§1.3 Vector Equations

A column vector (vector) is a matrix with only one column.

The set of all vectors with two entries is denoted by $\mathbb{R}^2$.

Vectors in $\mathbb{R}^2$ are ordered pairs of real numbers.
The geometric point (a, b) is equivalent to the column vector $\begin{bmatrix} a \\ b \end{bmatrix}$.
$\mathbb{R}^2$ is viewed as the set of all points in the plane.

Vectors in $\mathbb{R}^3$ are 3×1 column matrices with three entries.
Their geometric representations are points in $3-$space, sometimes with arrows from the origin for clarity.

Let n be a positive integer. The $\mathbb{R}^n$ represents all possible $n \times 1$ column matrices.

Example: $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

The zero vector is one whose entries are all zero and is denoted by $\mathbf{0}$.

If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are vectors in $\mathbb{R}^n$ with scalars $c_1, c_2, \dots, c_p$, then the vector $\mathbf{y}$ defined by $\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$ is called a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$ with weights $c_1, \dots, c_p$.

A vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$ has the same solution set as the linear system whose augmented matrix is $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}]$. Furthermore, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the above vector equation/augmented matrix.

If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ is denoted by $\operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ and is called the subset of $\mathbb{R}^n$ spanned (generated) by $\mathbf{v}_1, \dots, \mathbf{v}_p$. That is, span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$ with scalars $c_1, \dots, c_p$.

Asking whether a vector $\mathbf{b}$ is in span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is equivalent to asking whether the vector equation $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$ has a solution or, similarly, asking whether the linear system with augmented matrix $[\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_p \quad \mathbf{b}]$ has a solution.

Note that span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ contains every scalar multiple of $\mathbf{v}_1$, for example, since $c\mathbf{v}_1 = c\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p$. In particular, the zero vector must be in span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$. Then span$\{\mathbf{v}\}$ is the set of all scalar multiples of $\mathbf{v}$, which is the set of points on the line in $\mathbb{R}^3$ through $\mathbf{v}$ and $\mathbf{0}$.

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^3$, with $\mathbf{v}$ not a multiple of $\mathbf{u}$, then span$\{\mathbf{u}, \mathbf{v}\}$ is the plane in $\mathbb{R}^3$ that contains $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{0}$. In particular, span$\{\mathbf{u}, \mathbf{v}\}$ contains the line in $\mathbb{R}^3$ through $\mathbf{u}$ and $\mathbf{0}$ and the line through $\mathbf{v}$ and $\mathbf{0}$.