Appendix A

Boolean algebras

Boolean algebras is an important class of algebras which has been studied and used extensively for many purposes (see Section A.5). The **switching algebra,** used in the description of switching expressions discussed in Section 2.4, is an instance (an element) of the class of Boolean algebras. Consequently, theorems developed for Boolean algebras are also applicable to switching algebra, so they can be used for the transformation of switching expressions. Moreover, certain identities from Boolean algebra are the basis for the graphical and tabular techniques used for the minimization of switching expressions.

In this appendix, we present the definition of Boolean algebras as well as theorems which are useful for the transformation of Boolean expressions. We also show the relationship among Boolean and switching algebras; in particular, we show that the switching algebra satisfies the postulates of a Boolean algebra. We also sketch other examples of Boolean algebras, which are helpful to further understand the properties of this class of algebras.

A.1 Boolean algebra

A **Boolean algebra** is a tuple $\{B, +, \cdot\}$, wherein

- B is a set of elements;
- $+$ and $\cdot$ are binary operations applied over the elements of B,

which satisfies the following postulates:

P1: If $a, b \in B$, then

$$\begin{array}{ll} \text{(i)} & a + b = b + a \\ \text{(ii)} & a \cdot b = b \cdot a \end{array}$$

That is, $+$ and $\cdot$ are commutative.

P2: If $a, b, c \in B$, then

$$\begin{array}{ll} \text{(i)} & a + (b \cdot c) = (a + b) \cdot (a + c) \\ \text{(ii)} & a \cdot (b + c) = (a \cdot b) + (a \cdot c) \end{array}$$

P3: The set B has two distinct **identity elements,** denoted as 0 and 1, such that for every element in B

$$\begin{array}{ll} \text{(i)} & 0 + a = a + 0 = a \\ \text{(ii)} & 1 \cdot a = a \cdot 1 = a \end{array}$$

The elements 0 and 1 are called the **additive identity element** and the **multiplicative identity element**, respectively. (These elements should not be confused with the integers 0 and 1.)

P4: For every element $a \in B$ there exists an element a', called the **complement** of a, such that

$$\begin{array}{ll} \text{(i)} & a + a' = 1 \\ \text{(ii)} & a \cdot a' = 0 \end{array}$$

The symbols $+$ and $\cdot$ should not be confused with the arithmetic addition and multiplication symbols. However, for convenience $+$ and $\cdot$ are often called "plus" and "times," and the expressions $a + b$ and $a \cdot b$ are called "sum" and "product," respectively. Moreover, $+$ and $\cdot$ are also called "OR" and "AND," respectively.

The elements of the set B are called **constants.** Symbols representing arbitrary elements of B are **variables.** The symbols a, b and c in the postulates above are variables, whereas 0 and 1 are constants.

A **precedence ordering** is defined on the operators: $\cdot$ has precedence over $+$. Therefore, parentheses can be eliminated from products. Moreover, whenever single symbols are used for variables, the symbol $\cdot$ can be eliminated in products. For example,

$$a + (b \cdot c) \quad \text{can be written as} \quad a + bc$$

A.2 Switching algebra

The **switching algebra** is an algebraic system used to describe switching functions by means of switching expressions. In this sense, a switching algebra serves the same role for switching functions as the ordinary algebra does for arithmetic functions.

The switching algebra consists of the set of two elements $B = \{0, 1\}$, and two operations AND and OR defined as follows:

AND	0 1
0	0 0
1	0 1

OR	0 1
0	0 1
1	1 1

These operations are used to evaluate switching expressions, as indicated in Section 2.4.

Theorem 1 *The switching algebra is a Boolean algebra.*

Proof: We show that the switching algebra satisfies the postulates of a Boolean algebra.

P1: Commutativity of (+), ($\cdot$). This is shown by inspection of the operation tables. The commutativity property holds if a table is symmetric about the main diagonal.

P2: Distributivity of (+) and ($\cdot$). Shown by **perfect induction,** that is, by considering all possible values for the elements a, b, and c. Consider the following table:

abc	$a + bc$	$(a + b)(a + c)$
0 0 0	0	0
0 0 1	0	0
0 1 0	0	0
0 1 1	1	1
1 0 0	1	1
1 0 1	1	1
1 1 0	1	1
1 1 1	1	1

Since $a + bc = (a + b)(a + c)$ for all cases, P2(i) is satisfied. A similar proof shows that P2(ii) is also satisfied.

