Vectori și puncte: $\overrightarrow{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j}$

- $\overrightarrow{AB} \sim \overrightarrow{CD}$ → = modul, dir, sens
- $\|\lambda a\| := |\lambda| \cdot \|a\|$
  ↳ dir la fel, sens = pt $\lambda > 0$, inv pt $\lambda < 0$
- $a \cdot b = \|a\| \cdot \|b\| \cdot \cos\alpha \Rightarrow \cos\alpha = \dfrac{a \cdot b}{\|a\|\|b\|}$
- $a \cdot b = b \cdot a$
- $(\lambda a + \mu b) \cdot c = \lambda \cdot (a \cdot c) + \mu \cdot (b \cdot c)$
- $\vec{p} \perp \vec{a} \Longleftrightarrow p \cdot a = 0$

! $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$

Prod mixt: modulul p.m = V paralelipiped: $V = |a, b, c|$

(a,b,c) $<0 \to$ ST, $>0 \to$ DR

- V tetraedru: $V = \frac{1}{6} |(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD})|$

→ a, b, c copl $\Longleftrightarrow (a \times b) \cdot c = 0$ / $(a,b,c) = 0$

→ a, b, c nu-s copl $\Leftrightarrow (a, b, c) > 0$

Prod scalar în coord: $a \cdot b = a_1 \cdot b_1 + a_2 b_2 + a_3 \cdot b_3$

$\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$$\cos\alpha = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}}$$

$a \perp b \Longleftrightarrow a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$

$\vec{a}, \vec{b}$ coliniari sau $\vec{a} \parallel \vec{b} \Leftrightarrow \frac{x_1}{x_2} = \frac{y_1}{y_2}$
↳ $\Leftrightarrow \exists\, r \in \mathbb{R}$ a.î. $\vec{a} = \alpha \cdot \vec{b}$, $\vec{b} \neq \vec{0}$

Prod vectorial: $\vec{a} \times \vec{b}$

① a, b colin $\Longleftrightarrow \vec{a} \times \vec{b} = \vec{0}$

② $\|a \times b\| = \|a\| \cdot \|b\| \cdot \sin\alpha$
↳ $\vec{a} \times \vec{b} \perp \vec{a}$ & $\vec{a} \times \vec{b} \perp \vec{b}$

anticomutativ: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

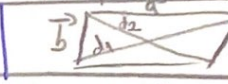
→ nu e asoc.: $(a \times b) \times c \neq a \times (b \times c)$

$(\lambda a + \mu b) \times c = \lambda (a \times c) + \mu (b \times c)$

$A_\Delta = \frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\|$ $\quad A_{\square} = \|\overrightarrow{AB} \times \overrightarrow{AC}\|$

$A_\Delta = \frac{1}{2} \|\overrightarrow{AC}\| \cdot h_B$ $\quad h = \dfrac{\|\overrightarrow{AB} \times \overrightarrow{AC}\|}{\|\overrightarrow{AC}\|}$ dim B pe AC

$(a, b, c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$



$d_1 = \vec{a} + \vec{b}$
$d_2 = \vec{a} - \vec{b}$

$m(\sphericalangle p, Oy) > 90^\circ \Rightarrow p \cdot \vec{j} < 0 \Rightarrow p < 0$

$m(\sphericalangle p, Ox) < 90^\circ \Rightarrow p \cdot \vec{i} > 0 \to p > 0$

Dreapta în plan: v dir: $\vec{a} \neq \vec{0}$, col cu dr

versori (v dir = 1): $v^{\pm} = \pm \dfrac{a}{\|a\|}$

Δ în plan, $\vec{a}$ v.dir., M_0 pct oarecare, n_0 v poz.

