ECE - 5984 : Homework 2
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Problem 1

We consider an application of MDP, where there is a car and its state i can be $0, 1, 2, \ldots$. Here the state could be an indicator of how good the car is. The state depends on various factors such as the days of operation, age of the car, etc. Everyday, we want to make a decision whether to bring the car to mechanic shops or not based on its observed states. If we decide to bring it, the car is repaired instantaneously and the state of the car returns to 0. Repairing the car (e.g., tune up) incurs a cost R, while maintaining it causes a cost $C(i)$ per day if the car is in state $i = 0, 1, \ldots$ Moreover, $C(i), i \geq 0$, is assumed to be an increasing bounded function in i, i.e., higher maintenance costs are associated with higher state indices. Given the current state i, let P_{ij} be the transition probability that the car will be at state j the next day.

Question (10 points): Let $\gamma \in (0, 1)$ and π is a policy. Write the Markov decision model for this problem, i.e., define the state and action space, the instantaneous reward $r(s, a)$, and the transition probability matrix. Also, formulate the optimization problem to find the best policy. Note that in this case we want to minimize the discounted cumulative cost.

Definitions:

- State space $= S = \{S(i)\} = \{S(0), S(1) \ldots\}$
- Actions $= A = \{\text{repair}, \text{maintain}\}$ (policy $\pi = \{\mu_i\}$, $a_i = \mu_i(S_i)$)
- Reward $= r(s(i), a)$

$$r(s(i),a) = \begin{cases} R & \text{if } a = \text{repair} \\ C(i) & \text{if } a = \text{maintain} \end{cases}$$

(for an unspecified maintenance cost function $C(i)$)

- Transition probability $= P^a_{ss'}$ (note that the reward is contingent on the action $a \in A$ taken)

$$\Rightarrow P^a_{ss'} = \begin{cases} P^{\text{repair}}_{ss'} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} & \text{if } a = \text{repair} \\ P^{\text{maintain}}_{ss'} = \begin{bmatrix} P^m_{00} & P^m_{01} & \cdots \\ P^m_{10} & P^m_{11} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} & \text{if } a = \text{maintain} \end{cases}$$

- where P^m_{01} = probability of transitioning from state $i=0$ to $i=1$, given action = maintain (m)

- Note that the reward is bounded, as $C(i)$ for $i \geq 0$ is increasing and bounded as given in the problem statement, and R (repair cost) is constant

$$\therefore \quad M = \max\{R, C(0), C(1), C(2) \ldots\}$$

$$\Rightarrow \text{reward} \leq M$$

· Optimization problem:

· The Bellman optimality equation defines the equation for the optimal policy π^* and corresponding value function V_{π^*}

$$V_{\pi^*}(s) = \max_{\mu(s)} \left(\mathbb{E}\left[r(s, \mu(s) \right] + \gamma \sum_{s' \in S} P_{ss'}(\mu(s)) \, V_{\pi}^*(s') \right), \quad s \in S$$

· similarly

$$\mu^*(s) = \arg\max_{\mu(s)} \left(\mathbb{E}\left[r(s, \mu(s) \right] + \gamma \sum_{s' \in S} P_{ss'}(\mu(s)) \, V_{\pi}^*(s') \right), \quad s \in S$$

with $\pi^* = \{\mu^*(s)\}, \; s \in S$

Problem 2

We recall here the policy iteration (PI) algorithm for the discounted MDP with discount factor γ. There are two main steps as follows.

1. (**Policy evaluation**) Given the current policy μ_k, estimate V_{μ_k}

$$(\mathbf{I} - \gamma \mathbf{P}_{\mu_k}) V_{\mu_k} = \mathbb{E}[r],$$

   or equivalently $V_{\mu_k} = T_{\mu_k} V_{\mu_k}$, where r is the vector of rewards.

