Name:

Homework 8
Due 4 Dec 2019

1. Find the polynomial form of an Airy stress function which can solve the cantilever beam shown. Carefully note which boundary conditions are exact and which have been replaced with a statically equivalent boundary using Saint Venant's principle. **Note:** You do not need to solve for the constants.

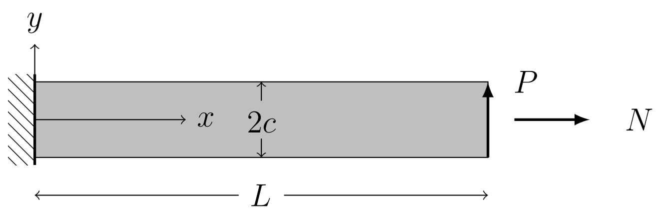


Figure 1: Cantilever beam for Problem 1

- We first replace the cantilever boundary conditions with a statically equivalent form (using Saint Venant's principle)

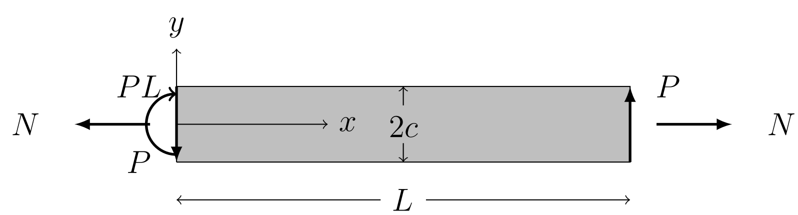


Figure 2: Replacing displacement boundaries with statically equivalent forces and moments

- We can now formulate all the boundary conditions in terms of stress

$$\sigma_y(x, \pm c) = 0 \tag{1}$$

$$\tau_{xy}(x, \pm c) = 0 \tag{2}$$

$$\int_{-c}^{c} \tau_{xy}(0, y)dy = P \tag{3}$$

$$\int_{-c}^{c} \tau_{xy}(L, y)dy = P \tag{4}$$

$$\int_{-c}^{c} \sigma_x(0, y)dy = N \tag{5}$$

$$\int_{-c}^{c} \sigma_x(L, y)dy = N \tag{6}$$

$$\int_{-c}^{c} \sigma_x(0, y)ydy = PL \tag{7}$$

$$\int_{-c}^{c} \sigma_x(L, y)ydy = 0 \tag{8}$$

- Note that while the direction of the applied load changes between ends of the beam, the direction of the load relative to the normal vector does not change (remember Cauchy's stress theorem $t_j = \sigma_{ij} n_i$)
- We are free to begin consideration of any stress term, we will begin with σ_x, let us consider a constant stress, $\sigma_x = A$
- This would satisfy conditions (5) and (6) when $A = \frac{N}{2c}$

$$\int_{-c}^{c} Ady = 2Ac = N$$

- It also satisfies (8), but not (9)

$$\int_{-c}^{c} Aydy = 0 \neq PL$$

- To have a positive moment, we need to include an odd function of y. We also note that to meet the condition at both ends, we need a function of x, so we consider $\sigma_x = A + Bxy$. (5) and (6) remain satisfied because

$$\int_{-c}^{c} Bxydy = 0$$

- So we consider (7) and (8)

$$\int_{-c}^{c} Bxyydy = \frac{2Bxc^3}{3}$$

- Since we need a positive moment at $x = 0$, but a zero-moment at $x = L$, we substitute $x - L$ for x, which satisfies (8) automatically and we can solve for B to ensure (7) is satisfied.

$$\begin{aligned}\frac{2B(0-L)c^3}{3} &= PL \\ B &= -\frac{3P}{2c^3}\end{aligned}$$

- We now have $\sigma_x = \frac{N}{2c} - \frac{3P}{2c^3}(x-L)y$. To find ϕ we recall that $\sigma_x = \frac{\partial^2\phi}{\partial y^2}$, so ϕ can be found by integrating twice with respect to y

$$\phi = \frac{N}{4c}y^2 - \frac{P}{4c^3}(x-L)y^3$$

- None of these terms has any power of x higher than 1, so $\sigma_y = 0$, this satisfies (1), so we proceed to check the boundary conditions for τ_{xy}.