⇒ v poz $\forall M$ de pe dr: $\vec{n} = \vec{n_0} + t \cdot \vec{a}$, $t \in \mathbb{R}$ (ec vect)

↳ $\begin{cases} x = x_0 + l \cdot t \\ y = y_0 + m \cdot t \end{cases}$, $t \in \mathbb{R}$, $M(x, y)$ ec param

ec. canonică: $\dfrac{x - x_0}{l} = \dfrac{y - y_0}{m}$, $l, m \neq 0$

↳ $\dfrac{x - x_0}{0} = \dfrac{y - y_0}{m} \Rightarrow x - x_0 = 0$ | $\vec{a}(l, m)$

• $M_0(x_0, y_0)$ $M_1(x_1, y_1) \in \Delta \Rightarrow \overrightarrow{M_0 M_1}$ v. dir al Δ

⇒ ec dr în 2 pct: $\dfrac{x - x_0}{x_1 - x_0} = \dfrac{y - y_0}{y_1 - y_0}$

→ $\Delta \in \Pi$, $M_0(x_0, y_0) \in \Delta$, $\vec{a}(l, m)$ v dir

ec canonică a dr: $y - y_0 = \dfrac{m}{l}(x - x_0)$, k (panta)

⇒ $\vec{n} - \vec{n_0} = t \cdot \vec{a}$, fie $\vec{m} \perp$ v dir al $\Delta \Rightarrow (n - n_0) \cdot m = 0$

⇒ $m_1(x - x_0) + m_2(y - y_0) = 0$

⇒ $Ax + By + C = 0$ [ec generală a dreptei]

⇒ $\vec{m}(A, B)$, vect normal, $\vec{a}(-B, A)$ v. dir și $\perp \vec{m}$

$k = \operatorname{tg}\varphi$

a, b coord unde dr taie Ox și Oy, $a, b \in \mathbb{R}$

ec dr prin tăieturi: $\dfrac{x}{a} + \dfrac{y}{b} - 1 = 0$

$d_1: A_1 x + B_1 y + C_1 = 0$ $\quad d_2: A_2 x + B_2 y + C_2 = 0$

d concurente

⇒ $\lambda \cdot d_1 + \mu \cdot d_2 = 0$ (o dr oarecare care trece prin pct $d_1 \cap d_2$)

$A_1 x + B_1 y + C_1 + \alpha(A_2 x + B_2 y + C_2) = 0$ → ec fasc de dr.

$d_1: A_1 x + B_1 y + C_1 = 0$, $d_2: A_2 x + B_2 y + C_2 = 0$

⇒ $\vec{m_1}(A_1, B_1)$, $\vec{m_2}(A_2, B_2)$

$y = k_1 x + b_1$, $y = k_2 x + b_2 \Rightarrow \operatorname{tg}\theta = \pm \dfrac{k_1 - k_2}{1 + k_1 k_2}$

$m_{AB} = \dfrac{y_B - y_A}{x_B - x_A} = \dfrac{-A}{B}$

$$\cos\theta = \pm \frac{A_1 A_2 + B_1 B_2}{\sqrt{A_1^2 + B_1^2} \cdot \sqrt{A_2^2 + B_2^2}}$$

$d_1 \perp d_2 \Longleftrightarrow A_1 A_2 + B_1 B_2 = 0$ sau $1 + k_1 k_2 = 0$

$d_1 \parallel d_2 \Longleftrightarrow A_1 B_2 - A_2 B_1 = 0$ sau $k_1 = k_2$

forma Hesse: v norm împărțit la versor ⇒ t lib < 0

$\cos\alpha \cdot x + \sin\alpha \cdot y - p = 0$, dist de la orig la d, $\alpha = (\sphericalangle d, Ox)$

ecuația gen. a dreptei: $\dfrac{A}{\pm\sqrt{A^2 + B^2}} \cdot x + \dfrac{B}{\pm\sqrt{A^2 + B^2}}\, y + \dfrac{C}{\pm\sqrt{A^2 + B^2}} = 0$, ↳ < 0

Distanța: Δ și $M(x_0, y_0, z_0)$

$$d(M_0, \Delta) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

Planul în spațiu: Π plan, $M_0(x_0, y_0, z_0)$ pct în plan; $\vec{a_1}(l_1, m_1, n_1)$, $\vec{a_2}(l_2, m_2, n_2)$ vect necol $\parallel \Pi$

vect de poz n pt $\forall M \in \Pi$: $n = n_0 + u\, a_1 + v\, a_2$ (ec vect), n_0 – vect poz al M_0, $u, v \in \mathbb{R}$

