§4.4 Coordinate Systems

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \dots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

Suppose $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for a vector space V and $\mathbf{x}$ is in V. The coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$ (or the $\mathcal{B}$-coordinates of $\mathbf{x}$) are the weights $c_1, \dots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

If $c_1, \dots, c_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$, then the vector in $\mathbb{R}^n$ $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is the coordinate vector of $\mathbf{x}$ (relative to $\mathcal{B}$), or the $\mathcal{B}$-coordinate vector of $\mathbf{x}$.

Given a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ let $\mathcal{P}_{\mathcal{B}} = [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n]$. Then the vector equation $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ is equivalent to $\mathbf{x} = \mathcal{P}_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$. We call $\mathcal{P}_{\mathcal{B}}$ the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$. Since the columns of $\mathcal{P}_{\mathcal{B}}$ form a basis for $\mathbb{R}^n$, $\mathcal{P}_{\mathcal{B}}$ is invertible. Hence $\mathcal{P}_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$.