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\frac{3P}{4c^3}y^2$$

- This cannot satisfy (2) as it stands, so we add a constant shear stress, C, and evaluate at $y = \pm c$

$$\begin{aligned}\tau_{xy} &= -\frac{3P}{4c^3}y^2 + C \\ 0 &= -\frac{3P}{4c^3}c^2 + C \\ C &= \frac{3P}{4c}\end{aligned}$$

- With (2) satisfied, we now check (3) and (4)

$$\begin{aligned}\int_{-c}^{c}\tau_{xy}(y)dy &= -\frac{P}{2} + \frac{3P}{2} \\ &= P\end{aligned}$$

- Which are both satisfied, this gives the final Airy stress function as

$$\phi = \frac{N}{4c}y^2 - \frac{P}{4c^3}(x-L)y^3 + \frac{3P}{4c}xy$$

- $\nabla^4\phi = 0$ for each of these terms, so compatibility is also satisfied. The stress field is

$$\begin{aligned}\sigma_x &= \frac{N}{2c} - \frac{3P}{2c^3}(x-L)y \\ \sigma_y &= 0 \\ \tau_{xy} &= -\frac{3P}{4c^3}y^2 + \frac{3P}{4c}\end{aligned}$$

2. Use the given polynomial Airy stress function to find the stress field for a cantilever beam with a distributed load.

$$\phi = Ax^2 + By^3 + Cx^2y + Dx^2y^3 + Ey^5 \tag{9}$$

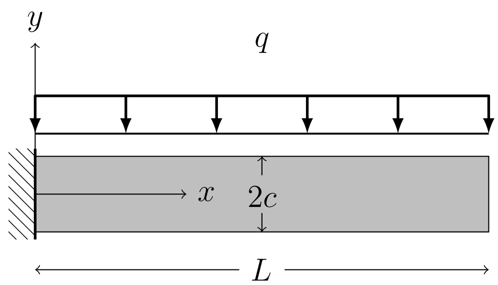


Figure 3: Cantilever beam for Problem 2

- We once again use Saint Venant's principle to formulate the boundary conditions in terms of stress. However, noticing that both σ_x and τ_{xy} should be zero at $x = L$, we consider a new coordinate system x' with $x' = 0$ at $x = L$, such that $x' = L - x$.

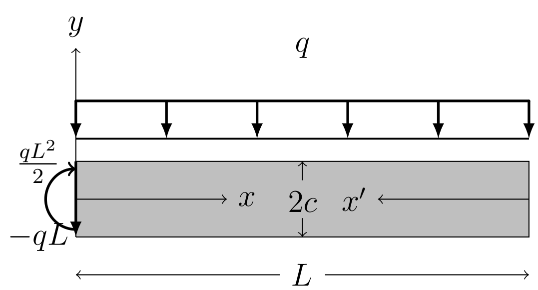


Figure 4: Statically equivalent boundary conditions

- Which gives boundary conditions of

$$\sigma_y(x', c) = -q \tag{10}$$
$$\sigma_y(x', -c) = 0 \tag{11}$$
$$\tau_{xy}(x', \pm c) = 0 \tag{12}$$
$$\int_{-c}^{c} \tau_{xy}(L, y)dy = -qL \tag{13}$$
$$\tau_{xy}(0, y) = 0 \tag{14}$$
$$\int_{-c}^{c} \sigma_x(L, y)dy = 0 \tag{15}$$
$$\sigma_x(0, y) = 0 \tag{16}$$
$$\int_{-c}^{c} \sigma_x(L, y)ydy = \frac{qL^2}{2} \tag{17}$$

