# Projectile motion

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## 1 Introduction

This projects considers projectile motion in two dimensions. The first question considers the simplified case of a projectile in vacuum, where the $4^{th}$ order Runge Kutta algorithm will be implemented and used in the further questions to analyze the situation with drag and air density models. The air density corrections used are the isothermal and the adiabatic models. Lastly this will be used to look at the Big Bertha, the German long-range cannon used in World War I, where predicted range, maximum projectile height and time of flight will be analyzed.

## 2 Projectile motion without air drag

The first problem consists of finding a numerical solution that can be used to analyze more complex systems which will be included later. Here the $4^{th}$ order Runge Kutta algorithm is implemented and compared to the analytical solution. As figure (1), demonstrates, the plot of the numerical solution is indistinguishable from the analytical solution in this plot.

To achieve the most informative results it was important to choose the right time step, $dt$. One method to choose the appropriate time step is to do a convergence test where one plots the difference in distance between starting point and landing position $\Delta r$ as a function of number of steps, N. This has been done in figure (2). Here $\Delta r = r_{analytical} - r_{numerical}$. Here we define $N = \frac{T}{dt}$, where $T$ is the total time the projectile uses before reaching the landing point $x_L$. The time can be found analytically from the equations of motion, where the time was found to be $T = 101$s. The number of points N=971 corresponds to an accuracy within 1m, furthermore this gives a time step of $dt \approx 0.1$s.

## 3 Projectile motion at high altitudes

Here the model is extended to include both air density and drag. The general formula for drag is as follows:

$$F_{drag} \approx -B_1 v - B_2 v^2 \tag{1}$$

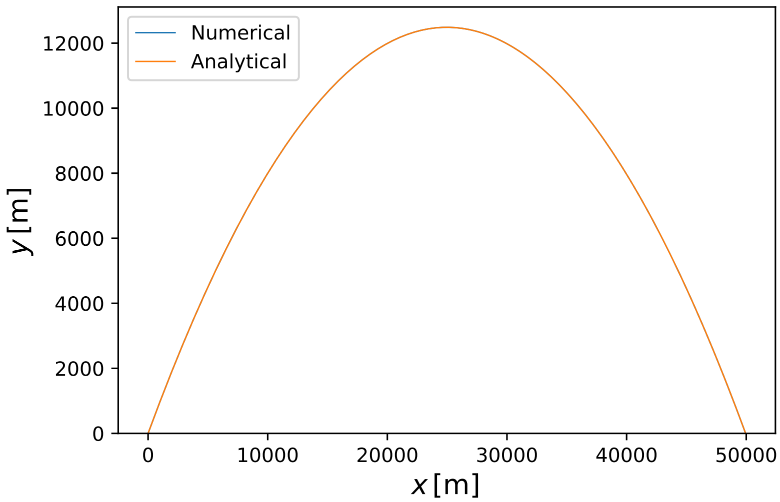



Figure 1: The analytical and numerical solution to the projectile motion in 2D without air drag. The start values are: $x = y = 0, \vec{v}_0 = 700 \text{ ms}^{-1}$. Furthermore the firing angle used is $\theta = 45°$ and the mass of the projectile is $m = 50$kg.

