Exercițiul 3 *Determinați soluțiile generale pentru ecuațiile:*

0,5 (a) **(0,5p)** $(x^2+1)y'+2x\cdot y=1$

1 (b) **(1p)** $y''+2y'+10y=10x+2$

$(x^2+1)y'+2xy=1 \mid :(x^2+1)$

$y'+P(x)\cdot y=Q(x)$

$y'+\underbrace{\frac{2x}{x^2+1}}_{P(x)}y=\underbrace{\frac{1}{x^2+1}}_{Q(x)}$

I $\quad y'+\frac{2x}{x^2+1}y=0$

$y'=\underbrace{\frac{-2x}{x^2+1}}_{f(x)}\underbrace{y}_{g(y)}$

$\frac{dy}{dx}=\frac{-2x}{x^2+1}y \mid : y,\ y\neq 0$

$\frac{dy}{y}=\frac{-2x}{x^2+1}dx \mid \int$

$y\equiv 0$ sol. singulară

$\ln|y|=-\ln(x^2+1)+C$

$-\int\frac{2x}{x^2+1}dx=-\int\frac{dt}{t}=-\ln|t|$
$=-\ln|x^2+1|$
$=-\ln(x^2+1)$

$x^2+1=t$
$2x\,dx=dt$

$e^{\ln|y|}=e^{-\ln(x^2+1)+C}$

$|y|=e^{-\ln(x^2+1)}\cdot e^C$

$-\ln a=\ln a^{-1}=\ln\frac{1}{a}$

$|y|=e^{\ln\left(\frac{1}{x^2+1}\right)}\cdot e^C$

$|y|=\frac{1}{x^2+1}\cdot e^C$

$y=\pm\frac{1}{x^2+1}\cdot e^C$

$y_0=C\cdot\frac{1}{x^2+1}$

$y_p=C(x)\cdot\frac{1}{x^2+1}$ $\qquad y_p'=\frac{C(x)}{x^2+1}$

$y'+\frac{2x}{x^2+1}y=\frac{1}{x^2+1}$ $\qquad =\frac{C'(x)(x^2+1)-2x\cdot C(x)}{(x^2+1)^2}$

$=\frac{C'(x)}{x^2+1}-\frac{2xC(x)}{(x^2+1)^2}$

$\frac{C'(x)}{x^2+1}-\cancel{\frac{2xC(x)}{(x^2+1)^2}}+\cancel{\frac{2xC(x)}{(x^2+1)^2}}=\frac{1}{x^2+1}$

$\frac{C'(x)}{x^2+1}=\frac{1}{x^2+1} \Rightarrow C'(x)=1 \mid \int$

$C(x)=x+C$

$C=0$

$c = 0$

$\Rightarrow c(x) = x$

$\Rightarrow y_p = \frac{x}{x^2+1}$

$$\boxed{y(x) = \frac{c}{x^2+1} + \frac{x}{x^2+1},\ c \in \mathbb{R}}$$

$y'' + 2y' + 10 = 10x + 2$

I y_0

$y'' + 2y' + 10 = 0$

$r^2 + 2r + 10 = 0$

$\Delta = 4 - 40 = -36 < 0$

$r_{1,2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$

$-1 = \alpha$

$3 = \beta$

$y_1(x) = e^{-x} \cdot \cos(3x)$

$y_2(x) = e^{-x} \sin(3x)$

$y_0 = c_1 \cdot y_1(x) + c_2 \cdot y_2(x)$

$c_1, c_2 \in \mathbb{R}$

$= c_1 \cdot e^{-x} \cos(3x) + c_2 e^{-x} \sin(3x)$

$c_1, c_2 \in \mathbb{R}$

$y_p \qquad P_m(x)$

$10x + 2$

P_1

$y'' + ay' + by = \ldots$

$y'' + 2y' + 10y$

$\neq$

0

$y_p = ax + b$

$y_p' = a$

$y_p'' = 0$

$0 + 2a + 10ax + 10b = 10x + 2$

$\boxed{10a}x + 10b + 2a = \boxed{10}x + 2$

$10a = 10 \Rightarrow a = 1$

$10b + 2 = 2$

$\Rightarrow b = 0$

$y_p = 1 \cdot x + 0 = x$

$y(x) = \underbrace{c_1 \cdot y_1(x) + c_2 \cdot y_2(x)}_{y_0} + \underset{y_p}{x}$

$y'' + 2y' = 7x + 2$

$b = 0 \Rightarrow y_p = x(ax+b)$

$= ax^2 + bx$

$= 2x^2 + 3x + 1$

$y_p = x(ax^2 + bx + c)$

Exercițiul 7 *(1p)* *Să se determine soluția generală a sistemului:*

$$\begin{cases} y_1' = 2y_1 - 5y_2 \\ y_2' = 5y_1 + 2y_2 \end{cases}$$

1p

$$\begin{cases} y_1' = 2y_1 - 5y_2 \Rightarrow 5y_2 = 2y_1 - y_1' \\ y_2' = 5y_1 + 2y_2 \end{cases}$$