- Once again, we may start with any stress term we choose to formulate a suitable polynomial. We choose to begin with τ_{xy}. To satisfy (11) and (12) requires an even function in y, but to satisfy both conditions requires that the function be at least quadratic. We also note that to satisfy (13), the function must also be linear in x'.
$$\tau_{xy} = (Ay^2 + B)x'$$
- Solving (12) and (11) gives
$$\tau_{xy} = \frac{3q}{4c^3}(y^2 - c^2)x'$$
- We now integrate to find ϕ as
$$\phi = -\frac{q}{8c^3}x'^2y^3 + \frac{3q}{8c}x'^2y$$
- We now consider the σ_x boundary conditions
$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = -\frac{3q}{4c^3}x'^2y$$
- (15) is satisfied, since the function is multiplied by x'^2. (14) is satisfied because this is an odd function in y, so we now consider (16).
$$\int_{-c}^{c} \sigma_x(L, y)ydy = \frac{q}{2}L^2$$
- With the boundary condition satisfied, we proceed to check σ_y
$$\sigma_y = \frac{\partial^2\phi}{\partial x'^2} = -\frac{q}{4c^3}y^3 + \frac{3q}{4c}y$$
- We can satisfy (9) and (10) by adding a constant $-\frac{q}{2}$ term, this does not affect σ_x or τ_{xy}, so we can now return to the Airy Stress function and check compatibility.
$$\phi = -\frac{q}{8c^3}x'^2y^3 + \frac{3q}{8c}x'^2y - \frac{q}{4}x'^2$$
- Compatibility gives
$$\nabla^4\phi = -\frac{3q}{c^3}y$$
- To avoid adding shear terms, we add $\frac{qy^5}{40c^3}$ to the Airy Stress function to satisfy compatibility
- We now return to σ_x to check boundary conditions
$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = -\frac{3q}{4c^3}x'^2y + \frac{q}{2c^3}y^3$$

- We cannot easily satisfy (15), so we relax the condition using Saint-Venant's principle

$$\int_{-c}^{c} \sigma_x(0,y)dy = 0$$
$$\int_{-c}^{c} \sigma_x(0,y)ydy = 0$$

- We see readily that we can add a simpler odd function of y-only to negate the added term. This is convenient because it does not affect compatibility or the other stress terms.

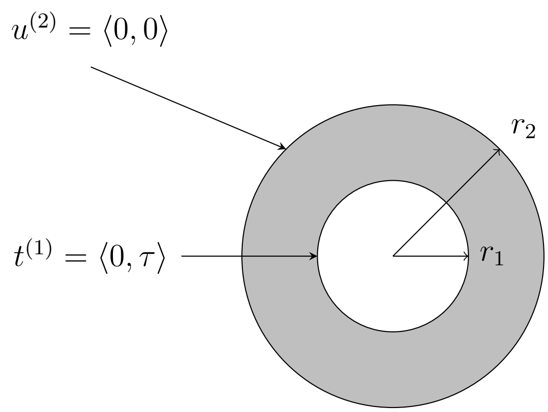
$$\sigma_x = -\frac{3q}{4c^3}x'^2 y + \frac{q}{2c^3}y^3 + Cy$$

- To cancel the effects of the compatibility term, we need

$$\int_{-c}^{c} (\frac{q}{2c^3}y^3 + Cy)ydy = 0$$

- And we find that $C = -\frac{3q}{10c}$
- We can find the stress field in terms of x by substituting $x = L - x'$ into the solution.

3. Consider the axisymmetric problem of an annular disk with a fixed outer radius under uniform shear loading at the inner radius. Find the stress and displacement solution.



Hint: Instead of our usual axisymmetric Airy stress function, which produces no shear stress, try to find a form for the Airy stress function which will have shear stress (thus θ in the Airy stress function), but will NOT have any θ dependence in the stress terms.

- When we write the boundary conditions for this problem, we see that

$$\sigma_r(r_1,\theta) = 0$$
$$\tau_{r\theta}(r_1,\theta) = \tau$$
$$u_r(r_2,\theta) = u_\theta(r_2,\theta) = 0$$

- Although this is an axisymmetric problem, we find all constants from the usual axisymmetric solution are zero. We note, however, that we may include the term $a_4\theta$, which although is a function of θ itself, gives stresses which are not a function of θ.

$$\begin{aligned}
\phi &= a_4\theta \\
\sigma_r &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \\
\sigma_\theta &= \frac{\partial^2\phi}{\partial r^2} = 0 \\
\tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = \frac{a_4}{r^2}
\end{aligned}$$

