§3.1 Introduction to Determinants

For $n \geq 2$, the determinant of an $n \times n$ matrix $A = \left[a_{ij}\right]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A.

$$\begin{aligned}\det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det A_{1j}\end{aligned}$$

Given $A = \left[a_{ij}\right]$, the (i, j)-cofactor of A is the number C_{ij} given by $C_{ij} = (-1)^{i+j} \det A_{ij}$. Then $\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$.

The determinant of an $n \times n$ matrix A can be computed by a cofactor expression across any row or down any column.

The expression across the i^{th} row is $\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$.

The expression down the j^{th} column is $\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$.

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.