Geometric Transformations

Lecturer: Rachel McDonnell

Assistant Professor in Creative Technologies

Rachel.McDonnell@cs.tcd.ie

Course www: https://www.scss.tcd.ie/~ramcdonn/

Credit: Some slides taken from Robb T. Koether, Hampden-Sydney College

Objectives

- Learn how to carry out transformations
  - Rotation
  - Translation
  - Scaling
  - Combinations!

Extra Reading

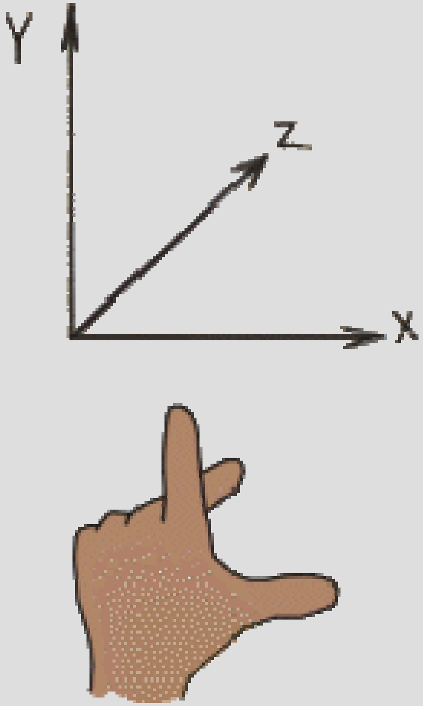
- **Chapter 3: Geometric Objects and Transformations**
- Interactive Computer Graphics: A Top Down Approach with OpenGL, 6th Edition (or other) Angel
- **Chapter 4: Math for 3D Graphics**
- OpenGL Superbible, 6th Edition
- **Elementary Linear Algebra**, Anton

Linear Algebra

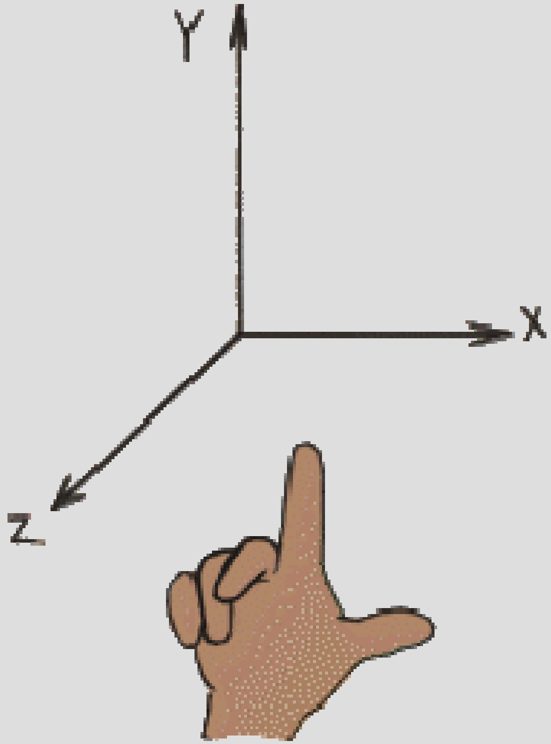
- Linear algebra is the cornerstone of computer graphics.
- Fundamentally, we need to be able to manipulate *points* and *vectors*.
  - these form the basis of all geometric objects & operations
- Geometric operations (*scale*, *rotate*, *translate*, *perspective projection*) are defined using matrix transformations.
- Optical effects (*reflect*, *refract*) defined using vector algebra.

Co-ordinate Systems

- By convention we usually employ a *Cartesian basis*:
  - basis vectors are *mutually orthogonal* and *unit length*
  - basis vectors named **x**, **y** and **z**
- We need to define the relationship between the 3 vectors: there are 2 possibilities:
  - *right handed systems*: **z** comes out of page
  - *left handed systems*: **z** goes into page
  - (note: OpenGL uses a right handed system)
- This affects direction of rotations and specification of *normal vectors*

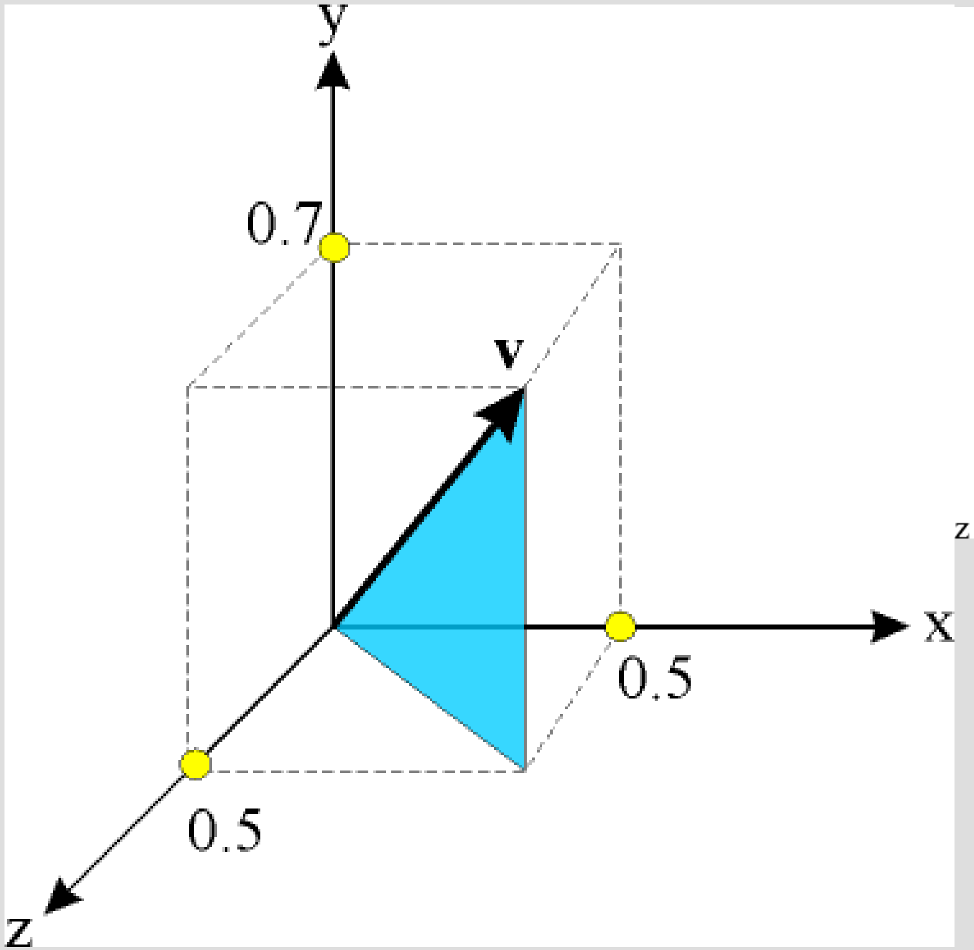


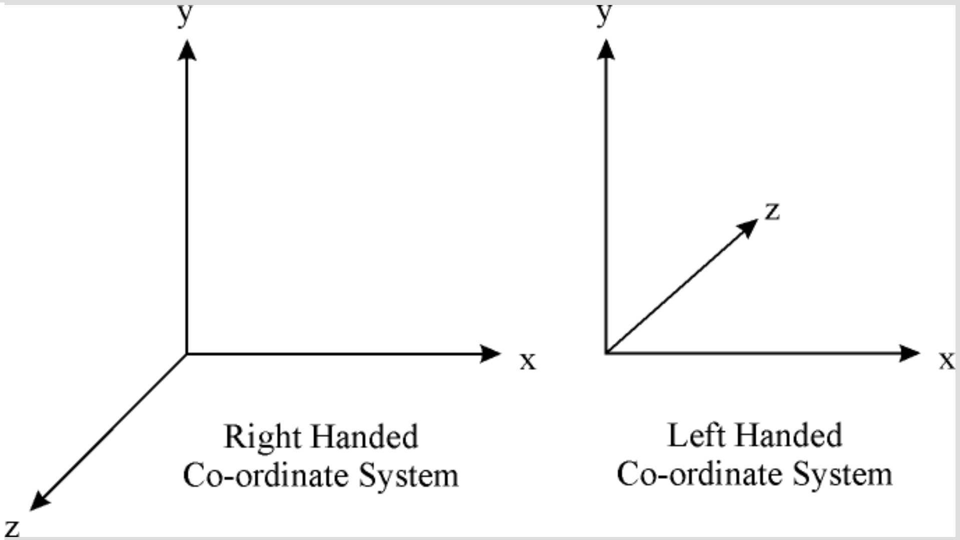
Left hand



Right hand

Cartesian co-ordinate System





$$\mathbf{v} = \begin{bmatrix} 0.5 \\ 0.7 \\ 0.5 \end{bmatrix} =$$

RHS LHS

Conventions

- Vector quantities denoted as **v** or $\vec{v}$
- Each vector is defined with respect to a set of *basis vectors* (which define a co-ordinate system).

- We will use *column format* vectors:

$$\mathbf{v} = \boxed{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}} \neq \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \quad \left(= \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T \right)$$

Row vs. Column Formats

- Both formats, though appearing equivalent, are in fact fundamentally different:
  - be wary of different formats used in textbooks

column format

row format

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$\mathbf{Mv} = \mathbf{v}^T \mathbf{M}^T$$

transposed

$$\begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} * \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

Math Notation

Result 3x3 Matrix 3d Vector

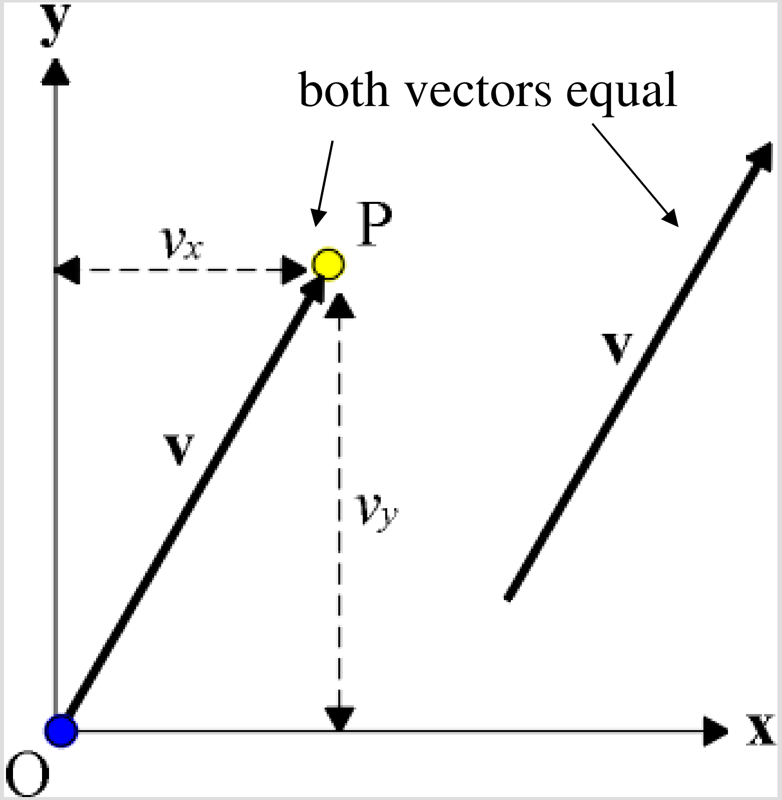
```
mat3 M;
vec3 v;

vec3 u = M * v;
```

GLSL

Vectors & Points

- Although *vectors* and *points* are often used inter-changeably in graphics texts, it is important to distinguish between them.
  - vectors represent directions
  - points represent positions
- Both are meaningless without reference to a *coordinate system*
  - vectors require a set of *basis vectors*
  - points require an *origin* and a *vector space*

$$\mathbf{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} \quad P = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$



Computer Graphics Problems

- Much of graphics concerns itself with the problem of displaying 3D objects in 2D screen
- We want to be able to:
  - rotate, translate, scale our objects
  - view them from arbitrary points of view
  - View them in perspective
- Want to display objects in coordinate systems that are convenient for us and to be able to reuse object descriptions
- Road example
  - Cars, tyres
  - View from a helicopter

Matrices

- If you need to rotate a million vertices representing a dinosaur object about some axis, you don't need to multiply each point by 5 different matrices
  - you simply multiply the 5 matrices together once and multiply each dinosaur point by that one matrix. Huge saving!

