ECE - 5984 : Homework 1
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Problem 1

Consider a finite discrete-time Markov chain (DTMC) $\{s_n\}$ taking values in $\{1,2\}$ with transition probability matrix

$$P = \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix},$$

where $P_{ij} = \mathbb{P}(s_{n+1} = j \mid s_n = i)$. Let $\{Y_n\}$ be a different random process defined as

$$Y_n = \begin{cases} s_n, & \text{with probability } 0.7 \\ s_n - 1 & \text{with probability } 0.3 \end{cases}$$

Questions:

1. Find the stationary distribution of P, i.e., find π such that $\pi P = \pi$. (5 points)

- let $\vec{\pi} = [\pi_1, \pi_2]$
- $\vec{\pi} P = \vec{\pi} = [\pi_1, \pi_2] \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix} = [\pi_1, \pi_2]$
- $0.3\pi_1 + 0.6\pi_2 = \pi_1$ $\quad \therefore \; 0.6\pi_2 = 0.7\pi_1$
  $0.7\pi_1 + 0.4\pi_2 = \pi_2$
- as $\pi_1 + \pi_2 = 1$
- $0.6\pi_2 = 0.7(1-\pi_2) \Rightarrow \pi_2 = 7/13, \quad \pi_1 = 6/13$

  $\therefore \; \vec{\pi} = [6/13, 7/13]$

2. Find $\lim_{n\to\infty} P(s_n = 1 \mid Y_n = 1)$. [Hint: Use Bayes rule formula]. (5 points)

By Bayes rule : $P(s_n=1 \mid Y_n=1) = \dfrac{P(Y_n=1 \mid S_n=1)\,P(S_n=1)}{P(Y_n=1)}$

- $P(S_n=1) = \pi_1 = 6/13$
- $P(S_n=2) = \pi_2 = 7/13$ $\Big\}$ (from stationary distribution)
- $P(Y_n=1) = P(S_n=2) \cdot P(Y_n = S_n - 1) + P(S_n=1) \times P(Y = S_n)$
  $= 7/13 \times 0.7 + 6/13 \times 0.3$
  $= 63/130$
- $P(Y_n=1 \mid S_n=1) = P(Y_n = S_n) = 0.7$

  $\therefore \; P(S_n=1 \mid Y_n=1) = \dfrac{0.7 \times \left(\frac{6}{13}\right)}{\left(63/130\right)} = \dfrac{2}{3}.$

Problem 2

We have the following facts

1. Let $\mathcal{S}$ be a bounded set of real numbers, i.e., $\exists D < \infty$ such that $|x| \le D$ for all $x \in \mathcal{S}$. Then there exists $\bar{D} < \infty$ such that

   - $x \le \bar{D}$ for all $x \in \mathcal{S}$
   - Given any $\epsilon > 0$ there exists $y \in \mathcal{S}$ s.t. $y \ge \bar{D} - \epsilon$

   In other words, $\bar{D}$ is the least upper bound or supremum of $\mathcal{S}$. Similar, there exists greatest lower bound or infimum of $\mathcal{S}$.

2. Consider an infinite sequence of real numbers $\{x_n\}_{n=1}^{\infty}$. Then there is a monotone subsequence of $\{x_n\}_{n=1}^{\infty}$, i.e., there exists $\{x_{n_1}, x_{n_2}, \ldots\}$, $n_1 \le n_2 \le \ldots$, that is either non-decreasing or non-increasing.

Questions:

1. Let $\{x_n\}_{n=1}^{\infty}$ be a non-decreasing upper bounded sequence of real numbers. Show that $\lim_{n\to\infty} x_n$ exists and finite. (10 points)

- for some $\varepsilon > 0$, there is a corresponding N : $x_N > D - \varepsilon$
- $\forall$ $n > N$, $x_n > D - \varepsilon$ (as S is non decreasing)
- S is bounded, $\therefore$ $x_n \leqslant D$

$$D - \varepsilon < x_n \leqslant D$$
$$-\varepsilon < x_n - D \leqslant 0$$
$$|x_n - D| < \varepsilon$$

$\therefore$ $\underline{\underline{\lim_{n\to\infty} x_n = D.}}$ (by limit definition)

[Note that the same method can be applied for a non increasing seq to show that it has a finite limit.]

2. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence, i.e., $\exists M < \infty$ such that $|x_n| \le M$. Show that $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequent. (10 points)

- Let the n^{th} term in a sequence be defined as dominant if it is greater then any term following it.
- $\therefore$ there can be either ∞, or finite dominant terms (2 cases)

- Infinite case:
  - Form a subsequence consisting only of dominant terms $\{x_k, x_{k+1} \ldots x_m\}$
  - $x_k > x_{k+1}$ (by the definition of dominant term)
  - $\therefore$ this subsequence is a decreasing monotone.

(continued)

- Finite case:
  - Select n_1 s.t. n_1 is beyond the last dominant term in the sequence.
  - as n_1 is not dominant, there exists some $m > n_1$ s.t. $x_m > x_{n_1}$
  - set $n_2 = m$
  - n_2 still not dominant, repeat for $n_3, n_4 \ldots\ldots$
  - $\{x_{n_1}, x_{n_2}, x_{n_3} \ldots\ldots\}$ is monotonic non decreasing.

$\therefore$
- all bounded sequences have a monotonic subsequence
- Q1 showed that all bounded monotonic sequences converge.
- all bounded sequences have a monotonic subsequence which converges.

3. A sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is Cauchy if given any $\epsilon > 0$ there exists N_ϵ (N depends on ϵ) s.t. $|x_n - x_m| < \epsilon$ for all $n, m > N_\epsilon$. Show that every Cauchy sequence is bounded. (5 points)

- let $\varepsilon > 0$ (set $\varepsilon = 1$ arbitarily)
- $\exists N_\varepsilon$ s.t. $\forall\, n, m > N_\varepsilon$, $|x_n - x_m| < 1$ (cauchy definition)
- Let $m = N+1$
- Then $\forall n > N$:
  - $|x_n| = |x_n - x_m + x_m| \leqslant |x_m - x_n| + |x_m| < 1 + |x_m| = 1 + |x_{N+1}|$ (triangle inequality)
  - $|x_n| < 1 + |x_{N+1}| \quad \forall n > N$
- Let $M = \max\{|x_1|, |x_2| \ldots\ldots |x_N|, |1 + x_{N+1}|\}$

$\therefore$ if $n > N$, $|x_n| < 1 + |x_{n+1}| \leqslant M$

if $n \leqslant N$, $|x_n| \leqslant \max\{|x_1|, |x_2| \ldots |x_n|\} \leqslant M$

$\therefore\ |x_n| \leqslant M \quad \forall n$ ($\{x_n\}$ bounded by M)

4. Show that if a Cauchy sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ has a convergent sub sequence, then the sequence $\{x_n\}_{n=1}^{\infty}$ must converge. (10 points)

- Let $\{x_{n_k}\}$ = the convergent subsequence of $\{x_n\}$.
- Let $\varepsilon > 0$
- $\exists K$ s.t. $\forall k > K$, $|x_{n_k} - c| < \frac{\varepsilon}{2}$ (bounded convergent subseq)
- $\exists N$ s.t. $\forall m, n > N$, $|x_n - x_m| < \frac{\varepsilon}{2}$ (cauchy definition)
- Pick $L > K$ s.t. $n_L > N$

$\therefore$ $\forall n > N$,

$$|x_n - c| = |(x_n - x_{n_L}) + (x_{n_L} - c)| \leqslant |x_n - x_{n_L}| + |x_{n_L} - c| < \varepsilon$$

$\therefore$ $\forall n > N$, $|x_n - c| < \varepsilon$

$\therefore$ $\lim_{n \to \infty} x_n = c$ (overall sequenc converges)

5. Show that every Cauchy sequence of real numbers is convergent. (5 points)

- Problem 2, Q 3 : all cauchy sequences are bounded. (of $\mathbb{R}$ numbers)
- Problem 2, Q 2 : all bounded sequences have a convergent subsequence
- Problem 2, Q 4 : all cauchy sequences with convergent subsequences converge themselves.

