Name:

Homework 7
Due 20 Nov 2019

1. Show which displacements will be functions of z in a full solution of the plane stress problem by integrating the strain-displacement relations

   - In plane stress we have $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$
   - Using Hooke's Law, we find that only $\epsilon_{xz} = \epsilon_{yz} = 0$
   - The strain-displacement relationships given by $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ are

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \tag{1}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} \tag{2}$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} \tag{3}$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{4}$$

$$\epsilon_{xz} = 0 = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{5}$$

$$\epsilon_{yz} = 0 = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{6}$$

   - From (3) we can see that w will be a function of z
   - In general, w will also be a function of x and y, since ϵ_{zz} is a function of σ_{xx} and σ_{yy}, which can both be functions of x and y
   - When w is a function of both x and y, then u and v must be functions of z to satisfy (5) and (6)
   - Thus all three displacements, u, v, and w may be functions of z in plane stress

2. Identify all nonzero compatibility relations for a full solution of the plane stress problem. What form must ϵ_{33} take to satisfy compatibility?

- The compatibility equations are

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} \tag{7}$$

$$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = 2\frac{\partial^2 \epsilon_{yz}}{\partial y \partial z} \tag{8}$$

$$\frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} = 2\frac{\partial^2 \epsilon_{zx}}{\partial z \partial x} \tag{9}$$

$$\frac{\partial^2 \epsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z}\right) \tag{10}$$

$$\frac{\partial^2 \epsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x}\right) \tag{11}$$

$$\frac{\partial^2 \epsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y}\right) \tag{12}$$

- (7) is clearly nonzero, as it includes only x and y derivatives
- None of the strains will be a function of z, so all z-derivatives will go to zero
- Taking this into account, we find (10) and (11) vanish, as the only terms without a z-derivative are ϵ_{yz} and ϵ_{xz}, which are zero
- This leaves the following simplified compatibility relations

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} \tag{13}$$

$$\frac{\partial^2 \epsilon_z}{\partial y^2} = 0 \tag{14}$$

$$\frac{\partial^2 \epsilon_z}{\partial x^2} = 0 \tag{15}$$

$$\frac{\partial^2 \epsilon_z}{\partial x \partial y} = 0 \tag{16}$$

- We find that to satisfy compatibility, ϵ_z must be of the form $Ax + By + C$
- In a plane stress problem, ϵ_z will be derived from σ_x and σ_y, thus this condition will not always be satisfied.
- This effect is generally neglected

3. Explicitly check the validity of the plane strain/plane stress transformation relations given in Table 1 by transforming:

   (a) Equation 17 from Plane Strain to Plane Stress

$$\mu\nabla^2 u + (\lambda + \mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_x = 0 \tag{17}$$

Table 1: Conversion between plane strain and plane stress

	E	ν
Plane stress to plane strain	$\frac{E}{1-\nu^2}$	$\frac{v}{1-\nu}$
Plane strain to plane stress	$\frac{E(1+2\nu)}{(1+\nu)^2}$	$\frac{v}{1+\nu}$

- We first rewrite (17) in terms of E and ν

$$\frac{E}{2(1+\nu)}\nabla^2 u + \left(\frac{\nu E}{(1+\nu)(1-2\nu)} + \frac{E}{2(1+\nu)}\right)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_x = 0$$

- We also simplify before applying the transformation

$$\frac{E}{2(1+\nu)}\nabla^2 u + \left(\frac{E}{2(1+\nu)(1-2\nu)}\right)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_x = 0$$

- Next we substitute $E = \frac{E(1+2\nu)}{(1+\nu)^2}$ and $\nu = \frac{v}{1+\nu}$

$$\frac{\frac{E(1+2\nu)}{(1+\nu)^2}}{2(1+\frac{v}{1+\nu})}\nabla^2 u + \left(\frac{\frac{E(1+2\nu)}{(1+\nu)^2}}{2(1+\frac{v}{1+\nu})(1-2\frac{v}{1+\nu})}\right)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_x = 0$$

- After simplification, we find

$$\frac{E}{2(1+\nu)}\nabla^2 u + \left(\frac{E}{2(1-\nu)}\right)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_x = 0$$

- Which is identical to the plane stress result, verifying this transformation

(b) Equation 18 from Plane Stress to Plane Strain

$$\mu\nabla^2 v + \frac{E}{2(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_y = 0 \tag{18}$$

- Once again we start by writing everything in terms of E and ν

$$\frac{E}{2(1+\nu)}\nabla^2 v + \frac{E}{2(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_y = 0$$

- Next we make the substitutions $E = \frac{E}{1-\nu^2}$ and $\nu = \frac{v}{1-\nu}$

$$\frac{\frac{E}{1-\nu^2}}{2(1+\frac{v}{1-\nu})}\nabla^2 v + \frac{\frac{E}{1-\nu^2}}{2(1-\frac{v}{1-\nu})}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_y = 0$$