Matrices

- Matrix addition
  - $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$+$\begin{bmatrix} e & g \\ f & h \end{bmatrix}$ = $\begin{bmatrix} a+e & c+g \\ b+f & d+h \end{bmatrix}$
- Matrix multiplication
  - $\begin{bmatrix} a & c \\ b & d \end{bmatrix}x\begin{bmatrix} e & g \\ f & h \end{bmatrix}$ = $\begin{bmatrix} ae+cf & ag+ch \\ be+df & bg+dh \end{bmatrix}$
  - Not commutative in most cases
    - **AB** /= **BA**
    - If **AB** = **AC,** it does not necessarily follow that **B** = **C**
  - It is associative and distributive
    - (AB)C = A(BC)
    - A(B+C) = AB + AC
    - (A+B)C = AC + BC
  - Transpose A^T of a matrix A is one whose rows are switched with its columns

Question

- $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$
- $B = \begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix}$

- What is $A+B^T$?

Answer

- $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$
- $B = \begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix}$

- What is $A+B^T$?
  - $AB^T = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}$

Geometric Transformations

- Many geometric transformations are *linear* and can be represented as a matrix multiplication.
- Function *f* is linear iff:

$$f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$$

- Implications:
  - to transform a line we transform the *end-points*. Points between are *affine combinations* of the transformed endpoints.
  - Given line defined by points *P* and Q, points along transformed line are affine combinations of transformed *P*´and Q´

$$L(t)=P+t(Q-P)$$
$$L'(t)=P'+t(Q'-P')$$

Homogeneous Co-ordinates

- Basis of the homogeneous co-ordinate system is the set of *n* basis vectors and the origin position:

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \text{ and } P_o$$

- All points and vectors are therefore compactly represented using their ordinates:

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \\ a_o \end{bmatrix} \text{or more usually} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Homogeneous Co-ordinates

- Vectors have no positional information and are represented using $a_o = 0$ whereas points are represented with $a_o = 1$:

$$\vec{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n + 0$$

$$P = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n + P_o$$

- Examples: $\begin{bmatrix} 0.2 \\ 1.3 \\ 2.2 \\ 1 \end{bmatrix} \begin{bmatrix} 1.0 \\ 1.0 \\ 0.0 \\ 1 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 0.2 \\ 1.3 \\ 2.2 \\ 0 \end{bmatrix} \begin{bmatrix} 1.0 \\ 1.0 \\ 0.0 \\ 0 \end{bmatrix}$

Points $\qquad$ Associated vectors

Homogenous Coordinates

- Using this scheme, every rotation, translation, and scaling operation can be represented by a matrix multiplication, and any combination of the operations corresponds to the products of the corresponding matrices
- Using homogeneous co-ordinates allows us to treat translation in the same way as rotation and scaling

Translation

- Simplest of the operations
  - Add a positive number – moves to the right
  - Add a negative number – moves to the left
- Addition of constant values, causes uniform translations in those directions
- Translations are independent and can be performed in any order (including all at once)
  - Object moved one unit to the right then up
  - Same as if moved one unit up and to the right
  - Net result is motion of sqrt(2) units to the upper-right

Translation

Definition (Translation)
A translation is a displacement in a particular direction

- A translation is defined by specifying the displacements a, b, and c

$$x' = x + a$$

$$y' = y + b$$

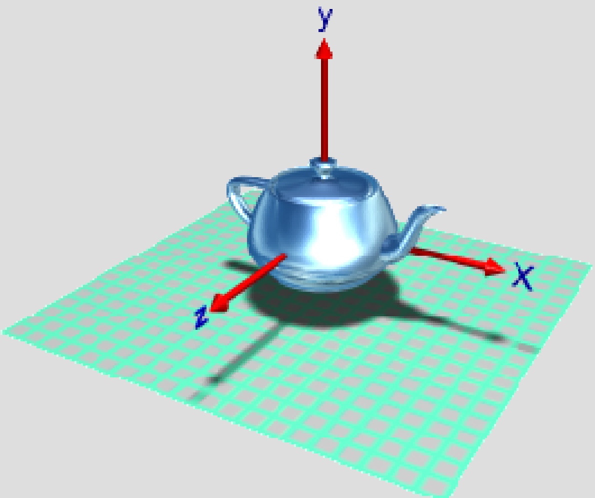
$$z' = z + c$$

Translation

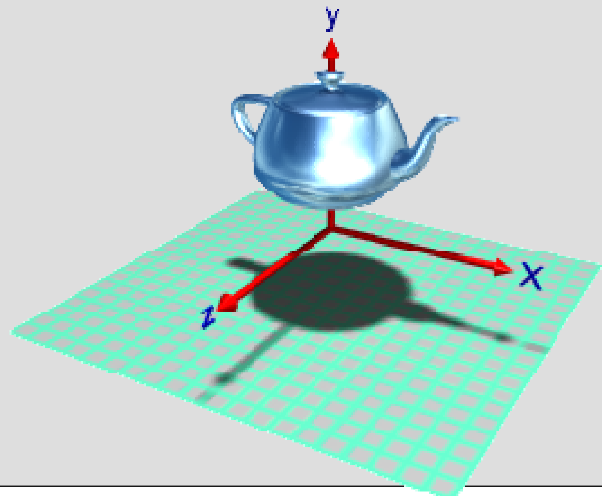
- Translation *only applies to points*, we never translate vectors.
- <u>Remember</u>: points have homogeneous co-ordinate *w = 1*

$$x' = x + a \qquad y' = y + b \qquad z' = z + c$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ z+c \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



translate along ***y***

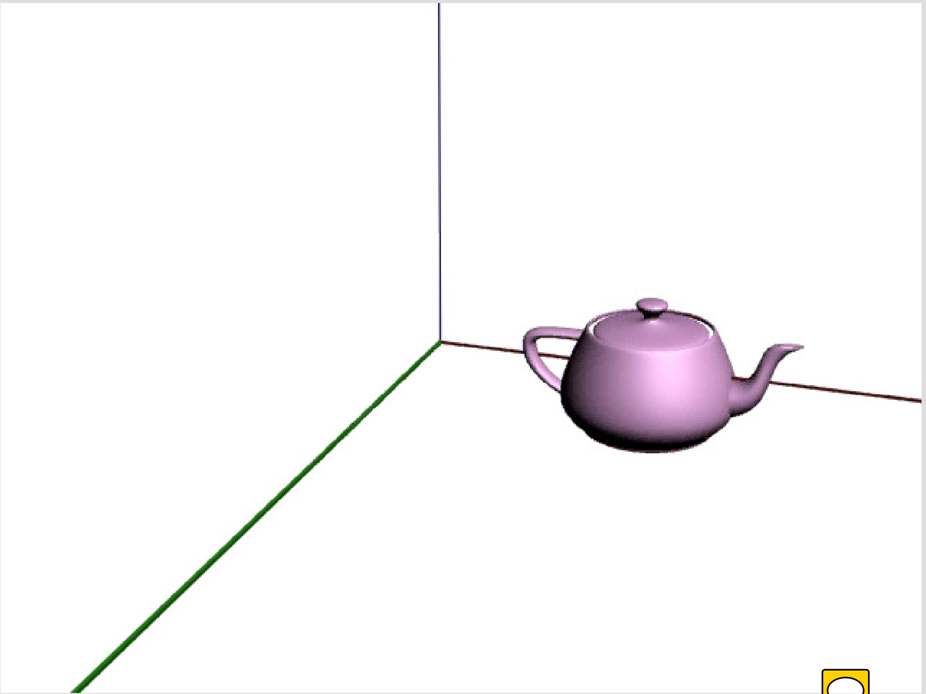


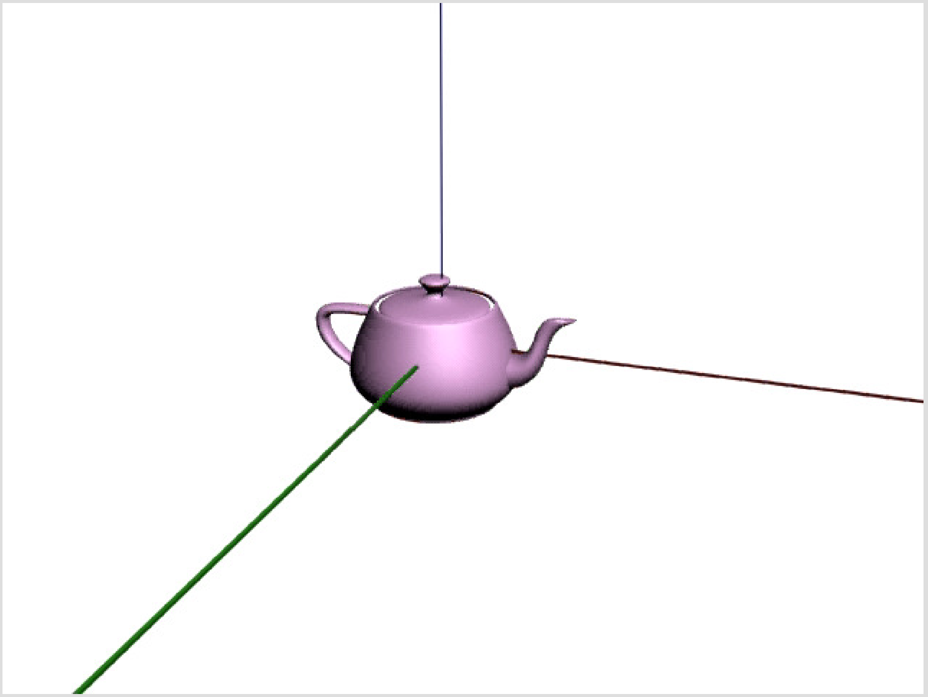
Scaling

- What if we want to make things larger or smaller?
- Have a car model
  - Want one 3 times smaller!