$y_2 = \boxed{\dfrac{2y_1 - y_1'}{5}}$

$y_1'' = 2y_1' - 5y_2'$

$y_2' = 5y_1 + 2y_2$

$= 5y_1 + \dfrac{4y_1 - 2y_1'}{5}$

$y_1'' = 2y_1' - 25y_1 - 4y_1 + 2y_1'$

$y_1'' = 4y_1' - 29y_1$

3
29
4
116

$y_1'' - 4y_1' + 29y_1 = 0$

$\Delta = 16 - 116 = -100$

$r^2 - 4r + 29 = 0$

$r_{1,2} = \dfrac{4 \pm 10i}{2} = 2 \pm 5i$

$\alpha = 2$

$\beta = 5$

$y_1(x) = c_1 \cdot \varphi_1(x) + c_2 \cdot \varphi_2(x)$

$= c_1 \cdot e^{2x} \cdot \cos(5x) + c_2 \cdot e^{2x} \sin(5x)$

$c_1, c_2 \in \mathbb{R}$

$y_2(x)$

Exercițiul 6 *Se consideră sistemul*

$$\begin{cases} x'(t) = xy - 1 \\ y'(t) = x^2 - y^2 \end{cases}$$

Se cere:

(a) **(0,5p)** *Să se determine punctele de echilibru.*

(b) **(1p)** *Să se studieze stabilitatea acestora.*

$$\begin{cases} xy - 1 = 0 \\ x^2 - y^2 = 0 \end{cases}$$

$x = \frac{1}{y}$

$\frac{1}{y^2} - y^2 = 0$

$\cancel{t = -1 \Rightarrow y^2 = -1}$

$t = 1 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

$1 - y^4 = 0 \Rightarrow y^4 = 1$

$y^2 = t \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$

$y = \pm 1$

$x = \frac{1}{y} \Rightarrow x = \pm 1$

Pct. de echilibru: $(1, 1)$, $(-1, -1)$

$f_1(x, y) = xy - 1$

$f_2(x, y) = x^2 - y^2$

$$J_f(x,y) = \begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix}$$

$$J_f(1,1) = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$$

$\det(J_f(1,1) - \lambda I_2) = 0$

$$\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1 \\ 2 & -2-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-2-\lambda) - 2 = 0$$

$$-(1-\lambda)(2+\lambda)$$

$$(\lambda-1)(2+\lambda) - 2 = 0$$

$$2\lambda + \lambda^2 - 2 - \lambda - 2 = 0$$

$$\lambda^2 + \lambda - 4 = 0$$

$$\Delta = 1 + 16 = 17$$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{17}}{2}$$

$$\lambda_1 = \frac{-1+\sqrt{17}}{2} > 0$$

$$\lambda_2 = \frac{-1-\sqrt{17}}{2} < 0$$

$\Rightarrow (1,1)$ pct. de echilibru instabil

Exercițiul 5 ***(0.5p)*** *Determinați ecuația orbitelor din portretul fazic, situate în cadranul pozitiv, pentru sistemul:*

$$\begin{cases} x'(t) = & -2xy \\ y'(t) = & -y + 3xy \end{cases}$$

$y(x)$

$\frac{dy}{dx}$

$$\frac{dx}{dt} = -2xy$$

$$\frac{dy}{dt} = -y + 3xy$$

$$\frac{dx}{dt} \cdot \frac{dt}{dy} = \frac{-2xy}{-y+3xy}$$

$$\frac{dx}{dy} = \frac{-2xy}{-y+3xy}$$

$$\frac{dx}{dy} = \frac{y(-2x)}{y(-1+3x)}$$

$$\frac{dx}{\ } = \frac{-2x}{3x-1}$$

$$\frac{dx}{dy} = \frac{-2x}{3x-1}$$

$$-\frac{3x-1}{2x}\,dx = dy \;\Big|\int$$

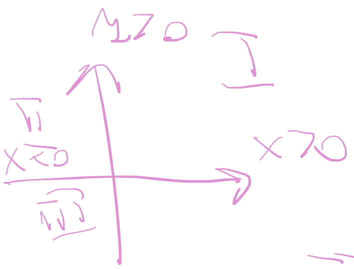
$$\int \frac{3x-1}{2x}\,dx = \int \frac{3x}{2x}\,dx - \int \frac{1}{2x}\,dx$$