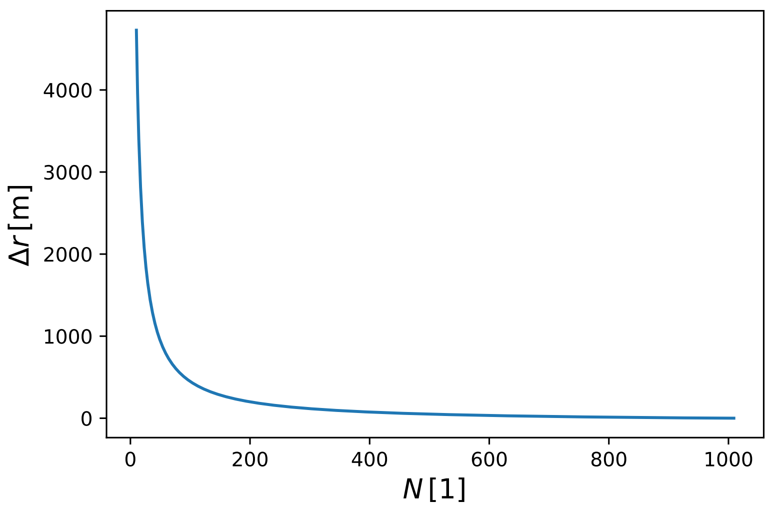



Figure 2: A plot which shows the convergence between the analytical and numerical value for the distance between start position $x_0$, and landing position $x_L$, (which is equal to $r$) for the projectile motion, against the number of points N. Here $\Delta r = r_{analytical} - r_{numerical}$.

where $B_1$ is given by Stokes' law, $B_1 = 6\pi\eta R$, and $B_2 = -\frac{1}{2}C\rho A$. Here $\eta$ is dynamic viscosity, $R$ is the radius of the spherical object, $C$ is the drag coefficient, $\rho$ is air density and $A$ is frontal area. Here $B_1 \ll B_2$ because the dynamic viscosity of air is very small, $\eta \sim 10^{-5}$. Therefore equation (1) becomes:

$$F_{drag} \approx -\frac{1}{2}C\rho A v^2. \tag{2}$$

When the air density is also considered for different altitudes, the air drag can be written as

$$F^*_{drag} = \frac{\rho}{\rho_0} F_{drag}(y=0). \tag{3}$$

Where $\rho$ is one of the following air density models. The two models which will be considered are the adiabatic and the isothermal ideal gas approximation. The equation for the adiabatic approximation is as follows:

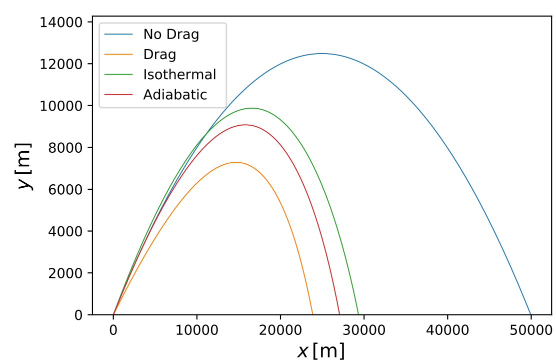
$$\rho = \begin{cases} \rho_0 (1 - \frac{ay}{T_0})^\alpha & y \leq T_0/a \\ 0 & y > T_0/a \end{cases}. \tag{4}$$

Where $a \approx 6.5 \cdot 10^{-3}$ Km$^{-1}$, $\alpha \approx 2.5$ in air and $T_0$ is the temperature at sea level(at $y = 0$). The equation for the isothermal ideal gas model is as follows:

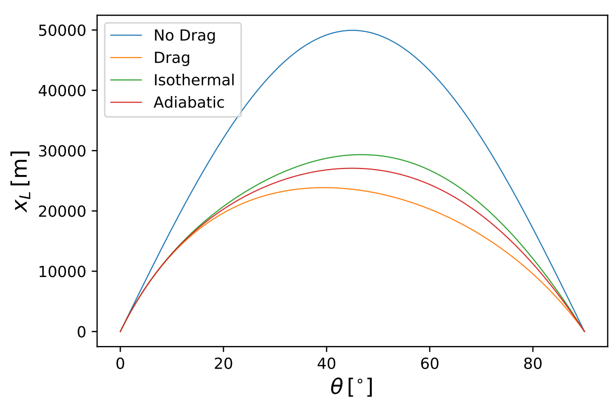
$$\rho = \rho_0 exp(-\frac{y}{y_0}). \tag{5}$$

Here $\rho_0$ is the pressure at sea level($y$ =0), and $y_0 = k_B T/mg$.

The calculation of the trajectory was performed for different firing angles $\theta$ to determine the optimal value of the angle, which in turn would give the maximum range. This has been done for all the different models presented, and demonstrated in figure (3).




