# Theory of Elasticity

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# upcoming schedule

- Oct 19 - Exam Return, Virtual Work
- Oct 21 - Airy Stress Functions
- Oct 22 - Homework 5 Self-Grade Due
- Oct 26 - Airy Stress
- Oct 28 - Airy Stress
- Oct 29 - Homework 6 Due

# outline

- integral theorems
- virtual work
- ritz method



# integral theorems

## clapeyron’s theorem

- If we return to the uniqueness derivation, the only non-general assumptions were

$$
\begin{aligned}
\sigma_{ij,j} &= 0 \\
T_i^n = \sigma_{ij} n_j &= 0 \qquad \text{Along traction boundary} \\
u_i &= 0 \qquad \text{Along displacement boundary}
\end{aligned}
$$



## clapeyron’s theorem

- This means that for any elastic body we can say

$$2\int_V U dV = \int_S \sigma_{ij} n_j u_i dS - \int_V \sigma_{ij,j} u_i dV$$

## clapeyron's theorem

- If we consider an elastic body in equilibrium, we can say that

$$\sigma_{ij,j} = -F_i$$

- We also know by Cauchy's stress theorem that

$$t_i = \sigma_{ij} n_j$$

- Both of these can be substituted to give

$$2\int_V U dV = \int_S t_i u_i dS + \int_V F_i u_i dV$$



## betti/rayleigh reciprocal theorem

- We can derive another theorem by returning to

$$2\int_V U dV = \int_S \sigma_{ij} n_j u_i dS - \int_V \sigma_{ij,j} u_i$$

- Consider two different sets of forces and displacements acting on the same body

$t_i^{(1)}$, $F_i^{(1)}$, $u_i^{(1)}$ and $t_i^{(2)}$, $F_i^{(2)}$, $u_i^{(2)}$

## reciprocal theorem

- We now consider the work done by the forces in the first system acting through the displacements of the second system

$$2\int_V U dV = \int_V \sigma_{ij}^{(1)}\epsilon_{ij}^{(2)} = \int_S t_i^{(1)}u_i^{(2)}dS + \int_V F_i^{(1)}u_i^{(2)}dV$$

- We can similarly write

$$\int_V \sigma_{ij}^{(2)}\epsilon_{ij}^{(1)} = \int_S t_i^{(2)}u_i^{(1)}dS + \int_V F_i^{(2)}u_i^{(1)}dV$$



## reciprocal theorem

- We can now use Hooke's Law and symmetry to say that

$$\sigma_{ij}^{(1)}\epsilon_{ij}^{(2)} = C_{ijkl}\epsilon_{kl}^{(1)}\epsilon_{ij}^{(2)} = \epsilon_{kl}^{(1)}\sigma_{kl}^{(2)}$$

- If $\sigma_{ij}^{(1)}\epsilon_{ij}^{(2)} = \sigma_{ij}^{(2)}\epsilon_{ij}^{(1)}$, then we can also say that the strain energies are equivalent, proving the Betti/Rayleigh Reciprocal Theorem

$$\int_S t_i^{(1)}u_i^{(2)}dS + \int_V F_i^{(1)}u_i^{(2)}dV = \int_S t_i^{(2)}u_i^{(1)}dS + \int_V F_i^{(2)}u_i^{(1)}dV$$

# integral elasticity

- The Betti/Rayleigh Reciprocal Theorem is used to derive the Integral Formulation of Elasticity
- Also known as Somigliana's Identity
- Used for Boundary Element Method (BEM) and Boundary Integral Equation methods (BIE), but we will not use it in this class



# virtual work

## virtual work

- The solution format we developed in Chapter 5 is known as *Strong Form*, and is not always a convenient solution form
- We can use energy and work principles to develop additional solution methods
- *Virtual Displacement* is a fictitious displacement such that the forces acting on the point remain unchanged
- The work done by these forces is known as *Virtual Work*



## virtual work

- If we consider the elastic boundary-value problem, with tractions applied over the boundary $S_t$ and displacements applied over the boundary $S_u$.
- Virtual displacements denoted by $\delta u_i$ and are arbitrary, but cannot violate the displacement boundary condition, thus $\delta u_i = 0$ on $S_u$.