P3: Existence of additive and multiplicative identity element. From the operation tables

$$0 + 1 = 1 + 0 = 1$$

Therefore, 0 is the additive identity. Similarly

$$0 \cdot 1 = 1 \cdot 0 = 0$$

so that 1 is the multiplicative identity.

P4: Existence of the complement. By perfect induction:

a	a'	$a + a'$	$a \cdot a'$
1	0	1	0
0	1	1	0

Consequently, 1 is the complement of 0 and 0 is the complement of 1.

Since all postulates are satisfied, the switching algebra is a Boolean algebra. As a result, all theorems true for Boolean algebras are also true for the switching algebra. ■

A.3 Important theorems in Boolean algebra

We now present some important theorems in Boolean algebra; these theorems can be applied to the transformation of switching expressions.

Theorem 2 Principle of Duality.

Every algebraic identity deducible from the postulates of a Boolean algebra remains valid if

- *the operations* $+$ *and* $\cdot$ *are interchanged throughout; and*
- *the identity elements 0 and 1 are also interchanged throughout.*

Proof: The proof follows at once from the fact that for each of the postulates there is another one (the dual) which is obtained by interchanging $+$ and $\cdot$, and 0 and 1. ■

This theorem is useful because it reduces the number of different theorems that must be proven: every theorem has its dual.

Theorem 3 *Every element in B has a* **unique** *complement.*

Proof: Let $a \in B$; let us assume that a_1' and a_2' are both complements of a. Then, using the postulates we can perform the following transformations:

$$\begin{array}{llll} a_1' & = a_1' \cdot 1 & \text{by P3(ii)} & \text{(identity)} \\ & = a_1' \cdot (a + a_2') & \text{by hypothesis} & (a_2' \text{ is the complement of } a) \\ & = a_1' \cdot a + a_1' \cdot a_2' & \text{by P2(ii)} & \text{(distributivity)} \\ & = a \cdot a_1' + a_1' \cdot a_2' & \text{by P1(ii)} & \text{(commutativity)} \\ & = 0 + a_1' \cdot a_2' & \text{by hypothesis} & (a_1' \text{ is complement of } a) \\ & = a_1' \cdot a_2' & \text{by P3(i)} & \text{(identity)} \end{array}$$

Changing the index 1 for 2 and vice versa, and repeating all steps for a_2' we get

$$\begin{array}{lll} a_2' & = & a_2' \cdot a_1' \\ & = & a_1' \cdot a_2' \qquad \text{by P1(ii)} \end{array}$$

and therefore $a_2' = a_1'$. ■

The uniqueness of the complement of an element allows considering $'$ as a unary operation called **complementation.**

Theorem 4 *For any $a \in B$:*

$$\begin{array}{ll} (1) & a + 1 = 1 \\ (2) & a \cdot 0 = 0 \end{array}$$

Proof: Using the postulates, we can perform the following transformations:

$$\begin{array}{llll} \text{Case (1):} & & & \text{by} \\ & a + 1 & = 1 \cdot (a + 1) & \text{P3 (ii)} \\ & & = (a + a') \cdot (a + 1) & \text{P4 (i)} \\ & & = a + (a' \cdot 1) & \text{P2 (i)} \\ & & = a + a' & \text{P3 (ii)} \\ & & = 1 & \text{P4 (i)} \end{array}$$

$$\begin{array}{llll} \text{Case (2):} & & & \text{by} \\ & a \cdot 0 & = 0 + (a \cdot 0) & \text{P3 (i)} \\ & & = (a \cdot a') + (a \cdot 0) & \text{P4 (ii)} \\ & & = a \cdot (a' + 0) & \text{P2 (ii)} \\ & & = a \cdot a' & \text{P3 (i)} \\ & & = 0 & \text{P4 (ii)} \end{array}$$

(2) can also be proven by means of (1) and the principle of duality. ■

Theorem 5 *The complement of the element 1 is 0, and vice versa. That is,*

$$\begin{array}{ll} (1) & 0' = 1 \\ (2) & 1' = 0 \end{array}$$

Proof: By Theorem 4,

$$\begin{array}{rcll} 0 + 1 & = & 1 & \text{and} \\ 0 \cdot 1 & = & 0 & \end{array}$$

Since, by Theorem 3, the complement of an element is unique, Theorem 5 follows. ■