- We can solve for a_4 by checking the boundary conditions at $r = r_1$

$$\begin{aligned}
\tau_{r\theta}(r_1, \theta) &= \tau \\
&= \frac{a_4}{r_1^2} \\
a_4 &= \tau r_1^2
\end{aligned}$$

- We must now check the displacement boundary conditions at $r = r_2$, we can do this by using Hooke's Law to find the strain, then integrating the strain-displacement relations to find the displacement.

$$\begin{aligned}
\epsilon_r &= 0 \\
\epsilon_\theta &= 0 \\
\epsilon_{r\theta} &= \frac{1+\nu}{E}\tau\frac{r_1^2}{r^2}
\end{aligned}$$

- From strain-displacement, we know that

$$\epsilon_r = \frac{\partial u_r}{\partial r} = 0$$

- Which we can integrate to find that $u_r = C + f(\theta)$, however as an axisymmetric problem, the radial displacement cannot change with θ, so $u_r = C$.
- Similarly we integrate

$$\epsilon_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial\theta}\right)$$

to find that $u_\theta = -C\theta + g(r)$. For the problem to be axisymmetric, C must be zero.

- We now consider the shear strain

$$
\begin{aligned}
\epsilon_{r\theta} &= \frac{1+\nu}{E}\tau\frac{r_1^2}{r^2} \\
&= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \\
&= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)
\end{aligned}
$$

- We can solve the differential equation for u_θ by consider the solution $u_\theta = \frac{A}{r} + Br$. We can find A from the strain-displacement relations and B from the boundary conditions, giving

$$
\begin{aligned}
A &= -\frac{1+\nu}{E}\tau r_1^2 \\
B &= \frac{1+\nu}{E}\tau\frac{r_1^2}{r_2^2}
\end{aligned}
$$

- Which we can combine to find

$$
u_\theta = \frac{1+\nu}{E}\tau r_1^2\left(\frac{r}{r_2^2} - \frac{1}{r}\right)
$$

4. Show that the curved beam with end loadings can be solved with a superposition of the Airy stress function

$$
\phi = \left[Ar^3 + \frac{B}{r} + Cr + Dr\log r\right]\cos\theta \tag{18}
$$

with the pure bending solution

$$
\phi = a_1\log r + a_2 r^2 + a_3 r^2\log r \tag{19}
$$

where

$$
\begin{aligned}
a_1 &= -\frac{4M}{N}a^2b^2\log\left(\frac{b}{a}\right) \\
a_2 &= \frac{M}{N}\left[b^2 - a^2 + 2(b^2\log b - a^2\log a)\right] \\
a_3 &= -\frac{2M}{N}(b^2 - a^2)
\end{aligned}
$$

and

$$
N = (b^2 - a^2)^2 - 4a^2b^2\left[\log\left(\frac{b}{a}\right)\right]^2
$$

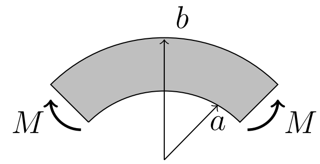


Figure 5: Pure bending problem

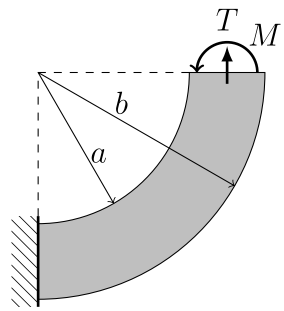


Figure 6: Curved beam with end loads

- We first replace the cantilever boundary conditions using Saint Venant's principle. Notice that we will have two bending moments, one as a reaction to the applied bending moment and one as a reaction to the applied load.