$\therefore$ all cauchy sequences are convergent (of $\mathbb{R}$ numbers)

- Note that Problem 2, Q1 assumes $\{x\}$ a sequence of real numbers, which is used in the proof of Problem 2, Q2. $\therefore$ The above has only been proved for sequences of real numbers.

Problem 3

Recall the definition of an MDP from the second lecture. Let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be an MC with transition probability $\mathbf{P}$. X is called a controlled MC if $\mathbf{P}$ can be controlled, i.e., $\mathbf{P} = [P_{ij}(a)]$ where a is a control action. At time k, the state is $s_k \in \mathcal{X}$, we take an action $a_k = \mu_k(s_k)$, and it incurs a (bounded) cost $r(s_k, a_k)$, where w.l.o.g we assume $c \geq 0$. Here μ_k is a mapping from state to action. The goal is to choose $\{a_k\}$ to maximize

$$V_\pi(i) = \lim_{N\to\infty} \mathbb{E}\left[\sum_{k=0}^{N} \gamma^k r(s_k, a_k) \mid s_0 = i\right],$$

where $\gamma \in (0,1)$ is called the discount factor and $\pi = [\mu_0, \mu_1, \ldots]$ is the policy. When μ_k does not depend on time k, i.e., $\mu_k = \mu$, we call the policy is stationary and with some abuse of notation denote it as μ.

Policy evaluation Let consider a subproblem, where we want to estimate the vector value function V_μ for a given stationary policy μ. We know from class that V_μ satisfies the so-called Bellman equation

$$V_\mu(i) = \mathbb{E}[r(i, \mu(i))] + \gamma \sum_j P_{ij}(\mu(i)) V_\mu(j),$$

or in vector form

$$V_\mu = \mathbb{E}[r] + \gamma \mathbf{P}_\mu V_\mu,$$

where $r = [r(i, \mu(i))]$ is a vector. In class, we have a theorem to show the existence and uniqueness of the solution of this Bellman equation. We mentioned that there are two ways to do it: using *algebra* or the *classic fixed point theorem*.

Questions of the algebra proof:

1. Consider matrix norm induced by the vector norm defined in class, i.e.,

$$\|\mathbf{P}\|_p = \max_{\|y\|_p=1} \|\mathbf{P}y\|_p.$$

Let λ_i be the eigenvalues of $\mathbf{P}$. Show that (10 points)

$$\max_i |\lambda_i| \leq \|\mathbf{P}\|_p, \qquad \forall p \geq 1.$$

<u>**Hint**</u>: Using the definition of the eigenvalues of a matrix.

- Let λ be an eigen values of P, and let $x \neq 0$ be a corresponding eigenvector.
- $\therefore$ $PX = \lambda X$ (where $X := [x | \ldots | x]$)
- $\Rightarrow$ $|\lambda| \, \|X\|_p = \|\lambda X\|_p = \|PX\|_p \leqslant \|P\|_p \|X\|_p$.
- $\Rightarrow$ $|\lambda| \, \|X\|_p \leqslant \|P\|_p \|X\|_p$
- $\Rightarrow$ $|\lambda| \leqslant \|P\|_p$ (as $\|X\|_p > 0$)

- taking the maximum eignvalue λ:

$$\max_i |\lambda_i| \leqslant \|P\|_p \quad (\forall p \geqslant 1)$$

note: ($p \geqslant 1$, as for $0 \leqslant p < 1$ the resulting function does not define a norm, as the triangle inequality is violated)

1. Let T be a continuous mapping from $\mathcal{S} \to \mathcal{S}$ where $\mathcal{S}$ is a closed set. Suppose that T satisfies a contraction property, i.e., $\exists\, \gamma \in (0,1)$ such that

$$\|T(x) - T(y)\| \leq \gamma \|x - y\|,$$

$\|\cdot\|$ can be any norm. Show that

(a) There **exists** a **unique** x^* s.t. $T(x^*) = x^*$ (10 points)

(b) The fixed point iteration starting with x_0 (10 points)

$$x_{k+1} = T(x_k),$$

converges to x^*, i.e., $\lim_{k\to\infty} x_k = x^*$.