↳ exprimă cond ca $n - n_0$, a_1, a_2 L.I. / coplanari

$\begin{cases} x = x_0 + l_1 \cdot u + l_2 \cdot v \\ y = y_0 + m_1 \cdot u + m_2 \cdot v \\ z = z_0 + n_1 \cdot u + n_2 \cdot v \end{cases}$, $u, v \in \mathbb{R}$ (ec param)

$(\vec{n} - \vec{n_0}, \vec{a_1}, \vec{a_2}) = 0$ (p.mixt) → ec v. reparam a plan care trece printr-un pct dat și $\parallel$ 2 vect necoliniari

x	0°	30°	45°	60°	90°	120°	135°	150°	180°	360°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	1

$\operatorname{tg}\alpha = \dfrac{\sin\alpha}{\cos\alpha}$ $\quad \operatorname{ctg} = \dfrac{\cos\alpha}{\sin\alpha}$

$\sin\alpha = \dfrac{C.OP}{IP}$ $\quad \cos\alpha = \dfrac{C.AL}{IP}$

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$M_1, M_2, M_3 \in \tilde{\pi}$, 2 v necol $\overrightarrow{M_1M_2} \parallel \tilde{\pi}$, $\overrightarrow{M_1M_3} \parallel \tilde{\pi}$

$\Rightarrow \tilde{\pi}$ det de 3 pct necol, trece prin M_1 și $\parallel \overrightarrow{M_1M_2}, \overrightarrow{M_1M_3}$

$$\Rightarrow \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$Ax + By + Cz + D = 0$, $A, B, C, D \in \mathbb{R}$

↳ $\vec{n}(A, B, C)$ vector normal în plan

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0$ ec planului prin tăieturi $a, b, c \to Ox, Oy, Oz$

→ pt versor: $\cos\alpha \cdot x + \cos\beta \cdot y + \cos\gamma \cdot z - p = 0$

↳ forma Hesse

ecuația normală: $\frac{Ax + By + Cz + D}{\pm\sqrt{A^2+B^2+C^2}} = 0$ → alegem semn a.î. $\frac{D}{-\|-} < 0$

$M(x_0, y_0, z_0)$ → abatere a unui pct relativ la plan $\tilde{\pi}$

$$\delta(M_0, \tilde{\pi}) = \frac{Ax_0 + By_0 + Cz_0 + D}{\pm\sqrt{A^2+B^2+C^2}}$$

$$d(M_0, \tilde{\pi}) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2+B^2+C^2}} = |\delta(M_0, \tilde{\pi})|$$

$\measuredangle(\tilde{\pi}_1, \tilde{\pi}_2) = \measuredangle(\vec{n}_1, \vec{n}_2)$

$\tilde{\pi}_1: A_1x + B_1y + C_1z + D_1 = 0 \Rightarrow \vec{n}_1(A_1, B_1, C_1)$

$\tilde{\pi}_2: A_2x + B_2y + C_2z + D_2 = 0 \Rightarrow \vec{n}_2(A_2, B_2, C_2)$

$$\cos\alpha = \frac{A_1 \cdot A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2} \cdot \sqrt{A_2^2+B_2^2+C_2^2}}$$

$\tilde{\pi}_1 \perp \tilde{\pi}_2: A_1A_2 + B_1B_2 + C_1C_2 = 0$

$\tilde{\pi}_1 \parallel \tilde{\pi}_2: \frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$

$\cos\tilde{\pi}_1 = \cos\tilde{\pi}_2$

↳ plane bisectoare

Dreapta în spațiu: vector director al Δ: $\vec{a}$

$r = r_0 + t \cdot a$, $t \in \mathbb{R}$, r_0: v. de poz al M_0,

↳ ec vect a dr care $\vec{a}$ v dir. trece prin M_0 și are v dir $\vec{a}$

$$\begin{cases} x = x_0 + l \cdot t \\ y = y_0 + m \cdot t \\ z = z_0 + n \cdot t \end{cases}$$ → ec param a dr Δ, care trece prin M_0, $\parallel \vec{a}$

ec. canonice: $\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$

Convenție: $\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{0} \Rightarrow \frac{x-x_0}{l} = \frac{y-y_0}{m}$, $z - z_0 = 0$