$$= \int \frac{3}{2}\,dx - \frac{1}{2}\int \frac{1}{x}\,dx$$

$$= \frac{3}{2}x - \frac{1}{2}\ln|x|$$

$$\boxed{\frac{1}{2}\ln|x| - \frac{3}{2}x = y + c}$$

$$\boxed{y = \frac{1}{2}\ln|x| - \frac{3}{2}x - c}$$



$$\Rightarrow \text{I} \quad y(x) = \frac{1}{2}\ln(x) - \frac{3}{2}x - c$$

$|x| = -x$

$x<0$

$$\text{II}, \text{III} \quad y(x) = \frac{1}{2}\ln(-x) - \frac{3}{2} - c$$

Exercițiul 4 *(1p)* *Determinați soluția problemei bilocale:*

1p

$$\begin{cases} y'' - \frac{1}{x\ln(x)}y' &= 12x^2\ln(x) \\ y(1) &= -\frac{1}{4} \\ y(2) &= e^4 \end{cases}$$

$$y'' - \text{ceva} \cdot y' = \text{ceva}$$

$$y'' = \underbrace{\frac{1}{x\ln x}y' + 12x^2\ln(x)}_{f(x,\,y')}$$

$z = y'$

$z' = y''$ $\Rightarrow$ $z' - \underbrace{\frac{1}{x\ln x}}_{P(x)} z = \underbrace{12x^2\ln(x)}_{Q(x)}$

I] $z' - \frac{1}{x\ln x} z = 0 \Rightarrow z' = \frac{z}{x\ln x} \Rightarrow \frac{dz}{dx} = \frac{z}{x\ln x} \; : z$

$z \equiv 0$ sol. sing. $\qquad \frac{dz}{z} = \frac{dx}{x\ln x}$

$\ln|y| = \ln|\ln(x)| + c$

$\int \frac{1}{x\ln x}dx = \int \frac{dt}{t} = \ln|t|$

$= \ln|\ln(x)|$

$\ln x = t$

$\frac{1}{x}dx = dt$

$\Rightarrow |z| = |\ln x| \cdot e^c$

$z_0 = \pm \ln x \cdot e^c$

$z_0 = c\ln(x)$

$z_p = c(x) \cdot \ln x$

$z' - \frac{1}{x\ln x} z = 12x^2\ln(x)$

$z_p' = c'(x)\ln x + \frac{c(x)}{x}$

$\Rightarrow c'(x)\ln x + \cancel{\frac{c(x)}{x}} - \cancel{\frac{1}{x\ln x} \cdot c(x)\ln x} = 12x^2\ln(x)$

$c'(x)\ln x = 12x^2\ln(x)$

$$\Rightarrow c'(x) = 12x^2$$

$$\left.\begin{array}{l} \Rightarrow c(x) = 12\frac{x^3}{3} + c \\ c = 0 \end{array}\right\} \Rightarrow c(x) = 4x^3$$

$$\Rightarrow z_p = 4x^3 \cdot \ln(x)$$

$$z(x) = c_1 \ln(x) + 4x^3 \ln(x)$$

$$y = \int c_1 \ln(x) + 4x^3 \ln x \, dx + c_2$$

z

$$\int \ln x \, dx$$

Exercițiul 2 *Determinați soluțiile generale pentru ecuațiile:*

(a) ***(0.5p)*** $x + y - (x - y) \cdot y' = 0$

$$x + y - (x - y) \cdot y' = 0 \quad | :x$$

$$1 + \frac{y}{x} - \left(1 - \frac{y}{x}\right) \cdot y' = 0$$

$$\frac{y}{x} = t \Rightarrow y = t \cdot x \qquad t(x) \cdot x$$

$$y' = t' \cdot x + t$$

$$1 + t - (1 - t)(t' \cdot x + t) = 0$$

$$1 + t + (t - 1)(t' \cdot x + t) = 0$$

$$1 + \cancel{t} + t' \cdot t \cdot x + t^2 - t' \cdot x - \cancel{t} = 0$$

$$t'(tx - x) + t^2 + 1 = 0$$

$$t'$$

$$x + y = (x - y)\, y' \quad | :(x - y)$$

$$y' = \frac{x+y}{x-y} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}}$$

$$\frac{y}{x} = t \Rightarrow y' = t' \cdot x + t$$

$$\frac{y}{x} = t \mid \cdot x$$

$$\frac{y \cdot \cancel{x}}{\cancel{x}} = t \cdot x$$

$$y = t \cdot x$$

$$y' = t' \cdot x + t$$

$$t' \cdot x + t = \frac{1+t}{1-t}$$

$$y = t \cdot x$$

$$t' \cdot x = \frac{1+t}{1-t} - t$$

$$t' \cdot x = \frac{1+t-t+t^2}{1-t} \quad \Big| : x \neq 0$$

y

$$t' = \underbrace{\frac{1}{x}}_{f(x)} \cdot \underbrace{\frac{t^2+1}{1-t}}_{g(t)}$$