Hint: In both questions a and b, first show that x_k is a Cauchy sequence. And then use results in Problem 2 and the fact that T is continuous.

• Showing that x_k is Cauchy:

• let $d(a,b) = \|a-b\|_p$ (an arbitary p norm)

• define $\{x_k\} := x_{k+1} = T(x_k)$ (generating a sequence based on T)

(1) • $d(x_{k+1}, x_k) = d(T(x_k), T(x_{k-1})) \leqslant \gamma\, d(x_k, x_{k-1})$ (by contraction definition)

(2) $\therefore\ d(x_{k+1}, x_k) \leqslant \gamma^n d(x_1, x_0),\ \forall k \geqslant 1$ (by induction)

• for $m > k \geqslant 1$:

(3) $d(x_m - x_k) \leqslant d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \dots + d(x_{k+1}, x_k)$ (by triangle inequality)

$\leqslant (\gamma^{m-1} + \gamma^{m-2} \dots + \gamma^k)\, d(x_1, x_0)$ (using equation (2))

• $(\gamma^{m-1} + \gamma^{m-2} + \gamma^{m-3} \dots \gamma^k) = \gamma^k(\gamma^{m-k-1} + \gamma^{m-k-2} \dots + 1)$

$= \gamma^k(1 + \gamma^2 + \gamma^3 \dots \gamma^{m-k-1})$

Simplifying via geometric sequence:

(4) $\Rightarrow \gamma^k(1+\gamma^2+\gamma^3 \dots \gamma^{m-k-1}) \leqslant \dfrac{\gamma^k}{1-\gamma}$ (as finite, and geometric series definition $1 + x^2 + x^3 \dots = \frac{1}{1-x}$)

(5) $\therefore\ d(x_m - x_k) \leqslant \dfrac{\gamma^k}{1-\gamma}\, d(a_1, a_0)$ (from equation (3) and (4))

$\therefore\ d(x_m - x_k) \to 0$ as $m, k \to \infty$

$\therefore\ \{x_k\}$ is cauchy (as above satisfies cauchy definition)

a) • Problem 2, Question 5 states that all cauchy sequences of real numbers converge, to a limit of x^*

$\therefore \{x_k\}$ converges to a point x^*

Showing x^* is a fixed point

$$x^* = \lim_{k\to\infty} x_k = \lim_{k\to\infty} T(x_{k-1}) = T\left(\lim_{k\to\infty}(x_{k-1})\right) = T(x^*)$$

• (note that the limit can be moved inside T, as it is continuous)

Uniqueness of x^*

• let x^* be a fixed point of the cauchy sequence $\{x_k\}$

• if y^* is another fixed point,

$$\Rightarrow |x^* - y^*| = |T(x^*) - T(y^*)| \leq \gamma |x^* - y^*|$$

• as $\gamma \in (0,1)$, $|x^* - y^*| = 0 \rightarrow x^* = y^*$

• $\therefore x^*$ is the unique fixed point

b) • The fixed point iteration $(x_{k+1} = T(x_k))$ is used inductively to generate terms in the sequence $\{x_k\}$, which was proved to be cauchy above.

• Cauchy sequences of real numbers converge (as per Problem 2, Question 5)

• $\therefore$ cauchy seq $\{x_k\}$ converges to x^*

Showing fixed point iteration $\rightarrow x^*$

• for the iteration: $x_k = T(x_{k-1})$

$$\lim_{k\to\infty}(x_k) = \lim_{k\to\infty}(T(x_{k-1}))$$ (limits of both sides of iteration)

$$= T\left(\lim_{k\to\infty}(x_{k-1})\right)$$ (as T is continuous, can bring in the limit)

$$= T(x^*)$$ (by cauchy seq converging to x^* noted above)

$$= x^*$$

$$\therefore \lim_{k\to\infty}(x_k) = x^*$$

(Shows that the iteration will eventually converge to a fixed point (x^*))