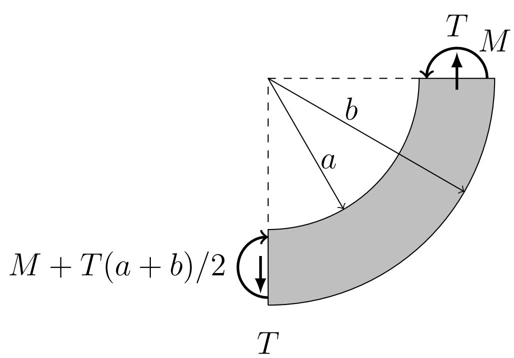
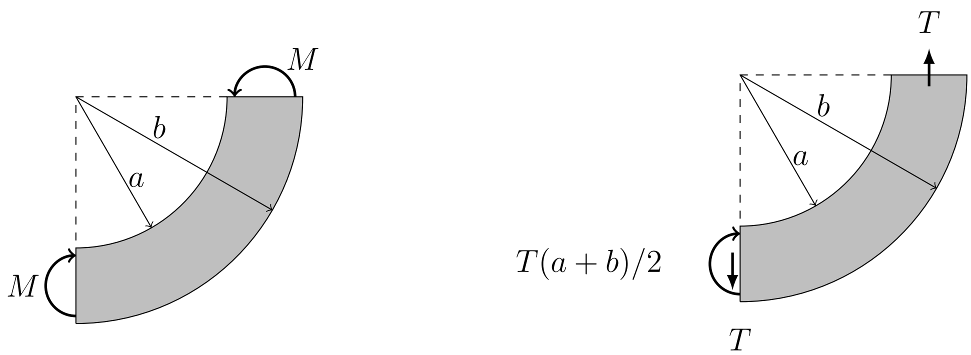


Figure 7: Statically equivalent loads

- We now use the principle of superposition to express this problem as the sum of two problems, one with pure bending (where the bending moments are equivalent at both ends) and one with the end load and a bending moment only at the previously fixed end.



- Since (18) has already been shown to provide the solution for a curved beam in pure bending, we need only show that (17) is a solution to the portion with a end loads and a bending moment only at one end.
- We formulate the boundary conditions as

$$\int_a^b \sigma_\theta(r,0)dr = T \tag{20}$$
$$\int_a^b \sigma_\theta(r,\pi/2)rdr = 0 \tag{21}$$
$$\int_a^b \sigma_{r\theta}(r,\pi/2)dr = T \tag{22}$$
$$\int_a^b \sigma_\theta(r,0)dr = 0 \tag{23}$$
$$\int_a^b \sigma_\theta(r,\pi/2)rdr = T(a+b)/2 \tag{24}$$
$$\sigma_r(a,\theta) = 0 \tag{25}$$
$$\sigma_r(b,\theta) = 0 \tag{26}$$
$$\sigma_{r\theta}(a,\theta) = 0 \tag{27}$$
$$\sigma_{r\theta}(b,\theta) = 0 \tag{28}$$

- We now consider whether (17) is capable of satisfying these conditions. We find the stresses as

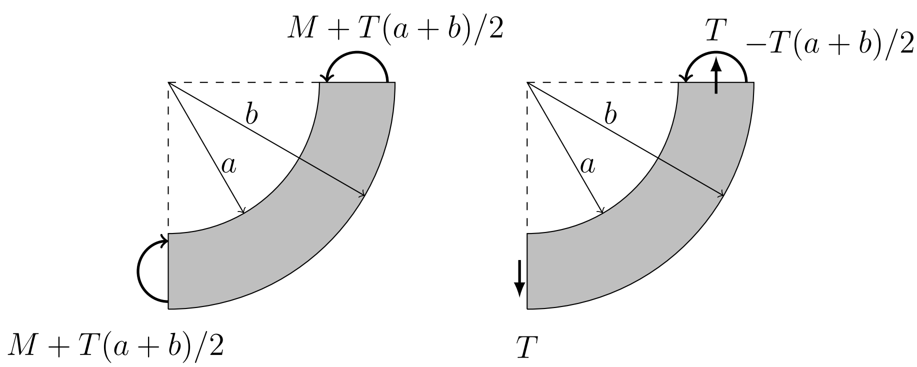
$$\sigma_r = \left[2Ar - \frac{2B}{r^3} + \frac{D}{r}\right]\cos\theta$$
$$\sigma_\theta = \left[6Ar + \frac{2B}{r^3} + \frac{D}{r}\right]\cos\theta$$
$$\tau_{r\theta} = \left[2Ar - \frac{2B}{r^3} + \frac{D}{r}\right]\sin\theta$$

- We note that C drops from all stress terms, so it may be disregarded from the solution.

