Theory of Elasticity

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upcoming schedule

- Nov 4 - Airy Stress
- Nov 5 - Homework 6 Self-Grade Due
- Nov 9 - Polar Coordinates
- Nov 11 - Micromechanics Project Presentation
- Nov 12 - Homework 7 Due
- Nov 16 - Airy Stress Review
- Nov 18 - Complex Methods

outline

- airy stress functions
- polynomial solutions
- polar coordinates

airy stress functions

airy stress function

- A stress function technique that can be used to solve many planar problems is known as the *Airy stress function*
- This method reduces the governing equations for a planar problem to a single unknown function

body forces

- We assume first that body forces are derivable from a *potential function*, V

$$F_x = -\frac{\partial V}{\partial x}$$
$$F_y = -\frac{\partial V}{\partial y}$$

body forces

- How restrictive is this assumption?
- Most body forces are linear (gravity) and can easily be represented this way
- Only a body force with some form of coupling between axes (a function of both x and y) would be difficult to represent this way

airy stress function

- Consider the following

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} + V$$
$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} + V$$
$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

- The function $\phi = \phi(x, y)$ is known as the Airy stress function
- Equilibrium automatically satisfied

compatibility

- Substituting the Airy Stress function and potential function into the relationships, we find

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = -\frac{1-2\nu}{1-\nu}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \quad \text{plane strain}$$
$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = -(1-\nu)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \quad \text{plane stress}$$

compatibility

- If there are no body forces, or the potential function satisfies Laplace's Equation

$$\nabla^2 V = 0$$

Then both plane stress and plane strain reduce to

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0$$

polynomial solutions

airy stress solutions

- To solve a problem using Airy stress functions, we need to solve this biharmonic equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

- One solution to this is the power series

$$\phi(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} x^m y^n$$

power series solution

- Note that terms for $m+n \leq 1$ do not contribute to the stress, and can be neglected
- Also note that for $m+n \leq 3$ compatibility is automatically satisfied
- For $m+n \geq 4$ the coefficients must be related for compatibility to be satisfied

example

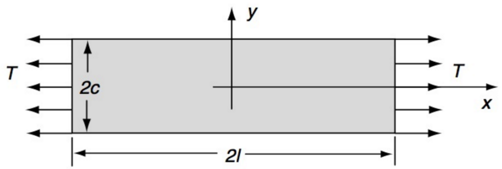

Figure 1: a simple tension example problem

example

- What are the boundary conditions in terms of the stress tensor?

$$\sigma_x(\pm l, y) = T$$
$$\sigma_y(x, \pm c) = 0$$
$$\tau_{xy}(\pm l, y) = \tau_{xy}(x, \pm c) = 0$$

example

- What is the simplest form of polynomial stress function that would satisfy these boundary conditions?

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} + V$$
$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} + V$$
$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

saint venant's principle

- Some boundary conditions are cumbersome to model exactly
- In this case we can use Saint Venant's principle to express a statically equivalent version of the boundary conditions

example

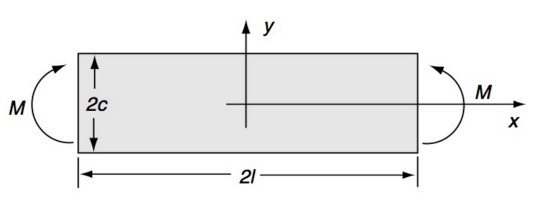

Figure 2: a simple bending example using saint venant's principle

example

- Locally along the ends, there will be some tractions in order to apply the bending moment
- These tractions will cancel out, however, so we can use Saint Venant's principle to avoid modeling them explicitly

example

$$
\begin{aligned}
\sigma_y(x, \pm c) &= 0 \\
\tau_{xy}(x, \pm c) &= \tau_{xy}(\pm L, y) = 0 \\
\int_{-c}^{c} \sigma_x(\pm l, y) dy &= 0 \\
\int_{-c}^{c} \sigma_x(\pm l, y) y dy &= -M
\end{aligned}
$$

example

- What is the simplest form of polynomial stress function that would satisfy these boundary conditions?