$$\frac{dt}{dx} = \frac{1}{x} \cdot \frac{t^2+1}{1-t}$$

$$\frac{1-t}{t^2+1}\, dt = \frac{dx}{x}$$

$$\int \frac{1}{t^2+1} - \frac{t}{t^2+1}\, dt = \ln|x|$$

$$t = x+5$$

$$y = x(x+5)$$

$$\operatorname{arctg}(t) - \frac{1}{2}\ln(t^2+1) = \ln|x| + c$$

sol. în formă implicită

$$t^2+1 = w$$

$$2t\, dt = dw$$

$$t\, dt = \frac{dw}{2}$$

$$t = \text{ceva}$$

$$\frac{1}{2}\int \frac{1}{w}\, dw = \frac{1}{2}\ln|w| = \frac{1}{2}\ln|t^2+1|$$

$$y = t \cdot x$$

6. (1p) prb Cauchy $\begin{cases} y' = x-1+4xy^3 \\ y(0)=1 \end{cases}$ Folosind met. Taylor, determinați polinomul lui Taylor de grad 3

$x_0 \quad y_0$

pb Cauchy $\begin{cases} \dots \\ y(0)=1 \end{cases}$ determinați polinomul lui Taylor de grad 3 ce aprox. sol problemei

$x_0 \quad y^0$

2) <u>Metoda seriei Taylor</u>

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y^0 \end{cases}$$

$y(x)$ sol. exactă

$$y(x) \simeq \boxed{\tilde{y}(x) = y(x_0) + \frac{y'(x_0)}{1!}(x-x_0) + \frac{y''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{y^{(n)}(x_0)}{n!}(x-x_0)^n}$$

$x_0 = 0$

$y^0 = 1$

$$y(x) \simeq \tilde{y}(x) = y(x_0) + \frac{y'(x_0)}{1!}(x-x_0) + \dots + \frac{y^{(n)}}{n!}(x-x_0)^n$$

$y' = x - 1 + 4xy^3$

$y' = x - 1 + 4xy^3$

$$\cancel{y'(x) = x - 1} + 4x \cdot y^3(x)$$

$$y'(0) = 0 - 1 + 4 \cdot 0 \cdot \underbrace{y^3(0)}_{1} = -1$$

$x_0 = 0$

$$\tilde{y}(x) = y(x_0) + \frac{y'(x_0)}{1!}(x-x_0)^1 + \frac{y''(x_0)}{2!}(x-x_0)^2 + \frac{y'''(x_0)}{3!}(x-x_0)$$

$$\tilde{y}(x) = \overbrace{y(0)}^{1} + \frac{\overbrace{y'(0)}^{-1}}{1!} \cdot x + \frac{\overbrace{y''(0)}^{5}}{2!} \cdot x^2 + \frac{\overbrace{y'''(0)}^{-24}}{3!} \cdot x^3$$

$$y'(x) = x - 1 + 4x \cdot y^3(x) \,\Big|'$$

$$y''(x) = 1 + 4\left(y^3(x) + x \cdot 3 \cdot y^2(x) \cdot y'(x)\right)$$

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

$$[y^3(x)]' = 3y^2(x) \cdot y'(x)$$

$$y''(0) = 1 + 4\left(1 + 0 \cdot 3 \cdot y^2(0) \cdot y'(0)\right)$$
$$= 1 + 4 = 5$$

$$y''(0) = 1+4(\ldots)$$
$$= 1+4 = 5$$

$$y''(x) = 1 + 4\left(y^3(x) + \underbrace{x}_{f(x)} \cdot \underbrace{3 \cdot y^2(x) \cdot y'(x)}_{g}\right)$$

$$y'''(x) = 4\left[3y^2(x)\cdot y'(x) + 3y^2(x)y'(x) + \underbrace{3 \cdot x\left(2y(x)\cdot [y'(x)]^2 + y^2(x)\cdot y''(x)\right)}_{0}\right]$$

$$\left(y^2(x)\cdot y'(x)\right)' = 2y(x)\cdot y'(x)\cdot y'(x) + y^2(x)\cdot y''(x)$$
$$= 2y(x)\cdot [y'(x)]^2 + y^2(x)\cdot y''(x)$$

$$y'''(0) = 4\left(3\underbrace{y^2(0)}_{1}\cdot \underbrace{y'(0)}_{-1} + 3\underbrace{y^2(0)}_{1}\underbrace{y'(0)}_{-1} + 0\right)$$

$$= 4(-3-3) = -24$$

$$\boxed{\hat{y}(x) = 1 - 1\cdot x + 5x^2 - 24x^3}$$