1 Method 1

Suppose we take time series data points, N (indexed by a) from each of S species.

$$X_i^a \rightarrow X^1 = \begin{bmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ x_S^1 \end{bmatrix}, X^2 = \begin{bmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_S^2 \end{bmatrix} ...., X^N = \begin{bmatrix} x_1^N \\ x_2^N \\ \vdots \\ x_S^N \end{bmatrix}$$

Goal: Maximizing $P(X|\mu, \sigma^2, d)$

Assumptions:

1). A is a symmetric S by S matrix

2). $A_{ij} = \dfrac{\mu}{S} + B_{ij}$, where B_{ij} $N(0, \dfrac{\sigma^2}{S})$, $i.e. A_{ij}$ $N(\dfrac{\mu}{S}, \dfrac{\sigma^2}{S})$ 3). $A_{ii} = -d + \dfrac{\mu}{S}$

4). For A sufficiently large, the determinants of A for different random entries are similar so we use the average determinat of A in place of A. $< det A >= D(\mu, \sigma^2, d)$

Since, x_i are independent, the average probability of observing x_i given μ, σ, d:

$$P(x|\mu, \sigma^2, d) = \int dA \prod (P(x^a|A)P(A|\mu, \sigma^2, d)$$

we can pull the product out and distribute it to each term

$$= \int (\prod_{i<j}(dA_{ij}) \frac{(detA)^N/2}{(2\pi)^N S/2} exp(\frac{1}{2}\sum_{a=1}^{S} N \sum_{ij} x_i^a A_{ij} x_j^a) P(A|\mu, \sigma^2, d)$$

Given our assumptions:

$$= \frac{< detA >^{N/2}}{(2\pi)^{NS/2}} \int (\prod_{i<j} dB_{ij}) exp(\frac{1}{2}\sum_{a=1}^{N}(-d\sum(x_i^a)^2 + \frac{\mu}{S}\sum_{ij} x_i^a x_j^a + \sum_{i<j} x_i B_{ij} x_j) P(B|\sigma^2))$$

pulling out the terms seperate from B_{ij}

$$= \frac{< detA >^{N/2}}{(2\pi)^{NS/2}} exp(\frac{-d}{2}\sum_{a=1}^{N}\sum_{i=1}^{S}(x_i^a)^2 + \frac{\mu}{2S}\sum_{a=1}^{N}(\sum_{i=1}^{S} x_i^a)^2 \times \int ((\prod_{ij} dB_{ij}) exp(\sum_{a=1}^{N}\sum_{i<j} x_i^a B_{ij} x_j^a) exp(-\sum_{i<j}\frac{B_{ij}^2}{2\sigma^2/S})$$

$$\times (\sqrt{\frac{S}{2\pi}}\frac{1}{\sigma}))$$

The firt part of the integral becomes

$$\prod_{i<j} dB_{ij} \rightarrow \prod_{i<j} exp(\frac{\sigma^2}{2S}(\sum_{a=1}^{N} x_i^a x_j^a)^2) = exp(\frac{\sigma^2}{4S}\sum_{ij}\sum_{a} x_i^a x_j^a \sum_{b} x_i^b x_j^b)$$

1

Then we can condense the summations and obtain:

$$exp(\frac{\sigma^2}{4S}\sum_{ab}(\sum_i x_i^a x_i^b)^2)$$

The portion in the exponent becomes

$$\prod_{i<j} exp(-\frac{B_{ij}^2}{2\sigma^2/S} + \sum_{a=1}^N x_i^a B_{ij} x_j^a)\sqrt{\frac{S}{2\pi}}\frac{1}{\sigma}$$

After rearranging the terms we get

$$\prod_{i<j} dB_{ij}\sqrt{\frac{S}{2\pi}}\frac{1}{\sigma} exp(-\frac{B_{ij}^2}{2\sigma^2/S} + \sum_{a=1}^N x_i^a B_{ij} x_j^a)$$

Let $y = B_{ij}, \tau = \frac{\sigma}{\sqrt{S}}, z = \sum_{a=1} x_i^a x_j^a$

Then our integral becomes :

$$\int dy\sqrt{\frac{1}{2\pi}}\frac{1}{\tau} exp(\frac{-y^2}{2\tau^2} + yz)$$

Which when solved, simplifies to:

$$exp(\frac{\tau^2 z^2}{2})$$

Now let us denote $P(x|\mu, \sigma, d)$ as P:

$$P = D \times exp(\frac{-d}{2}\sum_{a=1}^N\sum_{i=1}^S(x_i^a)^2 + \frac{\mu}{2S}\sum_{a=1}^N\sum_{i=1}^S(\sum_i x_i^a)^2 + \frac{\sigma^2}{4S}\sum_{ab}(\sum_i x_i^a x_i^b)^2)$$

$$\frac{2}{N}logP = logD - \frac{d}{N}\sum_{a=1}^N\sum_{i=1}^S(x_i^a)^2 - \frac{\mu}{SN}\sum_{a=1}^N\sum_{i=1}^S(\sum_i x_i^a)^2 + \frac{\sigma^2}{2SN}\sum_{ab}(\sum_i x_i^a x_i^b)^2$$

