Math 2010 – Introduction to Linear Algebra
Exam 3

Solutions/Scoring Guide
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Is $\begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} -1 & 0 & -2 \\ 2 & 5 & -4 \\ 0 & 2 & -2 \end{bmatrix}$? If so, find the eigenvalue.

$$A\mathbf{x} = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 5 & -4 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 15 \\ 6 \end{bmatrix} = 3\begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$$

+4 +2 +2

So $\begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 3$. +2

Find a basis for the eigenspace of $A = \begin{bmatrix} -1 & -3 \\ -3 & 7 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 8$.

$$(A - 8I)\mathbf{x} = \mathbf{0} \Rightarrow \left[\begin{array}{cc|c} -9 & -3 & 0 \\ -3 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

+2 +2 +2

$$\mathbf{x} = x_2 \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$ +2

A basis for the eigenspace corresponding to $\lambda = 8$ is $\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. +2

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Find the characteristic polynomial and the eigenvalues of the matrix $A = \begin{bmatrix} 8 & 2 \\ 2 & 8 \end{bmatrix}$.

$A - \lambda I = \begin{bmatrix} 8-\lambda & 2 \\ 2 & 8-\lambda \end{bmatrix}$ +2

$\det(A - \lambda I) = (8-\lambda)^2 - 4 = \lambda^2 - 16\lambda + 60 = (\lambda - 6)(\lambda - 10)$

+2 +2 +1 +1

$\det(A - \lambda I) = 0 \Rightarrow \lambda = 6, \quad \lambda = 10$

+1 +1

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Diagonalize the matrix $A = \begin{bmatrix} 3 & 3 & -6 \\ -3 & 13 & -18 \\ -1 & 3 & -2 \end{bmatrix}$ with eigenvalues $\lambda = 4$ and $\lambda = 6$.

$\underline{\lambda = 4}$:

$$A - 4I = \begin{bmatrix} -1 & 3 & -6 \\ -3 & 9 & -18 \\ -1 & 3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

+1 +1 +1

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}$$

+1 +1

$$\{\mathbf{v_1}, \mathbf{v_2}\} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix} \right\}$$ +1

$\underline{\lambda = 6}$:

$$A - 6I = \begin{bmatrix} -3 & 3 & -6 \\ -3 & 7 & -18 \\ -1 & 3 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

+1 +1 +1

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$ +1

$$\mathbf{v_3} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$$ +1

$$P = \begin{bmatrix} 3 & -6 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$ +2

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$ +2

Determine if the vectors $\begin{bmatrix} 7 \\ -5 \\ 3 \\ 7 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 7 \\ -7 \\ 9 \end{bmatrix}$, and $\begin{bmatrix} 14 \\ 46 \\ 44 \\ 0 \end{bmatrix}$ are orthogonal.

Since $\begin{bmatrix} 7 \\ -5 \\ 3 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 7 \\ -7 \\ 9 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 3 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 14 \\ 46 \\ 44 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -7 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 14 \\ 46 \\ 44 \\ 0 \end{bmatrix} = 0$, the set is orthogonal.

+3 +3 +3 +1

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Normalize the orthogonal set of vectors $\begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$ in order to produce an orthonormal set.

Let $\mathbf{u} = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$

$\mathbf{u} \cdot \mathbf{u} = \frac{2}{3}$ +3

$\mathbf{v} \cdot \mathbf{v} = \frac{1}{2}$ +3

$\|\mathbf{u}\| = \frac{\sqrt{6}}{3}$ +3

$\|\mathbf{v}\| = \frac{\sqrt{2}}{2}$ +3

So $\left\{ \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$ is an orthonormal set. +3

The given set is a basis for a subspace W. Use the Gram-Schmidt process to produce an orthogonal basis for W.

$$\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

Let $\mathbf{x_1} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ and $\mathbf{x_2} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$.

Set $\mathbf{v_1} = \mathbf{x_1}$. +3

$$\mathbf{v_2} = \mathbf{x_2} - \frac{\mathbf{x_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} \quad +3$$

$$= \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} - \frac{21}{14} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \quad +3$$

$$= \begin{bmatrix} 3 \\ \frac{5}{2} \\ \frac{3}{2} \end{bmatrix} \quad +3$$

So $\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ \frac{5}{2} \\ \frac{3}{2} \end{bmatrix} \right\}$ is an orthogonal basis for W. +3