6-2

Ex: Determine if $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$ are orthogonal.

$$\text{Since } \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

$$= 0,$$

the set is orthogonal.

6-2
2

Ex: Determine if $\begin{pmatrix} 2 \\ -5 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$ are orthogonal.

Since $\begin{pmatrix} 2 \\ -5 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} = 0$, the set is orthogonal.

6.2
3

Ex: Determine if $\begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix}$ are orthogonal.

Since $\begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix} = -32$, the set is not orthogonal.

6-2

4

Ex: Show that $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis for $\mathbb{R}^2$.

Then express $\vec{y}$ as a linear combination of the $\vec{u}$'s.

$$\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

Since $\vec{u}_1 \cdot \vec{u}_2 = (3)(-2) + (1)(6) = 0$, $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set and linearly independent since they are non-zero. Two such vectors in $\mathbb{R}^2$ automatically form a basis for $\mathbb{R}^2$, thus $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis for $\mathbb{R}^2$.

$$\vec{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2$$

$$= \frac{-15}{10} \vec{u}_1 + \frac{30}{40} \vec{u}_2$$

$$= -\frac{3}{2} \vec{u}_1 + \frac{3}{4} \vec{u}_2$$

Ex: Show that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

then express $\vec{p}$ as a linear combination of the $\vec{u}$'s.

$$\vec{u}_1 = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \quad \vec{p} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$$

Since $\vec{u}_1 \cdot \vec{u}_2 = \vec{u}_1 \cdot \vec{u}_3 = \vec{u}_2 \cdot \vec{u}_3 = 0$, $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set and linearly independent since they are non-zero. Three such vectors in $\mathbb{R}^3$ automatically form a basis for $\mathbb{R}^3$, thus $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

$$\vec{p} = \frac{\vec{p}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{p}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 + \frac{\vec{p}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3}\vec{u}_3$$

$$= \frac{24}{18}\vec{u}_1 + \frac{3}{9}\vec{u}_2 + \frac{6}{18}\vec{u}_3$$

$$= \frac{4}{3}\vec{u}_1 + \frac{1}{3}\vec{u}_2 + \frac{1}{3}\vec{u}_3$$

6-2

6

ex: Compute the orthogonal projection of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ onto the line through $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and the origin.

Let $\vec{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{-4}{10} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = -\frac{2}{5} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{5} \\ -\frac{6}{5} \end{bmatrix}$$

6-2
7

Ex: Let $\vec{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Write $\vec{y}$ as the sum of a vector in span$\{\vec{u}\}$ and a vector orthogonal to $\vec{u}$.

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{20}{50}\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{14}{5} \\ \frac{2}{5} \end{bmatrix}$$

$$\vec{y} = \hat{y} + \vec{z} = \hat{y} + (\vec{y} - \hat{y}) = \begin{bmatrix} \frac{14}{5} \\ \frac{2}{5} \end{bmatrix} + \begin{bmatrix} -\frac{4}{5} \\ \frac{28}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

6-7
8

Ex: Let $\vec{y} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute the distance from $\vec{y}$ to the line through $\vec{u}$ and the origin.

$$\vec{y} - \hat{y} = \vec{y} - \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$

$$= \begin{bmatrix} -3 \\ 9 \end{bmatrix} - \frac{15}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 9 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

$$\|\vec{y} - \hat{y}\| = \left[(-6)^2 + (3)^2 \right]^{\frac{1}{2}} = 3\sqrt{5}$$

6-2
9

Ex 1 Determine if $\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\end{bmatrix}\right\}$ is orthonormal. If a set is only orthogonal, normalize the vectors to produce an orthonormal set.

Let $\vec{u} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\-1\\0\end{bmatrix}$.

Since $\vec{u} \cdot \vec{v} = -1$, $\{\vec{u}, \vec{v}\}$ is not an orthogonal set.

6-2

10

Ex: Determine if $\left\{\begin{bmatrix}-\frac{2}{3}\\ \frac{1}{3}\\ \frac{2}{3}\end{bmatrix}, \begin{bmatrix}\frac{1}{3}\\ \frac{2}{3}\\ 0\end{bmatrix}\right\}$ is orthonormal. If a set is only orthogonal, normalize the vector to produce an orthonormal set.

Let $\vec{u} = \begin{bmatrix}-\frac{2}{3}\\ \frac{1}{3}\\ \frac{2}{3}\end{bmatrix}$, $\vec{v} = \begin{bmatrix}\frac{1}{3}\\ \frac{2}{3}\\ 0\end{bmatrix}$.

Since $\vec{u}\cdot\vec{v} = 0$, $\{\vec{u}, \vec{v}\}$ is an orthogonal set

$\vec{u}\cdot\vec{u} = \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = 1$

$\vec{v}\cdot\vec{v} = \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + (0)^2 = \frac{5}{9}$

We must normalize $\vec{v}$.

$\|\vec{v}\|^2 = \vec{v}\cdot\vec{v} = \frac{5}{9} \Rightarrow \|\vec{v}\| = \frac{\sqrt{5}}{3}$

$\frac{1}{\|\vec{v}\|}\vec{v} = \frac{3}{\sqrt{5}}\begin{bmatrix}\frac{1}{3}\\ \frac{2}{3}\\ 0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\\ 0\end{bmatrix}$

So $\left\{\begin{bmatrix}-\frac{2}{3}\\ \frac{1}{3}\\ \frac{2}{3}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\\ 0\end{bmatrix}\right\}$ is an orthonormal set

6-2
4

Ex: Determine if $\left\{ \begin{bmatrix} \frac{1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \right\}$ is orthonormal.

If a set is only orthogonal, normalize the vectors to produce an orthonormal set.

Let $\vec{u} = \begin{bmatrix} \frac{1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{bmatrix}$, $\vec{v} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$.

Since $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{w} = 0$, $\{\vec{u}, \vec{v}, \vec{w}\}$ is an orthogonal set.

$$\vec{u} \cdot \vec{u} = \left(\frac{1}{\sqrt{18}}\right)^2 + \left(\frac{4}{\sqrt{18}}\right)^2 + \left(\frac{1}{\sqrt{18}}\right)^2 = 1$$

$$\vec{v} \cdot \vec{v} = \left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = 1$$

$$\vec{w} \cdot \vec{w} = \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 = 1$$

So $\{\vec{u}, \vec{v}, \vec{w}\}$ is an orthonormal set.