Theory of Elasticity

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upcoming schedule

- Nov 2 - 2D Problems
- Nov 4 - Airy Stress
- Nov 5 - Homework 6 Self-Grade Due
- Nov 9 - Polar Coordinates
- Nov 11 - Micromechanics Project Presentation
- Nov 12 - Homework 7 Due

outline

- two-dimensional problems
- plane strain
- plane stress
- generalized plane stress

two-dimensional problems

2d problems

- As we learned in Chapter 5, it is often very difficult to solve full problems in 3D
- Some problems contain symmetry, or particular geometries which allow certain simplifications to be made
- In this chapter we will consider the following 2D formulations
  - Plane strain
  - Plane stress
  - Generalized plane stress
  - Antiplane strain

2d problems

- Airy stress functions provide a systematic method for solving 2D problems
- We will also develop Airy stress function solution methods in polar (cylindrical or spherical) coordinates

plane strain

plane strain

- Plane strain is a state we consider for very long bodies
- If the body is sufficiently long, then the deformation field can be considered to be a function of x and y only

$$
\begin{aligned}
u &= u(x, y) \\
v &= v(x, y) \\
w &= 0
\end{aligned}
$$

plane strain

- We can use the strain-displacement relations to find the corresponding strains from our assumptions on the displacement

$$\epsilon_{xx} = \frac{\partial u}{\partial x}$$
$$\epsilon_{yy} = \frac{\partial v}{\partial y}$$
$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$
$$\epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0$$

plane strain

- We can use Hooke's law to find the stresses

$$\sigma_{xx} = \lambda(\epsilon_{xx} + \epsilon_{yy}) + 2\mu\epsilon_{xx}$$
$$\sigma_{yy} = \lambda(\epsilon_{xx} + \epsilon_{yy}) + 2\mu\epsilon_{yy}$$
$$\sigma_{zz} = \lambda(\epsilon_{xx} + \epsilon_{yy})$$
$$\tau_{xy} = 2\mu\epsilon_{xy}$$
$$\tau_{xz} = \tau_{yz} = 0$$

- We can use these relationships to reduce the equilibrium equations.
- Recall that for equilibrium we have

$$\sigma_{ij,j} + F_i = 0$$

$$\tau_{xz} = \tau_{yz} = 0$$

, so those terms will vanish

- Although $\sigma_{zz} \neq 0$, it only appears with a derivative of z, and it is a function of x and y only, so σ_{zz} will not appear in any non-trivial equilibrium equation

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0$$

plane strain

- We can use the strain-displacement equations and Hooke's Law to write Navier's equations for plane strain

$$\mu\nabla^2 u + (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_x = 0$$
$$\mu\nabla^2 v + (\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_x = 0$$

compatibility

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}$$
$$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = 2\frac{\partial^2 \epsilon_{yz}}{\partial y \partial z}$$
$$\frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} = 2\frac{\partial^2 \epsilon_{zx}}{\partial z \partial x}$$
$$\frac{\partial^2 \epsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z}\right)$$
$$\frac{\partial^2 \epsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x}\right)$$
$$\frac{\partial^2 \epsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y}\right)$$

plane strain

- The only non-trivial term from the compatibility equations is

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}$$

- This can also be written in terms of stress (Beltrami-Mitchell)

$$\nabla^2(\sigma_x + \sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$

plane strain

- Plane strain is exact for a body of infinite length, but can also be useful for real shapes of finite length
- Consider a long body with fixed and frictionless ends.
- The boundary conditions for this case are

$$w(x, y, \pm L) = 0$$
$$\tau_{xz}(x, y, \pm L) = 0$$
$$\tau_{yz}(x, y, \pm L) = 0$$

plane stress

plane stress

- If the thickness of a body is small compared to the other dimensions, we assume that there can not be much variation in any of the stress components in that direction
- The assumptions for plane stress can be summarized as

$$\sigma_x = \sigma_x(x, y)$$
$$\sigma_y = \sigma_y(x, y)$$
$$\tau_{xy} = \tau_{xy}(x, y)$$
$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

plane stress

- To maintain these assumptions, there can be no body forces in the z-direction and no applied tractions in the z-direction.
- Other body forces must be independent of z, or distributed symmetrically such that the average may be used.

plane stress

- We can use Hooke's law to find the corresponding values of strain

$$
\begin{aligned}
\epsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) \\
\epsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) \\
\epsilon_z &= -\frac{\nu}{E}(\sigma_x + \sigma_y) \\
\epsilon_{xy} &= \frac{1+\nu}{E}\tau_{xy} \\
\epsilon_{xz} &= \epsilon_{yz} = 0
\end{aligned}
$$

strain-displacement

$$\epsilon_x = \frac{\partial u}{\partial x}$$
$$\epsilon_y = \frac{\partial v}{\partial y}$$
$$\epsilon_z = \frac{\partial w}{\partial z}$$
$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$
$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0$$
$$\epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0$$

plane stress

- Since strain in the z-direction is not zero, w becomes a linear function of z
- We also find that u and v will also be functions of z
- These effects are normally neglected, leading to an approximation in the formulation
- This is why we cannot use the full 3D compatibility equations to assess compatibility of a body with an assumed state of plane stress

plane stress

- The equilibrium equations reduce the same form in plane stress as they did for plane strain

