§4.5 The Dimension of a Vector Space

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

If a vector space V is spanned by a finite set, then V is said to be finite-dimensional, and the dimension of V, denoted dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be infinite-dimensional.

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded to a basis for H. Also, H is finite-dimensional and dim $H \leq$ dim V.

Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

The rank of an $m \times n$ matrix A is the dimension of the column space and the nullity of A is the dimension of the null space.

The rank of an $m \times n$ matrix A is the number of pivot columns and the nullity of A is the number of free variables. Since the dimension of the row space is the number of pivot rows, it is also equal to the rank of A.

The dimensions of the column space and the null space of an $m \times n$ matrix A satisfy the equation rank $A +$ nullity $A =$ number of columns in A.