Scaling an object

- 3 times smaller
- Multiply all our coordinates by 1/3
- We get a model that is 1/3 of the size
- However
  - If original coordinates described a car 1 mile from the origin
  - Miniature car would only be 1/3 mile from the origin
- Solution – translation to origin, and then scale, then translate back





Scaling

Definition (Scaling)
A scaling is an expansion or contraction in the x, y, and z directions by scale factors sx, sy, sz and centred at the point (a,b, c)

- Generally we centre the scaling at the origin

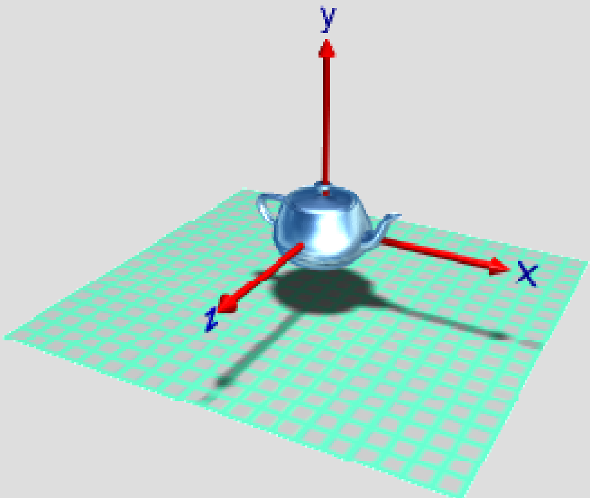
$$x' = s_x x$$
$$y' = s_y y$$
$$z' = s_z z$$

Non-Uniform Scaling

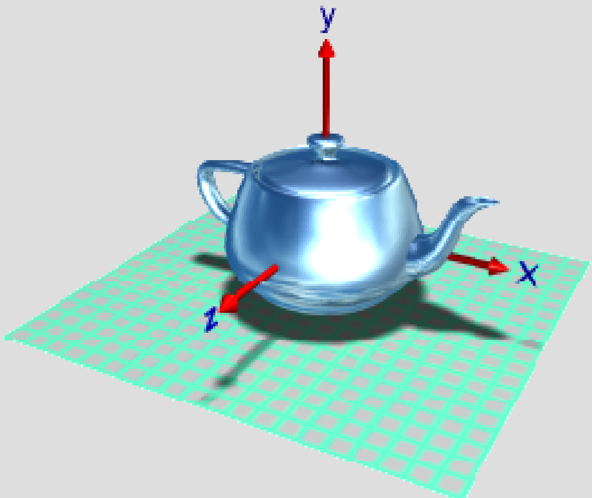
- Make an object twice as big in the x-direction
  - Multiply all x-coordinates by 2, leave y&z unchanged
- 3 times as large in the y-direction
  - Multiply all y-coordinates by 3, leave z&x unchanged

Scale

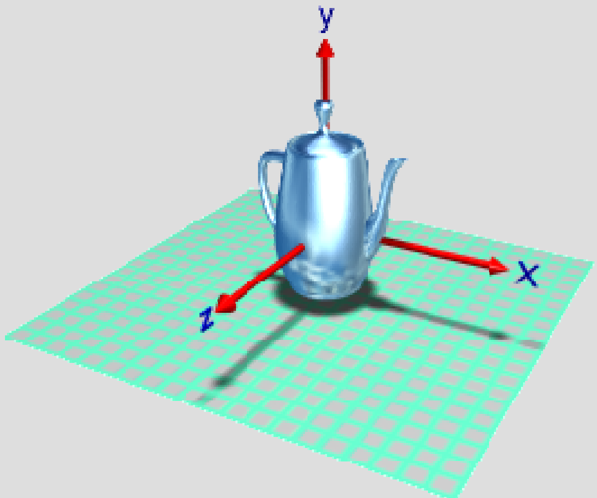
- all vectors are scaled from the origin:



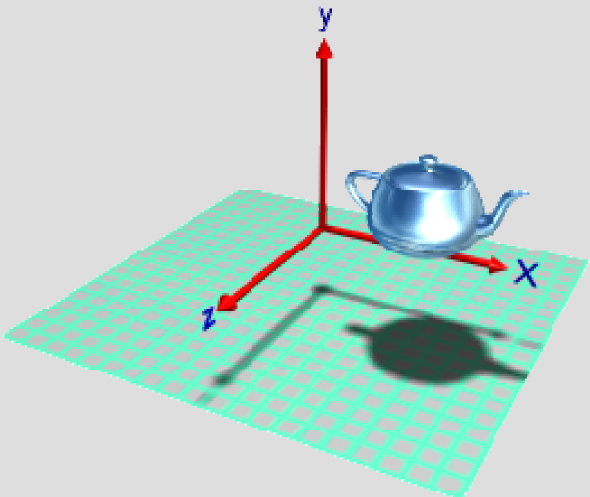
Original



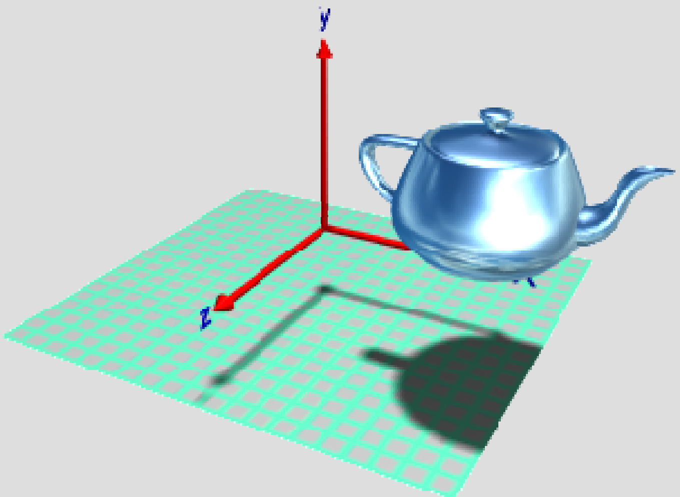
scale all axes



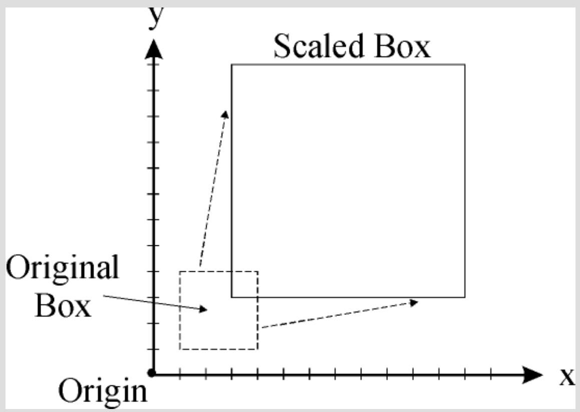
scale Y axis



offset from origin



distance from origin also scales



Scale

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \\ s_z z \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}' = \mathbf{S}\mathbf{v}$$

We would also like to scale points thus we need a *homogeneous transformation* for consistency:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \\ s_z z \\ w \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$\mathbf{S}^{-1} = \begin{bmatrix} 1/s_x & 0 & 0 & 0 \\ 0 & 1/s_y & 0 & 0 \\ 0 & 0 & 1/s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation

- Consider rotation in the x-y plane about the origin by an angle θ in counter-clockwise direction
  - Same as rotation about the z-axis

Rotation

Definition (Rotation)

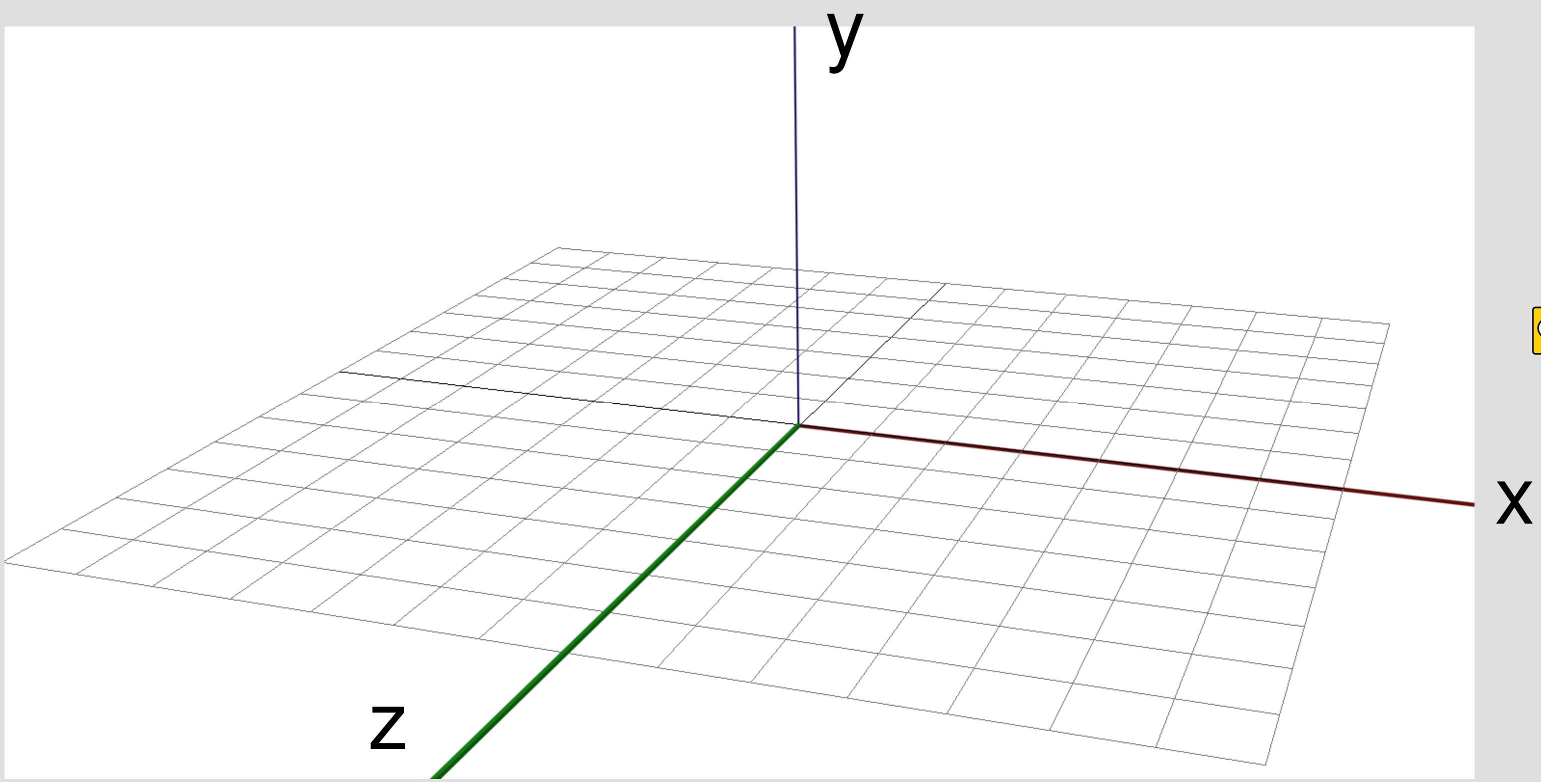
A rotation turns about a point (a,b) through an angle θ

