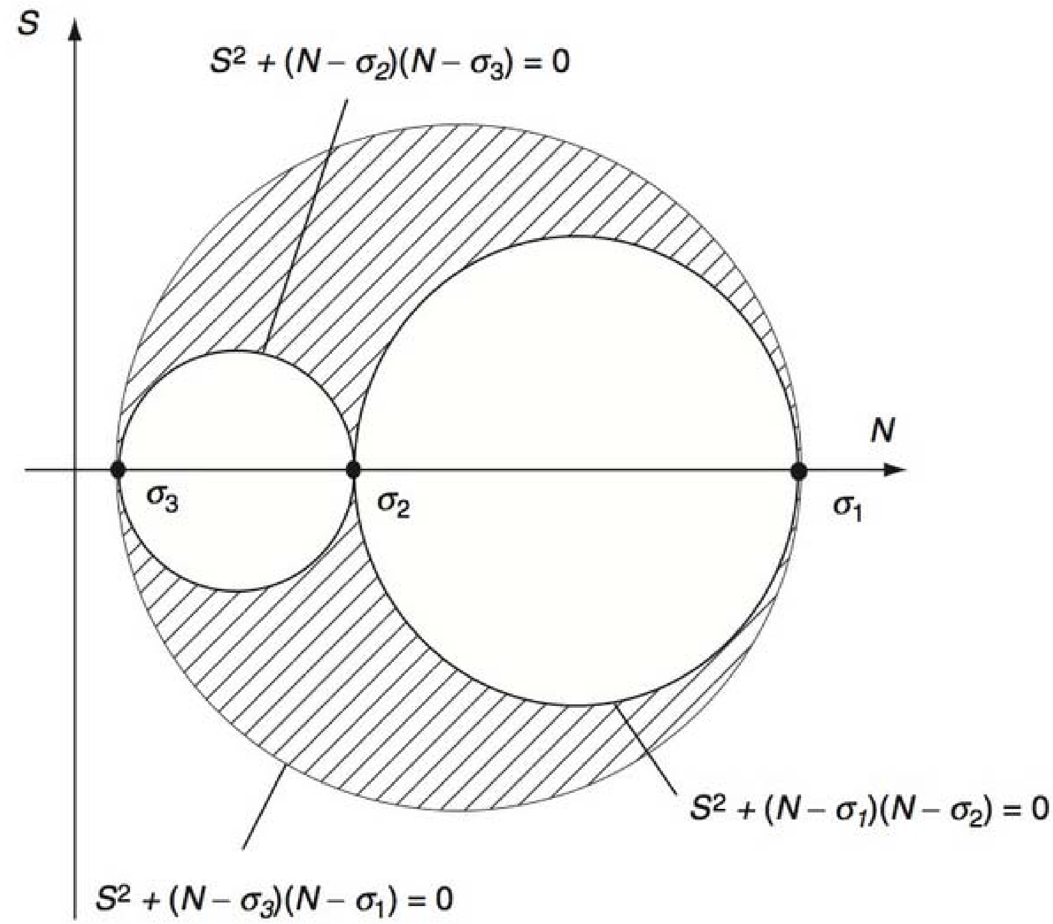
mohr's circle

- If we constrain the normal vector to be a unit vector we can formulate the following inequalities

$$
\begin{aligned}
S^2 + (N - \sigma_2)(N - \sigma_3) \geq 0 \\
S^2 + (N - \sigma_3)(N - \sigma_1) \leq 0 \\
S^2 + (N - \sigma_1)(N - \sigma_2) \geq 0
\end{aligned}
$$

- These inequalities form what is known as Mohr's circle

mohr’s circle



maximum shear stress

maximum shear stress

- From Mohr's circle, we can find the maximum shear stress in terms of the principal stresses

$$S_{max} = (\sigma_1 - \sigma_3)/2$$

maximum shear stress

- For plane stress problems, we can also use the stress transformation equations to find the maximum shear stress
- We desire to maximize this equation:

$$\tau'_{xy} = \frac{\sigma_y - \sigma_x}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

maximum shear stress

- Taking the derivative with respect to θ gives

$$\frac{\partial}{\partial\theta}(\tau'_{xy}) = (\sigma_y - \sigma_x)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