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} + V$$
$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} + V$$
$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

example

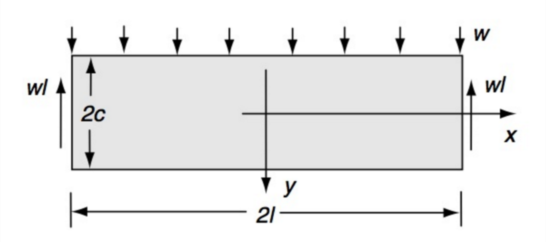

Figure 3: a distributed load example

boundary conditions

$$
\begin{aligned}
\tau_{xy}(x, \pm c) &= 0 \\
\sigma_y(x, c) &= 0 \\
\sigma_y(x, -c) &= -w \\
\int_{-c}^{c} \sigma_x(\pm l, y) dy &= 0 \\
\int_{-c}^{c} \sigma_x(\pm l, y) y dy &= 0 \\
\int_{-c}^{c} \tau_{xy}(\pm l, y) dy &= \mp wl
\end{aligned}
$$

example

- And find that the stress function

$$\phi = Ax^2 + Bx^2 y + Cx^2 y^3 + Dy^3 - \frac{1}{5} Cy^5$$

- Can satisfy the boundary conditions as well as compatibility

polar coordinates

strain-displacement

- Reduced strain-displacement:

$$
\begin{aligned}
\epsilon_r &= \frac{\partial u_r}{\partial r}, \epsilon_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right), \epsilon_z = \frac{\partial u_z}{\partial z} \\
\epsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \\
\epsilon_{\theta z} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \\
\epsilon_{zr} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)
\end{aligned}
$$

strain-displacement

- Which becomes

$$
\begin{aligned}
\epsilon_r &= \frac{\partial u_r}{\partial r} \\
\epsilon_\theta &= \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) \\
\epsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)
\end{aligned}
$$

integration

- When we change variables in integration, we also need to account for the proper change in dV
- $dV = dxdydz \neq drd\theta dz$
- We can find the correct dV by calculating the Jacobian

jacobian

$$dV = dxdydz = |\frac{\partial(x,y,z)}{\partial(r,\theta,z)}|drd\theta dz$$

$$dV = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} drd\theta dz = rdrd\theta dz$$

hooke's law

- The tensor equation for Hooke's Law is valid in polar/cylindrical/spherical coordinates too
- We only need special equations when differentiating or integrating

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} - (3\lambda + 2\mu)\alpha\Delta T\delta_{ij}$$
$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\delta_{ij}$$

equilibrium

- We have already found the equilibrium equations in polar coordinates, they are

$$\begin{aligned}\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r}(\sigma_r - \sigma_\theta) + F_r = 0 \\ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2}{r}\tau_{r\theta} + F_\theta = 0\end{aligned}$$

equilibrium

- The equilibrium equations can be written in terms of displacement (Navier equations)
- These are only useful when using a displacement formulation, but we are using stress functions
- Instead we need the Beltrami-Mitchell compatibility equations

compatibility

- Substituting stress-strain relations into the compatibility equations gives

$$\nabla^2(\sigma_r + \sigma_\theta) = -\frac{1}{1-\nu}\left(\frac{\partial F_r}{\partial r} + \frac{F_r}{r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}\right) \quad \text{(Plane Strain)}$$

$$\nabla^2(\sigma_r + \sigma_\theta) = -(1+\nu)\left(\frac{\partial F_r}{\partial r} + \frac{F_r}{r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}\right) \quad \text{(Plane Stress)}$$

airy stress functions

- When the body forces are zero, we find

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

airy stress functions

- When body forces are zero, we find the following biharmonic equation for the Beltrami-Mitchell equations

$$\nabla^4 \phi = 0$$

- Where the Laplacian is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

polar coordinates

- Recall that an Airy Stress function must satisfy the Beltrami-Mitchell compatibility equations

$$\nabla^4 \phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)^2 \phi = 0$$

- One method which gives several useful solutions assumes that the Airy Stress function has the form $\phi(r, \theta) = f(r)e^{b\theta}$

polar coordinates

- Substituting this into the compatibility equations (and canceling the common $e^{b\theta}$) term gives

$$f'''' + \frac{2}{r} f''' - \frac{1-2b^2}{r^2} f'' + \frac{1-2b^2}{r^3} f' + \frac{b^2(4+b^2)}{r^4} f = 0$$

polar coordinates

- To solve this, we perform a change of variables, letting $r = e^{\xi}$, which gives

$$r'''' = 4f''' + (2+2b^2)f'' - 4b^2 f' + b^2(4+b^2)f = 0$$

- We now consider f to have the form $f = e^{a\xi}$ which generates the characteristic equation

$$(a^2+b^2)(a^2-4a+4+b^2) = 0$$

polar coordinates

- This has solutions

$$\begin{aligned} a &= \pm ib, \pm 2ib \\ \text{OR} & \\ b &= \pm ia, \pm i(a-2) \end{aligned}$$

