§4.1 Vector Spaces and Subspaces

A vector space V is a nonempty set of objects (vectors) on which the operations of addition and scalar multiplication are defined, and hold, for the axioms below ($\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in V with c and d being scalars).

- $\mathbf{u} + \mathbf{v}$ is in V.
- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
- There exists a zero vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
- There exists a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- $c\mathbf{u}$ is in V.
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
- $c(d\mathbf{u}) = (cd)\mathbf{u}$.
- $1\mathbf{u} = \mathbf{u}$.

In addition to the previous axioms we have the following facts,

- $0\mathbf{u} = \mathbf{0}$
- $c\mathbf{0} = \mathbf{0}$
- $-\mathbf{u} = (-1)\mathbf{u}$

Basic examples of vector spaces include $\mathbb{R}^n$ and $\mathbb{P}_n$ (the set of all polynomials of degree at most n for $n \geq 0$).

A subspace of a vector space V is a subset H of V that satisfies the following:

- The zero vector of V is in H.
- H is closed under vector addition.
- H is closed under scalar multiplication.

If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space V, then span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subspace of V.

span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is called the subspace spanned (generated) by $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$. Given any subspace H of V, a spanning (generating) set for H is a set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in H such that $H = $ span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.