4.5

Ex: Find a basis and state the dimension of the subspace

$$\left\{ \begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix} \;\middle|\; s, t \text{ in } \mathbb{R} \right\}.$$

This subspace is $H = \text{span}\{\vec{v}_1, \vec{v}_2\}$ where $\vec{v}_1 = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$. Since $\vec{v}_1$ and $\vec{v}_2$ are not multiples of each other, $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and is thus a basis for H. Therefore $\dim H = 2$.

4.5
2

Ex: Find a basis and state the dimension of the subspace

$$\left\{ \begin{bmatrix} a+b \\ 2a \\ 3a-b \\ -b \end{bmatrix} \;\middle|\; a, b \text{ in } \mathbb{R} \right\}.$$

This subspace is $H = \text{span}\{\vec{v}_1, \vec{v}_2\}$ where $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$ and

$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$. Since $\vec{v}_1$ and $\vec{v}_2$ are not multiples of

each other, $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and is

thus a basis for H. Therefore $\dim H = 2$.

Ex: Find a basis and state the dimension of the subspace

$$\left\{ \begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{bmatrix} \;\middle|\; a,b,c \text{ in } \mathbb{R} \right\}.$$

This subspace is $H = \text{span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ where

$$\vec{v_1} = \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix}, \text{ and } \vec{v_3} = \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix}.$$

Since $\vec{v_1} = -3\vec{v_3}$, $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly dependent.

Remove $\vec{v_3}$ to obtain $H = \text{span}\{\vec{v_1}, \vec{v_2}\}$. Now $\{\vec{v_1}, \vec{v_2}\}$ is linearly independent since they are not multiples of each other. Thus $\{\vec{v_1}, \vec{v_2}\}$ is a basis for H and

$\dim H = 2$.

4-5

4

Ex: Find a basis and state the dimension of the subspace

$$\{(a, b, c, d) \mid a - 3b + c = 0\}.$$

From the equation $a-3b+c=0$, we have that $a = 3b - c$.

It follows that $\begin{bmatrix} 3b-c \\ b \\ c \\ d \end{bmatrix} = b\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

This subspace is $H = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where

$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

This set is linearly independent since

$$\left[\begin{array}{ccc|c} 3 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & -1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

has only the trivial solution. Thus $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for H and $\dim H = 3$.

4-5

5

Ex: Find the dimension of the subspace H of $\mathbb{R}^2$ spanned by

$$\begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}.$$

This subspace is $H = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -4 \\ 10 \end{bmatrix}, \text{ and } \vec{v}_3 = \begin{bmatrix} -3 \\ 6 \end{bmatrix}.$$

Since $\vec{v}_2 = -2\vec{v}_1$, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

Remove $\vec{v}_2$ to obtain $H = \text{span}\{\vec{v}_1, \vec{v}_3\}$. Now

$\{\vec{v}_1, \vec{v}_3\}$ is linearly independent since they are

not multiples of each other. Thus $\{\vec{v}_1, \vec{v}_3\}$ is a

basis for H and $\dim H = 2$.

45

6

Ex: Find the dimension of the subspace spanned by

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}.$$

The matrix A with these vectors as its columns is

$$\begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix}.$$

Matrix A row reduces to $\begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

There are three pivot columns, so the dimension of Col A (which is the dimension of the subspace spanned by the vectors) is 3.

4-5

7

Ex: Find the dimensions of Nul A and Col A for

$$A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

dim Col A = 3 since there are 3 pivot columns.

dim Nul A = 3 since there are 3 columns without pivots and hence 3 free variables.

4-5

8

Ex: Find the dimensions of Nul A and Col A for

$$A = \begin{bmatrix} 3 & 4 \\ -6 & 10 \end{bmatrix}.$$

dim Col $A = 2$ since there are 2 pivot columns.

dim Nul $A = 0$ since there are no columns without pivots.

Note that $\left[\begin{array}{cc|c} 3 & 4 & 0 \\ -6 & 10 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$.

Thus $A\vec{x} = \vec{0}$ has only the trivial solution and

Nul $A = \{\vec{0}\}$.

4-5
9

Ex: Find the dimensions of Nul A and Col A for

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

dim Col $A = 2$ since there are 2 pivot columns.

dim Nul $A = 1$ since there is one column without a pivot and hence one free variable.

4-5
10

Ex: Assume that matrix A is row equivalent to B.

List rank A and dim Nul A. Find bases for Col A, Row A, and Nul A.

$$A=\begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix} \quad B=\begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank A = dim Col A = 3 , dim Nul A = 2

Basis for Col A is $\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix} \right\}$.

$$B \sim \begin{bmatrix} 1 & -3 & 0 & 5 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}=\begin{bmatrix} 3x_2-5x_4 \\ x_2 \\ \frac{3}{2}x_4 \\ x_4 \\ 0 \end{bmatrix}$$

Basis for Nul A is $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$.

A basis for Row A is the pivot rows of B:

$\{(1,-3,0,5,-7), (0,0,2,-3,8), (0,0,0,0,5)\}$.

Ep: Assume that matrix A is row-equivalent to B.

List rank A and dim Nul A. Find bases for Col A, Row A, and Nul A.

$$A = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank A = dim Col A = 3, dim Nul A = 3

Basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ 1 \\ -5 \\ 0 \end{bmatrix} \right\}$.

$$B \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 9 & 2 \\ 0 & 1 & -1 & 0 & 7 & 3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2x_3 - 9x_5 - 2x_6 \\ x_3 - 7x_5 - 3x_6 \\ x_3 \\ x_5 + 2x_6 \\ x_5 \\ x_6 \end{bmatrix}$$

Basis for Nul A is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

A basis for Row A is the pivot rows of B:

$\{(1,1,-3,7,9,-9), (0,1,-1,3,4,-3), (0,0,0,1,-1,-2)\}$.