5-1
1

Ex: Is $\lambda = -2$ an eigenvalue of $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$?

$\lambda = -2$ is an eigenvalue if and only if $A\vec{x} = -2\vec{x}$ has a nontrivial solution. Now $A\vec{x} = -2\vec{x}$ implies $(A + 2I)\vec{x} = \vec{0}$. $A + 2I = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$ and the columns of $A + 2I$ are linearly dependent so $(A + 2I)\vec{x} = \vec{0}$ has a nontrivial solution.

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ex: Is $\begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$?

Does $A\vec{x} = \lambda\vec{x}$? that is, does $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix} = \lambda\begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}$?

Now $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -1+2\sqrt{2} \\ 3+\sqrt{2} \end{bmatrix}$ so if $A\vec{x} = \lambda\vec{x}$,

then $\lambda = 3+\sqrt{2}$. Now $(3+\sqrt{2})(-1+\sqrt{2}) = -1+2\sqrt{2}$

so that $\begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$

with eigenvalue $\lambda = 3+\sqrt{2}$.

5-1

3

Ex: Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$?

$$A\vec{x} = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

So $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue $\lambda = -2$.

5-1
4

Ex: Is $\lambda = 3$ an eigenvalue of $\begin{pmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$?

If so, find one corresponding eigenvector.

Does $(A - 3I)\vec{x} = \vec{0}$ have a nontrivial solution?

$$\left[\begin{array}{ccc|c} -2 & 2 & 2 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_3 \\ 2x_3 \\ x_3 \end{pmatrix}$$

Letting $x_3 = 1$ we obtain the eigenvector $\vec{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

5-1

5

Ex: Find a basis for the eigenspace corresponding to each listed eigenvalue.

$$A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}, \quad \lambda = 4$$

$$(A - 4I)\vec{v} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 6 & -9 & 0 \\ 4 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{9}{6} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v} = x_2 \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

A basis for the eigenspace corresponding to $\lambda = 4$ is

$$\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}.$$

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Ex: Find a basis for the eigenspace corresponding to each listed eigenvalue.

$$A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}, \quad \lambda = 1, \lambda = 5$$

$$(A - I)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 6 & 4 & 0 \\ -3 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = x_2 \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

A basis for the eigenspace corresponding to $\lambda = 1$ is $\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$.

$$(A - 5I)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 2 & 4 & 0 \\ -3 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

A basis for the eigenspace corresponding to $\lambda = 5$ is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

5-1
7

Ex: Find a basis for the eigenspace corresponding to each listed eigenvalue.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}, \quad \lambda = -2$$

$$(A+2I)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} 3 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 4 & -13 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = x_3 \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

A basis for the eigenspace corresponding to $\lambda = -2$ is $\left\{ \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} \right\}$.

5-1
8

<u>Ex</u>: Find a basis for the eigenspace corresponding to each listed eigenvalue.

$$A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad \lambda = 4$$

$$(A-4I)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cccc|c} -1 & 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = x_3 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the eigenspace corresponding to $\lambda = 4$ is

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

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Ex: Find the eigenvalues of the matrix $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$.

The given matrix is triangular so its eigenvalues are located along the main diagonal. They

$\lambda = 4$, $\lambda = 0$, and $\lambda = -3$.

Ex: Without calculation, find one eigenvalue and two linearly independent eigenvectors of

$$A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}.$$

The matrix A is not invertible because its columns are linearly dependent. Hence $\lambda = 0$ is an eigenvalue of A.

Eigenvectors for the eigenvalue $\lambda = 0$ are solutions of $A\vec{x} = \vec{0}$.

$$A\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} 5 & 5 & 5 & 0 \\ 5 & 5 & 5 & 0 \\ 5 & 5 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{So } \vec{x} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Let $x_2 = 1$, $x_3 = -2$. Then the eigenvector is $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

Let $x_2 = -1$, $x_3 = 0$. Then the eigenvector is $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.