Ex: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Basis, linearly independent, or span $\mathbb{R}^3$?

Not a basis or linearly independent since the set contains the zero vector.

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

A has only two pivot positions so it is not invertible and hence its columns do not span $\mathbb{R}^3$.

4-3

2

Ex: $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix}$

Basis, linearly independent, or span $\mathbb{R}^3$?

Let $A = \begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & 5 \\ 2 & 1 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 13 \\ 0 & -3 & -15 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 13 \\ 0 & 0 & 24 \end{bmatrix}$$

Reduced to I_3.

therefore invertible.

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 13 \\ 0 & 0 & 1 \end{bmatrix}$$

Columns are linearly independent and span $\mathbb{R}^3$.

Thus a basis.

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4-3
3

<u>Ex</u>: $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ -5 \\ 6 \end{bmatrix}$

Basis, linearly independent, or span $\mathbb{R}^3$?

Linearly independent since neither vector is a multiple of the other.

Let $A = \begin{bmatrix} 1 & -4 \\ 2 & -5 \\ -3 & 6 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -4 \\ 2 & -5 \\ -3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 3 \\ 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 1 \\ 0 & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

A does not have a pivot in each row so its columns do not span $\mathbb{R}^3$.

Not a basis.

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4

Ex: $\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$

Basis, linearly independent, or span $\mathbb{R}^3$?

Let $A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ -4 & 3 & -5 & 2 \\ 3 & -1 & 4 & -2 \end{bmatrix}$.

The matrix A has more columns than rows and will therefore produce a nontrivial solution.

Thus the columns are not linearly independent and hence not a basis.

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ -4 & 3 & -5 & 2 \\ 3 & -1 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & 7 & 2 \\ 0 & -1 & -5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 2 \\ 0 & 3 & 7 & 2 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

The matrix A has a pivot in each row implying that the columns span $\mathbb{R}^3$.

4-3
5

<u>Ex</u>: Find a basis for the null space.

$$A = \begin{pmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{pmatrix}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -5 & 1 & 4 & 0 \\ -2 & 1 & 6 & -2 & -2 & 0 \\ 0 & 2 & -8 & 1 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & -5 & 1 & 4 & 0 \\ 0 & 1 & -4 & 0 & 6 & 0 \\ 0 & 2 & -8 & 1 & 9 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -5 & 1 & 4 & 0 \\ 0 & 1 & -4 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & -5 & 0 & 7 & 0 \\ 0 & 1 & -4 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \end{array}\right]$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{bmatrix} 5x_3 - 7x_5 \\ 4x_3 - 6x_5 \\ x_3 \\ 3x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

Basis for Nul A is $\left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$

Ex: Matrix A is row equivalent to matrix B. Find bases for Nul A, Col A, and Row A.

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns for B are 1, 3, and 5. Since $A \sim B$, a basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$.

Complete the row reduction on B in order to find a basis for Nul A.

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -\frac{7}{5} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -2x_2 - 4x_4 \\ x_2 \\ \frac{7}{5}x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix}$$

Basis for Nul A is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix} \right\}$.

A basis for Row A can be taken from the pivot rows of B:

$\{ [1\ 2\ 0\ 4\ 5], [0\ 0\ 5\ -7\ 8], [0\ 0\ 0\ 0\ -9] \}$.

4.3

7

ex: Find a basis for the space spanned by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}.$$

This problem is equivalent to finding a basis for Col A where

$$A = \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -3 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1, 2, and 3 are pivot columns.

Basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix} \right\}$

4.3

8

Ex: Let $\vec{v}_1 = \begin{bmatrix} 7 \\ 4 \\ -9 \\ -5 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ -7 \\ 2 \\ 5 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ -5 \\ 3 \\ 4 \end{bmatrix}$.

It can be verified that $\vec{v}_1 - 3\vec{v}_2 + 5\vec{v}_3 = \vec{0}$.

Find a basis for $H = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Since $\vec{v}_1 - 3\vec{v}_2 + 5\vec{v}_3 = \vec{0}$, each of the vectors is a linear combination of the others and hence $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent. Since none of the three vectors is a multiple of any of the others, the sets $\{\vec{v}_1, \vec{v}_2\}$, $\{\vec{v}_1, \vec{v}_3\}$ and $\{\vec{v}_2, \vec{v}_3\}$ are linearly independent and thus each forms a basis for H.

4.3

9

Ex: In the vector space of all real-valued functions, find a basis for the subspace spanned by

$\{\sin t, \sin 2t, \sin t \cos t\}$.

Since $\sin 2t = 2 \sin t \cos t$, the set

$\{\sin t, \sin 2t\}$ spans the subspace.

Clearly $\{\sin t, \sin 2t\}$ is linearly independent and therefore it is a basis for the subspace.

Ex: Consider the polynomials $\vec{p}_1(t) = 1+t$, $\vec{p}_2(t) = 1-t$, and $\vec{p}_3(t) = 2$. Write a linear dependence relation among $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$. Then find a basis for span$\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$.

Since $\vec{p}_3 = \vec{p}_1 + \vec{p}_2$, span$\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ = span$\{\vec{p}_1, \vec{p}_2\}$.

Since neither $\vec{p}_1$ nor $\vec{p}_2$ is a multiple of the other, they are linearly independent and, hence, $\{\vec{p}_1, \vec{p}_2\}$ is a basis for span$\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$.