# virtual work

- Virtual work done by surface and body forces can be written as

$$\delta W = \int_{S_t} t_i \delta u_i dS + \int_V F_i \delta u_i dV$$

- Since the virtual displacement is zero over $S_u$, we can replace $S_t$ with $S$ in the integral.

$$\delta W = \int_S t_i \delta u_i dS + \int_V F_i \delta u_i dV$$



# virtual work

$$\begin{aligned}
\delta W &= \int_S T_i^n \delta u_i dS + \int_V F_i \delta u_i dV \\
&= \int_S \sigma_{ij} n_j \delta u_i dS + \int_V F_i \delta u_i dV \\
&= \int_V (\sigma_{ij} \delta u_i)_{,j} dV + \int_V F_i \delta u_i dV \\
&= \int_V (\sigma_{ij,j} \delta u_i + \sigma_{ij} \delta u_{i,j}) dV + \int_V F_i \delta u_i dV \\
&= \int_V (-F_i \delta u_i + \sigma_{ij} (\delta \epsilon_{ij} + \delta \omega_{ij})) dV + \int_V F_i \delta u_i dV \\
&= \int_V \sigma_{ij} \delta \epsilon_{ij} dV
\end{aligned}$$

## virtual work

- We can follow the procedure from the uniqueness derivation in reverse
- Notice that this gives the usual strain energy relationship, but without the factor of one-half.
- This is because stress is constant during virtual displacement



## virtual work

- The virtual strain energy follows the same relationships developed previously, namely

$$\int_V \delta U = \int_S t_i \delta u_i dS + \int_V F_i \delta u_i dV$$

- Because the external forces are unchanged during the virtual displacement, the $\delta$ operator can be placed outside the integrals.
- We can also move all terms to the same side of the equation to write

$$\delta \left( \int_V U - \int_S t_i u_i dS - \int_V F_i u_i dV \right)$$

## virtual work

- Or, written in terms of virtual work

$$\delta(U_T - W) = 0$$



## virtual work

- The total potential energy of an elastic solid is $U_T - W$, and must be zero for a virtual displacement
- These results are completely general, and apply to both linear and non-linear materials
- Special theories for rods, beams, plates, and shells use this principle
- Finite elements is also developed using virtual work
- We can even use virtual work to re-derive the continuum results we found previously

## virtual work

- If we start with this form

$$\int_V \sigma_{ij}\delta\epsilon_{ij}dV - \int_S t_i\delta u_i dS - \int_V F_i\delta u_i dV = 0$$

- We can replace the first term by writing it as

$$\sigma_{ij}\delta\epsilon_{ij} = \sigma_{ij}\delta u_{i,j} = (\sigma_{ij}\delta u_i)_{,j} - \sigma_{ij,j}\delta u_i$$

- Which leads to

$$\int_V [(\sigma_{ij}\delta u_i)_{,j} - \sigma_{ij,j}\delta u_i]dV - \int_S T_i^n\delta u_i dS - \int_V F_i\delta u_i dV = 0$$



## virtual work

- We can use the divergence theorem to say that

$$\int_V (\sigma_{ij}\delta u_i)_{,j}dV = \int_S \sigma_{ij}n_j\delta u_i dS$$

- This gives

$$\int_V [\sigma_{ij,j} + F_i]\delta u_i dV + \int_S (T_i^n - \sigma_{ij}n_j)\delta u_i dS = 0$$

## virtual work

- This will be satisfied if

$$\sigma_{ij,j} + F_i = 0 \text{(equilibrium)}$$

- And either

$$\delta u_i = 0 \text{(displacement boundary)}$$

- Or

$$t_i = \sigma_{ij} n_j \text{(traction boundary)}$$



# ritz method

## ritz method

- While we have showed previously how virtual work can be used to develop analytic solutions, it is also convenient for approximate solutions
- The Rayleigh-Ritz Method is an important approximation technique based on this method
- In this method, trial functions are used as approximate solutions which satisfy the boundary conditions, but not necessarily the differential equations.