- Generally, we rotate about the origin
- Using the z-axis as the axis of rotation, the equations are:

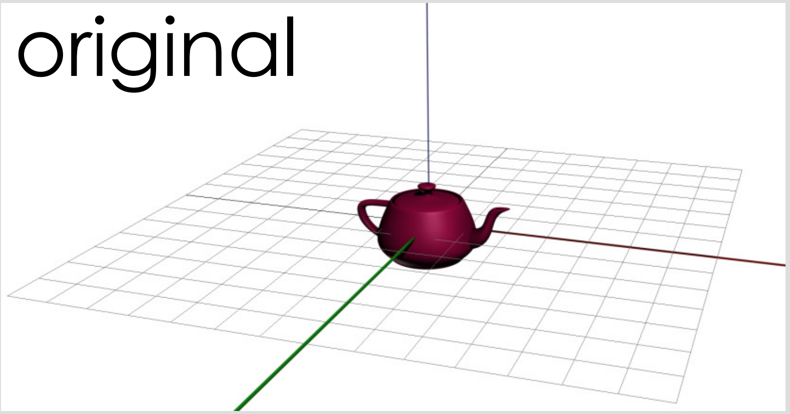
$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

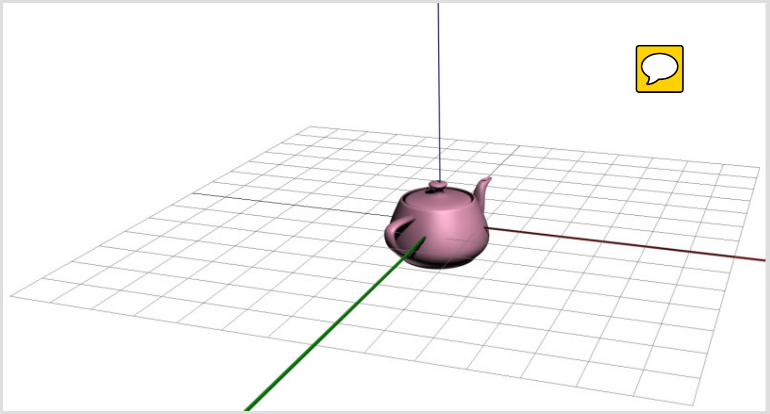
Rotation - idea

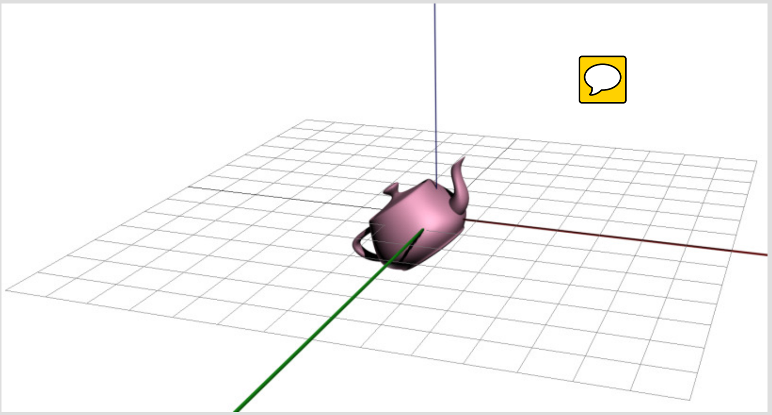
- Visualise rotation about an axis:
  - Put your eye on that axis in the positive direction and look towards the origin
  - Then, a positive rotation corresponds to a counter-clockwise rotation

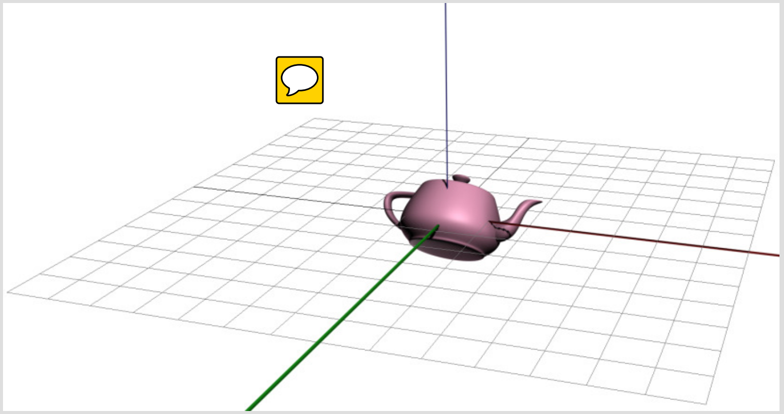


Which Axis?



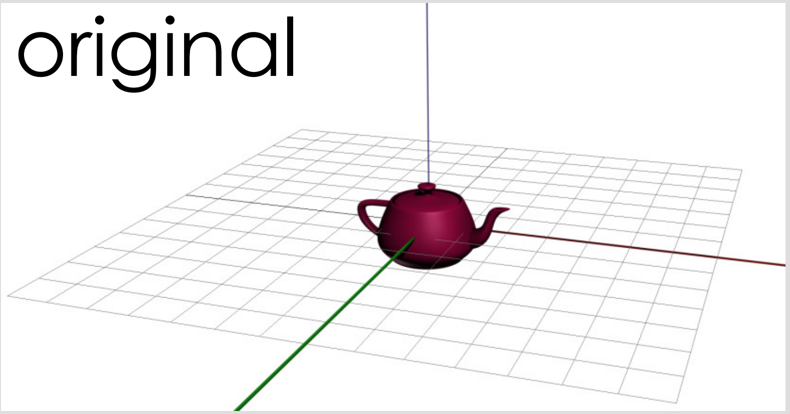




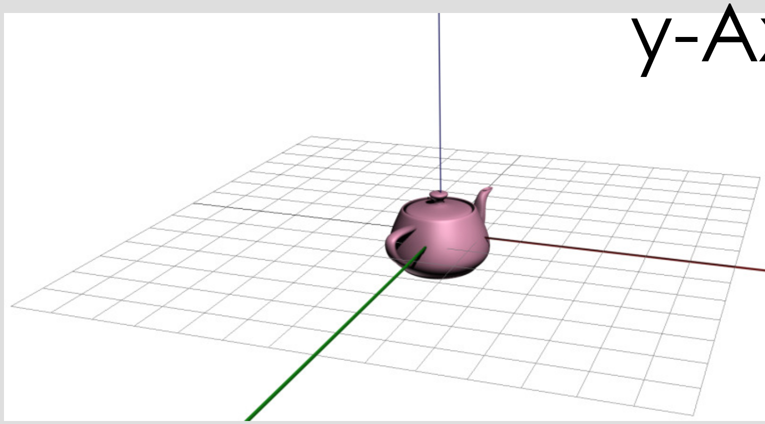


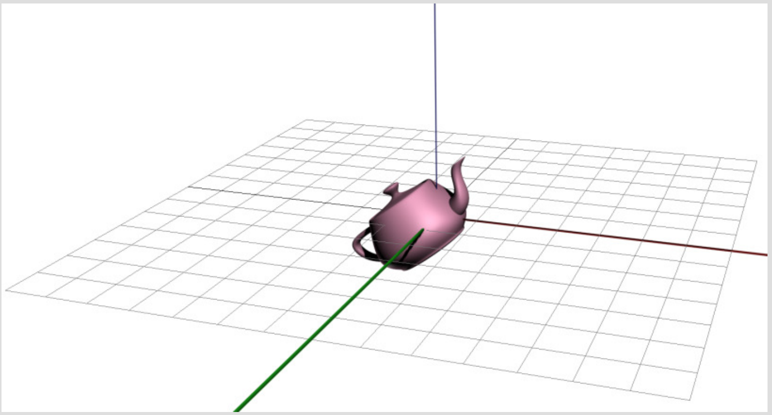
Which Axis?

