Name:

Homework 4
Due 9 Oct 2019

1. In one-dimensional problems, we can say that $\sigma = E\epsilon$. Explain in your own words, using Hooke's Law for an isotropic material, that $\sigma_{ij} \neq E\epsilon_{ij}$.

   - We know from Hooke's law that in 3D

   $$\sigma_{ij} = \frac{E}{1+\nu}\left[\epsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\epsilon_{kk}\right]$$

   - While it is clear from looking at this equation that $\sigma_{ij} \neq E\epsilon_{ij}$, we can examine the special case when $\epsilon_{11} = \epsilon^0$ and all other $\epsilon_{ij} = 0$ to see the stress in this simple case.
   - Substituting known values, and considering only the σ_{11} term, we find that

   $$\sigma_{11} = \frac{E\epsilon^0}{1+\nu} + \frac{\nu\epsilon^0}{1-2\nu}$$

   - Due to the Poisson's effect of contraction in the transverse directions under tension in the axial direction, a full 3D problem is much more complicated than the usual 1D relationship of $\sigma = E\epsilon$

2. Given the following displacements

   $$\begin{aligned} u &= -\frac{M(1-\nu^2)}{EI}xy \\ v &= \frac{M(1+\nu)\nu}{2EI}y^2 + \frac{M(1+\nu^2)}{2EI}\left(x^2 - \frac{l^2}{4}\right) \\ w &= 0 \end{aligned}$$

   Find the corresponding strain and stress fields. Assume the material is isotropic, and that M, E, I, and l are constants.

   - We begin by using the strain-displacement relations to find the strain field
   - Note: since there is no displacement in the z-direction, and u and v are not functions of z, we can consider the 2D problem using a 2x2 matrix

   $$u_{i,j} = \begin{bmatrix} -\frac{M(1-\nu^2)}{EI}y & -\frac{M(1-\nu^2)}{EI}x \\ \frac{2M(1+\nu^2)}{2EI}x & \frac{M(1+\nu)\nu}{EI}y \end{bmatrix}$$

- Since $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, we find the strain as

$$\epsilon_{ij} = \begin{bmatrix} -\frac{M(1-\nu^2)}{EI}y & \frac{M\nu^2}{EI}x \\ \frac{M\nu^2}{EI}x & \frac{M(1+\nu)\nu}{EI}y \end{bmatrix}$$

- We can find the stress field using Hooke's law

$$\sigma_{ij} = \frac{E}{1+\nu}\left[\epsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\epsilon_{kk}\right]$$

- Which gives

$$\sigma_{ij} = \frac{E}{1+\nu}\begin{bmatrix} -\frac{M(1-\nu^2)}{EI}y & \frac{M\nu^2}{EI}x \\ \frac{M\nu^2}{EI}x & \frac{M(1+\nu)\nu}{EI}y \end{bmatrix} + \frac{E\nu}{(1-2\nu)(1+\nu)}\begin{bmatrix} \frac{M(2\nu-1)(\nu+1)}{EI}y & 0 \\ 0 & \frac{M(2\nu-1)(\nu+1)}{EI}y \end{bmatrix}$$

- This can be simplified to

$$\sigma_{ij} = \begin{bmatrix} -\frac{M}{I}y & \frac{M\nu^2}{I(1+\nu)}x \\ \frac{M\nu^2}{I(1+\nu)}x & 0 \end{bmatrix}$$

3. For an isotropic material, the elastic modulus, E, shear modulus, G, and bulk modulus, K, must always be positive.

   (a) What limits must be imposed on the Poisson's ratio, ν?

   - We can find limits on the Poisson's ratio by using relationships with constants whose limits we know.
   - For example, we know that the Young's modulus must be greater than zero, $E > 0$.
   - We also know that $E = 3K(1-2\nu)$.
   - Relating these two:
   $$3K(1-2\nu) > 0$$
   - The bulk modulus, K, is greater than zero, so for this expression to be positive, we must have
   $$1 - 2\nu > 0$$
   - Which simplifies to
   $$\nu < \frac{1}{2}$$
   - This gives the upper bound for the Poisson's ratio, to find the lower bound, we consider the shear modulus, which must also be positive

   $$G = \frac{3K(1-2\nu)}{2(1+\nu)} > 0$$

- We already know that $K > 0$ and that $(1 - 2\nu) > 0$, therefore for the expression to be positive the denominator must also be positive

$$2(1 + \nu) > 0$$

- Which simplifies to

$$\nu > -1$$

- The limits for Poisson's ratio are $-1 < \nu < \frac{1}{2}$

(b) What is the physical meaning of a negative Poisson's ratio?

- Negative Poisson's ratio indicates expansion in the transverse direction under tensile loading, or contraction in the transverse direction under compressive loading.

Table 1: Selected material properties

	E (Gpa)	ν	μ, G (Gpa)	$\alpha(10^-6/°C)$
Aluminum	68.9	0.34	25.7	25.5
Nylon	28.3	0.4	10.1	102
Steel	207	0.29	80.2	13.5

4. Your good friend, who is getting a degree in "Humanities," thinks he has found some kind of super-material in his basement. What he shows you, however, looks a lot like a 20x20x5 (mm) block of anodized aluminum. To humor him, while simultaneously satisfying an obscure requirement for your Elasticity class, you set up an experiment to load it to 50 MPa in pure shear. Assuming it is aluminum, what deformation will you need to apply to obtain this state of stress?