- To satisfy (24) - (27) we need to satisfy the following equations

$$\left[2Aa - \frac{2B}{a^3} + \frac{D}{a}\right] = 0$$
$$\left[2Ab - \frac{2B}{b^3} + \frac{D}{b}\right] = 0$$
$$\left[2Aa - \frac{2B}{a^3} + \frac{D}{a}\right] = 0$$
$$\left[2Ab - \frac{2B}{b^3} + \frac{D}{b}\right] = 0$$

- Notice that the requirements for σ_r and $\tau_{r\theta}$ are equivalent, so there are only two equations to solve for two unknowns, this can readily be done, leaving a third variable un-solved.
- Noting that the only difference between σ_r and $\tau_{r\theta}$ is a sin or cos term, we note that both (19) and (21) will be solved with the same equation, which provides the solution for the third constant, D.
- We now check the bending moment conditions, where we notice that (23) will be 0, while (20) will be non-zero. This does not meet our boundary conditions, but we can re-formulate our superposition to give 0 bending moment at $\theta = \pi/2$ and a non-zero moment at $\theta = 0$

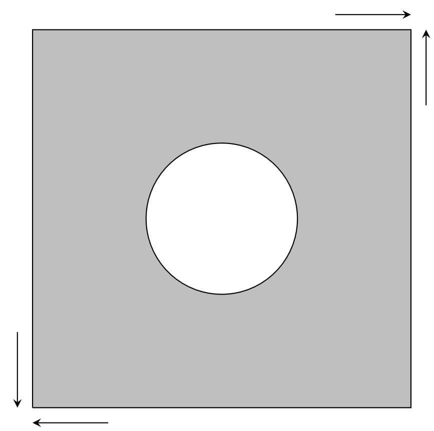


- Thus the Airy stress function has the correct form to satisfy the boundary conditions.

5. Use the Airy stress function

$$\phi = \left[Ar^2 + \frac{B}{r^2} + C \right] \sin 2\theta \tag{29}$$

to solve the problem of an infinite plate subjected to a pure shear stress at infinity. Plot σ_θ around the hole.



- We begin this problem by writing the boundary conditions for a rectangular plate with a pure shear stress applied

$$\begin{aligned} \sigma_x(\pm\infty, y) &= 0 \\ \sigma_y(\pm\infty, x) &= 0 \\ \tau_{xy}(\pm\infty, \pm\infty) &= \tau \end{aligned}$$

- Now since the traction-free boundary conditions around the hole are best expressed in polar coordinates, we transform these boundary conditions to polar coordinate form

$$\sigma_r(\infty, \theta) = 2\tau \sin\theta \cos\theta = \tau \sin 2\theta \tag{30}$$

$$\sigma_\theta(\infty, \theta) = -2\tau \sin\theta \cos\theta = -\tau \sin 2\theta \tag{31}$$

$$\tau_{r\theta}(\infty, \theta) = \tau(\cos^2\theta - \sin^2\theta) = \tau \cos 2\theta \tag{32}$$

- Traction-free boundary conditions around the hole require that

$$\sigma_r(a, \theta) = \tau_{r\theta}(a, \theta) = 0 \tag{33}$$

- The stresses from the Airy stress function (28) are

$$\begin{aligned} \sigma_r &= -2(A + 3B/r^4 + 2C/r^2) \sin 2\theta \\ \sigma_\theta &= 2(A + 3B/r^4) \sin 2\theta \\ \tau_{r\theta} &= 2(-A + 3B/r^4 + 2C/r^2) \cos 2\theta \end{aligned}$$

- To satisfy (32) for all θ we need

$$-2(A + 3B/a^4 + 2C/a^2) = 0$$
$$2(-A + 3B/a^4 + 2C/a^2) = 0$$

- Which gives

$$B = Aa^4$$
$$C = -2Aa^2$$

- We now consider (29)-(31) by letting $r \to \infty$

$$\sigma_r = -2A \sin 2\theta = \tau \sin 2\theta$$
$$\sigma_\theta = 2A \sin 2\theta = -\tau \sin 2\theta$$
$$\tau_{r\theta} = -2A \cos 2\theta = \tau \cos 2\theta$$