2. (**Policy improvement**) Obtain a new improved policy μ_{k+1}

$$T_{\mu_{k+1}} V_{\mu_k} = T V_{\mu_k}$$

Implementing these two steps exactly in many applications are expensive. We consider here an approximation of these two steps. In particular, given the current policy μ_k, the policy evaluation step only returns a δ-approximate of the value function V_{μ_k}, i.e., a vector V_k satisfies

$$\|V_k - V_{\mu_k}\| \leq \delta, \qquad \forall k.$$

In addition, using this value V_k the policy evaluation problem can only compute an ϵ-approximate of the mapping T, i.e., a new policy μ_{k+1} satisfies

$$\|T_{\mu_{k+1}} V_k - T V_k\| \leq \epsilon, \qquad \forall k.$$

In the sequel let $\mathbf{1}$ be the vector whose entries are 1. Given a policy μ_k and its value function V_{μ_k}, we have $V_{\mu_k} = T_{\mu_k} V_{\mu_k}$ where T_{μ_k} is given as

$$(T_{\mu_k} V_{\mu_k})(s) = \mathbb{E}[r(s, \mu_k(s))] + \gamma \sum_{s' \in \mathcal{S}} P_{ss'}(\mu_k(s)) V_{\mu_k}(s').$$

In addition, given a scalar c we have T_{μ_k} satisfies

$$T_{\mu_k}(V_{\mu_k} + c\mathbf{1}) = T_{\mu_k} V_{\mu_k} + \gamma c \mathbf{1}.$$

1. (10 points) Show that

$$\|T_{\mu_{k+1}}V_{\mu_k} - T_{\mu_{k+1}}V_k\| \leq \gamma\delta \qquad \text{and} \qquad \|TV_k - TV_{\mu_k}\| \leq \gamma\delta$$

$\|V_k - V_{\mu_k}\|_\infty \leqslant \delta \quad \forall k$

$-\delta 1 \leqslant V_k - V_{\mu_k} \leqslant \delta 1$

$\Rightarrow \quad V_{\mu_k} - \delta 1 \leqslant V_k \leqslant V_{\mu_k} + \delta 1$

$= \quad T_{\mu_{k+1}}(V_{\mu_k} - \delta 1) \leqslant T_{\mu_{k+1}} V_k \leqslant T_{\mu_{k+1}}(V_{\mu_k} + \delta 1)$

$\Rightarrow \quad T_{\mu_{k+1}} V_{\mu_k} - \gamma\delta 1 \leqslant T_{\mu_{k+1}} V_k \leqslant T_{\mu_{k+1}} V_{\mu_k} + \gamma \delta 1$

$\Rightarrow \quad -\gamma\delta 1 \leqslant T_{\mu_{k+1}} V_k - T_{\mu_{k+1}} V_{\mu_k} \leqslant +\gamma\delta 1$

$\Rightarrow \quad \|T_{\mu_{k+1}} V_{\mu_k} - T_{\mu_{k+1}} V_k\|_\infty < \gamma\delta$

$\|V_k - V_{\mu_k}\|_\infty \leqslant \delta \quad \forall k$

$-\delta 1 \leqslant V_k - V_{\mu_k} \leqslant \delta 1$

$\Rightarrow \quad V_{\mu_k} - \delta 1 \leqslant V_k \leqslant V_{\mu_k} + \delta 1$

$= \quad T(V_{\mu_k} - \delta 1) \leqslant TV_k \leqslant T(V_{\mu_k} + \delta 1)$

$\Rightarrow \quad TV_{\mu_k} - \gamma\delta 1 \leqslant TV_k \leqslant TV_{\mu_k} + \gamma\delta 1$

$\Rightarrow \quad -\gamma\delta 1 \leqslant TV_k - TV_{\mu_k} \leqslant +\gamma\delta 1$

$\Rightarrow \quad \|TV_k - TV_{\mu_k}\|_\infty \leqslant \gamma\delta$

2. (10 points) Show that

$$T_{\mu_{k+1}}V_{\mu_k} \geq TV_{\mu_k} - (\epsilon + 2\gamma\delta)\mathbf{1}$$

- $TV_k - T_{\mu_{k+1}} V_k \leqslant \varepsilon 1$ (from error definition given)

$\Rightarrow \quad 0 \leqslant T_{\mu_{k+1}} V_k - TV_k + \varepsilon 1$

$\Rightarrow \quad (TV_{\mu_k} - T_{\mu_{k+1}} V_{\mu_k}) \leqslant (TV_{\mu_k} - T_{\mu_{k+1}} V_{\mu_k}) + \overbrace{(T_{\mu_{k+1}} V_k - TV_k + \varepsilon 1)}^{\geqslant 0}$

$= (T_{\mu_{k+1}} V_k - T_{\mu_{k+1}} V_{\mu_k}) + (TV_{\mu_k} - TV_k) + \varepsilon 1$

$\leqslant \gamma \|V_k - V_{\mu_k}\|_\infty 1 + \gamma \|V_k - V_{\mu_k}\|_\infty 1 + \varepsilon 1$