original

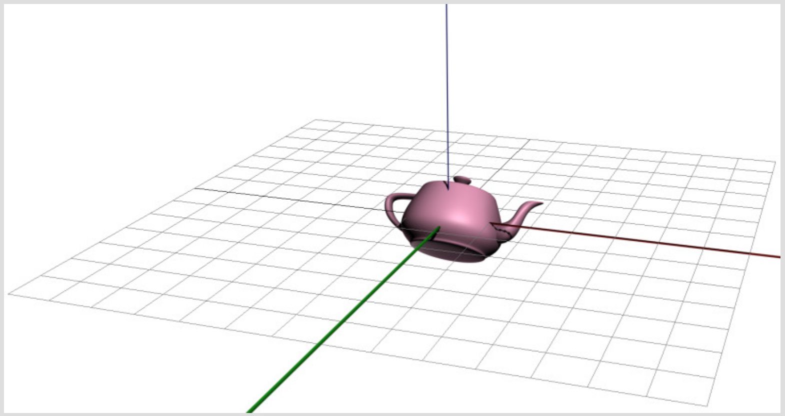


y-Axis





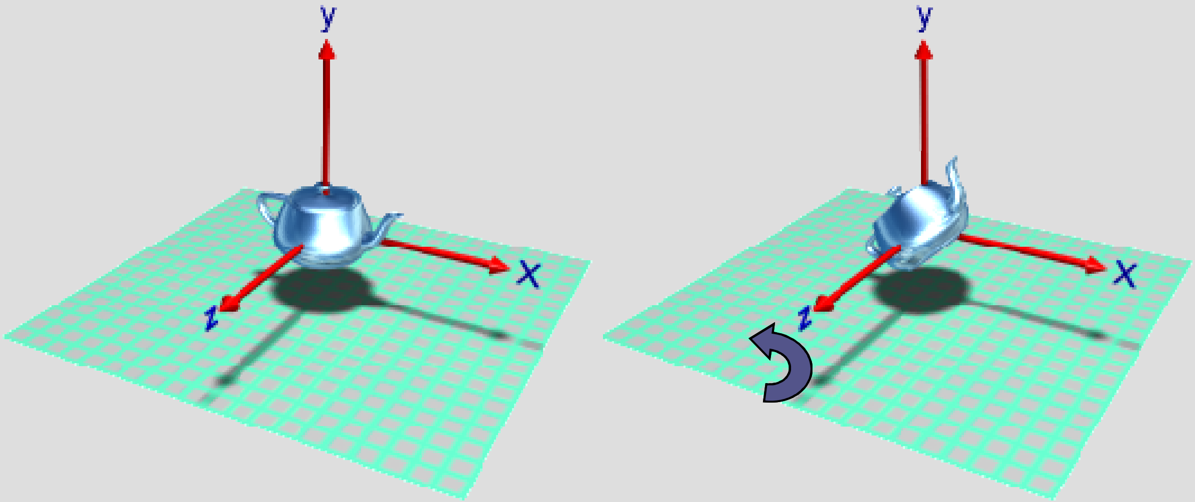
z-axis



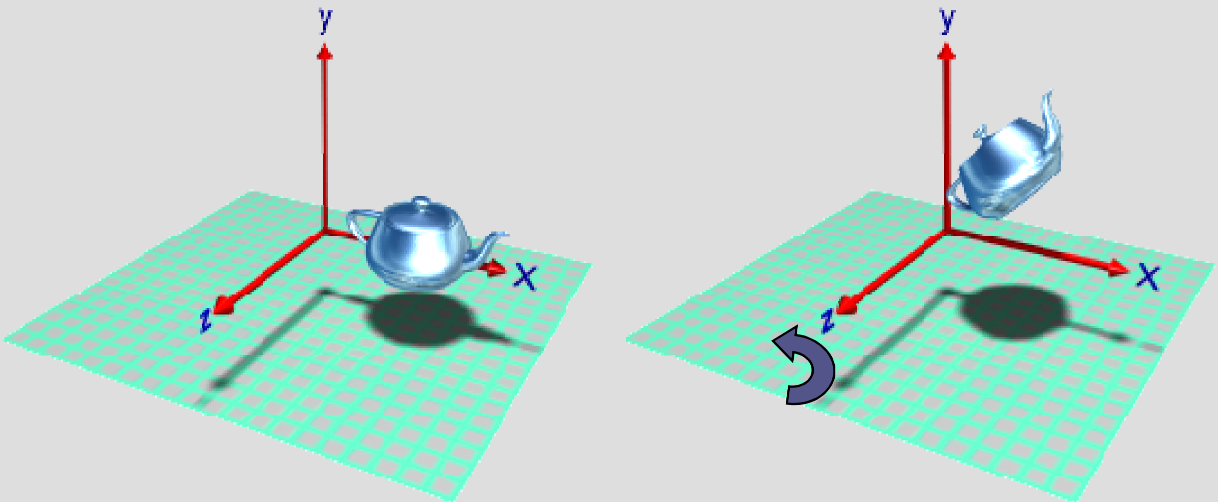
(-) x-axis

Rotation

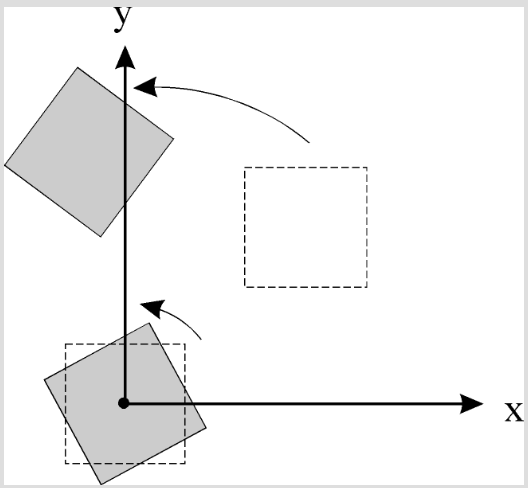
- Rotations are *anti-clockwise* about the *origin:*



rotation of 45º about the Z axis

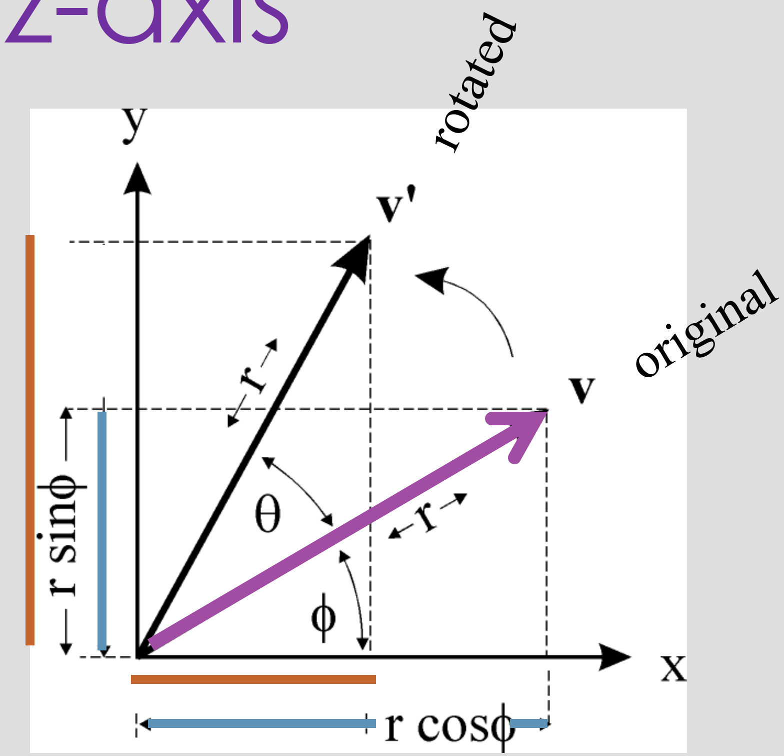


offset from origin rotation



Rotation about the z-axis

$$\mathbf{v} = \begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix} \quad \mathbf{v}' = \begin{bmatrix} \\ \end{bmatrix}$$



$$\text{expand } (\phi+\theta) \Rightarrow \begin{cases} x' = r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ y' = r\cos\phi\sin\theta + r\sin\phi\cos\theta \end{cases}$$

$$\text{but} \quad \begin{matrix} \underline{x = r\cos\phi} \\ \underline{y = r\sin\phi} \end{matrix} \Rightarrow \begin{matrix} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \end{matrix}$$

Rotation about the z-axis

- Rotation in the clockwise direction is the inverse of rotation in the counter-clockwise direction and vice versa

Rotation

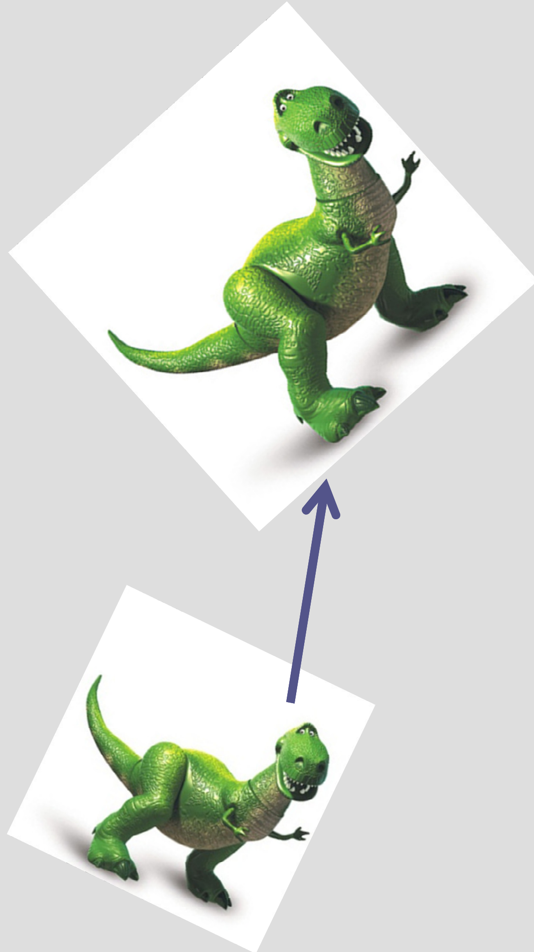
- 2D rotation of θ about origin: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
- 3D homogeneous rotations:

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{R}_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{R}_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Note: difference for rotation about y, due to RHS
- Note: $\begin{matrix} \cos(-\theta) = \cos\theta \\ \sin(-\theta) = -\sin\theta \end{matrix} \Rightarrow \mathbf{R}^{-1}(\theta) = \mathbf{R}(-\theta) = \mathbf{R}^T(\theta)$
- If $\mathbf{M}^{-1} = \mathbf{M}^T$ then **M** is *orthonormal*. All orthonormal matrices are rotations about the origin.

Combining Rotation, Translation, & Scaling

- Often advantageous to combine various transformations to form a more complex transformation
- If we do the algebra – things get complicated quickly
- Easier method – matrices

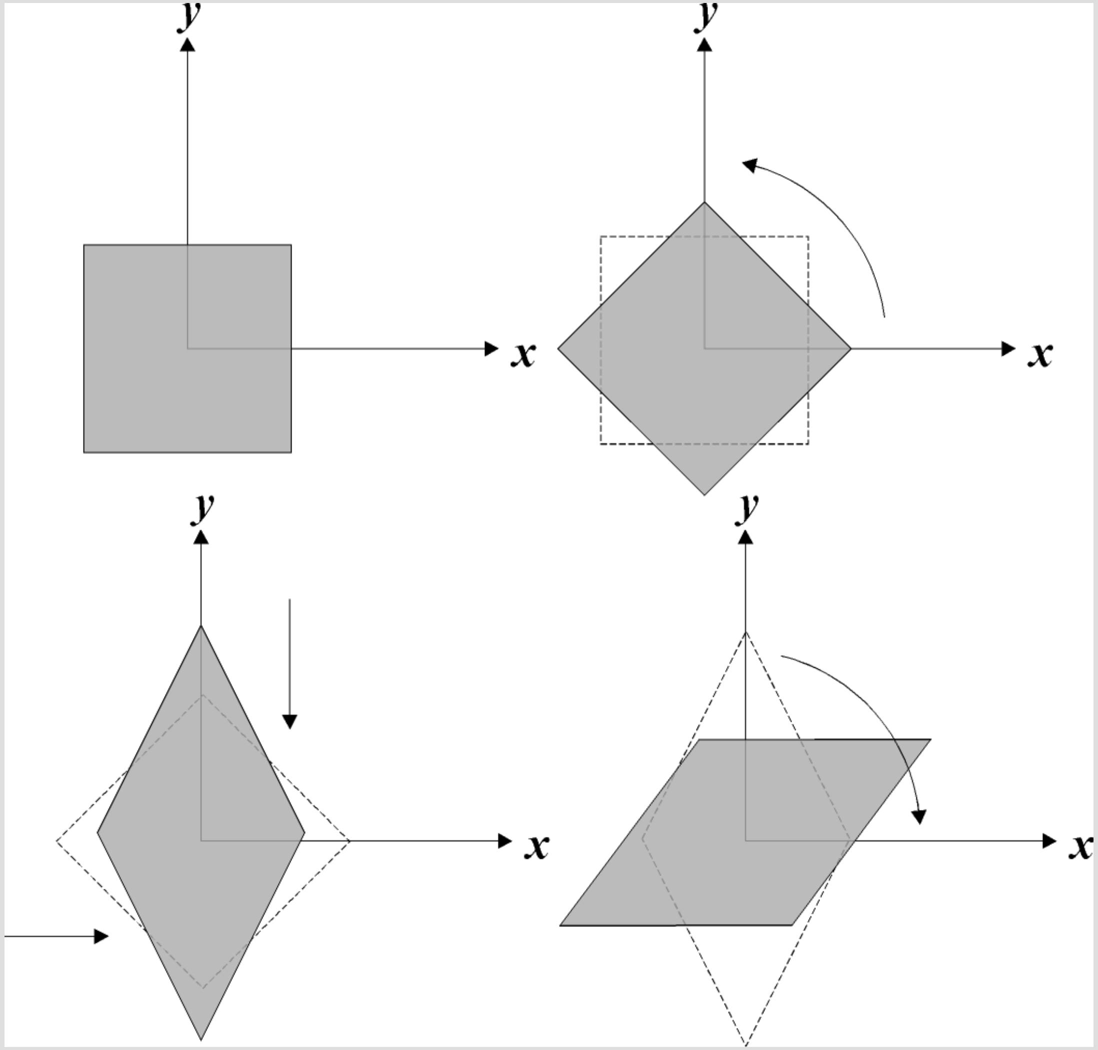


Homogenous Coordinates

- Using this scheme, every rotation, translation, and scaling operation can be represented by a matrix multiplication, and any combination of the operations corresponds to the products of the corresponding matrices

Affine Transformations

- All *affine transformations* are combinations of rotations, scaling and translations.