- Which we can use to find 2θ

$$2\theta = \tan^{-1}\left(\frac{(\sigma_y - \sigma_x)}{2\tau_{xy}}\right)$$

maximum shear stress

- Substituting back into the original equation gives

$$\tau'_{max} = \frac{\sigma_y - \sigma_x}{2}\sin\left[\tan^{-1}\left(\frac{(\sigma_y - \sigma_x)}{2\tau_{xy}}\right)\right] + \tau_{xy}\cos\left[\tan^{-1}\left(\frac{(\sigma_y - \sigma_x)}{2\tau_{xy}}\right)\right]$$

- Note that

$$\sin(\tan^{-1}(x)) = \frac{x}{\sqrt{1+x^2}}$$
$$\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$$

maximum shear stress

- We note that

$$\sqrt{1+\left(\frac{\sigma_y-\sigma_x}{2\tau_{xy}}\right)^2}=\frac{\sqrt{(\sigma_y-\sigma_x)^2+4\tau_{xy}^2}}{2\tau_{xy}}$$

- And thus we find

$$\tau_{max}=\frac{(\sigma_y-\sigma_x)^2}{2\sqrt{(\sigma_y-\sigma_x)^2+4\tau_{xy}^2}}+\frac{4\tau_{xy}^2}{2\sqrt{(\sigma_y-\sigma_x)^2+4\tau_{xy}^2}}$$

maximum shear stress

- Adding the terms and simplifying, we find

$$\tau_{max} = \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

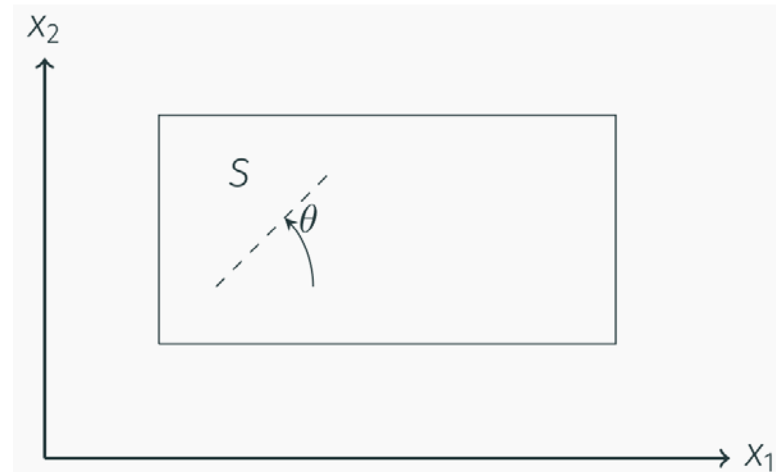
group problems

group one

- The stress state in a rectangle under biaxial loading is

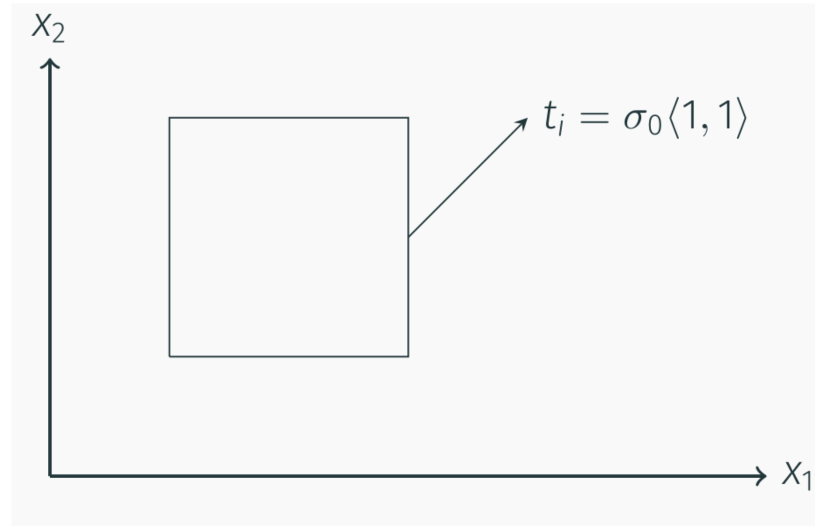
$$\sigma_{ij} = \begin{bmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Find the traction vector, as well as the normal and shearing stresses on some oblique plane, S



group two

- For the figure shown, what must the traction be on the other faces for the stress to be uniform and in equilibrium?



group three

- For the figure shown, find the (uniform) stress tensor. What must the traction be on the last face?