$\vec{a} = \overrightarrow{M_0M_1}(x_1-x_0, y_1-y_0, z_1-z_0)$

$\Rightarrow \frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0}$

Θ dr. ca $\tilde{\pi}_1 \cap \tilde{\pi}_2$: $\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix}$ → rang = 2

$M_1(x_1, y_1, z_1)$, Δ

$M_0(x_0, y_0, z_0) \in \Delta$, $\vec{a}$ dir

$\tilde{\pi}_1 \cap \tilde{\pi}_2 = d$: $\vec{a} = \vec{n}_1(A_1, B_1, C_1) \times \vec{n}_2(A_2, B_2, C_2)$

$$\vec{a} = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}\vec{i} + \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}\vec{j} + \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}\vec{k}$$

$d(M_1, \Delta) = \frac{\|(r_1 - r_0) \times a\|}{\|a\|}$, r_0 → poz M_0, r → poz M

$$\cos\alpha = \frac{|l_1 l_2 + m_1 \cdot m_2 + n_1 n_2|}{\sqrt{l_1^2 + m_1^2 + n_1^2} \cdot \sqrt{l_2^2 + m_2^2 + n_2^2}}$$

~~$d_1 \in \tilde{\pi}$, $d_2 \in \tilde{\pi}$~~

$d_1 \perp d_2 \Leftrightarrow l_1 \cdot l_2 + m_1m_2 + n_1n_2 = 0$

$d_1 \parallel d_2 \Leftrightarrow \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Dreapta în plan

$\tilde{\pi}_1: A_1x + B_1y + C_1z + D = 0$ $\quad \tilde{\pi}_2: A_2x + B_2y + C_2z + D_2 = 0$

→ taie după o dr: rang$\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = 2$

→ paralele: $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \neq \frac{D_1}{D_2}$

→ coincid: $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}$

3 plane: $\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3$, Δ = det, m, M

a) $\Delta \neq 0$, planele $\cap$ într-un pct

b) $\Delta = 0$, rang $m = 2$, rang $M = 3$

↳ v. norm: 2 câte 2 necol., $\tilde{\pi}$ inters 2-2 ⇒ 3 dr $\parallel$

c) rang $m = 2$, rang $M = 3$ ⇒ 2 din 3 v. norm. sunt coliniari ⇒ $\tilde{\pi}$ paralele între ele, $\tilde{\pi}_3$ inters după câte o dr.

d) rang $m = 2$, rang $M = 2$, v norm 2-2 necol ⇒ $\tilde{\pi}$ 2 câte 2 distincte și trec prin aceeași dr

e) rang $m = 2$, rang $M = 2$, 2/3 v norm col. ⇒ 2 din $\tilde{\pi}$ coincid, $\tilde{\pi}_3$ le inters.

f) rang $m = 1$, rang $M = 3$ ⇒ plane dist și $\parallel$

g) rang $m = 1$, rang $M = 2$ ⇒ 2/3 plane coincid $\tilde{\pi}_3 \parallel$ cu $\tilde{\pi}_1, \tilde{\pi}_2$

h) rang $m = 1$, rang $M = 1$ ⇒ $\tilde{\pi}$ coincid

Fie $\tilde{\pi}$, Δ

→ $Al + Bm + Cn \neq 0 \Rightarrow \Delta \cap \tilde{\pi} = P$

→ $Al + Bm + Cn = 0$, $Ax_0 + By_0 + Cz_0 + D \neq 0$ ⇒ $\Delta \parallel \tilde{\pi}$

→ $Al + Bm + Cn = 0$, $Ax_0 + By_0 + Cz_0 + D = 0$ ⇒ $\Delta \in \tilde{\pi}$

$\Delta_1, \Delta_2, M_0(x_0, y_0, z_0) \Rightarrow \begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

↳ 2 dr concurente

$\Delta: \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$, $M_2(x_2, y_2, z_2) \notin \Delta$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l & m & n \end{vmatrix} = 0$$

C și pt Δ_1, Δ_2, $\Delta_1 \parallel \Delta_2$, $\Delta_1 \neq \Delta_2$

Proiecția unui pct pe plan:

$$\begin{cases} Ax + By + Cz + D = 0 \\ \frac{x-x_0}{A} = \frac{y-y_0}{B} = \frac{z-z_0}{C} \end{cases}$$