2. Next let T be the right-hand side of the Bellman equation, i.e. for all i

$$(TV_\mu)(i) = \mathbb{E}[r(i,\mu(i))] + \gamma \sum_{j\in S} P_{ij}(\mu(i))V_\mu(j).$$

(a) Given any V, V' such that $V(i) \le V'(i)$ for all i. Show that the following **Monotonicity property** holds (5 points)

$$(TV)(i) \le (TV')(i).$$

- $(TV)(i) - (TV')(i) = \mathbb{E}(r(i,\mu(i))) + \gamma \sum_{j\in S} P_{ij}(\mu(i))V(j)$ (by T definition)

$$- (\mathbb{E}(r(i,\mu(i))) + \gamma \sum_{j\in S} P_{ij}(\mu(i))V'(j))$$

$$= \gamma \sum_{j\in S} P_{ij}(\mu(i))V(j) - \gamma \sum_{j\in S} P_{ij}(\mu(i))V'(j)$$

$$\le \gamma \sum_{j\in S} P_{ij}(\mu(i))V(j) - \gamma \sum_{j\in S} P_{ij}(\mu(i))V(j) \quad (\text{as } V<V')$$

$$= 0$$

$$\Rightarrow (TV)(i) - (TV')(i) \le 0$$

$$\Rightarrow \underline{\underline{(TV)(i) \le (TV')(i)}}$$

(b) Let q be a scalar. Show that (5 points)

$$(T(V+q))(i) = (TV)(i) + \gamma q.$$

$$T(V+q)(i) = \mathbb{E}\, r(i,\mu(i)) + \gamma \sum_{j\in S} P_{ij}(\mu(i))\,(V+q)$$

$$= \mathbb{E}\, r(i,\mu(i)) + \gamma \sum_{j\in S} P_{ij}(\mu(i))V + \gamma \sum_{j\in S} P_{ij}(\mu(i))q$$

$$= T(V)(i) + \gamma \sum_{j\in S} P_{ij}(\mu(i))q$$

$$= \underline{\underline{T(V)(i) + \gamma q}} \qquad (\text{as } \sum_{j\in S} P_{ij}(\mu(i)) = 1)$$

(c) Using the two properties above show that T is contractive under maximum-norm, i.e., for all V, V' (10 points)

$$\|TV - TV'\|_\infty \leq \gamma \|V - V'\|_\infty$$

Hint: Note that here the contraction only holds for the maximum norm. Then the first step is to consider $d = \max_i |V(i) - V'(i)|$. Recall that we consider finite-time MC, i.e., the set of states i is finite. Thus, d is well-defined.

- $T_\mu V_\mu(s_i) = \mathbb{E}(r(i, \mu(i))) + \gamma \sum_{j \in S} P_{ij}(\mu(i)) V_\mu(j)$ (RHS Bellman eq)

- Let $c = \max_{s \in S} |V_1(s) - V_2(s)|$

- $T_\mu V_1(s) - T_\mu V_2(s) = \bar{r}(s) + \gamma \sum_{s' \in S} P_{ss'}(\mu(s)) V_1(s') - \left[\bar{r}(s) + \gamma \sum_{s' \in S} P_{ss'}(\mu(s)) V_2(s')\right]$

$= \gamma \sum_{s' \in S} P_{ss'}(\mu(s)) \left[V_1(s') - V_2(s')\right]$

$\leq \gamma \sum_{s' \in S} P_{ss'}(\mu(s)) \left|V_1(s') - V_2(s')\right|$

$\leq \gamma c \sum_{s' \in S} P_{ss'}(\mu(s))$ (by definition of c above)

$= \gamma c$ (as $\sum_{s' \in S} P_{ss'}(\mu(s)) = 1$)

$\Rightarrow |T_\mu V_1(s) - T_\mu T_2(s)| \leq \gamma c$

$\Rightarrow \max_s |T_\mu V_1(s) - T_\mu V_2(s)| \leq \gamma \max_s |V_1(s) - V_2(s)|$

$\Rightarrow \|T_\mu V_1(s) - T_\mu V_2(s)\|_\infty \leq \gamma \|V_1(s) - V_2(s)\|_\infty$