- And simplify

$$\frac{E}{2(1+\nu)}\nabla^2 v + \frac{E}{2(1+\nu)(1-2\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_y = 0$$

- Which, as found previously, can also be written as

$$\mu\nabla^2 v + (\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_y = 0$$

  Which verifies this transformation

(c) Equation 19 from Plane Stress to Plane Strain

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = -(1+\nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) \tag{19}$$

- Where we only have one substitution to make

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = -\left(1 + \frac{v}{1-\nu}\right)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$

- After simplification

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = -\frac{1}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$

- Which is the Beltrami-Mitchell equation in Plain Strain, verifying this transformation.

4. The plane stress solution for pure bending is given by

$$u = -\frac{Mxy}{EI}$$
$$v = -\frac{M}{2EI}(\nu y^2 + x^2 - L^2)$$

Where $-L \leq x \leq L$.

Transform this result to plane strain and plot a comparison of the y-displacement (v) for the two solutions along the x-axis for various Poisson's ratios.

- After substituting the transformation relationships, we find

$$v_{strain} = \frac{-M(1-\nu^2)}{2EI}\left(\frac{\nu}{1-\nu}y^2 + x^2 - L\right)$$

- Along the x-axis we re-write the equation as

$$\frac{EI}{ML^2}v_{strain} = -\frac{1-\nu^2}{2}\left(\frac{x^2}{L^2} - 1\right)$$

- We can also re-write the original plane stress equation in the same form

$$\frac{EI}{ML^2}v_{stress} = -\frac{1}{2}\left(\frac{x^2}{L^2} - 1\right)$$

- Which gives the normalized plot

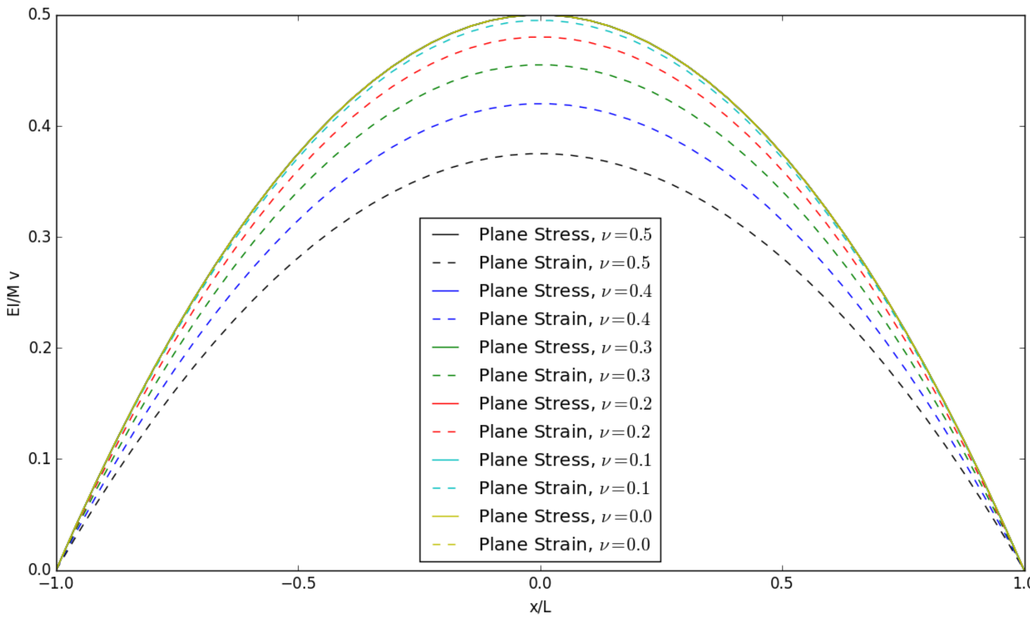

Figure 1: Normalized displacement along the x-axis of a beam in pure bending. The plane stress solution is not dependent on Poisson's ratio, but the plane strain solution is.

5. The plane strain radial displacement solution for a hole of radius R under uniform far-field loading, T, is

$$u_r = \frac{T(1+\nu)}{E}\left[(1-2\nu)r + \frac{R^2}{r}\right]$$

Transform this result to plane stress and plot the displacement versus $\frac{r}{R}$ for both solutions. Also plot the displacement along the hole ($r = R$) for varying Poisson's ratio. Comment on the results.