$\leqslant \gamma\delta 1 + \gamma\delta 1 + \varepsilon 1$ (by error definition given)

$= (\varepsilon + 2\gamma\delta) 1$

$\therefore \quad TV_{\mu_k} - T_{\mu_{k+1}} V_{\mu_k} \leqslant (\varepsilon + 2\gamma\delta) 1$

$\Rightarrow \quad T_{\mu_{k+1}} V_{\mu_k} \geqslant TV_{\mu_k} - (\varepsilon + 2\gamma\delta) 1$

3. (10 points) Show that

$$T_{\mu_{k+1}} V_{\mu_k} \geq V_{\mu_k} - (\epsilon + 2\gamma\delta)\mathbf{1}.$$

- $TV_k - T_{\mu_{k+1}} V_k \leq \varepsilon 1$ (from error definition given)

$\Rightarrow \quad 0 \leq T_{\mu_{k+1}} V_k - TV_k + \varepsilon 1$

$\leq T_{\mu_{k+1}} V_k - T_{\mu_k} V_k + \varepsilon 1$ (as $TV_k \geq T_{\mu_k} V_k$, as T finds the maximizing policy (π^*))

$$\Rightarrow (V_{\mu_k} - T_{\mu_{k+1}} V_{\mu_k}) \leq (V_{\mu_k} - T_{\mu_{k+1}} V_{\mu_k}) + \overbrace{(T_{\mu_{k+1}} V_k - T_{\mu_k} V_k + \varepsilon 1)}^{\geq 0}$$

$$= (T_{\mu_{k+1}} V_k - T_{\mu_{k+1}} V_{\mu_k}) + (V_{\mu_k} - T_{\mu_k} V_k) + \varepsilon 1$$

$$= (T_{\mu_{k+1}} V_k - T_{\mu_{k+1}} V_{\mu_k}) + (T_{\mu_k} V_{\mu_k} - T_{\mu_k} V_k) + \varepsilon 1 \quad (\text{as } T_{\mu_k} V_{\mu_k} = V_{\mu_k})$$

$$\leq \gamma \| V_k - V_{\mu_k} \|_\infty 1 + \gamma \| V_k - V_{\mu_k} \|_\infty 1 + \varepsilon 1$$

$$\leq \gamma \delta 1 + \gamma \delta 1 + \varepsilon 1 \quad (\text{by error definition given})$$

$$= (\varepsilon + 2\gamma\delta) 1$$

$$\therefore \quad V_{\mu_k} - T_{\mu_{k+1}} V_{\mu_k} \leq (\varepsilon + 2\gamma\delta) 1$$

$$\Rightarrow \quad \underline{\underline{T_{\mu_{k+1}} V_{\mu_k} \geq V_{\mu_k} - (\varepsilon + 2\gamma\delta) 1}}$$

4. (10 points) Show that given V_k if $T_{\mu_k} V_k \leq V_k + r\mathbf{1}$, then

$$V_{\mu_k} \leq V_k + \frac{r}{1-\gamma}\mathbf{1}.$$

Note that $T^0 V_k = V_k$ where $T^0 = \mathbf{I}$, an indentity matrix.

• For some V_k s.t. $T_{\mu_k} V_k \leqslant V_k + r\mathbb{1}$

$$\Rightarrow T_{\mu_k}^2 V_k = T_{\mu_k}(T_{\mu_k} V_k) \leqslant T_{\mu_k}(V_k + r\mathbb{1}) = T_{\mu_k} V_k + \gamma r \mathbb{1}$$

$$\Rightarrow T_{\mu_k}^3 V_k \leqslant T_{\mu_k}^2 V_k + \gamma^2 r \mathbb{1}$$

$$\vdots$$

$$\Rightarrow T_{\mu_k}^L V_k \leqslant T_{\mu_k}^{L-1} V_k + \gamma^{L-1} r \mathbb{1}$$

$$\Rightarrow T_{\mu_k}^L V_k - T_{\mu_k}^{L-1} V_k \leqslant \gamma^{L-1} r \mathbb{1}$$

