① $\begin{cases} x'(t) = r_0 x\left(1 - \frac{x}{k}\right) \\ x(0) = x_0 \end{cases}$

a) soluția modelului

b) $r_0 > 0$, $\lim\limits_{t \to \infty} x(t)$ - ? Concluzie.

$$x'(t) = r_0 \cdot x\left(1 - \frac{x}{k}\right)$$

$$\frac{dx}{dt} = r_0 \cdot x\left(1 - \frac{x}{k}\right)$$

$$\frac{dx}{x\left(1 - \frac{x}{k}\right)} = r_0 \, dt$$

$$\frac{A}{x} + \frac{B}{1 - \frac{x}{k}} = A - \frac{Ax}{k} + Bx = 1 \Rightarrow A = 1 \Rightarrow B = \frac{1}{k}$$

$$\int \frac{1}{-x} = -\ln x$$

$$\int \frac{dx}{x} + \frac{1}{k} \int \frac{1}{1 - \frac{x}{k}} \, dx$$

$$\int \frac{1}{k - x} = \frac{\ln(k - x)}{-1}$$

$$\ln|x| - \frac{1}{k} k \ln|k-x| = r_0 \cdot t + c$$

$x(t) = \frac{c \cdot k \cdot e^{r_0 \cdot t}}{1 + c \cdot e^{r_0 \cdot t}}$

$$\ln\left(\frac{x}{k-x}\right) = r_0 \cdot t + c$$

$$\frac{x}{k-x} = e^{r_0 \cdot t + c}$$

$$x = k \cdot e^{r_0 \cdot t + c} - x e^{r_0 \cdot t + c}$$

$$x(1 + e^{r_0 \cdot t + c}) = k \cdot e^{r_0 \cdot t + c}$$

$$x(t) = \frac{c k \cdot e^{r_0 t}}{1 + c \cdot e^{r_0 t}}$$

$$\lim_{t \to \infty} \frac{c k \cdot e^{r_0 t}}{1 + c \cdot e^{r_0 t}} = \frac{c k}{c} = \frac{k}{1}$$

$r_0 > 0$

③ a) $y' \sin(x) + 2y \cos(x) = 1$

b) $y'' - 4y' + 6y = 12x + 16$

a) $y' \sin x + 2y \cos(x) = 1 \quad | : \sin x$

d) $y' \sin x + 2y\cos(x) = 1 \quad | : \sin x$

$$y' + \underbrace{2\frac{\cos x}{\sin x}}_{P(x)} y = \underbrace{\frac{1}{\sin x}}_{Q(x)}$$

I $\quad y' + \frac{2\cos x}{\sin x} y = 0$

$$y' = -\frac{2\cos x}{\sin x} y \qquad \ln(\sin x)' = \frac{1}{\sin x} \cdot \cos x$$

$$\frac{dy}{y} = -2\operatorname{ctg} x \, dx$$

$$\ln|y| = -2\ln(\sin x) + c$$

$$y_0 = \frac{c}{\sin^2 x}, \quad c \in \mathbb{R}$$

$$y_p = \frac{c(x)}{\sin^2 x}$$

$$y_p' = \frac{c'(x) \cdot \sin^2 x - \sin(2x)\, c(x)}{\sin^4 x}$$

$$= \frac{c'(x)}{\sin^2 x} - \frac{2\cos x}{\sin^3 x} c(x)$$

$$\frac{c'(x)}{\sin^2 x} - \frac{2\cos x}{\sin^3 x}c(x) + \frac{2\cos x\, c(x)}{\sin^3 x} = \frac{1}{\sin x}$$

$$\frac{c'(x)}{\sin^2 x} = \frac{1}{\sin x}$$

$$c'(x) = \sin x$$

$$\left.\begin{array}{l} c(x) = -\cos x + c \\ c = 0 \end{array}\right\} \Rightarrow c(x) = -\cos x$$