Theorem 6 Idempotent Law.
For every $a \in B$

$$\begin{array}{ll} (1) & a + a = a \\ (2) & a \cdot a = a \end{array}$$

Proof:

$$\begin{array}{llll} & & & \text{by} \\ (1): & a + a & = (a + a) \cdot 1 & \text{P3 (ii)} \\ & & = (a + a) \cdot (a + a') & \text{P4 (i)} \\ & & = (a + (a \cdot a')) & \text{P2 (i)} \\ & & = a + 0 & \text{P4 (ii)} \\ & & = a & \text{P3 (i)} \\ & & & \\ (2): & & & \text{duality} \end{array}$$

■

Theorem 7 Involution Law.
For every $a \in B$,

$$(a')' = a$$

Proof: From the definition of complement $(a')'$ and a are both complements of a'. But, by Theorem 3, the complement of an element is unique, which proves the theorem. ■

Theorem 8 Absorption Law. *For every pair of elements $a, b \in B$,*

$$\begin{array}{ll} (1) & a + a \cdot b = a \\ (2) & a \cdot (a + b) = a \end{array}$$

Proof:

$$\begin{array}{llll} & & & \text{by} \\ (1): & a + ab & = a \cdot 1 + ab & \text{P3 (ii)} \\ & & = a(1 + b) & \text{P2 (ii)} \\ & & = a(b + 1) & \text{P1 (i)} \\ & & = a \cdot 1 & \text{Theorem 4 (1)} \\ & & = a & \text{P3 (ii)} \\ & & & \\ (2) & & & \text{duality} \end{array}$$

■

Theorem 9 *For every pair of elements $a, b \in B$,*

$$\begin{array}{ll} (1) & a + a'b = a + b \\ (2) & a(a' + b) = ab \end{array}$$

Proof:

$$\begin{array}{llll} & & & \text{by} \\ (1): & a + a'b & = (a + a')(a + b) & \text{P2 (i)} \\ & & = 1 \cdot (a + b) & \text{P4 (i)} \\ & & = a + b & \text{P3 (ii)} \\ & & & \\ (2): & & & \text{duality} \end{array}$$

■

Theorem 10 *In a Boolean algebra, each of the binary operations $(+)$ and $(\cdot)$ is associative. That is, for every $a, b, c \in B$,*

$$\begin{array}{ll} (1) & a + (b + c) = (a + b) + c \\ (2) & a(bc) = (ab)c \end{array}$$

The proof of this theorem is quite lengthy. The interested reader should consult the further readings suggested at the end of this appendix.

Corollary 1

1. *The order in applying the* $+$ *operator among* n *elements does not matter. For example*

$$\begin{aligned} a+(b+(c+(d+e))) &= (((a+b)+c)+d)+e \\ &= a+((b+c)+d))+e \\ &= a+b+c+d+e \end{aligned}$$

2. *The order in applying the* $\cdot$ *operator among* n *elements does not matter.*

Theorem 11 DeMorgan's Law.
For every pair of elements $a, b \in B$*:*

$$\begin{array}{ll} (1) & (a+b)' = a'b' \\ (2) & (ab)' = a'+b' \end{array}$$

Proof: We first prove that $(a+b)$ is the complement of $a'b'$. By the definition of complement (P4) and its uniqueness (Theorem 3), this corresponds to showing that $(a+b)+a'b' = 1$ and $(a+b)a'b' = 0$. We do this proof by the following transformations:

$$\begin{array}{lll} & & \text{by} \\ (a+b)+a'b' & = [(a+b)+a'][(a+b)+b'] & \text{P2(i)} \\ & = [(b+a)+a'][(a+b)+b'] & \text{P1(i)} \\ & = [b+(a+a')][a+(b+b')] & \text{associativity} \\ & = (b+1)(a+1) & \text{P4(i)} \\ & = 1 \cdot 1 & \text{Theorem 3(1)} \\ & = 1 & \text{idempotency} \\ & & \\ (a+b)(a'b') & = (a'b')(a+b) & \text{commutativity} \\ & = (a'b')a+(a'b')b & \text{distributivity} \\ & = (b'a')a+(a'b')b & \text{commutativity} \\ & = b'(a'a)+a'(b'b) & \text{associativity} \\ & = b'(aa')+a'(bb') & \text{commutativity} \\ & = b'\cdot 0+a'\cdot 0 & \text{P4(ii)} \\ & = 0+0 & \text{Theorem 3(2)} \\ & = 0 & \text{Theorem 5(1)} \end{array}$$

By duality,

$$(a \cdot b)' = a'+b'$$

■

Theorem 12 Generalized DeMorgan's Law.