- It is readily apparent that all three equations are redundant, and are solved when $A = -\frac{\tau}{2}$
- This gives a total solution of

$$\sigma_r = (1 + 3a^4/r^4 - 4a^2/r^2)\tau \sin 2\theta$$
$$\sigma_\theta = -(1 + 3a^4/r^4)\tau \sin 2\theta$$
$$\tau_{r\theta} = (\tau - 3\tau a^4/r^4 + 2\tau a^2/r^2) \cos 2\theta$$

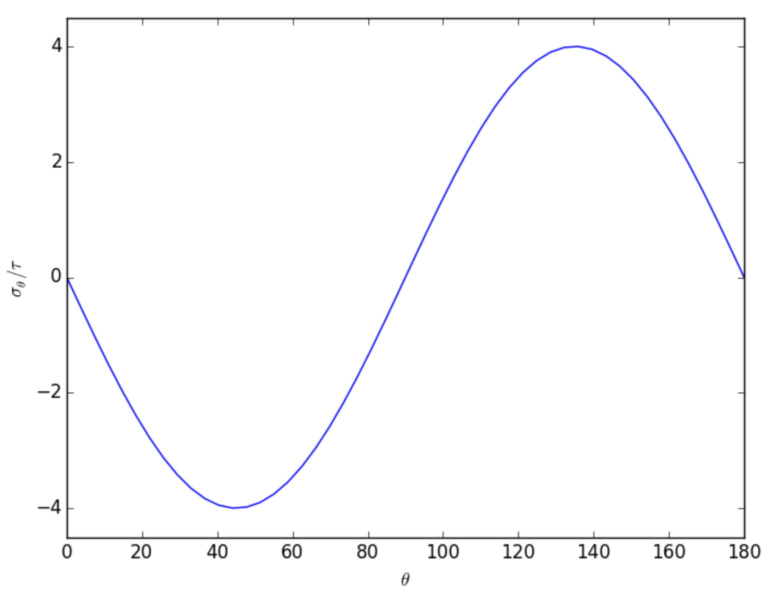


Figure 8: Hoop stress around a hole with remote shear stress, τ applied

6. An elastic disk is perfectly bonded to a rigid ring. This composite disk rotates at a constant angular velocity of Ω. Find the stress and displacement fields.

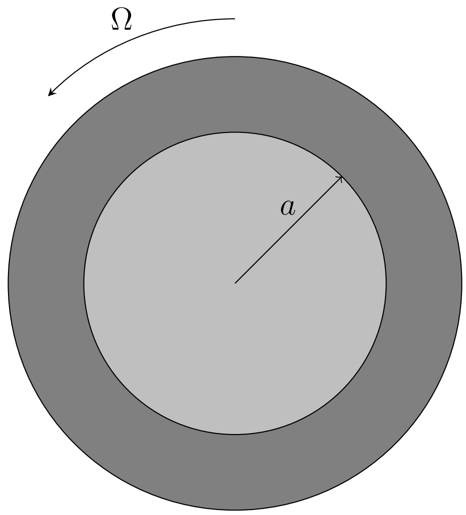


Figure 9: Diagram for an elastic disk bonded to a rigid ring. The light gray is the elastic disk, while the dark grey is the rigid ring.

Hint: Show that the particular solution,

$$\phi_P = \frac{\kappa - 1}{16(\kappa + 1)} \rho \Omega^2 r^4 \tag{34}$$

satisfies compatibility, then use it in the solution.