$$y = \frac{c}{\sin^2 x} - \frac{\cos x}{\sin^2 x}, \quad c \in \mathbb{R}$$

b) $y'' - 4y' + 6y = 12x + 6$

I y_0

$$r^2 - 4r + 6 = 0$$

$$\Delta = 16 - 24 = -8$$

$$r_{1,2} = \frac{4 \pm 2\sqrt{2}\,i}{2} = 2 \pm \sqrt{2}\,i$$

$$\alpha = 2 \quad \beta = \sqrt{2}$$

$$y_1(x) = e^{2x}\cos(\sqrt{2}x)$$

$$y_2(x) = e^{2x}\sin(\sqrt{2}x)$$

$$y = c_1 e^{2x}\cos(\sqrt{2}x) + c_2 e^{2x}\sin(\sqrt{2}x)$$

$$c_1, c_2 \in \mathbb{R}$$

$$y_p = ax + b$$

$$y_p' = a$$

$$y_p'' = 0$$

$$0 - 4a + 6ax + 6b = 12x + 6$$

$$a = 2$$

$$-8 + 6b = 6$$

$$6b = 14$$

$$b = \frac{7}{3}$$

$$y =$$

⑤ $\begin{cases} y' = 2x^2 + y \\ y(0) = 1 \end{cases}$

$$f(x, y) = 2x^2 + y$$

$x_0 = 0$

$$y_{n+1} = y_n + f(x_n, y_n) \cdot h$$

$h = 1$

$$y_1 = 1 + \underbrace{f(x_0, y_0)}_{1} \cdot 1 = 2$$

$x_1 = 1$

⑥ $x'(t) = 2x - x^3$

a) Puncte de echilibru, stabilitate.

b) Portret fazic.

$$x'(t) = 2x - x^3$$

$2x - x^3 = 0 \quad x(2 - x^2) = 0 \qquad x(\sqrt{2} - x)(\sqrt{2} + x)$

$-\quad + \quad -$

$x_1 = 0 \qquad (2x - x^3)' = 2 - 3x^2$

$x_2 = \sqrt{2} \qquad x_1 = 0 \Rightarrow 2 > 0$ instabil

$x_3 = -\sqrt{2} \qquad x_2 = \sqrt{2} \Rightarrow 2 - 6 < 0$ } asimptotic stabil

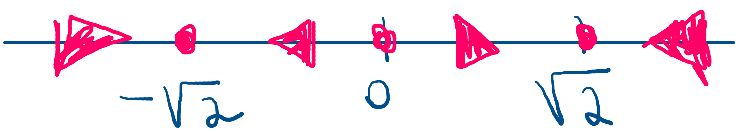
$--- \quad < 0$

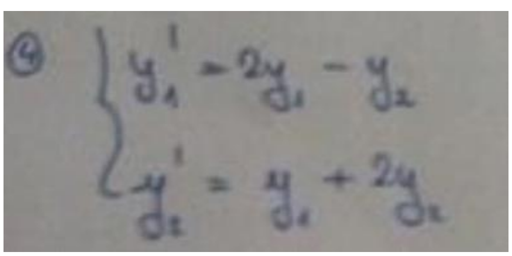
X	$-\sqrt{2}$	0	$\sqrt{2}$

x		$-\sqrt{2}$		0		$\sqrt{2}$	
$f(x)$	$+$	0	$+$	0	$+$	0	$+$
	$\rightarrow$		$\leftarrow$		$\rightarrow$		$\leftarrow$

$$\frac{dx}{dt} = 2x - x^3$$

PORTRET FAZIC





(4) $\begin{cases} y_1' = 2y_1 - y_2 \\ y_2' = y_1 + 2y_2 \end{cases}$

$\Rightarrow y_2 = -y_1' + 2y_1$

$y_2' = y_1 + 2y_2$

$y_1'' = 2y_1' - y_2'$

$y_2' = y_1 + 2y_2$

$y_1'' = 2y_1' - \left(y_1 + 2(-y_1' + 2y_1)\right]$

$y_1'' = 2y_1' - y_1 + 2y_1' - 4y_1$

$y_1'' - 4y_1' + 5y_1 = 0$

$r^2 - 4r + 5 = 0$

$\Delta = 16 - 25 = -9$

$$r_{1,2} = \frac{4 \pm 3i}{2} = 2 \pm \frac{3}{2} i$$

$$y_1(x) = c_1 \cdot e^{2x} \cdot \cos\left(\frac{3}{2}x\right) + c_2 \cdot e^{2x} \cdot \sin\left(\frac{3}{2}x\right)$$

$c_1, c_2 \in \mathbb{R}$

$$= e^{2x}\left[c_1 \cdot \cos\left(\frac{3}{2}x\right) + c_2 \cdot \sin\left(\frac{3}{2}x\right)\right]$$

$$y_1'(x) = 2e^{2x}\left[c_1 \cdot \cos\left(\frac{3}{2}x\right) + c_2 \cdot \sin\left(\frac{3}{2}x\right)\right]$$

$$+ e^{2x}\left(-\frac{3}{2} c_1 \sin\left(\frac{3}{2}x\right) + \frac{3}{2} c_2 \cdot \cos\left(\frac{3}{2}x\right)\right)$$

$y_2(x) = -y_1'(x$