Let $\{a, b, \ldots, c, d\}$ be a set of elements in a Boolean algebra. Then, the following identities hold:

$$\begin{array}{ll} (1) & (a + b \ldots + c + d)' = a'b' \ldots c'd' \\ (2) & (ab \ldots cd)' = a' + b' + \ldots + c' + d' \end{array}$$

Proof: By the method of **finite induction.** The basis is provided by Theorem 11, which corresponds to the case with two elements.
Inductive step: Let us assume that DeMorgan's law is true for n elements, and show that it is true for $n + 1$ elements. Let $a, b, \ldots, c$ be the n elements, and d be the $(n + 1)$st element. Then, by associativity and the basis,

$$\begin{aligned} (a + b + \ldots + c + d)' &= [(a + b + \ldots + c) + d]' \\ &= (a + b + \ldots + c)'d' \end{aligned}$$

By the induction hypothesis

$$(a + b + \ldots c)' = a'b' \ldots c'$$

Thus

$$(a + b + \ldots c + d)' = a'b' \ldots c'd'$$

■

DeMorgan's theorems are useful in manipulating switching expressions. For example, finding the complement of a switching expression containing parentheses is achieved by applying DeMorgan's law and the Involution law repeatedly to bring all $(')$ inside the parentheses. That is,

$$\begin{aligned} [(a + b')(c' + d') + (f' + g)']' &= [(a + b')(c' + d')]'[(f' + g)']' \\ &= [(a + b')' + (c' + d')'](f' + g) \\ &= (a'b + cd)(f' + g) \end{aligned}$$

The symbols $a, b, c, \ldots$ appearing in theorems and postulates are **generic variables.** That is, they can be substituted by complemented variables or expressions (formulas) without changing the meaning of these theorems. For example, DeMorgan's law can read as

$$(a' + b')' = ab$$

or

$$((a + b)' + c')' = (a + b)c$$

We have described a general mathematical system, called Boolean algebra, and established a basic set of algebraic identities, true for any Boolean algebra, without actually specifying the nature of the two binary operations, $(+)$ and $(\cdot)$. In Chapter 2, we present an algebra useful for the representation of switching functions by switching expressions.

A.4 Other examples of Boolean algebras

There are other algebras that are also instances of Boolean algebras. We now summarize the two most commonly used ones.

Algebra of Sets. The elements of B are all subsets of a set S (the set of all subsets of S is denoted by $P(S)$), and the operations are set-union ($\cup$) and set-intersection ($\cap$). That is,

$$M = (P(S), \cup, \cap)$$

The additive identity is the empty set, denoted by ϕ, and the multiplicative identity is the set S. The set $P(S)$ has $2^{|S|}$ elements, wherein $|S|$ is the number of elements of S.

It can be shown that every Boolean algebra is isomorphic to an algebra of sets. Consequently, every Boolean algebra has 2^n elements for some value $n > 0$.

Venn diagrams are used to represent sets as well as and the operations of union and intersection. Consequently, since the algebra of sets is a Boolean algebra, Venn diagrams can be used to illustrate the theorems of a Boolean algebra.

Algebra of Logic (Propositional Calculus). In this algebra, the elements are T and F (true and false), and the operations are LOGICAL AND and LOGICAL OR. It is used to evaluate logical propositions. This algebra is isomorphic with the switching algebra.

A.5 Further readings

The topic of Boolean Algebras has been extensively studied, and many good books on the subject exist. The following is a partial list, in which the reader can obtain additional material that goes significantly beyond the limited treatment of this appendix: *Boolean Reasoning: The Logic of Boolean Equations* by F.M. Brown, Kluwer Academic Publishers, Boston, MA, 1990; *Introduction to Switching and Automata Theory* by M.A. Harrison, McGraw-Hill, New York, NY 1965; *Switching and Automata Theory* by Z. Kohavi, 2nd. edition, McGraw-Hill, New York, NY, 1978; *Switching Theory* by R.E. Miller, Vols. 1 and 2, J. Wiley & Sons, New York, NY, 1965; *Introduction to Discrete Structures* by F. Preparata and R. Yeh, Addison-Wesley, Reading, MA, 1973; and *Discrete Mathematical Structures* by H.S. Stone, Science Research Associates, Chicago, IL, 1973.