1.1 Optimization

To optimize P we take the partial derivatives wrt μ, σ, and d and set them all equal to 0

1). $\frac{2}{N}\frac{\partial logP}{\partial d} = \frac{1}{D}\frac{\partial D}{\partial d} - \frac{1}{N}\sum_{a=1}^N\sum_{i=1}^S(x_i^a)^2 = 0$

2). $\frac{2}{N}\frac{\partial logP}{\partial \sigma^2} = \frac{1}{D}\frac{\partial D}{\partial \sigma^2} + \frac{1}{2SN}\sum_{ab}(\sum_i x_i^a x_i^b)^2 = 0$

3). $\frac{2}{N}\frac{\partial logP}{\partial \mu} = \frac{1}{D}\frac{\partial D}{\partial \mu} - \frac{1}{SN}\sum_{a=1}^N(\sum_{i=1}^S x_i^a)^2 = 0$

1.2 Determinant of A

The average determinant as given to us by the semicircular law in Random Matrix Theory-Robert Wiegner

$$< logD >= \sum_i log(\lambda_i) = log(d + \mu) + (S - 1) \int d\lambda \frac{\sqrt{4\sigma^2 - (\lambda + d)^2}}{2\pi\sigma} log(\lambda)$$

We evaluate this integral from $-d - 2\sigma : -d + 2\sigma$ as that is the radius in which all of our eigenvalues lie according to the semicircular law. Let $z = \dfrac{\lambda - d}{2\sigma}, b = \dfrac{2d}{\sigma}$ so $\lambda \to 2\sigma z + d, d\lambda \to 2\sigma dz$. Now we divide everything by 2σ and the integral:

$$\int_{-d/2\sigma-1}^{-d/2\sigma+1} d\lambda \frac{\sqrt{1 - (\frac{\lambda - d}{2\sigma})^2}}{2\pi\sigma} log(\frac{\lambda}{2\sigma}) + log(2\sigma)$$

becomes:

$$I = \frac{2\sigma}{\pi}[log(2\sigma) \int_{-b-1}^{-b+1} dz \sqrt{1 - z^2} + \int_{-b-1}^{-b+1} dz log(z + b)\sqrt{1 - z^2}]$$

By using mathematica we solve the integral, where $F_{3,1}$ is the Hypergeometric function $HypegeometricPFQ[(1, 1, \frac{3}{2}), (2, 3), \frac{1}{b^2}]$:

$$I = log(d) - \frac{\sigma^2}{2d^2}[F_{3,11}(\frac{4\sigma^2}{d^2})]$$

Now we can rewrite the average log determinant as:

$$\frac{1}{S} < logD >= \frac{1}{S}log(d + \mu) + (1 - \frac{1}{S})I$$

1.3 Solving the Optimization Equations

Now that we have the $< logD >$ we can solve optimization equations 1)., 2). and 3).

So first let $g(x) = \dfrac{1}{2}xF_{3,1}(4x)$, Then $g'(x) = \dfrac{1}{1 + \sqrt{1 - 4x} - 2x}$

Optimization equation 3). becomes

$$\frac{2}{N}\frac{\partial logP}{\partial \mu} = \frac{1}{d + \mu} - \frac{1}{SN}\sum_{a=1}^{N}(\sum_{i=1}^{S} x_i^a)^2 = 0$$

If we solve optimization equation 3). for $d + \mu$ we get $d + \mu = \dfrac{SN}{\sum_{a=1}^{N}(\sum_{i=1}^{S} x_i^a)^2}$.
We replace $d + \mu$ in optimization equation 1). with this.

Optimization equation 1). becomes:

$$\frac{2}{N}\frac{\partial logP}{\partial d} = \frac{1}{d} + \frac{2\sigma^2}{d^3}g'(\frac{\sigma^2}{d^2}) - \frac{1}{SN}\sum_{a=1}^{N}\sum_{i=1}^{S}(x_i^a)^2 + \frac{1}{S^2N}\sum_{a=1}^{N}(\sum_{i=1}^{S}x_i^a)^2 = 0$$

$$\rightarrow \frac{1}{d} + \frac{2\sigma^2}{d^3}g'(\frac{\sigma^2}{d^2}) = \frac{1}{SN}\sum_{a=1}^{N}\sum_{i=1}^{S}(x_i^a)^2 - \frac{1}{S^2N}\sum_{a=1}^{N}(\sum_{i=1}^{S}x_i^a)^2$$

Optimization equation 2). becomes:

$$\frac{2}{N}\frac{\partial logP}{\partial \mu} = -\frac{1}{d^2}g'(\frac{\sigma^2}{d^2}) + \frac{1}{2SN}\sum_{ab}(\sum_{i}x_i^a x_i^b)^2 = 0$$

$$\rightarrow \frac{1}{S}\frac{1}{2SN}\sum_{ab}(\sum_{i}x_i^a x_i^b)^2 = \frac{1}{d^2}g'(\frac{\sigma^2}{d^2})$$

To solve for μ, σ and d we use rootsolvers in R...