- For a rotating disk of constant angular velocity, the body force is

$$\rho b_r = \rho r \Omega^2$$
$$\rho b_\theta = 0$$

- We can formulate a suitable potential function V, using

$$\rho b_r = -\frac{\partial V}{\partial r}$$
$$\rho b_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta}$$

  which gives

$$V = -\frac{1}{2}\rho r^2 \Omega^2$$

- Since this problem has a body force, compatibility gives the following equation

$$\nabla^4 \phi = -\frac{2(\kappa - 1)}{\kappa + 1} \nabla^2 V \tag{35}$$

- Since right side is non-zero, we must find a particular solution to add to the already-known homogeneous solution. We will show that (33) is a particular solution.

$$
\begin{aligned}
\nabla^4\phi &= \nabla^2\nabla^2\phi \\
&= \nabla^2\left(\frac{\kappa-1}{(\kappa+1)}\rho\Omega^2 r^2\right) \\
&= \frac{\kappa-1}{(\kappa+1)}4\rho\Omega^2
\end{aligned}
$$

and

$$
\begin{aligned}
-\frac{2(\kappa-1)}{\kappa+1}\nabla^2 V &= -\frac{2(\kappa-1)}{\kappa+1}\left(-4\rho\Omega^2\right) \\
&= \frac{\kappa-1}{(\kappa+1)}4\rho\Omega^2
\end{aligned}
$$

- Thus (33) is the particular solution to this differential equation. Since this is an axisymmetric problem, we consider only the axisymmetric terms from the homogeneous solution, giving

$$
\phi = a_0 + a_1\log r + a_2 r^2 + a_3 r^2\log r + \frac{\kappa-1}{16(\kappa+1)}\rho\Omega^2 r^4
$$

- a_0 produces no stress and $a_1\log r$ and $a_3 r^2\log r$ both give singular stresses when $r=0$, which is not acceptable in this problem, thus we consider each of these coefficients to be zero, giving

$$
\phi = a_2 r^2 + \frac{\kappa-1}{16(\kappa+1)}\rho\Omega^2 r^4
$$

- We are left with only one unknown coefficient, with displacement boundary conditions it is desirable to calculate displacements from stress, at which point we will solve for the unknown coefficient.

$$
\begin{aligned}
\sigma_r &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + V = 2a_2 - \frac{\kappa+3}{4(\kappa+1)}\rho\Omega^2 r^2 \\
\sigma_\theta &= \frac{\partial^2\phi}{\partial r^2} + V = 2a_2 + \frac{\kappa-5}{4(\kappa+1)}\rho\Omega^2 r^2
\end{aligned}
$$

$$
\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = 0
$$

- Which gives as strains

$$
\begin{aligned}
\epsilon_r &= \frac{1}{E}\left(2a_2(1-\nu) + \frac{-(1+\nu)(\kappa+3)-2\nu}{4(\kappa+1)\rho\Omega^2 r^2}\right) \\
\epsilon_\theta &= \frac{1}{E}\left(2a_2(1-\nu) + \frac{(1+\nu)(\kappa-5)-2\nu}{4(\kappa+1)\rho\Omega^2 r^2}\right) \\
\epsilon_{r\theta} &= 0
\end{aligned}
$$

- Next we integrate to find the displacement

$$u_r = \int \epsilon_r dr = \frac{1}{E}\left(2a_2(1-\nu)r + \frac{-(1+\nu)(\kappa+3) - 2\nu}{12(\kappa+1)\rho\Omega^2 r^3}\right) + C$$

- Where C is a constant, not a function of θ, because the problem is axisymmetric and cannot be a function of θ.
- Next we know that

$$\begin{aligned}
\epsilon_\theta &= \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \\
\frac{\partial u_\theta}{\partial \theta} &= r\epsilon_\theta - u_r \\
\frac{\partial u_\theta}{\partial \theta} &= u_r - C - u_r \\
\frac{\partial u_\theta}{\partial \theta} &= -C \\
u_\theta &= -C\theta + f(r)
\end{aligned}$$

- Since $u_\theta = 0$ at $r = a$, $C = 0$ and $f(a) = 0$. As since displacement will only be radial in this problem, $u_\theta = 0$, therefore $f(r) = 0$.
- Solving the boundary conditions for $u_r = 0$ at $r = a$ gives

$$a_2 = \frac{\rho\Omega^2 a^2}{4(1+\kappa)}$$

- Which can be substituted into the above results to give the final stress and displacement field