- To load the block of material in pure shear, we would have a stress state of

$$\sigma_{ij} = \begin{bmatrix} 0 & 50 & 0 \\ 50 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

- We know from Hooke's law that

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

- Since the diagonal (normal) stresses are zero, the second term will be zero, and the strain in the material will be

$$\epsilon_{ij} = \begin{bmatrix} 0 & 9.72 & 0 \\ 9.72 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-4}$$

- Using the strain-displacement relationships to integrate, we find

$$u = 9.72 \times 10^{-4} y$$
$$v = 9.72 \times 10^{-4} x$$
$$w = 0$$

5. Somewhat humbled to learn about his not-so-super material, your determined friend decides to use some of the material he has in his basement (Aluminum, Nylon, and Steel) to make bi-material rings. A revolved schematic drawing for his ring design is shown in Figure 1.

Figure 1: Revolved schematic of your friend's ring design

Your friend has the most steel, so he prefers to use steel as the main ring (white fill in schematic), with either aluminum or nylon as the "highlight." Using your knowledge of thermoelasticity, compare the maximum operating temperature of a ring with aluminum to a ring with nylon. Assume that the inside ring radius is 10 mm, the outside radius is 13 mm, and that the highlight has an inside radius of 12 mm. The ring has a height of 9 mm and the highlight has a height of 3 mm.

Note: The textbook summarizes field equations in cylindrical coordinates in Appendix A

- In this case the maximum operating temperature occurs when the highlight expands at a faster rate than the ring to the point where it no longer will fit in the recess, and falls out.
- If we label the interface of the ring and the highlight as point A, while the outside edge of the ring can be labeled point B. We can say that the highlight will fall out when $u_{rH}^{(A)} - u_{rR}^{(B)} \geq 1$mm (where r indicates radial displacement, H indicates the Highlight, R indicates the Ring, and the superscript indicates at which point we are considering displacement)
- We can use the strain-displacement relationships to relate this inequality to the strain, and we can use the thermoelastic relationship to relate strain to a change in temperature, giving us a maximum operating temperature for the ring

- Thermal strains are given as

$$\epsilon_{ij} = \alpha \Delta T \delta_{ij}$$

- and the radial displacement can be found from

$$\frac{\partial u_r}{\partial r} = \epsilon_r$$

- Substituting and integrating gives

$$u_r = \int \alpha \Delta T dr = \alpha \Delta T r \tag{1}$$

- We can substitute this relationship into our inequality to find

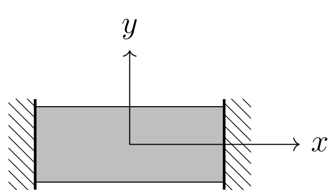
$$\alpha_H \Delta T r^{(A)} - \alpha_R \Delta T r^{(B)} = 1mm$$

- Since we are concerned with the displacement at the interface, we let $r^{(A)} = 12$mm and $r^{(B)} = 13$mm, and we can solve for the change in temperature, ΔT, giving

$$\Delta T = \frac{1}{12\alpha_H - 13\alpha_R}$$

- For a steel ring with an aluminum highlight, this gives $\Delta T = 7663$°C, or a maximum operating temperature of about 7690° C
- For a steel ring with a nylon highlight, this gives $\Delta T = 954$°C, or a maximum operating temperature of about 980° C

6. Consider a bar with a square cross-section, constrained in the x-direction, but free in the y and z directions. Assuming it is isotropic, find the stress and strain for some change in temperature, ΔT.



- If the bar were unconstrained, and thermal strains were the only strains present, the thermal expansion would be given as

$$\epsilon_{ij} = \alpha(T - T_0)\delta_{ij}$$

with no stress.

- The constraint of $\epsilon_{11} = 0$ at the ends of the bar, however, mean that some stresses must be developed to prevent the expansion of the bar in the x-direction.

- We can use the full thermoelastic strain relation to find this stress

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha(T - T_0)\delta_{ij} \tag{2}$$

- Since the material is free to expand in all other directions, we know $\sigma_{ij} = 0$ for all $i, j \neq 1$.
- Using this knowledge, we can use the condition $\epsilon_{11} = 0$ to find σ_{11}.

$$\epsilon_{11} = \frac{1+\nu}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{11} + \alpha(T - T_0) = 0 \tag{3}$$

- Simplifying and solving for σ_{11} gives

$$\sigma_{11} = -E\alpha(T - T_0) \tag{4}$$

- This gives the stress tensor as

$$\sigma_{ij} = \begin{bmatrix} -E\alpha(T - T_0) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- The total strain can be found using the thermoelastic form of Hooke's law

$$\begin{aligned} \epsilon_{ij} &= \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha(T - T_0)\delta_{ij} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & (1+\nu)\alpha(T - T_0) & 0 \\ 0 & 0 & (1+\nu)\alpha(T - T_0) \end{bmatrix} \end{aligned}$$