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1

Ex: Verify that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set, and then find the orthogonal projection of $\vec{y}$ onto span$\{\vec{u}_1, \vec{u}_2\}$.

$$\vec{y} = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}, \quad \vec{u}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$$

$$\vec{u}_1 \cdot \vec{u}_2 = (3)(-4) + (4)(3) + (0)(0) = 0$$

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}\vec{u}_2$$

$$= \frac{30}{25}\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \frac{-15}{25}\begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$$

$$= \frac{6}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} - \frac{3}{5}\begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

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2

Ex: Verify that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set, and then find the orthogonal projection of $\vec{y}$ onto span$\{\vec{u}_1, \vec{u}_2\}$.

$$\vec{y} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}, \quad \vec{u}_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{u}_1 \cdot \vec{u}_2 = (-4)(0) + (-1)(1) + (1)(1) = 0$$

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2$$

$$= \frac{-27}{18} \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= -\frac{3}{2} \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

Ex: Let W be the subspace spanned by the $\vec{u}$'s, and write $\vec{y}$ as the sum of a vector in W and a vector orthogonal to W.

$$\vec{y} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}, \quad \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$$

$\vec{u}_1 \cdot \vec{u}_2 = (1)(-1) + (1)(3) + (1)(-2) = 0$ so $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set.

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}\vec{u}_2 = \frac{6}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{7}{14}\begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{7}{2} \\ 1 \end{pmatrix}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{pmatrix} -\frac{5}{2} \\ \frac{1}{2} \\ 2 \end{pmatrix}$$

$$\vec{y} = \hat{y} + \vec{z} = \begin{pmatrix} \frac{3}{2} \\ \frac{7}{2} \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{5}{2} \\ \frac{1}{2} \\ 2 \end{pmatrix}$$

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Ex: Let W be the subspace spanned by the $\vec{u}$'s, and write $\vec{y}$ as the sum of a vector in W and a vector orthogonal to W.

$$\vec{y} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \quad \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$\vec{u}_1 \cdot \vec{u}_2 = \vec{u}_1 \cdot \vec{u}_3 = \vec{u}_2 \cdot \vec{u}_3 = 0$ so $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set.

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 + \frac{\vec{y}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3}\vec{u}_3$$

$$= \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \frac{14}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \frac{-5}{3}\begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

$$\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{y} = \hat{y} + \vec{z} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

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Ex: Find the closest point to $\vec{y}$ in the subspace W spanned by $\vec{v}_1$ and $\vec{v}_2$.

$$\vec{y} = \begin{pmatrix} 3 \\ -1 \\ 1 \\ 13 \end{pmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

$\vec{v}_1 \cdot \vec{v}_2 = 0$ so $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal set.

$$\hat{y} = \frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \frac{\vec{y} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$= \frac{30}{10} \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + \frac{26}{26} \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}$$

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Ex: Find the best approximation to $\vec{z}$ by vectors of the form $c_1\vec{v}_1 + c_2\vec{v}_2$.

$$\vec{z} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}$$

$$\hat{z} = \frac{\vec{z}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{z}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$$

$$= \frac{7}{14}\begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} + \frac{0}{49}\begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$$

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Ex: let $\vec{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$.

Find the distance from $\vec{y}$ to the subspace of $\mathbb{R}^4$ spanned by $\vec{v}_1$ and $\vec{v}_2$.

$$\hat{y} = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}$$

$$\vec{y} - \hat{y} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\|\vec{y} - \hat{y}\| = \left[(4)^2 + (4)^2 + (4)^2 + (4)^2\right]^{\frac{1}{2}} = 8$$