- Substituting to transform from plane strain to plane stress, we find

$$u_{r,stress} = \frac{T(1+\nu)}{E}\left[\frac{1-\nu}{1+\nu}r + \frac{R^2}{r}\right]$$

- And normalizing

$$\frac{E}{TR}u_{r,stress} = (1+\nu)\left[\frac{1-\nu}{1+\nu}\frac{r}{R} + \frac{R}{r}\right]$$

- We also normalize the plane strain solution

$$\frac{E}{TR}u_{r,strain} = (1+\nu)\left[(1-2\nu)\frac{r}{R} + \frac{R}{r}\right]$$

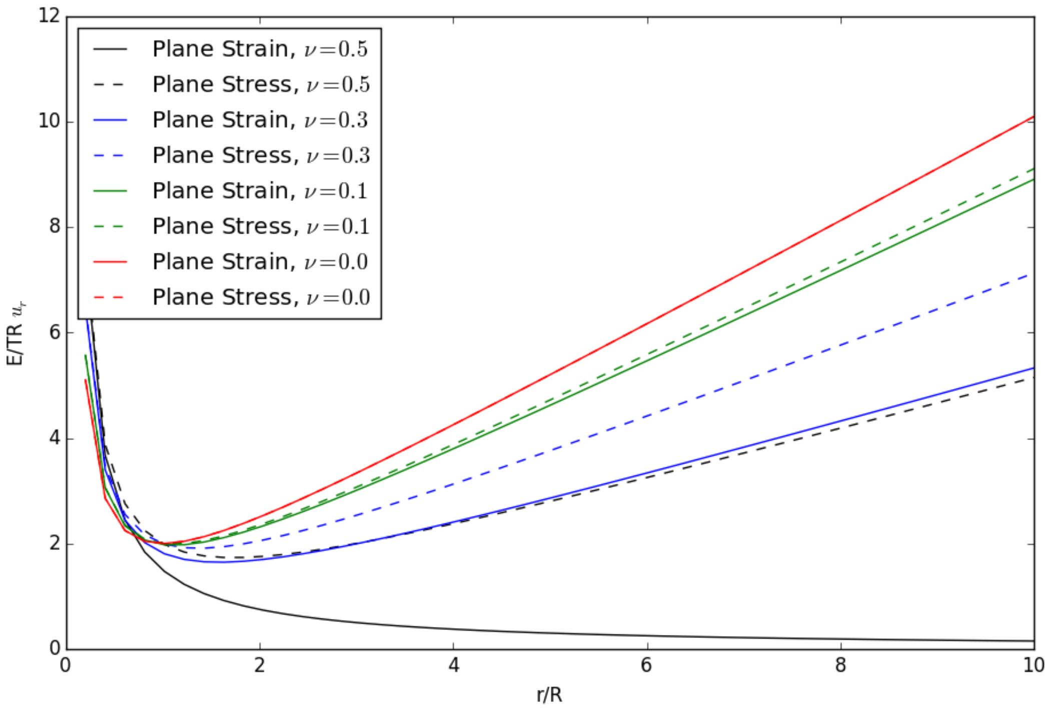

Figure 2: Displacement vs. normalized radial location for various Poisson's ratios.

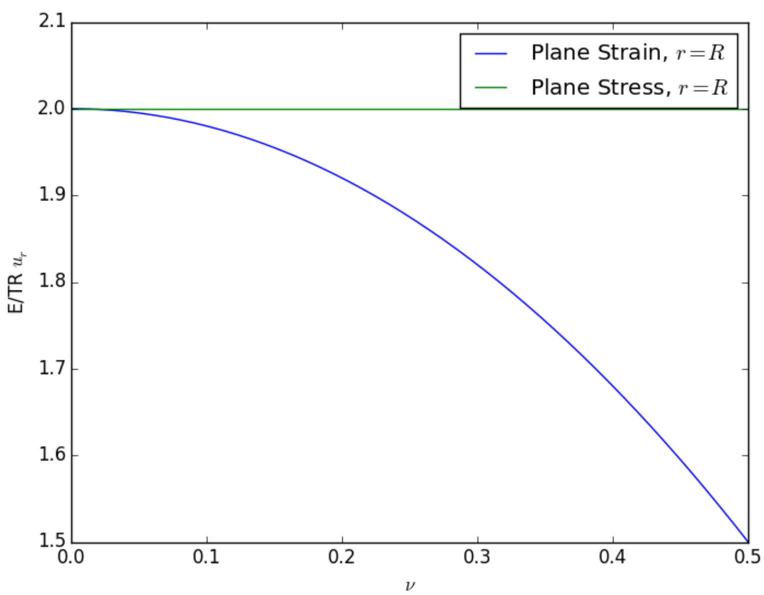

Figure 3: Radial displacement at the hole boundary plotted vs. Poisson's ratio.

- Notice in Figure 2 that as Poisson's ratio increases, the displacement solutions for plane strain and plane stress get farther apart, especially at radial distances very far away from the hole