Notes on MPC Math in Kernel Module

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1 Model

$l(t)$ denotes the RTT and $r(t)$ denotes the pacing rate at time t. Let N be the current time.

1.1 Basic Parameterized Model

$$\begin{aligned}
\Delta l(n) &= l(n) - \hat{l}(n) \\
\hat{l}(n) &= \sum_{i=0}^{p} a_i l(n-i) + \sum_{j=0}^{q} b_j r(n-j)
\end{aligned}$$

$$\begin{aligned}
a_i' &= a_i + \frac{l(N-i)\Delta l}{\sum l^2 + \sum r^2} \\
b_i' &= b_i + \frac{r(N-i)\Delta l}{\sum l^2 + \sum r^2}
\end{aligned}$$

$$\begin{aligned}
\sum_{i=0}^{p} a_i' l(N-i) + \sum_{j=0}^{q} b_j' r(N-j) &= \sum_{i=0}^{p} a_i l(N-i) + \sum_{j=0}^{q} b_j r(N-j) \\
&\quad + \sum_{i=0}^{p} \frac{l(N-i)^2 \Delta l}{\sum l^2 + \sum r^2} + \sum_{j=0}^{q} \frac{r(N-j)^2 \Delta l}{\sum l^2 + \sum r^2} \\
&= \sum_{i=0}^{p} a_i l(N-i) + \sum_{j=0}^{q} b_j r(N-j) + \frac{(\sum l^2 + \sum r^2)\Delta l}{\sum l^2 + \sum r^2}
\end{aligned}$$

$$
\begin{aligned}
&= \hat{l}(N) + \Delta l(N) \\
&= l(N)
\end{aligned}
$$

1.2 Square Root Parameterized Model

$$
\begin{aligned}
\Delta l(n) &= l(n) - \hat{l}(n) \\
\hat{l}(n) &= \sum_{i=0}^{p} a_i \sqrt{l(n-i)} + \sum_{j=0}^{q} b_j \sqrt{r(n-j)}
\end{aligned}
$$

$$
\begin{aligned}
a_i' &= a_i + \frac{\sqrt{l(N-i)}\Delta l}{\sum l + \sum r} \\
b_i' &= b_i + \frac{\sqrt{r(N-i)}\Delta l}{\sum l + \sum r}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=0}^{p} a_i' \sqrt{l(N-i)} + \sum_{j=0}^{q} b_j' \sqrt{r(N-j)} &= \sum_{i=0}^{p} a_i \sqrt{l(N-i)} + \sum_{j=0}^{q} b_j \sqrt{r(N-j)} \\
&\quad + \sum_{i=0}^{p} \frac{l(N-i)\Delta l}{\sum l + \sum r} + \sum_{j=0}^{q} \frac{r(N-j)\Delta l}{\sum l + \sum r} \\
&= \sum_{i=0}^{p} a_i \sqrt{l(N-i)} + \sum_{j=0}^{q} b_j \sqrt{r(N-j)} + \frac{(\sum l + \sum r)\Delta l}{\sum l + \sum r} \\
&= \hat{l}(N) + \Delta l(N) \\
&= l
\end{aligned}
$$

Considering the control $r = r(N)$, we want to minimize

$$
\begin{aligned}
\frac{(1-\alpha)\hat{l}(N)}{\mu_l(N)} - \frac{\alpha r}{\mu_r(N-1)} &= \frac{1-\alpha}{\mu_l(N)} \left(\sum_{i=0}^{p} a_i \sqrt{l(N-i)} + \sum_{j=0}^{q} b_j \sqrt{r(N-j)} \right) - \frac{\alpha r}{\mu_r(N-1)} \\
&= \frac{(1-\alpha) b_0 \sqrt{r(N)}}{\mu_l(N)} - \frac{\alpha r}{\mu_r(N-1)} \\
&\quad + \frac{1-\alpha}{\mu_l(N)} \left(\sum_{i=0}^{p} a_i \sqrt{l(N-i)} + \sum_{j=1}^{q} b_j \sqrt{r(N-j)} \right)
\end{aligned}
$$

$$= \frac{(1-\alpha) b_0 \sqrt{r(N)}}{\mu_l(N)} - \frac{\alpha r}{\mu_r(N-1)} + \frac{(1-\alpha)\left[\hat{l}\right]_{r(N)=0}}{\mu_l(N)}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\left(\hat{l}(N) - r\right) &= 0 \\
\frac{(1-\alpha) b_0}{2\mu_l(N)\sqrt{r}} - \frac{\alpha}{\mu_r(N-1)} &= 0 \\
r &= \left(\frac{(1-\alpha) b_0 \mu_r(N-1)}{2\alpha\mu_l(N)}\right)^2
\end{aligned}$$

α is a parameter that weighs which parameter we want to minimize more. $\alpha = 0$ only weighs the pacing rate ($r \to \infty$), and $\alpha = 1$ only weighs RTT ($r = 0$).