Transformation Composition

- It is common for graphics programs to apply more than one transformation to an object
  - Take vector v_1, Scale it (**S**), then rotate it (**R**)
  - First, $v_2 = \mathbf{S}v_1$, then, $v_3 = \mathbf{R}v_2$
  - $v_3 = \mathbf{R}(\mathbf{S}v_1)$
  - Since matrix multiplication is associative: $v_3 = (\mathbf{RS})v_1$
- In other words, we can represent the effects of transforms by two matrices in a single matrix of the same size by multiplying the two matrices: $\mathbf{M} = \mathbf{RS}$

Transformation Composition

- More complex transformations can be created by *concatenating* or *composing* individual transformations together.

$$\mathbf{M} = \mathbf{T} \circ \mathbf{R} \circ \mathbf{S} \circ \mathbf{T} = \mathbf{TRST} \qquad \mathbf{v}' = \mathbf{T}[\mathbf{R}[\mathbf{S}[\mathbf{Tv}]]] = \mathbf{Mv}$$

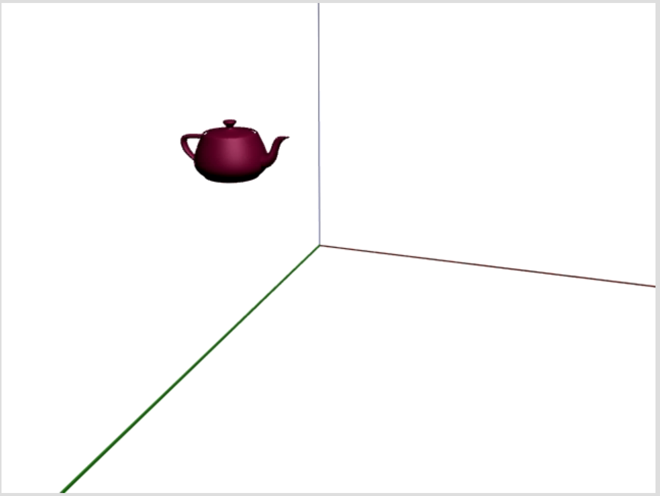
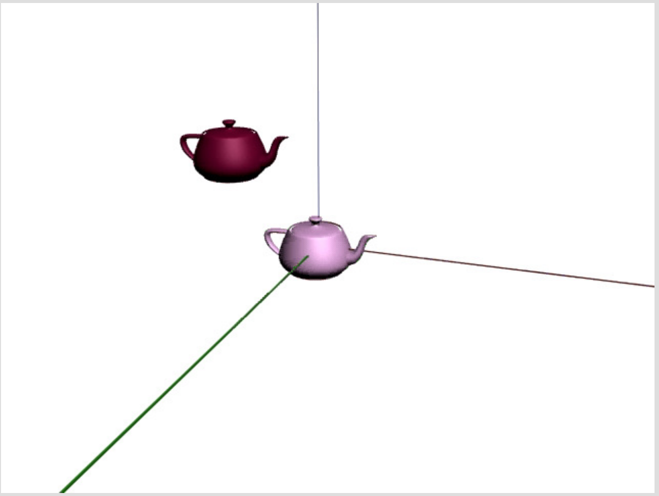
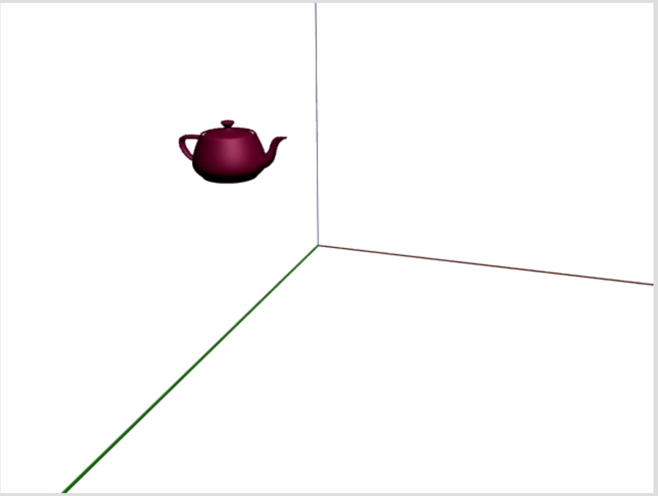
- Matrix multiplication is *non-commutative* $\Rightarrow$ **order is vital**
- We can create an affine transformation representing rotation about a point P_R:

$$\mathbf{M} = \mathbf{T}(P_R)\mathbf{R}(\theta)\mathbf{T}(-P_R)$$

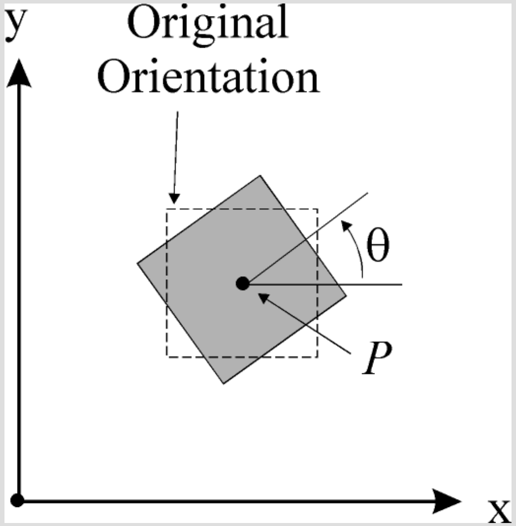
= *translate to origin, rotate about origin, translate back to original location*

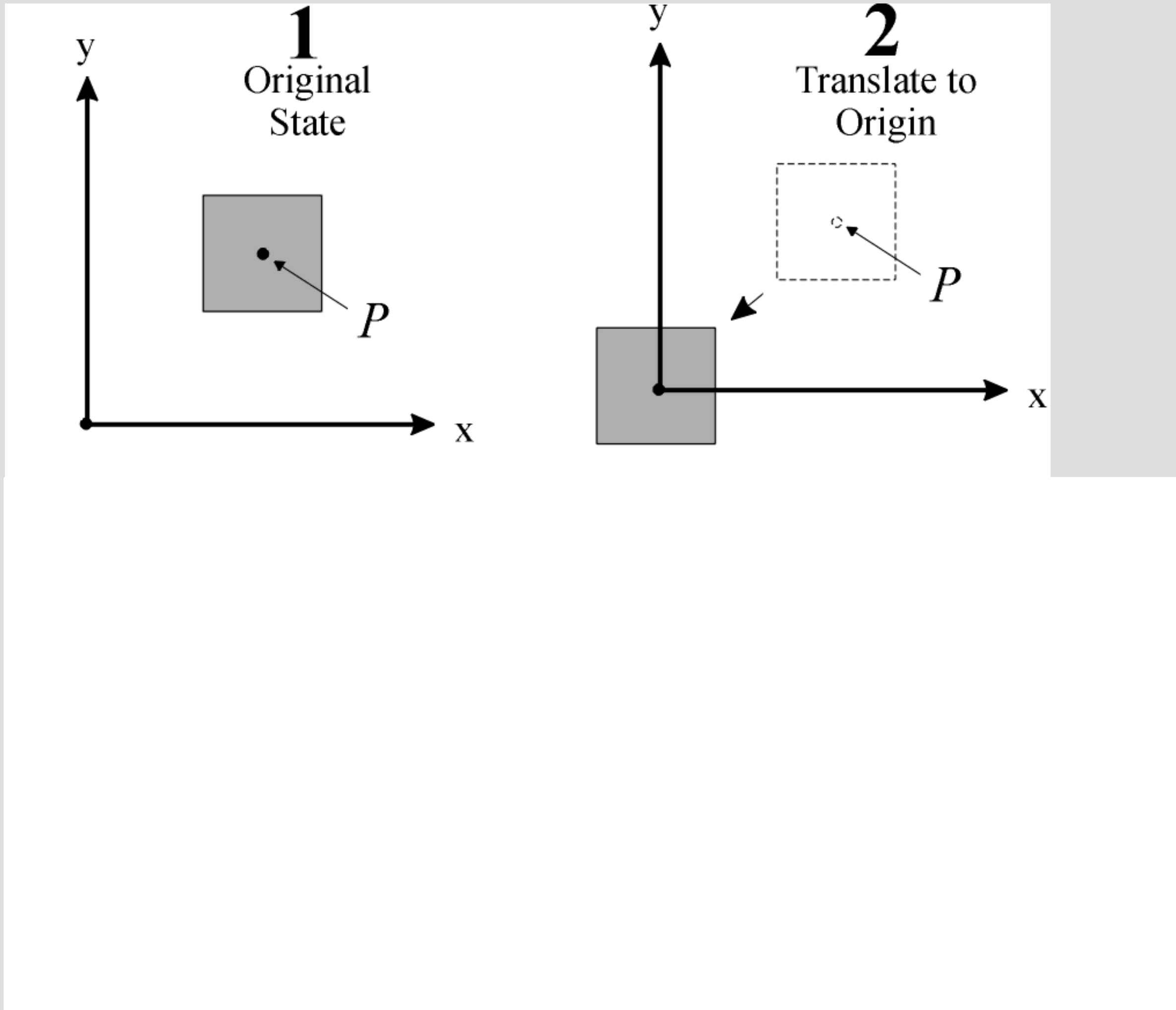
Rotation about a point

- What if rotation is not about the origin?
  - Translate the centre of rotation to the origin,
  - Perform the rotation
  - Translate back