## ritz method

- For the elasticity displacement formulation, trial functions take the form

$$u = u_0 + \sum_{j=1}^{N} a_j u_j$$

$$v = v_0 + \sum_{j=1}^{N} b_j v_j$$

$$w = w_0 + \sum_{j=1}^{N} c_j w_j$$

# ritz method

- Where the unknown constants are chosen to minimize the total potential energy.

$$\frac{\partial \Pi}{\partial a_j} = 0$$
$$\frac{\partial \Pi}{\partial b_j} = 0$$
$$\frac{\partial \Pi}{\partial c_j} = 0$$



# example

**Figure 1:** an end-loaded cantilever beam

## example

- We recall that the total potential energy is

$$\Pi = U_T - W$$

- In a simple (Euler-Bernoulli) beam, we assume that the stress is a function of the vertical displacement, $w$ and the cross-sectional area
- All stress terms other than $\sigma_{11}$ are zero



## example

- The strain energy density is

$$U = \frac{\sigma_{11}^2}{2E} = \frac{M^2 y^2}{2EI^2} = \frac{E}{2}\left(\frac{d^2 w}{dx^2}\right)^2 y^2$$

- We integrate over the volume to find the total strain energy in the beam

$$\begin{aligned} U_T &= \int_0^L \left[\iint_A \frac{E}{2}\left(\frac{d^2 w}{dx^2}\right)^2 y^2 dA\right] dx \\ &= \int_0^L \frac{EI}{2}\left(\frac{d^2 w}{dx^2}\right)^2 dx \end{aligned}$$

## example

- The work done by external forces is quite simple in this case

$$W = Pw(L)$$

- We now consider a trial function for $w$, let us consider a polynomial function

$$w = a_0 + a_1 \left(\frac{x}{L}\right) + a_2 \left(\frac{x}{L}\right)^2$$



## example

- We first ensure the trial solution satisfies the essential boundary conditions

$$\begin{aligned} w(0) &= 0 \\ 0 &= a_0 + a_1 \left(\frac{0}{L}\right) + a_2 \left(\frac{0}{L}\right)^2 \end{aligned}$$

## example

- And

$$\frac{dw(0)}{dx} = 0$$
$$0 = a_1\left(\frac{1}{L}\right) + 2a_2\left(\frac{0}{L}\right)$$



## example

- This gives $a_0 = a_1 = 0$
- $a_2$ is to be determined
- The total potential energy is

$$\Pi = U_t - W = \int_0^L \frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 dx - Pw(L)$$

- After differentiation and substitution, we find

$$\Pi = \frac{EI}{2}\int_0^L \left(\frac{2a_2}{L^2}\right)^2 dx - Pa_2$$

## example

- We minimize the potential energy by letting $\frac{\partial \Pi}{\partial a_j} = 0$

$$\Pi = \frac{2EIa_2^2}{L^3} - Pa_2$$
$$\frac{\partial \Pi}{\partial a_2} = \frac{4EIa_2}{L^3} - P = 0$$
$$a_2 = \frac{PL^3}{4EI}$$



## example

- Thus our approximate solution is

$$w = \frac{PL}{4EI} x^2$$

## example

- A simple cantilever beam of this form can be solved for exactly
- The exact solution is

$$w = \frac{Px^2}{6EI}(3L - x)$$



## example

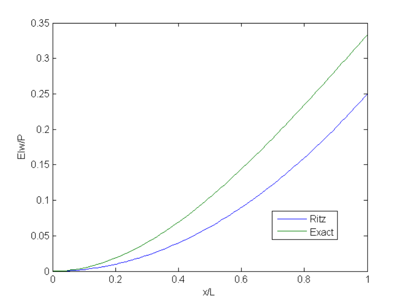


**Figure 2:** comparison of ritz displacement solution to exact solution

## example

- If we considered one more term in our trial, we would have recovered the exact solution
- In this case, more terms would be redundant
- We could have also considered a trigonometric function
- A worked example with more terms considered is here[1]

[1] https://mybinder.org/v2/gh/ndaman/live-examples/master?filepath=example/ritz.ipynb