Chain Rule Assignment

1. Given $f(z) = \log_e (1+z)$ where $z = X^T X$, $X \in R^d$

Ans: if $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$ then $X^T = [x_1, x_2, \cdots x_d]$

$X^T X = [x_1^2 + x_2^2 + \cdots + x_d^2]$

Applying chain rule,

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{d}{dz}(\log(1+z)) \cdot \frac{d}{dx}(X^T . X)$$

$$= \frac{1}{1+z} \cdot \frac{d}{dz}(z) \cdot \frac{d}{dx}(x_1^2 + x_2^2 + \cdots + x_d^2)$$

$$= \frac{1}{1+z}(2x_1 + 2x_2 + \cdots + 2x_d)$$

$$= \frac{1}{1+z} \cdot 2(x_1 + x_2 + \cdots + x_d)$$

$$= \frac{2}{1+z} \sum_{i=1}^{d} x_i$$

2. $f(z) = e^{-\frac{z}{2}}$; where $z = g(z)$, $g(y) = y^T S^{-1} y$, $y = h(x)$, $h(x) = x - \mu$

<u>Ans:</u> Using ~~chai~~ chain rule,

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

here, $\frac{df}{dz} = \frac{d}{dz}\left(e^{-\frac{z}{2}}\right) = -\frac{e^{-\frac{z}{2}}}{2}$

$$\frac{dz}{dy} = \frac{d}{dy}\left(y^T S^{-1} y\right)$$

$$= \lim_{h \to 0} \frac{g(y+h) - g(y)}{h}$$

$$= \lim_{h \to 0} \frac{(y^T + h) S^{-1} (y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{(y^T S^{-1} + h S^{-1})(y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{y^T S^{-1} y + y^T S^{-1} h + h S^{-1} y + h^2 S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{h(y^T S^{-1} + S^{-1} y + h S^{-1})}{h}$$

$$= \lim_{h \to 0} (y^T S^{-1} + S^{-1} y + h S^{-1})$$

$$= y^T S^{-1} + S^{-1} y$$

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$$\frac{\partial y}{\partial x} = \frac{\partial (x-\mu)}{\partial x} = 1$$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

$$= -\frac{e^{-\frac{z}{2}}}{2} \left(y^{T} S^{-1} + S^{-1} y\right) \cdot I$$

$$= -\frac{e^{-\frac{z}{2}}}{2} \cdot \frac{1}{S} \left(y^{T} + y\right)$$