Transformation Composition





Transformation Composition

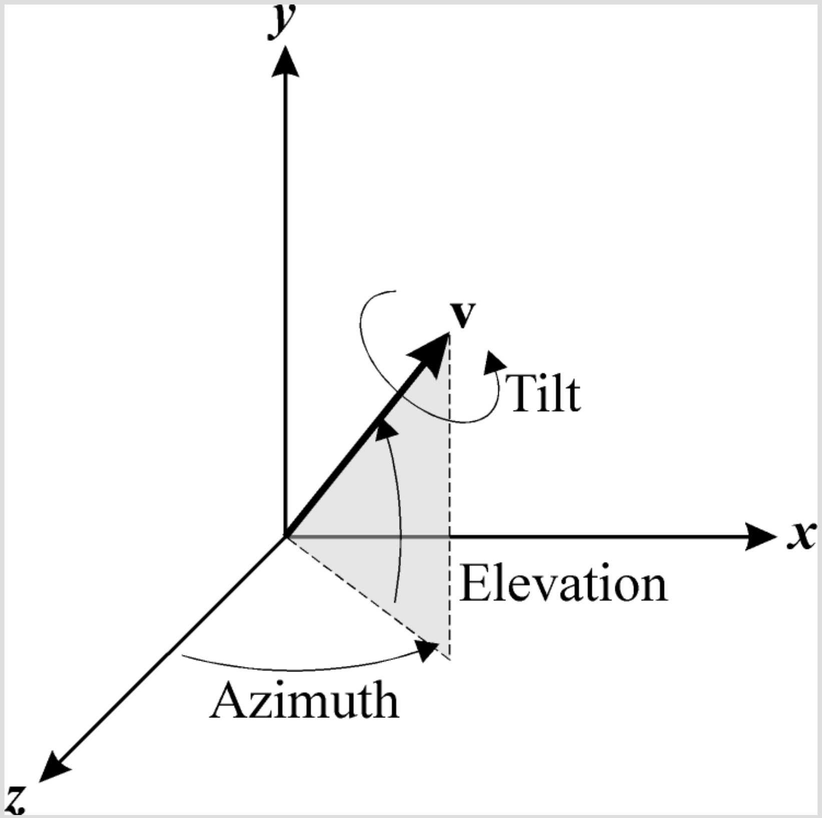
Rotation in **XY** plane by *q* degrees anti-clockwise about point *P*

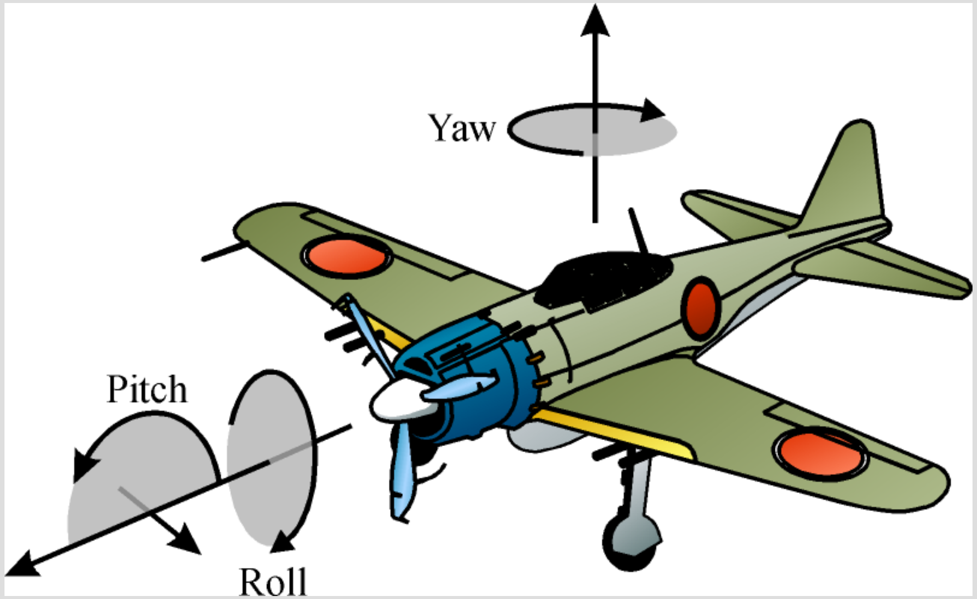
$$
\begin{aligned}
\mathbf{M} &= \mathbf{T}(P)\mathbf{R}(\theta)\mathbf{T}(-P) \\
&= \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & -P_x \\ 0 & 1 & 0 & -P_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & P_x - P_x\cos\theta + P_y\sin\theta \\ \sin\theta & \cos\theta & 0 & P_y - P_x\sin\theta - P_y\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Euler Angles

- *Euler angles* represent the angles of rotation about the co-ordinate axes required to achieve a given orientation (θ_x, θ_y, θ_z)
- The resulting matrix is: $\mathbf{M} = \mathbf{R}(\theta_x)\mathbf{R}(\theta_y)\mathbf{R}(\theta_z)$
- Any required rotation may be described (*though not uniquely*) as a composition of 3 rotations about the coordinate axes.
- Remember rotation *does not commute* $\Rightarrow$ order is important

Rotational DOF





Sometimes known as *roll*, *pitch* and *yaw*

OpenGL - Uniforms

- Pass data into a shader that stays the same – is uniform
  - e.g., transformation matrix
- Get data directly from application to shaders
- Simply place the keyword **uniform** at beginning of variable definition
  - uniform float fTime
  - uniform mat4 modelMatrix

Using Uniforms to Transform Geometry

- Now it is time to put all our knowledge together and build a program that does a little more than pass vertices through un-transformed

The Old Vertex Shader

```
in vec4 vPosition;

void main () {
  // The value of vPosition should be between -1.0 and +1.0
  gl_Position = vPosition;
}
-----------------------------------------
out vec4 fColor ;

void main () {
  // No matter what, color the pixel red!
  fColor = vec4 (1.0, 0.0, 0.0, 1.0);
}
```

A Better Vertex Shader

```
in vec4 vPosition; // the vertex in local coordinate system
uniform mat4 mM;  // the matrix for the pose of the model
uniform mat4 mV;  // The matrix for the pose of the camera
uniform mat4 mP; // The projection matrix (perspective)


void main () {
   // The value of vPosition should be between -1.0 and +1.0
   gl_Position = mP * mV * mM * vPosition;
}
```

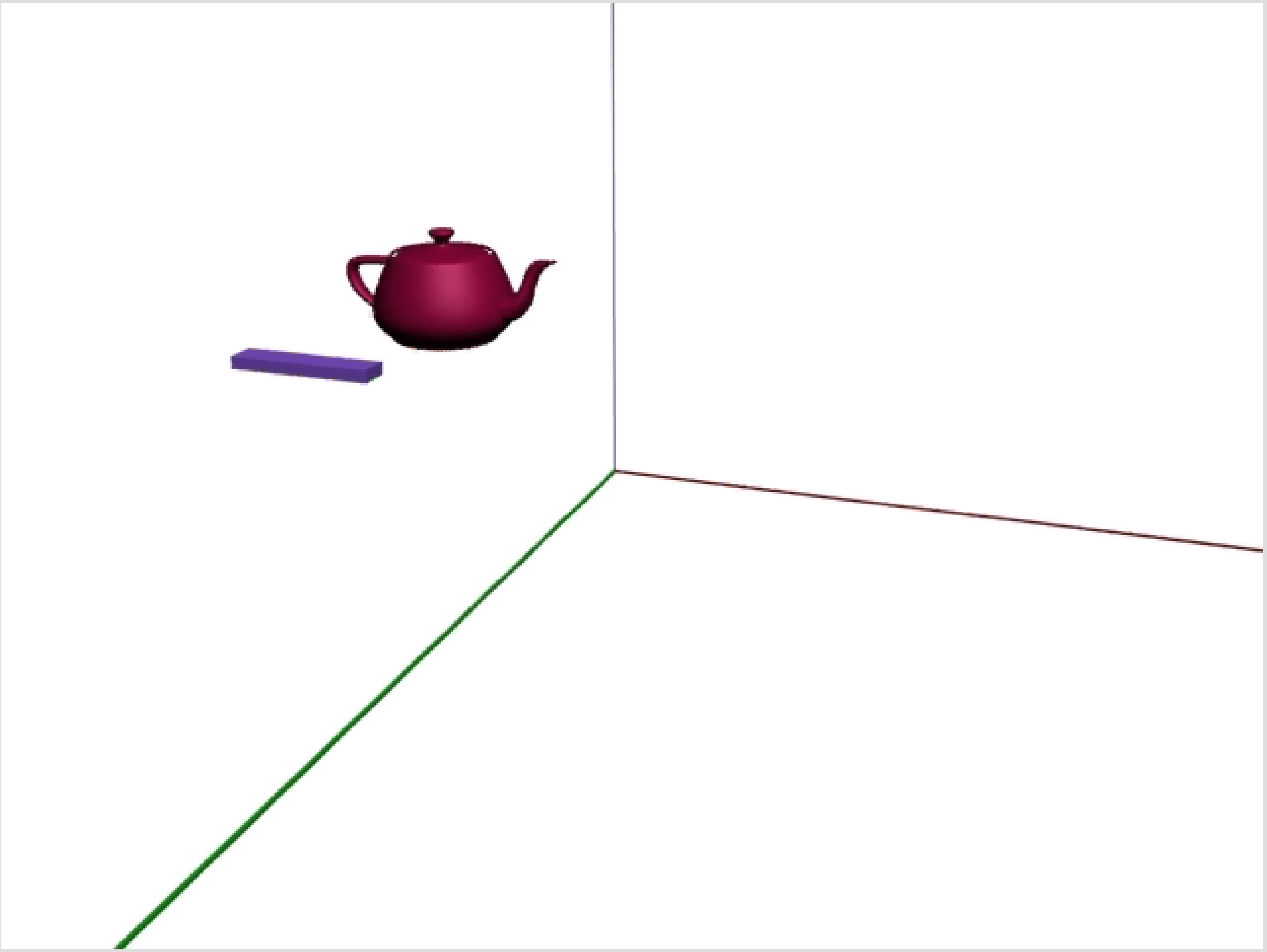
New position in NDC

Original (local) position

General Rotation

- What if you want to rotate about an axis that does not happen to be one of the 3 principle axes?
  - Can do this using operations we already have
- Strategy:
  - Do one or two rotations about the principal axes to get the axis we want aligned with the z-axis
  - Then, rotate about the z-axis
  - Undo the rotations we did to align your axis with the z-axis

