Math 2010 – Introduction to Linear Algebra
Exam 2

Solutions/Scoring Guide
20 Points Automatic

Using Cramer's Rule, solve the following system of equations.

$$6x_1 + 5x_2 = 8$$
$$6x_1 + 3x_2 = 10$$

$$x_1 = \frac{\begin{vmatrix} 8 & 5 \\ 10 & 3 \end{vmatrix}}{\begin{vmatrix} 6 & 5 \\ 6 & 3 \end{vmatrix}} = \frac{-26}{-12} = \frac{13}{6}$$

+3 +2

$$x_2 = \frac{\begin{vmatrix} 6 & 8 \\ 6 & 10 \end{vmatrix}}{\begin{vmatrix} 6 & 5 \\ 6 & 3 \end{vmatrix}} = \frac{12}{-12} = -1$$

+3 +2

Let W be the set of all vectors of the form $\begin{bmatrix} 4a+9b \\ -9 \\ 4a-3b \end{bmatrix}$, where a and b represent arbitrary real numbers. Find a set S of vectors that spans W or give an example to show that W is *not* a vector space.

Since a vector in W may be written as $\mathbf{w} = a\begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} + b\begin{bmatrix} 9 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ -9 \\ 0 \end{bmatrix}$ +5, we conclude that the zero vector is not in W and therefore W is not a vector space. +5

Let W be the set of all vectors of the form $\begin{bmatrix} 5a+5b \\ 9b-9c \\ 8c-3a \\ 2b \end{bmatrix}$, where a, b, and c represent arbitrary real numbers. Find a set S of vectors that spans W or give an example to show that W is *not* a vector space.

Since a vector in W may be written as $\mathbf{w} = a\begin{bmatrix} 5 \\ 0 \\ -3 \\ 0 \end{bmatrix} + b\begin{bmatrix} 5 \\ 9 \\ 0 \\ 2 \end{bmatrix} + c\begin{bmatrix} 0 \\ -9 \\ 8 \\ 0 \end{bmatrix}$ +5, we conclude that

$S = \left\{ \begin{bmatrix} 5 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -9 \\ 8 \\ 0 \end{bmatrix} \right\}$ is a set that spans W. +5

Assume that matrix A is row equivalent to matrix B. Find bases for Col A and Nul A.

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -5 & 2 \\ -3 & 8 & 4 & -4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 8 & 4 \\ 0 & 2 & 7 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns for B are 1 and 2. Since $A \sim B$, a basis for Col A is $\left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}$. +4

Complete the row reduction on B in order to find a basis for Nul A.

$$\begin{bmatrix} 1 & 0 & 8 & 4 \\ 0 & 2 & 7 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 8 & 4 \\ 0 & 1 & \frac{7}{2} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ +2

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -8 \\ -\frac{7}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$ +2

A basis for Nul A is $\left\{ \begin{bmatrix} -8 \\ -\frac{7}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. +2

Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -9 \\ 7 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 7 \\ 3 \\ -8 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 21 \\ -16 \\ 3 \end{bmatrix}$. It can be verified that $7\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$.
Use this information to find three bases for $H = \text{span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Since $7\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$, each of the vectors is a linear combination of the others +1 and hence $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent +2. Since none of the three vectors is a multiple of any of the others +1, the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$ +2, $\{\mathbf{v}_1, \mathbf{v}_3\}$ +2, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ +2 are linearly independent and thus each forms a basis for H.

The set $\mathcal{B} = \{1 - t^2, t - t^2, 1 - 2t - t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 3 + 6t - 11t^2$ relative to $\mathcal{B}$.

$3 + 6t - 11t^2 = c_1(1 - t^2) + c_2(t - t^2) + c_3(1 - 2t - t^2)$ +1

$= (c_1 + c_3) + t(c_2 - 2c_3) + t^2(-c_1 - c_2 - c_3)$ +2

$c_1 + c_3 = 3$ +1

$c_2 - 2c_3 = 6$ +1

$-c_1 - c_2 - c_3 = -11$ +1

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 6 \\ -1 & -1 & -1 & -11 \end{array}\right] \sim$$ +1

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 6 \\ 0 & -1 & 0 & -8 \end{array}\right]$$ +2

Therefore $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 8 \\ 1 \end{bmatrix}$ +1

Determine the rank and nullity of $A = \begin{bmatrix} 1 & -5 & 3 & -3 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The rank of A is 2 since there are two pivot columns. +5

The nullity of A is 3 since there are three columns without a pivot and hence three free variables. +5

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Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V, and suppose $\mathbf{b}_1 = 3\mathbf{c}_1 - 8\mathbf{c}_2$ and $\mathbf{b}_2 = -4\mathbf{c}_1 + 7\mathbf{c}_2$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and find $[\mathbf{x}]_\mathcal{C}$ for $\mathbf{x} = -6\mathbf{b}_1 + 8\mathbf{b}_2$.

$\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} = \begin{bmatrix} 3 & -4 \\ -8 & 7 \end{bmatrix}$ +4

$\mathbf{x} = -6\mathbf{b}_1 + 8\mathbf{b}_2$

$= -6(3\mathbf{c}_1 - 8\mathbf{c}_2) + 8(-4\mathbf{c}_1 + 7\mathbf{c}_2)$ +2

$= -50\mathbf{b}_1 + 104\mathbf{b}_2$ +2

Thus $[\mathbf{x}]_\mathcal{C} = \begin{bmatrix} -50 \\ 104 \end{bmatrix}$ +2