1.3 Step Model

$$
\begin{aligned}
y_0 &= \bar{l} \\
\Delta l(i) &= l(i) - l(i-1) \\
\Delta r(i) &= r(i) - r(i-1)
\end{aligned}
$$

$$
\begin{aligned}
\delta l(N) &= l(N) - \hat{l}(N) \\
\hat{l}(N) &= y_0 + \sum_{i=0}^{p} a_i \Delta l(N-i) + \sum_{j=0}^{q} b_j \Delta r(N-j)
\end{aligned}
$$

$$
\begin{aligned}
a_i' &= a_i + \frac{\Delta l(N-i)\delta l(N)}{\sum(\Delta l)^2 + \sum(\Delta r)^2} \\
b_i' &= b_i + \frac{\Delta r(N-i)\delta l(N)}{\sum(\Delta l)^2 + \sum(\Delta r)^2}
\end{aligned}
$$

$$
\begin{aligned}
y_0 + \sum_{i=1}^{p} a_i' \Delta l(N-i) + \sum_{j=1}^{q} b_j' \Delta r(N-j) &= y_0 + \sum_{i=1}^{p} a_i \Delta l(i) + \sum_{j=1}^{q} b_j \Delta r(j) \\
&\quad + \sum_{i=1}^{p} \frac{(\Delta l(N-i))^2 \delta l(N)}{\sum(\Delta l)^2 + \sum(\Delta r)^2} + \sum_{j=1}^{q} \frac{(\Delta r(N-j))^2 \delta l(N)}{\sum(\Delta l)^2 + \sum(\Delta r)^2} \\
&= y_0 + (\hat{l}(N) - y_0) + \frac{(\sum(\Delta l)^2 + \sum(\Delta r)^2)\delta l(N)}{\sum(\Delta l)^2 + \sum(\Delta r)^2} \\
&= \hat{l}(N) + \delta l(N) \\
&= l(N)
\end{aligned}
$$

1.3.1 Rate Prediction

Choose $r(N)$ such that $m(r(N))$ is minimized.

$$
\begin{aligned}
m(r(N)) &= \frac{\left[\hat{l}(N)\right]_{\Delta r(N) = r(N) - r(N-1)}}{\mu_l(N)} - \frac{r(N)}{\mu_r(N-1)} \\
&= \frac{b_0 \Delta r(N)}{\mu_l(N)} - \frac{r(N)}{\mu_r(N-1)} + \left[\hat{l}(N)\right]_{\Delta r(N)=0}
\end{aligned}
$$

$$= \frac{b_0(r(N) - r(N-1))}{\mu_l(N)} - \frac{r(N)}{\mu_r(N-1)} + \left[\hat{l}(N)\right]_{\Delta r(N)=0}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r(N)} m(r(N)) &= 0 \\
\frac{b_0}{\mu_l(N)} - \frac{1}{\mu_r(N-1)} &= 0 \\
b_0 &= \frac{\mu_l(N)}{\mu_r(N-1)}
\end{aligned}$$

We want to move b_0 towards this value, so consider b_0'.

$$\begin{aligned}
b_0' &= \frac{\mu_l(N)}{\mu_r(N-1)} \\
b_0 + \frac{\Delta r(N)\delta l(N)}{\sum(\Delta l)^2 + \sum(\Delta r)^2} &= \frac{\mu_l(N)}{\mu_r(N-1)} \\
b_0 + \frac{\Delta r(N)\delta l(N)}{\Delta r(N)^2 + \sum(\Delta l)^2 + \sum_{j=1}^{N}(\Delta r(N-j))^2} &= \frac{\mu_l(N)}{\mu_r(N-1)}
\end{aligned}$$