5-3
1

Ex: Diagonalize $A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$.

A is triangular so its only eigenvalue is $\lambda = 5$.

$\lambda = 5$: $A - 5I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

x_1 is free

$x_2 = 0$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Since we cannot generate an eigenvector basis for $\mathbb{R}^2$,

A is not diagonalizable.

5-3
2

Ex: Diagonalize $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)(1-\lambda) - 12 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$$

$$\det(A - \lambda I) = 0 \Rightarrow \lambda = 5,\ \lambda = -2$$

$\underline{\lambda = 5}$: $A - 5I = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $x_1 = x_2$, x_2 is free

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda = -2}$: $A + 2I = \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{4} \\ 1 & \frac{3}{4} \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{4} \\ 0 & 0 \end{bmatrix}$ $x_1 = -\frac{3}{4} x_2$, x_2 is free

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{3}{4} \\ 1 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$$

5-3

3

Ex: diagonalize $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$ with eigenvalues $\lambda = 2$, $\lambda = 8$.

$$\lambda = 2: \quad A - 2I = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\lambda = 8: \quad A - 8I = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

5-3
4

Ex! Diagonalize $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ with eigenvalues $\lambda = 5$, $\lambda = 4$.

$\lambda = 5$: $A - 5I = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\lambda = 4$! $A - 4I = \begin{bmatrix} 0 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \Rightarrow \vec{v}_3 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$

$P = \begin{bmatrix} 0 & -2 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ $\qquad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Ex: Diagonalize $A = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$ with eigenvalues $\lambda = 2, \lambda = 1$.

$\underline{\lambda = 1}$: $A - I = \begin{bmatrix} -1 & -4 & -6 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

$\underline{\lambda = 2}$: $A - 2I = \begin{bmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\{\vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$P = \begin{bmatrix} -2 & -2 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

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6

Ex: Diagonalize $A = \begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}$

with one eigenvalue $\lambda = 5$ and one eigenvector $\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$.

$\lambda = 5$: $A - 5I = \begin{bmatrix} -12 & -16 & 4 \\ 6 & 8 & -2 \\ 12 & 16 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{4}{3} & -\frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3}v_2 + \frac{1}{3}v_3 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

$$\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$$

$$\begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} = -3 \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = -3\vec{v}_3$$

Thus $\lambda = -3$ is the eigenvalue that corresponds to $\vec{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$.

$$P = \begin{bmatrix} -4 & 1 & -2 \\ 3 & 0 & 1 \\ 0 & 3 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Ex: Diagonalize $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$.

The matrix A is triangular so its eigenvalues are $\lambda = 4$ and $\lambda = 2$.

$\underline{\lambda = 4}$: $A - 4I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_4 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\underline{\lambda = 2}$: $A - 2I = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\{\vec{v}_3, \vec{v}_4\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad P = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$