polar coordinates

- If we consider only solutions which are periodic in θ, we find

$$\begin{aligned} \phi = {} & a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r \\ & + (a_4 + a_5 \log r + a_6 r^2 + a_7 r^2 \log r)\theta \\ & + \left(a_{11} r + a_{12} r \log r + \frac{a_{13}}{r} + a_{14} r^3 + a_{15} r\theta + a_{16} r\theta \log r \right) \cos\theta \\ & + \left(b_{11} r + b_{12} r \log r + \frac{b_{13}}{r} + b_{14} r^3 + b_{15} r\theta + b_{16} r\theta \log r \right) \sin\theta \\ & + \sum_{n=2}^{\infty} (a_{n1} r^n + a_{n2} r^{2+n} + a_{n3} r^{-n} + a_{n4} r^{2-n}) \cos n\theta \\ & + \sum_{n=2}^{\infty} (b_{n1} r^n + b_{n2} r^{2+n} + a_{n3} r^{-n} + b_{n4} r^{2-n}) \sin n\theta \end{aligned}$$

polar coordinates

- For axisymmetric problems, all field quantities are independent of θ
- This reduces the general solution to

$$\phi = a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 logr$$

polar coordinates

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$\log r$	$1/r^2$	0	$-1/r^2$
θ	0	$1/r^2$	0
$r^2 \log r$	$2\log r + 1$	0	$2\log r + 3$
$r^2\theta$	2θ	-1	2θ
$r^3 \cos\theta$	$2r\cos\theta$	$2r\sin\theta$	$6r\cos\theta$
$r^3 \sin\theta$	$2r\sin\theta$	$-2r\cos\theta$	$6r\sin\theta$
$r\theta\sin\theta$	$2\cos\theta/r$	0	0
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\log r\cos\theta$	$\cos\theta/r$	$\sin\theta/r$	$\cos\theta/r$
$r\log r\sin\theta$	$\sin\theta/r$	$-\cos\theta/r$	$\sin\theta/r$
$\cos\theta/r$	$-2\cos\theta/r^3$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$2\sin\theta/r^3$

Figure 4: table with pre-calculated airy stress terms in polar coordinates

$r^4 \cos 2\theta$	0	$6r^2 \sin 2\theta$	$12r^2 \cos 2\theta$
$r^4 \sin 2\theta$	0	$-6r^2 \cos 2\theta$	$12r^2 \sin 2\theta$
$r^2 \cos 2\theta$	$-2 \cos 2\theta$	$2 \sin 2\theta$	$2 \cos 2\theta$
$r^2 \sin 2\theta$	$-2 \sin 2\theta$	$-2 \cos 2\theta$	$2 \sin 2\theta$
$\cos 2\theta$	$-4 \cos 2\theta / r^2$	$-2 \sin 2\theta / r^2$	0
$\sin 2\theta$	$-4 \sin 2\theta / r^2$	$2 \cos 2\theta / r^2$	0
$\cos 2\theta / r^2$	$-6 \cos 2\theta / r^4$	$-6 \sin 2\theta / r^4$	$6 \cos 2\theta / r^4$
$\sin 2\theta / r^2$	$-6 \sin 2\theta / r^4$	$6 \cos 2\theta / r^4$	$6 \sin 2\theta / r^4$
$r^n \cos n\theta$	$-n(n-1)r^{n-2} \cos n\theta$	$n(n-1)r^{n-2} \sin n\theta$	$n(n-1)r^{n-2} \cos n\theta$
$r^n \sin n\theta$	$-n(n-1)r^{n-2} \sin n\theta$	$-n(n-1)r^{n-2} \cos n\theta$	$n(n-1)r^{n-2} \sin n\theta$
$r^{n+2} \cos n\theta$	$-(n+1)(n-2)r^n \cos n\theta$	$(n+1)nr^n \sin n\theta$	$(n+2)(n+1)r^n \cos n\theta$
$r^{n+2} \sin n\theta$	$-(n+1)(n-2)r^n \sin n\theta$	$-(n+1)nr^n \cos n\theta$	$(n+2)(n+1)r^n \sin n\theta$
$\cos n\theta / r^n$	$-(n+1)n \cos n\theta / r^{n+2}$	$-(n+1)n \sin n\theta / r^{n+2}$	$(n+1)n \cos n\theta / r^{n+2}$
$\sin n\theta / r^n$	$-(n+1)n \sin n\theta / r^{n+2}$	$(n+1)n \cos n\theta / r^{n+2}$	$(n+1)n \sin n\theta / r^{n+2}$
$\cos n\theta / r^{n-2}$	$-(n+2)(n-1) \cos n\theta / r^n$	$-n(n-1) \sin n\theta / r^n$	$(n-1)(n-2) \cos n\theta / r^n$
$\sin n\theta / r^{n-2}$	$-(n+2)(n-1) \sin n\theta / r^n$	$n(n-1) \cos n\theta / r^n$	$(n-1)(n-2) \sin n\theta / r^n$

Figure 5: continued table of polar coordinate airy stress terms