§1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

If A is an $m \times n$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the product of A and $\mathbf{x}$, denoted $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights. That is,

$$A\mathbf{x} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

Note that $A\mathbf{x}$ is only defined when the number of columns of A equals the number of entries in $\mathbf{x}$.

If A is an $m \times n$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, then the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution set as the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$ which has the same solution set as the system of linear equations whose augmented matrix is $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}]$.

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent.

- For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
- Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
- The columns of A span $\mathbb{R}^m$.
- A has a pivot position in every row.