$\therefore$ note that :

$$T_{\mu_k}^m V_k - V_k = \sum_{L=1}^{m} \left(T_\mu^L V_k - T_\mu^{L-1} V_k\right) \leqslant \sum_{L=1}^{m} \gamma^{L-1} r \mathbb{1}$$

letting $m \to \infty$

• $$\lim_{m\to\infty}\left(T_{\mu_k}^m V_k - V_k\right) = V_{\mu_k} - V_k$$

• $$\lim_{m\to\infty}\left(\sum_{L=1}^{m} \gamma^{L-1} r \mathbb{1}\right) = \frac{r\mathbb{1}}{1-\gamma}$$

$$\therefore \quad V_{\mu_k} - V_k \leqslant \frac{r}{1-\gamma}\mathbb{1}$$

or $$V_{\mu_k} \leqslant V_k + \frac{r}{1-\gamma}\mathbb{1}$$

Problem 3

We consider here the deterministic version of almost supermartingale convergence theorem study in the class. In particular, let $\{y_k\}$, $\{z_k\}$, and $\{w_k\}$ be nonnegative sequence satisfying

$$y_{k+1} \leq y_k - z_k + w_k,$$

where w_k satisfies

$$\sum_{k=0}^{\infty} w_k < \infty.$$

Show that

1. y_k converges. [Hint: Show that the sequence $V_k = y_k + \sum_{t=k}^{\infty} w_t$ is nonincreasing] [15 points]

Defining V_k and showing that it is nonincreasing.

- let $V_k = y_k + \sum_{t=k}^{\infty} w_t$
- as $y_{k+1} \leq y_k - z_k + w_k$ (given equation)

$\Rightarrow$ $y_{k+1} - y_k + z_k - w_k \leq 0$

$\Rightarrow$ $y_{k+1} - y_k - w_k \leq 0$ (as z_k is non negative)

$\therefore$ $V_{k+1} - V_k = y_{k+1} - y_k - w_k \leq 0$

$\therefore$ V_k is non increasing for all k.

Showing that V_k converges via monotone convergence theorem

- additionally V_k bounded below by 0 as:
  - y_k is non negative.
  - $\sum_{t=k}^{\infty} w_t$ is non negative, as w_t is non negative.

$\therefore$ $V_k \geq 0$.

- as V_k is non increasing, and bounded below, by the monotone convergence theorem V_k must converge.

Inspecting terms of V_k for convergence.

$$V_k = y_k + \sum_{t=k}^{\infty} w_t$$

- V_k converges
- $\sum_{t=k}^{\infty} w_t$ is non negative and upper bounded (as $< \infty$) $\therefore$ converges. by monoton convergence theorm.
- y_k is non negative.

$\therefore$ for V_k to converge as shown, y_k must also converge.

$\therefore$ y_k converges.

2. $\lim_{k\to+\infty} w_k = 0$ [15 points]

• Showing convergence of $\sum_{k=0}^{\infty} w_k$:

• as $\sum_{k=0}^{\infty} w_k < \infty$ (bounded above) and w_k is non negative ($\therefore \sum_{k=0}^{\infty} w_k$ is non decreasing)

$\Rightarrow \sum_{k=0}^{\infty} w_k$ must converge to a finite number by the monotone convergence theorem.

• Partial sums:

$$S_{k-1} = \sum_{i=0}^{k-1} w_i = w_0 + w_1 + w_2 \ldots\ldots + w_{k-1}$$

$$S_k = \sum_{i=0}^{k} w_i = w_0 + w_1 + w_2 \ldots\ldots + w_k.$$

$$\Rightarrow w_k = S_k - S_{k-1}$$

• Letting $k \to \infty$:

• as $\sum_{k=0}^{\infty} w_k$ is convergent, $\{S_k\}_{k=0}^{\infty}$ is also convergent.

• let $\lim_{k\to\infty} S_k = S$

• Since $k-1 \to \infty$ as $k \to \infty$, $\lim_{k\to\infty} S_{k-1} = S$.

$$\therefore \lim_{k\to\infty} (w_k) = \lim_{k\to\infty} (S_k - S_{k-1}) = \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} = S - S = 0$$

$$\therefore \lim_{k\to\infty} (w_k) = 0$$

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Problem 4

In this question, you are asked to prove the almost supermartingale convergence theorem. The question is optional, however, if you work on it you'll have extra points. We first introduce the notion of martingale.

