22.2.1 Richardson's Extrapolation

Recall that, in Sec. 10.3.3, we used iterative refinement to improve the solution of a set of simultaneous linear equations. Error-correction techniques are also available to improve the results of numerical integration on the basis of the integral estimates themselves. Generally called *Richardson's extrapolation,* these methods use two estimates of an integral to compute a third, more accurate approximation.

The estimate and error associated with a multiple-application trapezoidal rule can be represented generally as

$$I = I(h) + E(h)$$

where I = the exact value of the integral, $I(h)$ = the approximation from an n-segment application of the trapezoidal rule with step size $h = (b - a)/n$, and $E(h)$ = the truncation error. If we make two separate estimates using step sizes of h_1 and h_2 and have exact values for the error,

$$I(h_1) + E(h_1) = I(h_2) + E(h_2) \tag{22.1}$$

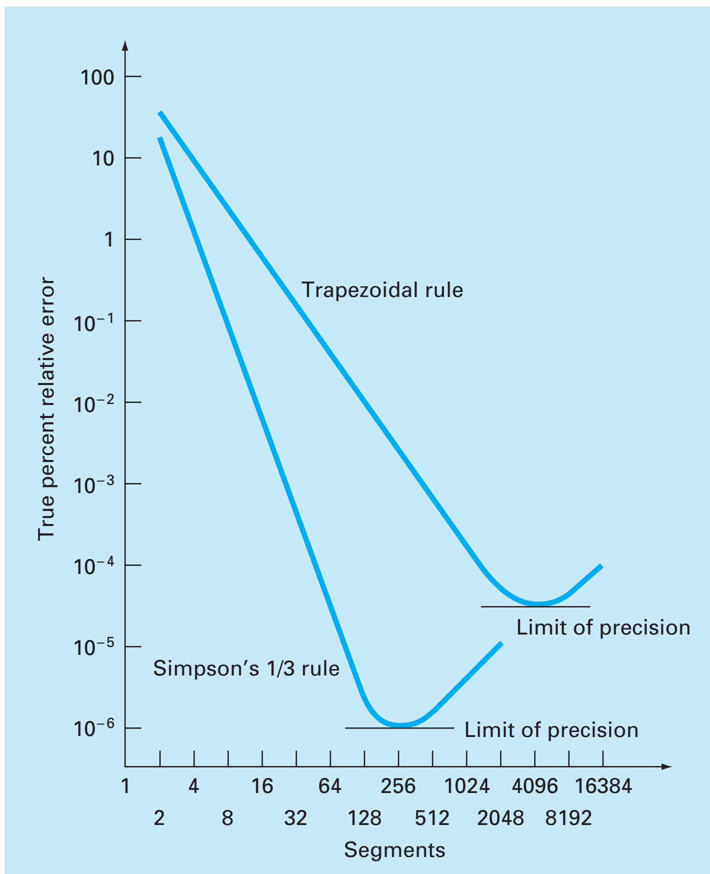

FIGURE 22.2
Absolute value of the true percent relative error versus number of segments for the determination of the integral of $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$, evaluated from $a = 0$ to $b = 0.8$ using the multiple-application trapezoidal rule and the multiple-application Simpson's 1/3 rule. Note that both results indicate that for a large number of segments, round-off errors limit precision.

Now recall that the error of the multiple-application trapezoidal rule can be represented approximately by Eq. (21.13) [with $n = (b - a)/h$]:

$$E \cong -\frac{b-a}{12} h^2 \bar{f}'' \tag{22.2}$$

If it is assumed that $\bar{f}''$ is constant regardless of step size, Eq. (22.2) can be used to determine that the ratio of the two errors will be

$$\frac{E(h_1)}{E(h_2)} \cong \frac{h_1^2}{h_2^2} \tag{22.3}$$

This calculation has the important effect of removing the term $\bar{f}''$ from the computation. In so doing, we have made it possible to utilize the information embodied by Eq. (22.2) without prior knowledge of the function's second derivative. To do this, we rearrange Eq. (22.3) to give

$$E(h_1) \cong E(h_2)\left(\frac{h_1}{h_2}\right)^2$$

which can be substituted into Eq. (22.1):

$$I(h_1) + E(h_2)\left(\frac{h_1}{h_2}\right)^2 \cong I(h_2) + E(h_2)$$

which can be solved for

$$E(h_2) \cong \frac{I(h_1) - I(h_2)}{1 - (h_1/h_2)^2}$$

Thus, we have developed an estimate of the truncation error in terms of the integral estimates and their step sizes. This estimate can then be substituted into

$$I = I(h_2) + E(h_2)$$

to yield an improved estimate of the integral:

$$I \cong I(h_2) + \frac{1}{(h_1/h_2)^2 - 1}[I(h_2) - I(h_1)] \tag{22.4}$$

It can be shown (Ralston and Rabinowitz, 1978) that the error of this estimate is $O(h^4)$. Thus, we have combined two trapezoidal rule estimates of $O(h^2)$ to yield a new estimate of $O(h^4)$. For the special case where the interval is halved ($h_2 = h_1/2$), this equation becomes

$$I \cong I(h_2) + \frac{1}{2^2 - 1}[I(h_2) - I(h_1)]$$

or, collecting terms,

$$I \cong \frac{4}{3} I(h_2) - \frac{1}{3} I(h_1) \tag{22.5}$$

EXAMPLE 22.1

Error Corrections of the Trapezoidal Rule

Problem Statement. In the previous chapter (Example 21.1 and Table 21.1), we used a variety of numerical integration methods to evaluate the integral of $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $a = 0$ to $b = 0.8$. For example, single and multiple