§6.1 Inner Product, Length, and Orthogonality

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then $\mathbf{u}$ and $\mathbf{v}$ are both $n \times 1$ matrices. The transpose of $\mathbf{u}$, $\mathbf{u}^T$, is a $1 \times n$ matrix so that $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix. The number $\mathbf{u}^T\mathbf{v}$ is called the inner product (dot product) of $\mathbf{u}$ and $\mathbf{v}$ and is written as $\mathbf{u} \cdot \mathbf{v}$.

If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, then

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

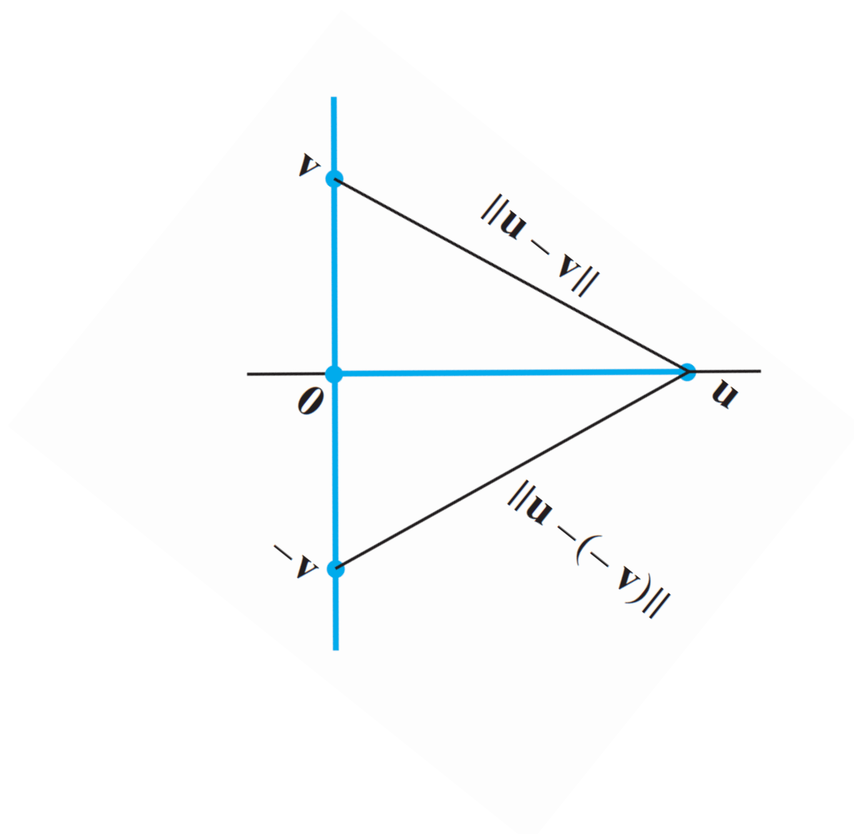
The length (norm) of $\mathbf{v}$ is the nonnegative scalar, $\|\mathbf{v}\|$, defined by $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{{v_1}^2 + {v_2}^2 + \cdots + {v_n}^2}$. Clearly, $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$.

A unit vector is a vector whose length is one. Dividing a nonzero vector $\mathbf{v}$ by its length $\left(\text{multiplying by } \frac{1}{\|\mathbf{v}\|}\right)$ produces a unit vector $\mathbf{u}$ because the length of $\mathbf{u}$ is $\left(\frac{1}{\|\mathbf{v}\|}\right)\|\mathbf{v}\|$. The process of creating $\mathbf{u}$ from $\mathbf{v}$ is called normalizing $\mathbf{v}$ and we consider $\mathbf{u}$ to be in the same direction as $\mathbf{v}$.

If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^n$, then the distance between $\mathbf{u}$ and $\mathbf{v}$, denoted $\text{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$. That is, $\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.

Extending the concept of perpendicular lines in traditional Euclidean geometry to $\mathbb{R}^n$.

In $\mathbb{R}^2$ or $\mathbb{R}^3$ consider two lines through the origin determined by vectors $\mathbf{u}$ and $\mathbf{v}$. The two lines are geometrically perpendicular if and only if the distance from $\mathbf{u}$ to $\mathbf{v}$ is the same as the distance from $\mathbf{u}$ to $-\mathbf{v}$. This is equivalent to the squares of the distances being the same.



Thus

$$\begin{aligned}[\text{dist}(\mathbf{u},-\mathbf{v})]^2 &= \|\mathbf{u}-(-\mathbf{v})\|^2 \\ &= \|\mathbf{u}+\mathbf{v}\|^2 \\ &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v}+2\mathbf{u}\cdot\mathbf{v} \\ &= \|\mathbf{u}\|^2+\|\mathbf{v}\|^2+2\mathbf{u}\cdot\mathbf{v}\end{aligned}$$

and

$$\begin{aligned}[\text{dist}(\mathbf{u},\mathbf{v})]^2 &= \|\mathbf{u}-\mathbf{v}\|^2 \\ &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v}-2\mathbf{u}\cdot\mathbf{v} \\ &= \|\mathbf{u}\|^2+\|\mathbf{v}\|^2-2\mathbf{u}\cdot\mathbf{v}\end{aligned}$$

So the two squared distances are equal if and only if $2\mathbf{u}\cdot\mathbf{v}=-2\mathbf{u}\cdot\mathbf{v}$ which occurs only if $\mathbf{u}\cdot\mathbf{v}=0$.

We now generalize the notion of perpendicularity (traditional geometry) to orthogonality (linear algebra) with the following:

Vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal if $\mathbf{u}\cdot\mathbf{v}=0$.