(a) Graph which shows the path traveled by the projectile for each of the models. The optimal firing angles are used here.




(b) Plot of landing point $x_L$ against the firing angle $\theta$.

Figure 3: Graphical representation of the trajectory with different models. No drag refers to projectile motion without air drag or air density correction. Drag refers to the projectile motion with air drag, i.e using equation (2). Isothermal and adiabetic is referring to a trajectory which is affected by air density corrections, i.e equation (5) and equation (4) respectivly.

As one can deduct from figure (3b); For no drag the optimal angle is $\theta = 45°$, which in turn gives $x_L = 49949$m. For drag, the optimal firing angle is $\theta = 39°$, and gives $x_L = 23861$m. For the isohermal model, the values are $\theta = 47°$ and $x_L$= 29335. Wile for the adiabatic model we get $\theta = 45°$ and $x_L$ =27076m. This can also be seen in figure (3a).

# 4 Big Bertha (Paris Gun)

The Germans used a long-range cannon in World War I to bomb Paris called "Big Bertha". The muzzle velocity of it's 106 kg shells was 1640 $\mathrm{ms}^{-1}$ [1]. The adiabatic air drag approximation was used in this model. Therefore equation (3) with $\rho = \rho_{adiabatic}$ will be used to analyze the system. Figure (4) shows the numerical plot of the projectile path with the air drag and adiabatic air density model used before.

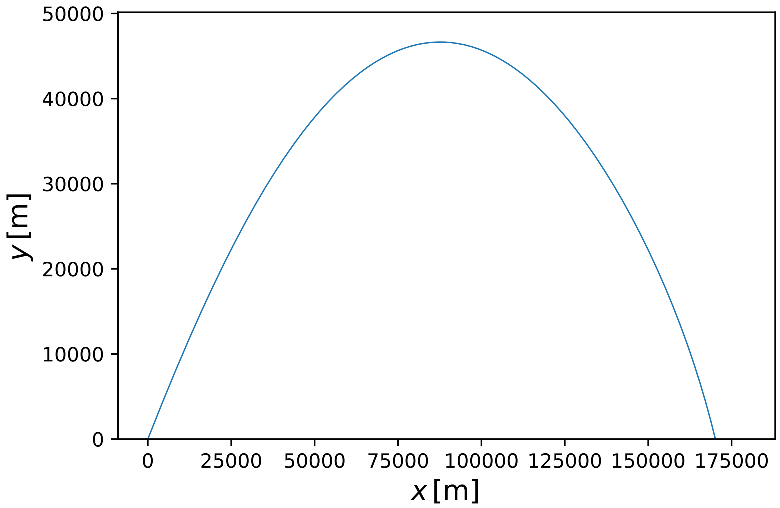



Figure 4: The numerical projectile path of a 106 kg shell fired from the Big Bertha cannon, with an angle of $\theta = 45°$ and a muzzle velocity of $\vec{v}_0 = 1640\ \mathrm{ms}^{-1}$, using an adiabatic air density model.

To find the cannon shell's maximum range, height and travel time, the projectile range, height and travel time was plotted for angles $0 - 90°$ which can be seen respectively in Figure 5,6 and 7. The maximum range was found to be $x_L \approx 174054$m with the optimal angle $\theta = 51°$. The optimal angle of Big Bertha is different from the angle found for the adiabatic model before, which can be explained by that the adiabatic model(4), has no air drag when heights exceed $y = T_0/a \approx 44308$m which the shells do. This might also explain why the shape of the plot differs too. The maximum height of a shell $y_{max} \approx 89155$m can be achieved with a firing angle of $\theta = 90°$, this angle also gives the longest total travel time $t_{tot} \approx 267$s.

