Rubik's Cube Group

Akil Marshall

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1 Introduction

The goal of this document is to build up a framework that links group theory to techniques in artificial intelligence, first search techniques, with luck more sophisticated approaches later.

2 What is a Group

A group is a set of objects with method of combining them called the group operation $\{G, \cdot\}$. The set cannot be empty and the group operation must obey 4 conditions:

- For all $a, b \in G$,

  $$a \cdot b \in G$$

  The group operation is **closed** in G.

- For all $a, b, c \in G$,

  $$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

  The group operation is **associative** in G.

- There exists $e \in G$ such that for any $a \in G$,

  $$a \cdot e = a = e \cdot a.$$

  There exists an **identity** element of the binary operation in G.

- For all $a \in G$ there exists $a^{-1} \in G$ such that

  $$a \cdot a^{-1} = e = a^{-1} \cdot a.$$

  There exists **inverse** elements of the binary operation in G.

3 The Rubik's Cube Group

It turns out that the Rubik's cube puzzle forms a group. The set consists of the physical manipulations you can do to the puzzle and the binary operation is the composition of those manipulations. [1] To put it plainly if you turn a turn a Rubik's cube it results in a cube (closure), the "identity" is not manipulating the cube, and each turn and be undone (inverse). With these realizations we will build up a theory to describe manipulations (elements of the groups).

Before continuing I would like to note that I am a math enthusiast and not a math professor and will be rigorous as I can. However I will not be proving everything I do.

3.1 Notation

From the group theoretic point of view the coloring and orientation of a Rubik's cube do not matter. With this we will define two major groups of faces, the **vertical** faces and the **horizontal** faces denoted V_n and H_n respectively. There are 3 vertical and 3 horizontal faces, The subscript n describes which layer of the cube the face makes up, V_1 if the cube is sitting on a table V_1 is the layer that touch the table top. The orientation of the horizontal face is picked arbitrarily any of the faces that sits orthogonal to V_n will do.

We now can interpret V_n to be a 90° clockwise rotation of V_n and H_n to be a 90° clockwise rotation of H_n, their inverses $V_n^{-1}, H_n^{-1}$are counter-clockwise turns.

With this language we can now what the online Rubik's cube community calls algorithms, although we now know these to be compositions of the elements in of our group we will continue to use this it in this manner for the time being.

3.2 Algorithms and Algebra with the Rubik's Cube Group

The following is a popular algorithm to rotate the centers of a opposing hemispheres of the cube.

$$H_2V_2H_2V_2H_2V_2H_2V_2 \tag{1}$$

Because the binary operation is composition the algorithm is applied right-to-left. If this doesn't feel quite right hopefully writing the group members will function notation will convince you. Suppose we wanted to do V_2 and then H_1 this is denoted,

$$V_2(H_1()).$$

[1]The turns are also associative, suppose τ and σ are some turns of the cube you'll find that $(\sigma\tau\tau^{-1})\sigma^{-1} = \sigma(\tau\tau^{-1}\sigma^{-1})$.

3.2.1 Exponent Rules

Recall our algorithm described in (1), it is a very simple two step string of moves that is repeated 3 times, we can use exponents to describe it compactly,

$$H_2V_2H_2V_2H_2V_2H_2V_2 = (H_2V_2)^3 \tag{2}$$

The usual exponent rules hold however it is critical that you under stand that the exponent is simple short hand for repeated applications of the group operation (composition) and not try to apply the rules willy nilly. Usually $(ab)^2 = a^2b^2$ however this is not the case within the Rubik's cube group. [2]

4 Sources

- https://en.wikipedia.org/wiki/Rubik%27s_Cube_group
- https://www.quora.com/Group-Theory-mathematics/Why-are-commutators-useful-for-solving-answer/Mark-Eichenlaub
- http://kociemba.org/cube.htm

[2] This occurs because the Rubik's cube group is not an abelian group $ab \neq ba$. If $a, b \in \mathbb{R}$ then $(ab)^2 = abab = aabb = a^2b^2$ with abelian groups we can shuffle the order of multiplication.