AE731

Theory of Elasticity

Dr. Nicholas Smith

Wichita State University, Department of Aerospace Engineering
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upcoming schedule

- Aug 24 - Principal Values
- Aug 26 - Tensor Calculus
- Aug 27 - Homework 1 Due
- Aug 31 - Displacement and Strain
- Sep 2 - Strain Transformation

outline

- group problems
- coordinate transformation
- examples
- principal directions
- examples

- My office - WH 200D
- If you cannot make it to those hours just e-mail me to set an appointment

principal directions

principal directions

- Principal directions are defined as

$$(a_{ij} - \lambda\delta_{ij})n)j = 0$$

- λ are the principal values and n_{ij} are the principal directions
- For each eigenvalue there will be a principal direction
- We find the principal direction by substituting the solution for λ back into this equation

example

$$\begin{bmatrix} 1-\lambda_1 & 2 \\ 2 & 4-\lambda_1 \end{bmatrix} \left\{ n_1 \;\; n_2 \right\} = 0$$

- This gives

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \left\{ n_1 \;\; n_2 \right\} = 0$$

example

- We proceed to solve the equations using row-reduction, but we find

$$\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \left\{ n_1 \ n_2 \right\} = 0$$

- This means we cannot uniquely solve for n_j
- We are only concerned with the direction, magnitude is not important
- Choose $n_2 = 1$, solve for n_1
- $n^{(1)} = \langle \frac{1}{2}, 1 \rangle$

example

- Similarly, for $\lambda_2 = 0$, we find

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \left\{ n_1 \ n_2 \right\} = 0$$

- Which, after row-reduction, becomes

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \left\{ n_1 \ n_2 \right\} = 0$$

- If we choose $n_2 = 1$, we find $n_1 = -2$
- This gives $n^{(2)} = \langle -2, 1 \rangle$

example

- We can assemble a transformation matrix, Q_{ij}, from the principal directions
- First we need to normalize them to unit vectors
- $||n^{(1)}|| = \sqrt{\frac{5}{4}}$
- $\hat{n}^{(1)} = \frac{2}{\sqrt{5}}\langle \frac{1}{2}, 1\rangle = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle$
- $||n^{(2)}|| = \sqrt{5}$
- $\hat{n}^{(2)} = \langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle$

example

- This gives

$$Q_{ij} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

- And we find

$$A_{mn} = Q_{mi}Q_{nj}A_{ij}$$

$$A'_{ij} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$$

examples

example

- Find principal values, principal directions, and invariants for the tensor

$$c_{ij} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

example

- Characteristic equation simplifies to
- $-\lambda^3 + 14\lambda^2 - 56\lambda + 64 = 0$
- Which has the solutions $\lambda = 2, 4, 8$

example

- To find the principal direction for $\lambda_1 = 8$

$$\begin{bmatrix} 8-8 & 0 & 0 \\ 0 & 3-8 & 1 \\ 0 & 1 & 3-8 \end{bmatrix} \left\{ n_1 \;\; n_2 \;\; n_3 \right\} = 0$$

example

- After row-reduction, we find

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -24 \\ 0 & 1 & -5 \end{bmatrix} \left\{ n_1 \ n_2 \ n_3 \right\} = 0$$

- This means that $n_3 = 0$ and, as a result, $n_2 = 0$.
- n_1 can be any value, we choose $n_1 = 1$ to give a unit vector.
- $n^{(1)} = \langle 1, 0, 0 \rangle$

example

- To find the principal direction for $\lambda_2 = 4$

$$\begin{bmatrix} 8-4 & 0 & 0 \\ 0 & 3-4 & 1 \\ 0 & 1 & 3-4 \end{bmatrix} \left\{ n_1 \ n_2 \ n_3 \right\} = 0$$

example

- After row-reduction, we find

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \left\{ n_1 \;\; n_2 \;\; n_3 \right\} = 0$$

- This means that $n_1 = 0$
- We also know that $n_2 = n_3$ so we choose $n_2 = n_3 = 1$
- This gives $n^{(2)} = \frac{1}{\sqrt{2}}\langle 0, 1, 1 \rangle$ after normalization

example

- To find the principal direction for $\lambda_3 = 2$

$$\begin{bmatrix} 8-2 & 0 & 0 \\ 0 & 3-2 & 1 \\ 0 & 1 & 3-2 \end{bmatrix} \left\{ n_1 \;\; n_2 \;\; n_3 \right\} = 0$$

example

- After row-reduction, we find

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \left\{ n_1 \ n_2 \ n_3 \right\} = 0$$

- This means that $n_1 = 0$
- We also know that $n_2 = -n_3$, so we choose $n_2 = 1$ and $n_1 = -1$
- This gives $n^{(3)} = \frac{1}{\sqrt{2}}\langle 0, 1, -1\rangle$ after normalization

example

- In summary, for c_{ij} we have:
- $\lambda_1 = 8$ and $n^{(1)} = \langle 1, 0, 0\rangle$
- $\lambda_2 = 4$ and $n^{(2)} = \langle 0, 1, 1\rangle$
- $\lambda_3 = 2$ and $n^{(3)} = \langle 0, 1, -1\rangle$
- We can assemble $n^{(i)}$ into a transformation tensor

$$Q_{ij} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

example

- What is c'_{ij}?
- $c'_{ij} = Q_{im} Q_{jn} c_{mn}$

$$c'_{ij} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

example

- We can also find the invariants for

$$c_{ij} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

- Recall:

$$\begin{aligned} I_\alpha &= a_{ii} \\ II_\alpha &= \frac{1}{2}(a_{ii} a_{jj} - a_{ij} a_{ij}) \\ III_\alpha &= \det[a_{ij}] \end{aligned}$$

example

- First invariant

$$I_\alpha = a_{ii} = 8 + 3 + 3 = 14$$

- Second invariant

$$II_\alpha = \frac{1}{2}(a_{ii}a_{jj} - a_{ij}a_{ij})$$
$$a_{ii}a_{jj} = 14 \times 14$$
$$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + ... + a_{33}a_{33}$$
$$II_\alpha = \frac{1}{2}(196 - 84) = 56$$

example

- Third invariant

$$III_\alpha = \det[a_{ij}]$$
$$III_\alpha = 8(3 \times 3 - 1 \times 1) = 64$$