Given an increasing sequence of σ-field F_k, a sequence $\{X_k\}$ is said to be adapted to F_k if $X_k \in F_k$. $\{X_k\}$ is said to be a martingale if

1. $E|X_k| < \infty$
2. X_k is adapted to F_k
3. $\mathbb{E}[X_{k+1} \,|\, F_k] = X_k$ for all k

In (3) if $\mathbb{E}[X_{k+1} \,|\, F_k] \leq X_k$ for all k then X_k is said to be a supermartingale. The supermartingale theorem says that if $X_k \geq 0$ and X_k is a supermartingale then

$$\lim_{k\to\infty} X_k = X \text{ a.s.},$$

for some random variable X with $\mathbb{E}[X] \leq \mathbb{E}[X_0]$. Using this result we now prove the "almost" supermartingale theorem given in class. Let $\{y_k\}$, $\{z_k\}$, $\{w_k\}$ and $\{\gamma_k\}$ be nonnegative sequences of random variables satisfying

$$\mathbb{E}[y_{k+1} \,|\, \mathcal{F}_k] \leq (1+\gamma_k)y_k - z_k + w_k,$$

where γ_k and w_k satisfy

$$\sum_{k=0}^{\infty} \gamma_k < \infty \qquad \text{and} \qquad \sum_{k=0}^{\infty} w_k < \infty.$$

Here $y_k \in \mathcal{F}_k$ for all k and "almost" because of the extra term $\gamma_k y_k$.

Question (20 points): Show that $\{y_k\}$ is convergent.

Hint: Consider

$$P_k = \prod_{t=k}^{\infty}(1+\gamma_t)y_k + \sum_{t=k}^{\infty} w_t \prod_{\ell=t+1}^{\infty}(1+\gamma_\ell).$$

Note that since $\sum_k \gamma_k < \infty$, we have $\prod_{\ell=k}^{\infty}(1+\gamma_\ell)$ converges. Show that P_k is a supermartingale.

Defining P_k and showing $\mathbb{E}(P_{k+1} \,|\, F_k) \leq P_k \ \forall k.$

Let $\quad P_k = \prod_{t=k}^{\infty}(1+\gamma_t)\, y_k + \sum_{t=k}^{\infty} w_t \prod_{\ell=t+1}^{\infty}(1+\gamma_\ell)$

$$\rightarrow \quad P_{k+1} - P_k = \prod_{t=k+1}^{\infty}(1+\gamma_t)\, y_{k+1} - \prod_{t=k}^{\infty}(1+\gamma_t)\, y_k + \sum_{t=k+1}^{\infty} w_t \prod_{\ell=t+1}^{\infty}(1+\gamma_\ell) - \sum_{t=k}^{\infty} w_t \prod_{\ell=t+1}^{\infty}(1+\gamma_\ell)$$

$$\prod_{t=k+1}^{\infty}(1+\gamma_t)\, y_{k+1} - \prod_{t=k}^{\infty}(1+\gamma_t)\, y_k = y_{k+1}(1+\gamma_{k+1})(1+\gamma_{k+2})(1+\gamma_{k+3})\ldots\ldots - y_k(1+\gamma_k)(1+\gamma_{k+1})(1+\gamma_{k+2})\ldots\ldots$$

$$= \prod_{t=k+1}^{\infty}(1+\gamma_t)\,\big(y_{k+1} - (1+\gamma_k)\,y_k\big)$$

$$\sum_{t=k+1}^{\infty} \omega_t \prod_{\ell=t+1}^{\infty} (1+\gamma_\ell) - \sum_{t=k}^{\infty} \omega_t \prod_{\ell=t+1}^{\infty} (1+\gamma_\ell) = \Big(\omega_{k+1} (1+\gamma_{k+2})(1+\gamma_{k+3}) \cdots$$
$$+ \omega_{k+2} (1+\gamma_{k+3})(1+\gamma_{k+4}) \cdots$$
$$+ \omega_{k+3} (1+\gamma_{k+4})(1+\gamma_{k+5}) \cdots$$
$$+ \cdots \Big)$$
$$- \Big(\omega_k (1+\gamma_{k+1})(1+\gamma_{k+2}) \cdots$$
$$+ \omega_{k+1} (1+\gamma_{k+2})(1+\gamma_{k+3}) \cdots$$
$$+ \omega_{k+2} (1+\gamma_{k+3})(1+\gamma_{k+4}) \cdots$$
$$+ \cdots \Big)$$

$$= -\omega_k (1+\gamma_{k+1})(1+\gamma_{k+2}) \cdots$$
$$= -\omega_k \prod_{t=k+1}^{\infty} (1+\gamma_t)$$

$$\therefore P_{k+1} - P_k = \prod_{t=k+1}^{\infty} (1+\gamma_t) \left(y_{k+1} - (1+\gamma_k) y_k - \omega_k \right)$$

- Since $\sum_{k}^{\infty} \gamma_k < \infty$, $\prod_{t=k+1}^{\infty} (1+\gamma_t)$ converges (as per hint) and is positive as γ_k non negative $\forall k$.

