§4.3 Bases

Let A be an $n \times n$ matrix. Then the following statements are logically equivalent.

- A is an invertible matrix.
- A is row equivalent to I_n.
- A has n pivot positions.
- The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- The columns of A form a linearly independent set.
- The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
- The columns of A span $\mathbb{R}^n$.

Let H be a subspace of a vector space V. The set $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a basis for H if

- $\mathcal{B}$ is a linearly independent set, and
- the subspace spanned by $\mathcal{B}$ coincides with H; that is, $H = \text{span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$.

The previous applies to the case when $H = V$, because any vector space is a subspace of itself. Hence a basis of V is a linearly independent set that spans V.

Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the columns of the $n \times n$ matrix I_n. That is,

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \mathbf{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The set $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is called the standard basis for $\mathbb{R}^n$.

The set $S = \{1, t, t^2, \dots, t^n\}$ is called the standard basis for $\mathbb{P}_n$.

Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a set in V with $H = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

- If one of the vectors in S, $\mathbf{v}_k$, is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.
- If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

The pivot columns of a matrix A form a basis for Col A.

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.