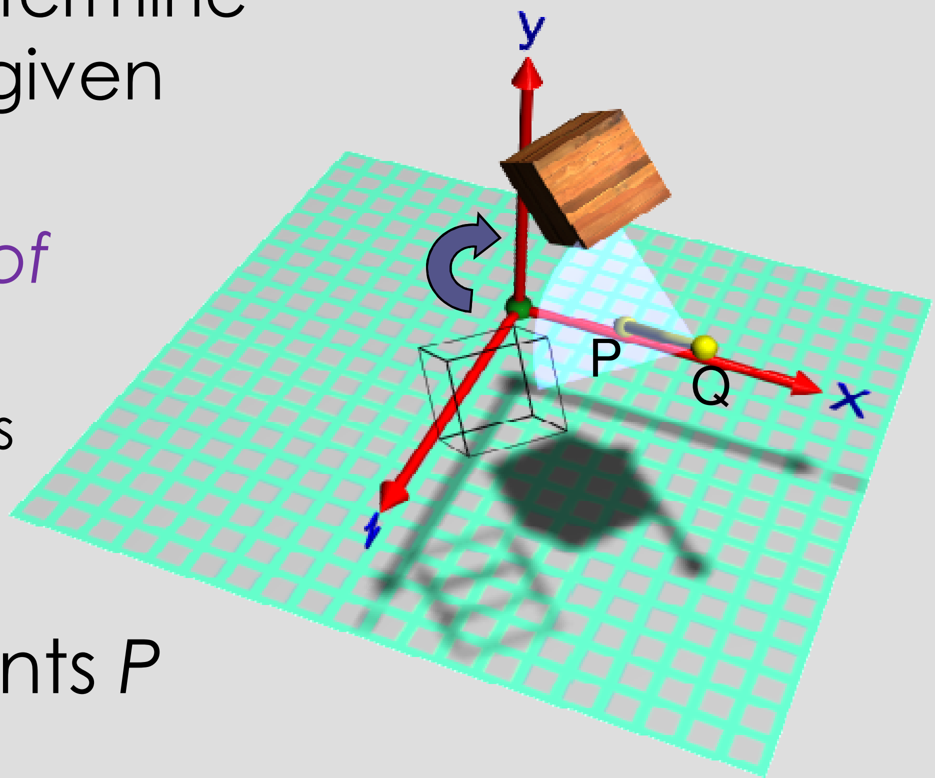
Rotation about an arbitrary axis



Rotation about an arbitrary axis

- A frequent requirement is to determine the matrix for rotation about a given axis.
- Such rotations have 3 *degrees of freedom* (DOF):
  - 2 for spherical angles specifying axis orientation
  - 1 for twist about the rotation axis
- Assume axis is defined by points *P* and Q
- Pivot point is *P* and rotation axis vector is:

$$\mathbf{v} = \frac{P - Q}{|P - Q|}$$



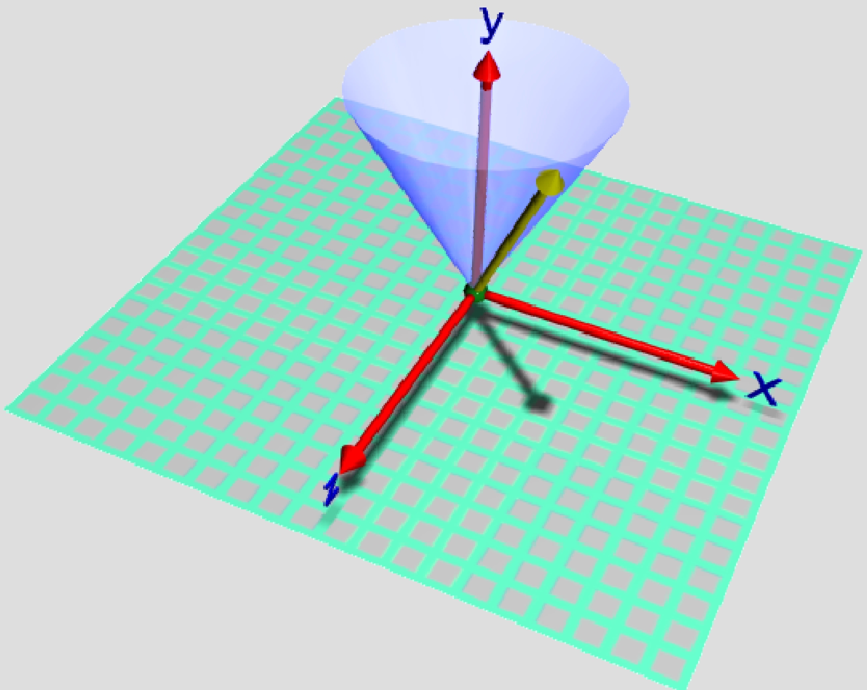
Rotation about an arbitrary axis

1. Translate the pivot point of the axis to the origin $\Rightarrow$ $\mathbf{T}(-P)$
2. Determine a series of rotations to achieve the required rotation about the desired vector.
3. Rotate the axis and object so that the axis lines up with **z** say $\Rightarrow$ $\mathbf{R}(\theta_y)\mathbf{R}(\theta_x)$
4. Rotate about **z** by the required angle $\theta \Rightarrow \mathbf{R}(\theta)$
5. Undo the first 2 rotations to bring us back to the original orientation $\Rightarrow$ $\mathbf{R}(-\theta_x)\mathbf{R}(-\theta_y)$
6. Translate back to the original position $\Rightarrow$ $\mathbf{T}(P)$
7. The final rotation matrix is:

$$\mathbf{M} = \mathbf{T}(P)\mathbf{R}(-\theta_y)\mathbf{R}(-\theta_x)\mathbf{R}(\theta)\mathbf{R}(\theta_x)\mathbf{R}(\theta_y)\mathbf{T}(-P)$$

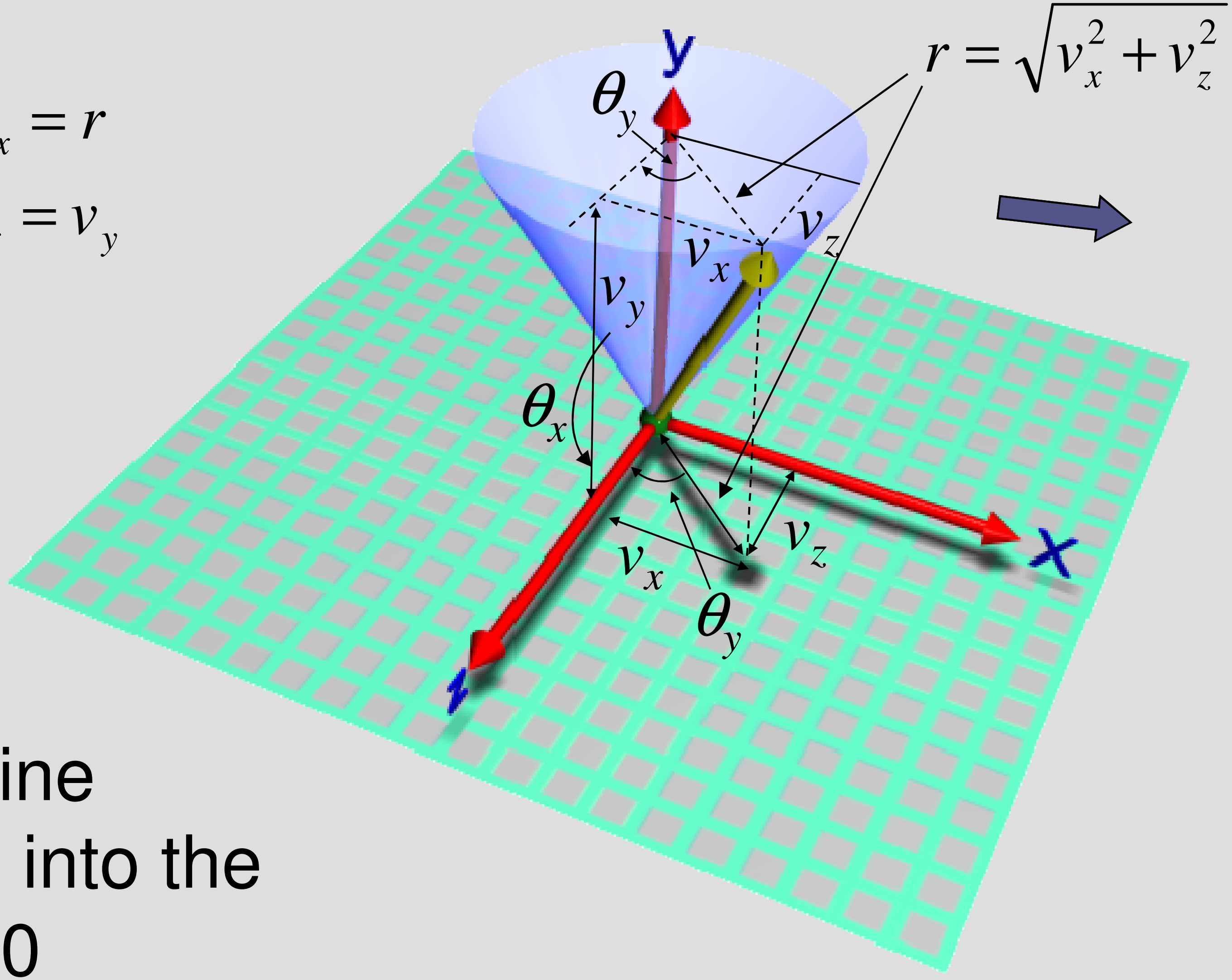
Rotation about an axis

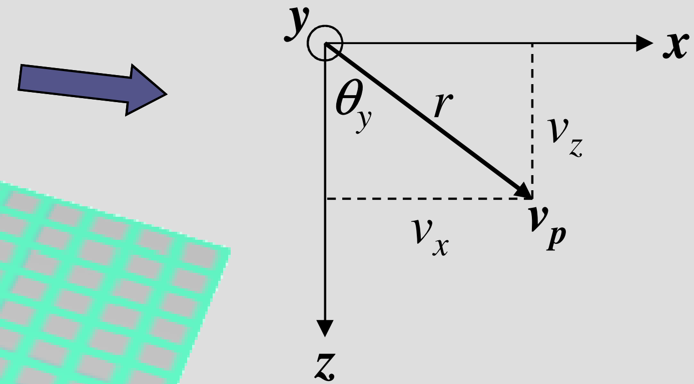
- We need the Euler angles θ_x and θ_y which will orient the rotation axis along the **z** axis.
- We determine these using simple trigonometry.



Aligning axis with z

$$\cos\theta_x = r$$
$$\sin\theta_x = v_y$$





$$\cos\theta_y = \frac{v_z}{r}$$
$$\sin\theta_y = \frac{v_x}{r}$$

-Rotate line segment into the plane y=0

Aligning axis with z

- Note that as shown the rotation about the **x** axis is anti-clockwise but the **y** axis rotation is *clockwise*.
- Therefore the required **y** axis rotation is -$\theta_y \Rightarrow$

$$\mathbf{R}(\theta_x)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & r & -v_y & 0\\ 0 & v_y & r & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\Rightarrow \mathbf{R}(-\theta_x)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & r & v_y & 0\\ 0 & -v_y & r & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\mathbf{R}(\theta_y)=\begin{bmatrix}{v_z}/{r} & 0 & {v_x}/{r} & 0\\ 0 & 1 & 0 & 0\\ -{v_x}/{r} & 0 & {v_z}/{r} & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\Rightarrow \mathbf{R}(-\theta_y)=\begin{bmatrix}{v_z}/{r} & 0 & -{v_x}/{r} & 0\\ 0 & 1 & 0 & 0\\ {v_x}/{r} & 0 & {v_z}/{r} & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\mathbf{M}=\mathbf{T}(P)\mathbf{R}(\theta_y)\mathbf{R}(-\theta_x)\mathbf{R}(\theta)\mathbf{R}(\theta_x)\mathbf{R}(-\theta_y)\mathbf{T}(-P)$$

Recommended Reading

- "Homogeneous Coordinates and Computer Graphics" by Tom Davis
- http://www.geometer.org/mathcircles/cghomogen.pdf
- Interactive Computer Graphics: A Top-down approach with OpenGL, by Edward Angel