- $$\mathbb{E}(y_{k+1} | F_k) \leq (1+\gamma_k) y_k - z_k + \omega_k$$
$$\leq (1+\gamma_k) y_k + \omega_k \qquad (\text{as } z_k \text{ non negative } \forall k)$$

$$\Rightarrow \mathbb{E}(y_{k+1} | F_k) - (1+\gamma_k) y_k - \omega_k \leq 0$$

$$\therefore \mathbb{E}(P_{k+1} - P_k | F_k) = \underbrace{\prod_{t=k+1}^{\infty} (1+\gamma_t)}_{\text{converges and is positive.}} \Big[\underbrace{\mathbb{E}(y_{k+1} | F_k) - (1+\gamma_k) y_k - \omega_k}_{\leq 0} \Big]$$

$$\leq 0$$

Confirming 3 listed criteria of supermartingale. for P_k

- P_k is therefore a supermartingale as:

  1) $P_k = \prod_{t=k}^{\infty}(1+\gamma_t)\,y_k + \sum_{t=k}^{\infty} w_t \prod_{\ell=t+1}^{\infty}(1+\gamma_\ell)$

     - $\prod_{t=k}^{\infty}(1+\gamma_t)$ converges and is positive as shown above.
     - y_k non negative
     - $\sum_{t=k}^{\infty} w_t < \infty$ .. bounded above, and non decreasing $\therefore$ converges.
     - $\prod_{\ell=k+1}^{\infty}(1+\gamma_\ell)$ converges as $\prod_{t=k}^{\infty}(1+\gamma_k)$ converges.
     - $\mathbb{E}(P_{k+1} - P_k \mid F_k) \leq 0$

     $\therefore \quad E|P_k| < \infty$ (condition #1)

  2) P_k is adapted to F_k (condition #2)

  3) $\mathbb{E}(P_{k+1} \mid F_k) \leq P_k$ (as show above) (condition #3)

- additionally note that P_k is bounded below by 0, as y_k, w_k and $\prod_{t=k}^{\infty}(1+\gamma_t)$ are non negative. $\therefore$ $P_k \geq 0$.

$\therefore$ as $P_k \geq 0$ and P_k is a supermartingale, we can apply the supermartingale convergence theorem:

$\lim_{k\to\infty} P_k = P$ a.s. , for some random variable P with $\mathbb{E}(P) \leq \mathbb{E}(P_0)$

Inspecting terms of P_k with respect to convergence.

$$P_k = \prod_{t=k}^{\infty}(1+\gamma_t)\,y_k + \sum_{t=k}^{\infty} w_t \prod_{l=t+1}^{\infty}(1+\gamma_l)$$

- $\sum_{t=k}^{\infty} w_t \prod_{l=t+1}^{\infty}(1+\gamma_l)$ converges (shown below):
  - as $\sum_{t=k}^{\infty} w_t$ converges (as is $< \infty$ and non decreasing)
  - and $\prod_{l=k+1}^{\infty}(1+\gamma_l)$ converges (as stated in hint)
  - and both $\sum_{t=k}^{\infty} w_t$ and $\prod_{l=k+1}^{\infty}(1+\gamma_l) \geq 0 \ \forall k$.

  $\Rightarrow \therefore \sum_{t=k}^{\infty} w_t \prod_{l=t+1}^{\infty}(1+\gamma_l)$ converges.

- $\lim_{k \to \infty} P_k = P$ a.s. (shown previously)
- $\therefore \prod_{t=k}^{\infty}(1+\gamma_t)\,y_k$ must converge.

Inspecting $\prod_{t=k}^{\infty}(1+\gamma_t)\,y_k$ with respect to convergence.

- $\prod_{t=k}^{\infty}(1+\gamma_t)$ • converges (as shown in hint, as $\sum_{k=0}^{\infty}\gamma_k < 0$)
  - non negative (as γ_k is non negative)

  $\Rightarrow \prod_{t=k}^{\infty}(1+\gamma_t) \geq 1 \ \forall k$.

- y_k is non negative $\therefore$ $y_k \geq 0 \ \forall k$.

$\therefore$ Since $\prod_{t=k}^{\infty}(1+\gamma_t)\,y_k$ converges as $k \to \infty$, y_k must converge.

$\therefore y_k$ converges.