Ex: Let $\mathcal{B} = \{\vec{b_1}, \vec{b_2}\}$ and $\mathcal{C} = \{\vec{c_1}, \vec{c_2}\}$ be bases for a vector space V, and suppose $\vec{b_1} = -\vec{c_1} + 4\vec{c_2}$ and $\vec{b_2} = 5\vec{c_1} - 3\vec{c_2}$.

Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

$$[\vec{b_1}]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \quad [\vec{b_2}]_{\mathcal{C}} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}$$

Find $[\vec{x}]_{\mathcal{C}}$ for $\vec{x} = 5\vec{b_1} + 3\vec{b_2}$.

$$\vec{x} = 5\vec{b_1} + 3\vec{b_2} = 5(-\vec{c_1} + 4\vec{c_2}) + 3(5\vec{c_1} - 3\vec{c_2})$$
$$= 10\vec{c_1} + 11\vec{c_2}$$

So $[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$.

OR

$$\begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}.$$

Ex: Let $\mathcal{D} = \{\vec{d_1}, \vec{d_2}, \vec{d_3}\}$ and $\mathcal{F} = \{\vec{f_1}, \vec{f_2}, \vec{f_3}\}$ be bases for a vector space V, and suppose $\vec{f_1} = 2\vec{d_1} - \vec{d_2} + \vec{d_3}$, $\vec{f_2} = 3\vec{d_2} + \vec{d_3}$, and $\vec{f_3} = -3\vec{d_1} + 2\vec{d_3}$.

Find the change-of-coordinates matrix from $\mathcal{F}$ to $\mathcal{D}$.

$$[\vec{f_1}]_{\mathcal{D}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad [\vec{f_2}]_{\mathcal{D}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \quad [\vec{f_3}]_{\mathcal{D}} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

$$\underset{\mathcal{D} \leftarrow \mathcal{F}}{P} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Find $[\vec{x}]_{\mathcal{D}}$ for $\vec{x} = \vec{f_1} - 2\vec{f_2} + 2\vec{f_3}$.

$$\vec{x} = \vec{f_1} - 2\vec{f_2} + 2\vec{f_3} = 2\vec{d_1} - \vec{d_2} + \vec{d_3} - 2(3\vec{d_2} + \vec{d_3}) + 2(-3\vec{d_1} + 2\vec{d_3})$$

$$= -4\vec{d_1} - 7\vec{d_2} + 3\vec{d_3}$$

So $[\vec{x}]_{\mathcal{D}} = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}$ OR $\begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}$

4-6
3

<u>Ex</u>: Let $\mathcal{B} = \{\vec{b_1}, \vec{b_2}\}$ and $\mathcal{C} = \{\vec{c_1}, \vec{c_2}\}$ be bases for $\mathbb{R}^2$.

Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

$\vec{b_1} = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$, $\vec{b_2} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\vec{c_1} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $\vec{c_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} : \begin{bmatrix} 1 & 1 & -1 & 1 \\ 4 & 1 & 8 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -3 & 12 & -9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -4 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -4 & 3 \end{bmatrix}$$

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$$

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P^{-1}} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}.$$

Ex: Let $\mathcal{B} = \{\vec{b_1}, \vec{b_2}\}$ and $\mathcal{C} = \{\vec{c_1}, \vec{c_2}\}$ be bases for $\mathbb{R}^2$.

Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$

and the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

$\vec{b_1} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$, $\vec{b_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\vec{c_1} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\vec{c_2} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} : \left[\begin{array}{cc:cc} 4 & 5 & 7 & 2 \\ 1 & 2 & -2 & -1 \end{array}\right] \sim \begin{bmatrix} 1 & 2 & -2 & -1 \\ 4 & 5 & 7 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & -3 & 15 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & -5 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 8 & 3 \\ 0 & 1 & -5 & -2 \end{bmatrix}$$

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}$$

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P^{-1}} = \begin{bmatrix} 2 & 3 \\ -5 & -8 \end{bmatrix}.$$

4-6
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Ex: In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis $\mathcal{B} = \{1-3t^2, 2+t-5t^2, 1+2t\}$ to the standard basis. Then write t^2 as a linear combination of the polynomials in $\mathcal{B}$.

Let $\mathcal{C} = \{1, t, t^2\}$.

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$$

Let $\vec{p} = t^2$. Then $[\vec{p}]_{\mathcal{B}}$ satisfies $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\vec{p}]_{\mathcal{B}} = [\vec{p}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

So $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -3 & -5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$ and $[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

Thus $t^2 = 3(1-3t^2) - 2(2+t-5t^2) + (1+2t)$.