# References


[1] Projectile Motion , 2018

[2] Appendix Projectile Motion , 2018

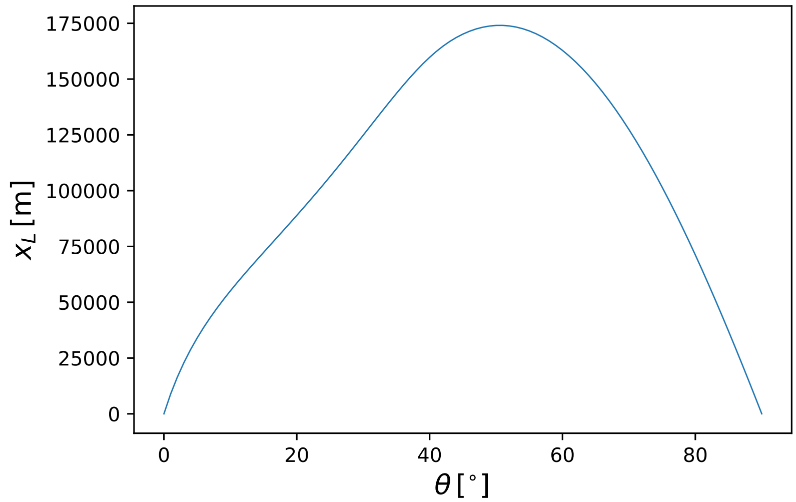



Figure 5: The landing distance $x_L$ of a shell fired from a Big Bertha cannon for different firing angles.

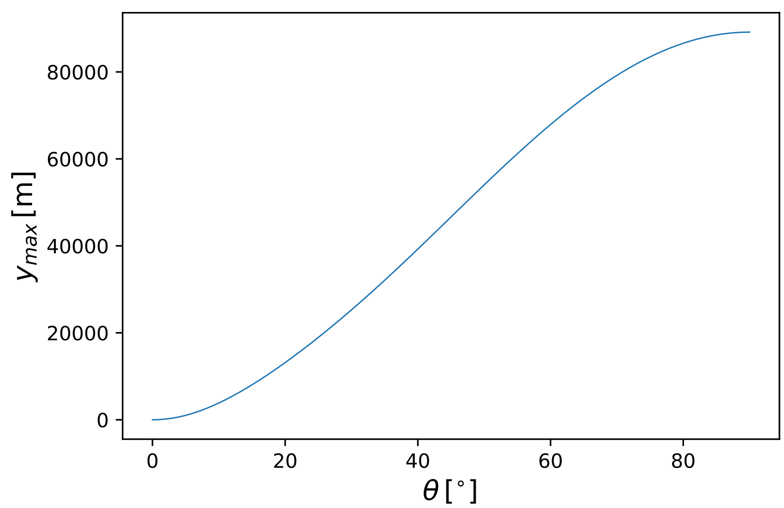



Figure 6: The maximum height $y_{max}$ of a shell fired from a Big Bertha cannon for different firing angles.

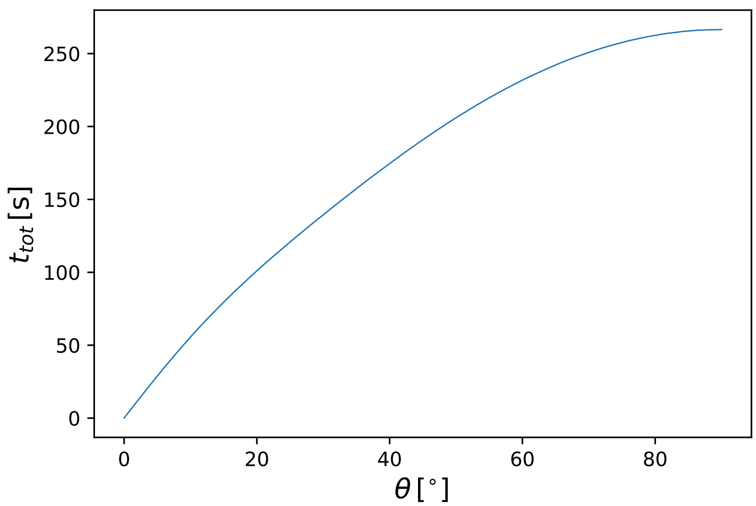



Figure 7: The total flight time $t_{tot}$ of a shell fired from a Big Bertha cannon for different firing angles.