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0$$

- But the Navier equations in terms of displacement do not reduce to exactly the same form

$$\mu \nabla^2 u + \frac{E}{2(1-\nu)} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + F_x = 0$$
$$\mu \nabla^2 v + \frac{E}{2(1-\nu)} \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + F_y = 0$$

navier equations

- The factor in the plane strain Navier equations is

$$(\lambda + \mu)$$

- We can convert this to E, ν to better compare with the plane stress equation

navier equations

$$\begin{aligned}\lambda + \mu &= \frac{\nu E}{(1+\nu)(1-2\nu)} + \frac{E}{2(1+\nu)} \\ &= \frac{2\nu E}{2(1+\nu)(1-2\nu)} + \frac{E(1-2\nu)}{2(1+\nu)(1-2\nu)} \\ &= \frac{2\nu E + E - 2\nu E}{2(1+\nu)(1-2\nu)} \\ &= \frac{E}{2(1+\nu)(1-2\nu)}\end{aligned}$$

compatibility

- Due to the approximations we made earlier, we neglect all compatibility equations with ϵ_z, even though these may not be zero

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}$$

- or in terms of stress

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = -(1+\nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$

conversion

- While plane strain and plane stress give similar results, they are not identical
- We can convert between plane strain and plane stress by replacing E and ν

	E	ν
Plane stress to plane strain	$\frac{E}{1-\nu^2}$	$\frac{\nu}{1-\nu}$
Plane strain to plane stress	$\frac{E(1+2\nu)}{1+\nu^2}$	$\frac{\nu}{1+\nu}$

- When $\nu = 0$, plane strain and plane stress give identical results

generalized plane stress

generalized plane stress

- Some approximations introduced inconsistencies in the plane stress formulation
- Generalized plane stress is based on averaging the field quantities through the thickness

$$\bar{\psi} = \frac{1}{2h}\int_{-h}^{h} \psi(x, y, z)dz$$

generalized

- We again assume that the thickness, $2h$, is much smaller than the other dimensions
- We also assume that tractions on the surfaces $z = \pm h$ are zero
- Edge loadings must have no z component and are independent of z
- Body forces also cannot have a z component and must be independent of z or symmetrically distributed through the thickness
- This gives w as a linear function of z which means

$$w(x, y, z) = -w(x, y, -z)$$

average field variables

$$
\begin{aligned}
\bar{u} &= \bar{u}(x, y) \\
\bar{v} &= \bar{v}(x, y) \\
\bar{w} &= \bar{w}(x, y) \\
\bar{\sigma_z} &= \bar{\tau_{xz}} = \bar{\tau_{yz}} = 0 \\
\bar{\sigma_x} &= \lambda^*(\bar{\epsilon_x} + \bar{\epsilon_y}) + 2\mu\bar{\epsilon_x} \\
\bar{\sigma_y} &= \lambda^*(\bar{\epsilon_x} + \bar{\epsilon_y}) + 2\mu\bar{\epsilon_y} \\
\bar{\tau_{xy}} &= 2\mu\bar{\epsilon_{xy}} \\
\bar{\epsilon_z} &= -\frac{\lambda}{\lambda + 2\mu}(\bar{\epsilon_x} + \bar{\epsilon_y})
\end{aligned}
$$

- Where $\lambda^* = \frac{2\lambda\mu}{\lambda+2\mu}$

generalized plane stress

- We can also write the equilibrium equations in terms of the averaged values

$$
\begin{aligned}
\frac{\partial\bar{\sigma_x}}{\partial x} + \frac{\partial\bar{\tau_{xy}}}{\partial x} + \bar{F}_x = 0 \\
\frac{\partial\bar{\tau_{xy}}}{\partial x} + \frac{\partial\bar{\sigma_y}}{\partial y} + \bar{F}_y = 0
\end{aligned}
$$

generalized plane stress

- Or in terms of displacements

$$\mu\nabla^2\bar{u} + (\lambda^* + \mu)\frac{\partial}{\partial x}\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) + \bar{F}_x = 0$$
$$\mu\nabla^2\bar{u} + (\lambda^* + \mu)\frac{\partial}{\partial y}\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) + \bar{F}_y = 0$$

compatibility

- The compatibility relations reduce to

$$\nabla^2(\bar{\sigma_x} + \bar{\sigma_y}) = -\frac{2(\lambda^* + \mu)}{\lambda^* + 2\mu}\left(\frac{\partial \bar{F}_x}{\partial x} + \frac{\partial \bar{F}_y}{\partial y}\right)$$

compatibility

- When we write the coefficient $\frac{2(\lambda^*+\mu)}{\lambda^*+2\mu}$ in terms of E and ν, we find

$$\frac{2(\lambda^* + \mu)}{\lambda^* + 2\mu} = 1 + \nu$$

- Which means this is an identical result to the simple plane stress derivation
- Thus the generalized plane stress method is not particularly useful

beam example

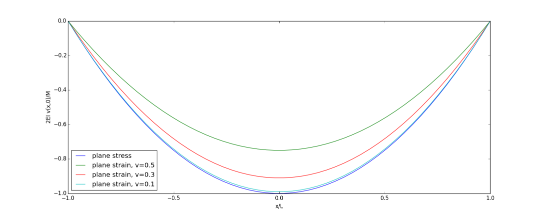

Figure 1: beam bending example