4-2
1

ex: Determine if $\vec{w} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$ is in Nul A, where

$$A = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix}.$$

$$A\vec{w} = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore $\vec{w}$ is in Nul A.

4-2
2

ex: Find an explicit description of Nul A by listing vectors that span the null space.

$$A = \begin{bmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & -6 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6x_2 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

spanning set for Nul A is $\left\{ \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

4-2

3

Ex: Find an explicit description of Nul A by listing vectors that span the null space.

$$A = \begin{bmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccccc|c} 1 & 5 & -4 & -3 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 6 & -8 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -6x_3 + 8x_4 - x_5 \\ 2x_3 - x_4 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

spanning set for Nul A is $\left\{ \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Ex: Show that the given set, W, is a vector space, or find a specific example to the contrary.

$$\left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} \middle| \; 5r-1 = s+2t \right\}$$

$$r = \frac{1}{5}s + \frac{2}{5}t + \frac{1}{5}$$

$$s = 5r - 2t - 1$$

$$t = \frac{5}{2}r - \frac{1}{2}s - \frac{1}{2}$$

$$\begin{bmatrix} \frac{1}{5}s + \frac{2}{5}t + \frac{1}{5} \\ 5r - 2t - 1 \\ \frac{5}{2}r - \frac{1}{2}s - \frac{1}{2} \end{bmatrix} = r\begin{bmatrix} 0 \\ 5 \\ \frac{5}{2} \end{bmatrix} + s\begin{bmatrix} \frac{1}{5} \\ 0 \\ -\frac{1}{2} \end{bmatrix} + t\begin{bmatrix} \frac{2}{5} \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{5} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$

W is not a vector space since it doesn't contain the zero vector.

Ex: Show that the given set, W, is a vector space, or find a specific example to the contrary.

$$\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| \begin{array}{l} a+3b=c \\ \\ b+c+a=d \end{array} \right\}$$

W is the set of all solutions to the homogeneous system $a+3b-c=0$, $a+b+c-d=0$. So

$W = \mathrm{Nul}\, A$ where $A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

Thus W is a subspace of $\mathbb{R}^4$ and hence a vector space.

4-3

6

Ex: Show that the given set, W, is a vector space, or find a specific example to the contrary.

$$\left\{ \begin{bmatrix} b-5d \\ 2b \\ 2d+1 \\ d \end{bmatrix} \;\middle|\; b, d \text{ are real} \right\}$$

Since a vector in W may be written as

$$\vec{w} = b\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

we conclude that the zero vector is not in W

and therefore W is not a vector space.

4-2

7

Ex: Show that the given set, W, is a vector space, or find a specific example to the contrary.

$$\left\{ \begin{bmatrix} -a+2b \\ a-2b \\ 3a-6b \end{bmatrix} \middle| \; a, b \text{ real} \right\}$$

Since a vector in W may be written as

$$\vec{w} = a\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 2 \\ -2 \\ -6 \end{bmatrix},$$

we conclude that $W = \text{Col}\, A$ where

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \\ 3 & -6 \end{bmatrix}.$$

Hence W is a subspace of $\mathbb{R}^3$ and therefore a vector space.

4-2
8

Ex: Find A such that the given set is Col A.

$$\left\{ \begin{bmatrix} b-c \\ 2b+c+d \\ 5c-4d \\ d \end{bmatrix} \;\middle|\; b, c, d \text{ real} \right\}$$

Since an arbitrary vector in the given set may be written as

$$b\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix},$$

we conclude that $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}$.

4-2
9

Ex1: If $A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 0 & -5 \\ 0 & -5 & 7 \\ -5 & 7 & -2 \end{bmatrix}$, find K such that

Nul A is a subspace of $\mathbb{R}^K$, and find K such that

Col A is a subspace of $\mathbb{R}^K$.

The matrix A is a 4×3 matrix.

Nul A is a subspace of $\mathbb{R}^K$ if $K=3$.

Col A is a subspace of $\mathbb{R}^K$ if $K=4$.

4-5

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Ex: If $A = [1 \ -3 \ 9 \ 0 \ -5]$, find K such that

Nul A is a subspace of $\mathbb{R}^K$, and find K such that

Col A is a subspace of $\mathbb{R}^K$.

The matrix A is a 1×5 matrix.

Nul A is a subspace of $\mathbb{R}^K$ if $K=5$.

Col A is a subspace of $\mathbb{R}^K$ if $K=1$.

Ex: With $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$, find a nonzero vector in Nul A and a nonzero vector in Col A.

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & 0 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -7 & 6 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7x_3 - 6x_4 \\ -4x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} \quad \text{set } x_3 = x_4 = 1$$

Any column of A is a nonzero vector in Col A.

4-2

12

__ex:__ Let $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. Determine if

$\vec{w}$ is in Col A and if $\vec{w}$ is in Nul A.

$$\left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -\frac{1}{2} \\ 6 & 4 & 8 & 1 \\ -8 & -2 & -9 & 2 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -\frac{1}{2} \\ 0 & 4 & 2 & 4 \\ 0 & -2 & -1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

the system is consistent and $\vec{w}$ is in Col A.

$$\begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

therefore $\vec{w}$ is in Nul A.