§4.6 Change of Basis

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$ such that $[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}[\mathbf{x}]_{\mathcal{B}}$.

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}} = [[\mathbf{b}_1]_{\mathcal{C}} \quad \cdots \quad [\mathbf{b}_n]_{\mathcal{C}}]$.

The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$ is called the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$ are linearly independent because they are the coordinate vectors of the linearly independent set $\mathcal{B}$. Since $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$ is square, it must be invertible and $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}\right)^{-1}[\mathbf{x}]_{\mathcal{C}} = [\mathbf{x}]_{\mathcal{B}}$. Thus $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}\right)^{-1}$ is the matrix that converts $\mathcal{C}$-coordinates into $\mathcal{B}$-coordinates. That is, $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{\mathcal{P}}$.

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases of a vector space V. Then $[\mathbf{c}_1 \quad \cdots \quad \mathbf{c}_n \quad \vdots \quad \mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n] \sim [I \quad \vdots \quad \underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathcal{P}}]$.